STAT 460: Assignment 6

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For this homework assignment, please complete the three exercises below. These exercises will require you to write Markov chain Monte Carlo algorithms. You may use the sample code from lecture slides, previous homework solutions, or BDA3 as a guide, but you should not simply take code from the internet or rely on R packages (or similar for other languages) for implementing the algorithms.

Please also note that this homework assignment is worth 15 points, not 10 points as for previous assignments, and will thus be weighted slightly more when computing your final grade.

For this assignment, we will consider the simplified model for high-energy spectral analysis in astrophysics that was presented during lecture (see the slides for Bayesian Computing: Part 3). The data are photon counts in equally spaced bins, and we will infer the parameters of a simple power-law model for the observed data using Markov chain Monte Carlo techniques for model fitting.

The spectral data set in `spectral-1.txt` is a simulated high-energy spectrum and consist of photon counts from a number of energy bins. The first column reports the mean energy (in keV) of each bin and the second bin the photon count. I refer to the mean energy in bin i as E_i, and the count in bin i as Y_i for $i = 1, ..., n$. The energy range (in keV) for the simulated spectra is $[1.905, 10.730]$.

We model the photon counts as independent Poisson random variables, $Y_i \sim \text{Poisson}(\Lambda_i)$, where Λ_i is the expected photon count in bin i and is constrained according to a parameterized model:

$$\Lambda_i(\theta) = \alpha E_i^{-\beta},$$

where $\theta = (\alpha, \beta)$ is a model parameter of direct scientific interest with $\alpha > 0$ and $\beta > 0$. For a prior distribution, we specify $p(\alpha, \beta) \propto 1$. (Although this prior distribution is improper, the posterior will be a proper distribution.)

Note that you do not need to understand the astrophysics jargon to complete this assignment; it is provided only for additional context. Also note that the *true values for α and β may not be the same as those used for the simulation in the lecture notes.* This makes the exercise more realistic as you would not know the "true" values in the real world!

Exercise 1 (7 points): Construct a random-walk Metropolis algorithm (not Metropolis-Hastings!) with a symmetric bivariate jumping distribution to sample from the joint posterior distribution of α and β. Use $m = 5$ chains with overdispersed starting points and use the Gelman-Rubin $\hat{R}$ statistic to check convergence. Run the chains until you have a total effective sample size $n_{\text{eff}} > 200$ for each parameter. (This may take some trial/error!)

For the overdispersed starting values, you can draw the starting values for α and β from a Uniform$(0, 100)$ distribution.

Please turn in the following:

1. Trace plots for α and β after burn in, with the five chains plotted for each using different colors.
2. The values of $\hat{R}$ using your 5 chains for each of α and β, and a brief explanation as to why you believe the chains have converged.
3. Your values of n_{eff} corresponding to α and β.
4. The two-dimensional scatterplot of the draws for (α, β).

5. Estimates of the posterior means and equal-tailed 95% posterior intervals for α and β.
6. A description, in words, of your Metropolis algorithm (as was done for Assignment 5).
7. A brief discussion about your process for constructing the Metropolis algorithm and obtaining a total $n_{\text{eff}} > 200$ for each parameter. For example, how did you choose a specific jumping distribution? Did you start with an initial sampler prior to tune your algorithm? Was it difficult to obtain $n_{\text{eff}} > 200$? Etc.

Exercise 2 (7 points): Redo Exercise 1 using a single-site Metropolis algorithm. Turn in parts (1) through (7) above for this new algorithm.

Exercise 3 (1 points): Which algorithm would you prefer to sample from $p(\theta|Y)$, and why? (Note that your inferences should be very similar for both algorithms, or else there is a mistake.)