Lesson 2: Practice with Algebra and Relationships

Waterloo Math Team Youth Outreach

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Note: This is a continuation of Lesson 1. Refer to the summary in the Lesson 1 handout if you are still uncomfortable with any of the concepts we are using.

Summary

- With what we learned last class, we now know that *variables* can be used as stand-ins to represent quantitative traits of certain *objects*.
- An *equation* describes how two objects are the same, and can help us to find information about one object if we know something about the other.
- However, sometimes there are multiple objects to take into consideration; in these cases, we do not know anything specific about either object, but can figure out the *relationship* between them.
  - e.g. If Tony has 20 apples and I have none, no matter how many apples I steal from him there will still be 20 in total. You might not know the exact number either of us have, but you do know they must add up to 20. Therefore, if you saw how many apples Tony has left you will also know how many apples I have.
  - e.g. If I know that ten stones weighs as much as a boulder, I would be able to express weight of any pile of stones in terms of boulders and vice versa. For example, three boulders is the same weight as thirty stones, and fifteen stones is the same weight as one and a half boulders.
  - e.g. If I know the area of a rectangular field is 100 m^2, no matter what the lengths of the length and width are, they must multiply to 100 m^2. If you measure one side, you would always be able to determine the other side.
- If we have a relationship between two variables, we are able to replace one variable with the other in other relationships for the sake of solving problems.
  - e.g. In the apple example, if Tony gets 10 points for every apple he stops me from stealing and I get 5 points for every apple I steal, then I tell you that the total number of points we have is 165, you can figure out two relationships:

$$\begin{aligned} x + y &= 20 && \text{[This is about the number of apples.]} \\ 10x + 5y &= 165 && \text{[This is about the number of points.]} \end{aligned}$$

(where x and y are the number of apples Tony and I have, respectively.)

If I were to then note that $y = 20 - x$ and replace y with $20 - x$ in the second equation:

$$\begin{aligned}
10x + 5(20 - x) &= 165 \\
10x + 100 - 5x &= 165 \\
5x &= 65 \\
x &= 13 \\
\implies y &= 20 - 13 \\
&= 7
\end{aligned}$$

So Tony has $x = 13$ apples and I have $y = 7$ apples.

(If you don't understand the math behind this, look back at Lesson 1's summary again to grasp how we can replace one object with another if they are "functionally equivalent".)

- e.g. In the rectangular field example, let's assume we know the total fencing used to surround the field (i.e. the perimeter of the field) is 58 m. Now we should have two relationships once again:

$$\begin{aligned}
2(x + y) = 2x + 2y &= 58 && \text{[This describes the perimeter.]} \\
xy &= 100 && \text{[This describes the area.]}
\end{aligned}$$

(where x and y are the length and width of the rectangular field.)

There are two ways to *substitute* the equations into one another; I will substitute (or "plug") the second equation into the first after a bit of manipulation:

$$\begin{aligned}
xy &= 100 \\
y &= \frac{100}{x} \\
\implies 2x + 2(\frac{100}{x}) &= 58 \\
2x + \frac{200}{x} &= 58 \\
2x^2 + 200 &= 58x \\
2x^2 - 58x + 200 &= 0 \\
x^2 - 29x + 100 &= 0 \\
(x - 4)(x - 25) &= 0 && \text{[How do we get this?]}
\end{aligned}$$

(The last step requires a technique called factoring, which we have not touched upon. Remember the $(a + b)(c + d)$ problem on last week's homework? You can use that information to find the result of $(x - r)(x - s)$.)

The last line is satisfied for both $x = 4$ and for $x = 25$. These values correspond to $y = 25$ and $y = 4$. Note that this shows the perimeter and area is not changed depending on which we call the length or the width.

Problems

1. You take a deck of 52 cards (no jokers) and split it into two piles of 26.
   (a) If there are 15 red cards in the first pile, how many black cards are in the second pile?
   (b) Show that the number of red cards in the first pile is always equal to the number of black cards in the second pile.
2. The units digit in a two-digit number is three times the tens digit. If the digits are reversed, the resulting number is 54 more than the original number. Find the original number.
3. Here are some rectangle problems! Remember that area is length times width!
   (a) A rectangle has length twice its width, and area 72 m^2. What is its width?
   (b) A rectangle has perimeter 30cm, and area 56 cm^2. What is its length and width?
4. You start with a number, multiply it by 3, subtract 9, then divide what you get by 3. What number do you have to add now, if you want to get your original number back?
5. Five consecutive even numbers add up to 100. What is the largest of these numbers?
6. Pipe A can fill a pool in 5 hours, while Pipe B can fill it in four. How long will it take for the two to fill the pool if both are operating at the same time? (Assume that the rate at which water flows out of each pipe is constant.)
7. A canoeist paddled upstream for 2 hours, then downstream for 3. The rate of the current was 2 km/h. When she stopped, the canoeist realized she was 20 km downstream from her starting point. How many hours will it take her to paddle back to her starting point? Assume every time she paddles, she has the same speed.
8. Every day, you wear either your red shirt, blue shirt, or yellow shirt. However, you also have n different colours of ties, and every day, you choose a different tie to wear. If you have 24 different possibilities for how to dress, what is n? (This can be solved with a concept we will be learning next lecture, but you can still work it out!)

(Bonus Question)

- On a one-lane street, a get-away car is stuck between two police cruisers approaching it from opposite ends, with one traveling at 40 m/s and one traveling at 30 m/s. The cars start off 560 m apart. The car starts at one of the police cruisers and runs toward the other one. It turns around instantly when it hits the other vehicle (because he has "mad skills"). It repeatedly drives between the cruisers, turning around when it hits one of them, until the two cruisers trap the car to arrest the criminals inside. If the car drives at 60 m/s, what is the total distance it drives before finally stopping?

Interesting Problem of the Week Three guests decide to stay the night at a lodge whose rate they are initially told is \$30 per night. However, after the guests have each paid \$10 and gone to their room, the proprietor discovers that the correct rate should actually be \$25. As a result, he gives the bellboy the \$5 that was overpaid, together with instructions to return it to the guests. Upon consideration of the fact that \$5 will be problematic to split three ways, the bellboy decides to pocket \$2 and return \$1 each, or a total of \$3, to the guests. Upon doing so, the guests have now each paid a total of \$9 for the room, for a total of \$27, and the bellboy has retained \$2. So where has the remaining \$1 from the initial \$30 paid by the guests gone?!