Lesson 1: Introduction to Algebra

Waterloo Math Team Youth Outreach

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Note: Make sure that you are comfortable with all the common operations and have a solid understanding of negative numbers.

Summary

- Any noun (person, place, thing, or idea) can be categorized as an *object*.
- In the physical world, it is rare to find two objects that are exactly the same; instead, we think of objects as functionally equivalent in some quantitative way.
  - e.g. It does not matter if we count with our fingers starting from the thumb or the pinky; we get to the same number. In this case, each of our fingers are the same as the others, and we are using each of them to represent 1. In this case, all of our fingers (physical entities) and the number 1 (a mathematical concept) are essentially equivalent to one another!
- Mathematicians use " $=$ " to show that two objects are functionally equivalent, or e*qual*. These objects are on both sides of an *equation*.
- When two objects are equal, they are still equal after performing the same *operation* (what mathematicians call structured procedures) to both of them.
  - e.g. If I have two identical apples then slice, dice, and mash them in the same way, I should theoretically get two identical servings of applesauce. I am performing the same operation (the preparations for the apples) on two equivalent objects (the identical apples), so I should get the same object back out.
  - e.g. If Alice and Bob have the same number of toys and I give them each one more, they still have the same number of toys. In this case, the objects are the numbers describing the amount of physical objects (toys), and the operation is me giving each of them another toy.
    - Note: even though Alice and Bob's piles of toys have the same number of toys inside (ergo, the amount of toys are equal), I do not know if the toys themselves are equivalent (marbles vs. puzzles), so I cannot say that the two piles of toys are equal.
  - e.g. In the finger example, the idea of "quantity of fingers held up" is equal to some number; if we hold up one more finger, we should also add 1 to our number.
- *Algebra* seeks to describe how operations affect mathematical objects.
- The most common operations used in algebra are addition ($+$) and multiplication ($\times$). (Note that subtraction and division are simply forms of addition and multiplication!)

- Often, to solve problems, we use *variables* to take the place of mathematical objects; this is helpful when the mathematical object being replaced refers to another unknown value.
  - e.g. Can you immediately think of the number that is 6 smaller than its triple? No worries if you can't! Let x be this number; since we are told that this number is equal to its triple minus 6:

$$\begin{aligned}
x &= 3x - 6 && \text{[Set the two equal.]}\\
(x) - x &= (3x - 6) - x && \text{[*Subtract } x \text{ from both sides.]}\\
0 &= 2x - 6 && \text{[Simplify into equivalent expression.]}\\
(0) + 6 &= (2x - 6) + 6 && \text{[Add 6 to both sides.]}\\
6 &= 2x && \text{[Simplify into equivalent expression.]}\\
6\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right) && \text{[Divide 2 from both sides.]}\\
3 &= x && \text{[Simplify into equivalent expression.]}
\end{aligned}$$

  - * Yes, we can subtract variables if they describe the same thing! I can say that one finger plus two fingers is three fingers, or that taking away three apples from four leaves me with one apple. In the same way, we had three x's, and then we simply took away one of them.
  - From this, we see that $x = 3$, and since x was describing the number that we wanted to find, we now know that 3 is 6 smaller than its triple!
  - We converted a word problem into an equation (and replacing values we do not know with variables), then conducted identical operations to both sides until we finally get the equality expression we need (in this case, something in the form $x =?$).

- A hint for some of the word problems: equations can be thought of as objects too! In fact, this is a huge part of computer science. (The following example might be strange, but bear with me.)
  - Let's say we know that $a + b = c + d$, and we also know that $a = d$. Notice that since a and d are equal, subtracting a is the same as subtracting d. Subtract a from the left and d from the right to get that $b = c$! Another way to see this is that you are stacking the two equations, then just carrying out a "long subtraction" problem:

$$\begin{aligned}
a + b &= c + d\\
a &= d\\
---- &- ----\\
(a + b) - a &= (c + d) - d\\
b &= c
\end{aligned}$$

Problems

1. Find the value of the variable in the following equations. Try to work it out step by step!

   (a) $x = 5 - 7$

   (b) $k + 18 = 25$ (Yes, a variable doesn't have to be x! I will talk about conventions for variable naming in another lesson.)

   (c) $22 + 3k = 16$

   (d) $-\frac{5}{a} = 10$

   (e) $aa = 16$ (There are actually two answers to this; try to find both! Note that I could have also written this as $a^2 = 16$. The superscript 2, the *exponent*, means you multiply two of the *base* together. If I had 3^5, that's the same as $3 \times 3 \times 3 \times 3 \times 3$.)

   (f) $2b + 7 = 3b - 2$

   (g) $x(3) - 5 = 3x + 6x - 5x + 7$

   (h) $\frac{n+1}{n} = -3$

   (i) $x^2 = x$ (Careful! There are two answers. HINT: You cannot just divide out a value unless you know either it cannot be 0, or until you already took the answer 0 into account. This is because in mathematics, dividing by 0 is not defined!)

   (j) $5b = 7a - 2$, solve for a. (Your expression for a will contain b, but remember b can just be treated as a value! We call this "solving for a in terms of b.)

   (Bonus Questions)

   - $((3x - 7)4 + 2)5 = 3$ (Remember order of operations! You also need to know distributive property, which is touched on in question 3(b).)
   - $10x - 6 = 2(5x - 2) - 2$ (You will get an interesting result! Just how many answers are there?)
   - $10x - 7 = 2(5x - 2) - 2$ (Another interesting result!)
   - $(x - 6)(x + 2)(x - 3) = 0$ (Three answers!)
   - $(x - 5)^2 = 49$ (Two answers!)
   - $2^{x+3} = 16$ (There is a note about what this means in a later question if you have never seen this before.)

2. The following problems will be word-based. Set up the appropriate equations yourself!

   (a) Janette spent \$42 for shoes. This was \$14 less than twice what she spent for a blouse. How much was the blouse?

   (b) The sum of two numbers is 84, and one of them is 12 more than the other. What are the two numbers? (Tip: if n is the smaller number, what would the bigger one be? What should the two add up to?)

   (c) Divide \$80 among three people so that the second will have twice as much as the first, and the third will have \$5 less than the second.

   (d) Two small pitchers and one large pitcher can hold 8 cups of water. One large pitcher holds 2 cups of water more than a small pitcher. How many cups of water can each pitcher hold? (Try "replacing" all of one type of pitcher with an equivalent amount of the other plus some amount of cups of water. Note that one large pitcher holds as much as a small pitcher plus two cups of water.)

(e) A test worth 100 points has twenty questions. The test consists of True/False questions worth 3 points each and multiple choice questions worth 11 points each. How many multiple choice questions are on the test? (You have enough information to make two equations! Play around with them until you can solve for the number of one type of test question, then use that to find the number of the other type.)

(f) A mental math trick I learned from one of my math team coaches: if you want to multiply a two-digit number that ends with 5 by itself (so we would find 25^2 if the number is 25), take the first digit, multiply it by itself plus 1, then stick 25 to the end of it (25 starts with 2, 2 times 3 is 6, then sticking 25 on the end we get 625 which is equal to 25^2). Figure out which number I started with if I ended up with 3025. (This question is easy, but it illustrates how you can get answers by trying to work backwards.)

- (Bonus Question) Here is another mental math trick: if a number is in the fifties (50 to 59), you can find is square (that is, the result of multiplying it by itself) with the following method: first, add the units digit to 25. Then, take the value we get, and stick the two-digit form of the square of the units digit onto the end of that value.

  So as an example, 52 has 2 in its units digit; add that to 25 to get 27, then stick 04 (which is the two-digit form of $2^2 = 4$) on the end to get that $52^2 = 2704$.

  This also works in a similar way for numbers slightly below 50 or above 59. With this information, what is the square root (the number whose square is equal to the given number) of 2116? Of 3721? Tip: when we use this squaring trick on a value from 50 to 59, the first step can be seen as a shortcut for finding the distance from that value from 50 and adding it to 25, while the second step is a shortcut for multiplying the value from step by 100 then adding the square of the distance from 50 we already found.

3. These questions will test your understanding. Take some time to think!

   (a) Let $x = 2$. So $(x-2)(3) = (x-2)(4)$, since $x - 2 = 0$ (both sides equal 0). Now, divide both sides by $x - 2$. We're left with $3 = 4$, which is mathematical nonsense! What went wrong? (What are we really dividing by?)

   (b) Imagine that there are many bags containing five apples. If I take two bags of apples one day and three bags the next, how many apples should I have in total? Is this different from me buying five bags of apples at once? With this logic in mind, try to explain what $(a + b) \times c$ would be equal to. What about $(a + b)(c + d)$?

Interesting Problem of the Week Four people come to a bridge at night, but it can only hold two people at once. They have one torch and the torch has to be used when crossing. Person A can cross the bridge in one minute, B in two minutes, C in five minutes, and D in eight minutes. When two people cross together, they must move at the slower person's pace. Can they all get across the bridge in fifteen minutes or less?