Algebraic Topology from a Homotopical Viewpoint

Marcelo Aguilar
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To my parents
To Danny

To Viola
To Sebastian and Adrian

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Preface

This book introduces the basic concepts of algebraic topology using homotopy-theoretical methods. We believe that this approach allows us to cover the material more efficiently than the more usual method using homological algebra. After an introduction to the basic concepts of homotopy theory, using homotopy groups, quasifibrations, and infinite symmetric products, we define homology groups. Furthermore, with the same tools, Eilenberg–Mac Lane spaces are constructed. These, in turn, are used to define the ordinary cohomology groups. In order to facilitate the computation, cellular homology and cohomology are defined.

In the second half of the book, vector bundles are presented and then used to define K-theory. We prove the classification theorems for vector bundles, which provide a homotopy approach to K-theory. Later on, K-theory is used to solve the Hopf invariant problem and to analyze the existence of multiplicative structures in spheres. The relationship between cohomology and vector bundles is established introducing characteristic classes and related topics. To finish the book, we unify the presentation of cohomology and K-theory by proving the Brown representation theorem and giving a short account of spectra.

In two appendices at the end of the book the proof of the Dold–Thom theorem on quasifibrations and infinite symmetric products is given in detail, and a new proof of the complex Bott periodicity theorem, using quasifibrations, is presented.

It is expected that the reader has a basic knowledge of general topology and algebra. In any case, the book is mainly aimed at advanced undergraduates and at graduate students and researchers for whose work algebraic-topological concepts are needed.

This text originated in a preliminary version in Spanish, which was a joint edition of the Mathematics Institute of the National University of Mexico and McGraw-Hill Interamericana Editores. To both institutions the authors are grateful. The translation of the main body of the text was the excellent job of Stephen Bruce Sontz, to whom we express our deep thanks. Our gratitude goes also to Springer-Verlag, particularly to Ms. Ina Lindemann for her interest in our work, and to the referees for their valuable comments which certainly helped to improve the English version of the book. Its title is, of course, a tribute to John Milnor, from whose books and papers we have learnt many important concepts, which are included in this text.

Last, but not least, we wish to acknowledge the support of Professor Albrecht Dold, who after reading the Spanish manuscript gave various important comments to make some parts better.

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Autumn 2001

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Contents

Introduction

The fundamental idea of algebraic topology is to associate to each topological space X a group $h(X)$ and to each map $f : X \longrightarrow Y$ a homomorphism $h(f) : h(X) \longrightarrow h(Y)$ with the property that whenever X and Y are homotopy equivalent (in particular, if they are homeomorphic), then $h(X)$ is isomorphic to $h(Y)$. In other words, we consider functors h (both covariant and contravariant) from the category of (pointed) topological spaces to the category of (abelian) groups such that $h(f) = h(g)$ if the maps $f, g : X \longrightarrow Y$ are homotopic. The easiest way to construct such a covariant functor is to consider a fixed space X_0 and then to define the functor (on objects) by $h(Y) = [X_0, Y]$, where the brackets denote the set of (pointed) homotopy classes of maps from X_0 to Y. Similarly, we define such a contravariant functor by considering a fixed space Y_0 and setting $h(X) = [X, Y_0]$. In order to have a group structure on these sets of homotopy classes the spaces X_0 and Y_0 must have certain properties (see Sections 2.7 and 2.9), which are satisfied if $X_0 = \mathbb{S}^n$ or if Y_0 is an H-group. When $X_0 = \mathbb{S}^n$ we obtain the homotopy groups $\pi_n(Y) = [\mathbb{S}^n, Y]$. However the homotopy groups of an arbitrary space Y are extremely difficult to calculate due to the fact that they do not satisfy the excision axiom (see statement 5.3.15 and Section 6.2). But one could try to associate to Y another space whose homotopy groups are easier to calculate. It is known (see 6.4.15) that a topological abelian monoid has a simple homotopical structure. So we associate to Y the free topological abelian monoid generated by its points (with the base point of Y acting as the zero element). This monoid is the same as the infinite symmetric product $\mathrm{SP}\,Y$. Furthermore, since a topological abelian monoid is completely characterized by its homotopy groups (see 6.4.16), we are led to associate to Y the groups $H_n(Y) = \pi_n(\mathrm{SP}\,Y)$. These groups turn out to satisfy the excision axiom and thus are easier to calculate. Similarly, when we study the contravariant functors $[-, Y_0]$ with Y_0 an H-group, we shall consider spaces Y_0 with a simple homotopical structure, namely spaces $K(\mathbb{Z}, n)$ with only one nonzero homotopy group, which is $\mathbb{Z}$ in dimension n. These are called Eilenberg–Mac Lane spaces. To construct these spaces we shall also use a suitable symmetric product. Then we set $H^n(X) = [X, K(\mathbb{Z}, n)]$.

The purpose of this book is to introduce algebraic topology from the homotopical point of view. The basic concepts of homotopy theory, such as fibrations and cofibrations, are used to construct singular homology and cohomology, as well as K-theory.

In particular, the presentation of homology, using the homotopy groups of an infinite symmetric product, is nowadays adequate for the purposes of algebraic geometry, specifically for the definition of the Lawson homology theory (see [42, 43]). On the other hand, Voevodsky [79] and others, using the homotopical point of view of this book, translated many concepts of algebraic topology into algebraic geometry. This is the foundation for Voevodsky's proof of the Milnor conjecture, concerning a certain relationship between Milnor's K-theory groups of a field F and the Galois cohomology groups of F. More specifically, Voevodsky constructed a stable homotopy category of *schemes* in algebraic geometry, analogous to the stable homotopy category in algebraic topology. He defines spectra and the associated cohomology and homology theories. To construct the Eilenberg–Mac Lane spectrum he uses a suitable analogue of the symmetric products. He also constructed spectra for K-theory and cobordism in this setting.

A leitmotif of this book is to pursue the proof of one of the most remarkable results of algebraic topology: J. Frank Adams' theorem solving the Hopf invariant problem, implying that the only spheres that admit a multiplicative structure, converting them into H-spaces, are precisely $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$ or, equivalently, that the only real division algebras are the reals, the complex numbers, the quaternions, and the Cayley numbers. Throughout the text many other fundamental concepts are introduced, including the construction of the characteristic classes of vector bundles, to which a full chapter is devoted.

The book is adequate for use in a two-semester course, either at the end of an undergraduate program or at the graduate level. In order to understand its contents, a basic knowledge of point set topology as well as group theory is required. Although functors appear constantly throughout the text, no knowledge about category theory is expected from the reader; on the contrary, every time categorical or functorial properties appear, the categorical ideas are stressed in order to obtain the functorial properties of the introduced invariants.

The design of the text is as follows. We start with a chapter devoted to basic concepts and notation, followed by twelve substantial chapters, each of which is divided into several sections that are distinguished by their double numbering (1.1, 1.2, 2.1, ...). Definitions, propositions, theorems, remarks, formulas, exercises, etc., are designated with triple numbering (1.1.1, 1.1.2, …). Exercises are an important part of the text, since many of them are intended to carry the reader further along the lines already developed in order to prove results that are either important by themselves or relevant for future topics. Most of these are numbered, but occasionally they are identified inside the text by italics (*exercise*). On the other hand, two important theorems, whose proof somehow goes beyond the horizons of this book (the Dold–Thom theorem on quasifibrations and infinite symmetric products and the complex Bott periodicity theorem) are proved in two appendices. In the appropriate chapters these results are then freely used after some explanation to let the reader understand the scope and meaning of the results and to give their applications.

The chapter on basic concepts and notation, as its name suggests, presents most of the notation used throughout the text as well as some concepts that are not necessarily standard in the regular basic courses on point set topology or algebra.

Chapter 1 deals with the elements of the topology of function spaces, emphasizing the compact-open topology, and discusses the exponential law. Chapter 2 introduces the basic notions of homotopy theory, such as path connectedness and homotopy of maps. The former is, in a way, the basic concept on which all ideas in the book are built. We study the degree of maps of the circle into itself, and introduce the fundamental group. Finally, we define the concepts of topological groups and H-spaces, and the dual concept of H-cospace. As examples of H-spaces and H-cospaces, loop spaces and suspensions are carefully studied.

Chapter 3 contains a study of homotopy groups including the proof of the Seifert–van Kampen theorem. Special emphasis is put on the long exact sequences of homotopy groups. Then in Chapter 4, homotopy extension and lifting properties are analyzed, particularly the concepts of cofibration and fibration.

In order to prepare for the study of cohomology groups, CW-complexes are introduced in Chapter 5, and their homotopy properties are analyzed. The concepts of quasifibrations and infinite symmetric products are also reviewed. These are used to introduce the homology groups. Further homotopy topics are studied in Chapter 6, among which is the proof of the Blakers–Massey homotopy excision theorem. This is an invaluable tool in the study of homotopy aspects of the Moore and the Eilenberg–Mac Lane spaces.

Cohomology groups are introduced in Chapter 7, and their multiplicative structure is defined. After cellular homology and cohomology are introduced, some specific groups are computed. Further on in the same chapter we construct the exact sequences of Künneth, of universal coefficients, and of Mayer–Vietoris among others. Later on, in Chapter 8, vector bundles are studied in detail, building up to their classification. For that purpose, Grassmann manifolds and universal vector bundles over them are defined, and some classification results are proved.

Complex K-theory is introduced in Chapter 9 starting from complex vector bundles. Using their classification, various theorems are proved, which allow us to realize the K-theory of a space as a set of homotopy classes of mappings from the space into a classifying space, much in the same spirit as the cohomology groups were defined earlier. In order to exploit K-theory as much as possible the Bott periodicity theorem in the complex case is presented, but not yet proved. Later on, in Chapter 10, the Adams operations in complex K-theory are introduced to solve the Hopf invariant problem and thereby to study the existence of the structures of normed and division algebras in $\mathbb{R}^n$ as well as to prove Adams' theorem on multiplicative structures on the spheres $\mathbb{S}^{n-1}$.

In Chapter 11 the relationship between line bundles and cohomology is given, using the fact that the classifying spaces of real and complex line bundles, namely $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$, are Eilenberg–Mac Lane spaces. A simple proof of the existence of the Thom class of an oriented vector bundle and of Thom's isomorphism theorem is given to be used later on to define the Stiefel–Whitney classes of real vector bundles and the Chern classes of complex vector bundles. We finish the main part of the book with Chapter 12, where we present a short account of generalized cohomology and homology and prove the Brown representability theorem. Some remarks on the theory of spectra end the chapter.

The proof of the Dold–Thom theorem on quasifibrations and infinite symmetric products is postponed to Appendix A, and a topological proof of the complex Bott periodicity theorem is given in Appendix B. In the appendices the sections are doubly numbered (X.1, X.2, ...), and the items are triply numbered (X.1.1, X.1.2, ... where X is either A or B).

An effort was made to include a very complete alphabetical index; the reader should feel free to use it, even to look for simple concepts. A list of symbols containing much of the notation used in the book is also included.

Basic Concepts and Notation

In this section we present some of the basic concepts and notations that will be used in the text.

Basic Symbols

Throughout the text we shall use the following basic symbols, among others. The symbol $\approx$ between two topological spaces means that they are homeomorphic, $\simeq$ between continuous functions or topological spaces means that they are homotopic or homotopy equivalent, and $\cong$ between groups (abelian or nonabelian) means they are isomorphic. The symbol $\circ$ denotes composition of functions (maps, homomorphisms) and will be omitted ocasionally, if doing so does not lead to confusion. The term *map* invariably means a continuous function between topological spaces, and the term *function* is reserved either for functions between sets or for those maps whose codomain is $\mathbb{R}$ or $\mathbb{C}$.

And now a final note about some additional notation that will be used in the text. If X is a topological space and $A \subset X$, in agreement with the special cases mentioned below we shall use the notation $\overset{\circ}{A}$ to denote the topological interior of A in X, and the notation ∂A to denote its boundary. $X \sqcup Y$ denotes the topological sum of X and Y. On the other hand, if V is a vector space provided with a scalar product (or Hermitian product, if the space is complex), which we usually denote by $\langle -,- \rangle$, then we use the notation $\| \cdot \|$ or $| \cdot |$ to denote the norms in V associated to the inner product, that is, $\|x\|$ or $|x| = \sqrt{\langle x,x \rangle}$. Likewise, if $A \subset V$ is a subspace, we use $A^\perp = \{x \in V \mid \langle x,a \rangle = 0 \text{ for all } a \in A\}$ to denote the *orthogonal complement* of A in V with respect to the inner product.

Some Basic Topological Spaces

Euclidean spaces, various of its subspaces, and spaces derived from these will all play an important role for us.

$\mathbb{R}$ will represent the set (as well as the topological space and the real vector space) of *real numbers*. $\mathbb{R}^0$ will denote the singleton set (of only one point) $\{0\} \subset \mathbb{R}$. Frequently, we shall use the notation $*$ for an (arbitrary) singleton set. $\mathbb{R}^n$ will be the notation for *Euclidean space of dimension* n, or *Euclidean n-space*, such that

$$\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, \quad 1 \leq i \leq n\}\,.$$

Using the equality

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) = (x_1, \dots, x_m, y_1, \dots, y_n)$$

we identify the Cartesian product $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$. Likewise, we identify $\mathbb{R}^n$ with the closed subspace $\mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$. We give $\bigcup_{n=0}^{\infty} \mathbb{R}^n = \mathbb{R}^\infty$ the topology of the union (which is the colimit topology, as we shall see shortly). $\mathbb{R}^\infty$ consists, therefore, of infinite sequences of real numbers $(x_1, x_2, x_3, \dots)$ almost all of which are zero, that is to say, such that $x_k = 0$ for k sufficiently large. $\mathbb{R}^n$ is identified with the subspace of sequences $(x_1, \dots, x_n, 0, 0, \dots)$. The topology of $\mathbb{R}^\infty$ is such that a set $A \subset \mathbb{R}^\infty$ is closed if and only if $A \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$ for all n.

Topologically we identify the set (as well as the topological space and the complex vector space) $\mathbb{C}$ of *complex numbers* with $\mathbb{R}^2$ using the equality $x + \mathrm{i}y = (x, y)$, where i represents the imaginary unit, that is $\mathrm{i} = \sqrt{-1}$. Analogously with the real case, we have the *complex space of dimension* n, $\mathbb{C}^n = \{z = (z_1, \dots, z_n) \mid z_i \in \mathbb{C},\ 1 \leq i \leq n\}$, or *complex n-space*.

In $\mathbb{R}^n$ we define for every $x = (x_1, \dots, x_n)$ its *norm* by

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}\,;$$

likewise, in $\mathbb{C}^n$ we define the *norm* by

$$|z| = \sqrt{z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n}\,,$$

where $\bar{z}$ denotes the *complex conjugate* $x - \mathrm{i}y$ of $z = x + \mathrm{i}y$. Up to the natural identification between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, it is an *exercise* to show that the two norms coincide.

For $n \geq 0$ we shall use from now on the following subspaces of Euclidean space:

$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, the *unit disk* of dimension n.

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, the *unit sphere* of dimension $n - 1$.

$\overset{\circ}{\mathbb{D}}{}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$, the *unit cell* of dimension n.

$I^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\, 1 \leq i \leq n\}$, the *unit cube* of dimension n.

$\partial I^n = \{x \in I^n \mid x_i = 0$ or 1 for some $i\}$, the *boundary* of I^n in $\mathbb{R}^n$.

$I = I^1 = [0,1] \subset \mathbb{R}$, the *unit interval.*

Briefly, we usually call $\mathbb{D}^n$ the *unit n-disk*, $\mathbb{S}^{n-1}$ the *unit $(n-1)$-sphere*, $\overset{\circ}{\mathbb{D}}{}^n$ the *unit n-cell*, and I^n the *unit n-cube.* It is worth mentioning that all of the spaces just defined are connected (in fact, pathwise connected), except for $\mathbb{S}^0$ and ∂I, these being homeomorphic, of course. The disks, the spheres, the cubes, and their boundaries also are compact (but not the cells, except for the 0-cell $\overset{\circ}{\mathbb{D}}{}^0 = *$).

The *group of two elements* $\mathbb{Z}/2 = \mathbb{Z}_2 = \{-1, 1\}$ (which can also be seen as the quotient of the *group of the integers* $\mathbb{Z}$ modulo $2\mathbb{Z}$) acts on $\mathbb{S}^n$ by the antipodal action, that is, $(-1)x = -x \in \mathbb{S}^n$. The *orbit space* of the action, which is the result of identifying each $x \in \mathbb{S}^n$ with its antipode $-x$, is denoted by $\mathbb{RP}^n$ and is called *real projective space* of dimension n.

The *infinite-dimensional sphere* $\mathbb{S}^\infty = \bigcup_{n=0}^\infty \mathbb{S}^n$, where the inclusion $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ is defined by the inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, is a subspace of $\mathbb{R}^\infty$. The action of $\mathbb{Z}_2$ in $\mathbb{S}^n$ induces an action in $\mathbb{S}^\infty$, whose orbit space is denoted by $\mathbb{RP}^\infty$ and is called *infinite-dimensional real projective space.* In fact, the inclusion $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ induces an inclusion $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$ and the union $\bigcup_{n=0}^\infty \mathbb{RP}^n$ coincides topologically with $\mathbb{RP}^\infty$.

On the other hand, the *circle group* $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid \|\zeta\| = 1\}$ acts on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ by multiplication on each coordinate, namely, $\zeta(z_1, \ldots, z_{n+1}) = (\zeta z_1, \ldots, \zeta z_{n+1})$. The orbit space of this action, which is the result of identifying $z \in \mathbb{S}^{2n+1}$ with $\zeta z \in \mathbb{S}^{2n+1}$, for all $\zeta \in \mathbb{S}^1$, is denoted by $\mathbb{CP}^n$ and is called *complex projective space* of dimension n (in fact, its *real dimension* is $2n$). The action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ induces an action on $\mathbb{S}^\infty$, whose orbit space is denoted by $\mathbb{CP}^\infty$ and is called *infinite-dimensional complex projective space.* In analogy with the real case, the inclusion $\mathbb{S}^{2n-1} \subset \mathbb{S}^{2n+1}$, defined by the inclusion $\mathbb{C}^n \subset \mathbb{C}^{n+1}$, induces an inclusion $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ and the union $\bigcup_{n=0}^\infty \mathbb{CP}^n$ coincides topologically with $\mathbb{CP}^\infty$.

The *group of $n \times n$ invertible matrices* with real (complex) coefficients is denoted by $\mathrm{GL}_n(\mathbb{R})$ ($\mathrm{GL}_n(\mathbb{C})$) and consists of the matrices whose determinants are not zero. The subgroup $\mathrm{O}_n \subset \mathrm{GL}_n(\mathbb{R})$ ($\mathrm{U}_n \subset \mathrm{GL}_n(\mathbb{C})$) consisting of the *orthogonal matrices* (*unitary matrices*), that is, such that the matrix sends orthonormal bases to orthonormal bases with respect to the canonical scalar product in $\mathbb{R}^n$ (the canonical Hermitian product in $\mathbb{C}^n$) or, equivalently, such that its column vectors form an orthonormal basis, is called the *orthogonal group* (*unitary group*) of $n \times n$ matrices. In particular, $\mathrm{O}_1 = \mathbb{Z}_2$ and $\mathrm{U}_1 = \mathbb{S}^1$.

SOME GENERAL BASIC CONCEPTS

If $f : G \longrightarrow H$ is a homomorphism of groups, then $\ker(f) = \{g \in G \mid f(g) = 1\} \subset G$ represents the *kernel* of f and $\operatorname{im}(f) = \{f(g) \mid g \in G\} \subset H$ its *image*. An arrow of the form $\hookrightarrow$ represents an inclusion or an embedding of topological spaces, while one of the form $\rightarrowtail$ indicates a group monomorphism, and finally, one of the form $\twoheadrightarrow$ represents an epimorphism or, possibly, a surjective (quotient) map between topological spaces.

A sequence of homomorphisms (of groups, rings, modules, etc.)

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* at B if $\operatorname{im}(f) = \ker(g)$.

As we have already done in the case of $\mathbb{R}^n$ or $\mathbb{C}^n$ for defining $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$, we shall make frequent use of the general concept of *infinte union* or *colimit*. In the case of topological spaces let

$$X_1 \subset X_2 \subset X_3 \subset \cdots$$

be a chain of closed inclusions of topological spaces. We define its *union* $\bigcup_{i\geq 1} X_i$ as the union of the sets X_i, and we define its topology by declaring a subset $C \subset \bigcup_{i\geq 1} X_i$ to be closed if and only if its intersection $C \cap X_i$ is closed in X_i for all $i \geq 1$. This topology is called the *union topology*; frequently it is also called the *weak topology* with respect to the subspaces. It is an *exercise* to show that the union has the following *universal property*. If we have a family $\{f^i : X_i \longrightarrow Y \mid i \geq 0\}$ of continuous maps such that $f^{i+1}|X_i = f^i : X_i \longrightarrow Y$, then there exists a unique map $f : \bigcup X_i \longrightarrow Y$ such that $f|X_i = f^i : X_i \longrightarrow Y$. In a commutative diagram we write this as

$$\begin{tikzcd} X_i \arrow[rr, hook] \arrow[dr, "f^i"'] & & \bigcup_{i\geq 1} X_i \arrow[dl, dashed, "f"] \\ & Y. & \end{tikzcd}$$

It is an *exercise* to prove that the spaces $\mathbb{S}^\infty = \bigcup_{n=0}^\infty \mathbb{S}^n$, $\mathbb{RP}^\infty = \bigcup_{n=0}^\infty \mathbb{RP}^n$, $\mathbb{CP}^\infty = \bigcup_{n=0}^\infty \mathbb{CP}^n$ defined above have the union topology.

Limits and Colimits

In a slightly more general context, given a sequence of *closed embeddings*, that is, of maps that are homeomorphisms onto their range, which itself is closed,

$$X_1 \overset{j_2^1}{\hookrightarrow} X_2 \overset{j_3^2}{\hookrightarrow} X_3 \hookrightarrow \cdots ,$$

its *colimit* is a topological space denoted by $\operatorname{colim} X_i$, provided with maps $j^i : X_i \longrightarrow \operatorname{colim} X_i$ such that $j^k \circ j_k^i = j^i : X_i \longrightarrow \operatorname{colim} X_i$, where $j_k^i = j_k^{k-1} \circ \cdots \circ j_{i+1}^i : X_i \longrightarrow X_k$, $k > i$, and which has the following *universal property*. If $\{f^i : X_i \longrightarrow Y \mid i \geq 1\}$ is a family of maps such that $f^{i+1} \circ j_{i+1}^i = f^i : X_i \longrightarrow Y$ for all $i \geq 1$ or, equivalently, $f^k \circ j_k^i = f^i : X_i \longrightarrow Y$ for all $k > i \geq 1$, then there exists a unique map $f : \operatorname{colim} X_i \longrightarrow Y$ such that $f \circ j^i = f^i$. Diagrammatically this says

$$\begin{tikzcd} X_i \arrow[r, "j^i"] \arrow[dr, "f^i"'] & \operatorname{colim} X_i \arrow[d, dashed, "f"] \\ & Y. \end{tikzcd}$$

The space $\operatorname{colim} X_i$ can be defined by taking the quotient of the topological sum

$$\operatorname{colim} X_i = \left(\coprod X_i\right) / \sim$$

by the relation $X_i \ni x \sim j_{i+1}^i(x) \in X_{i+1}$ for all i. The maps $j^i : X_i \longrightarrow \operatorname{colim} X_i$ are defined as the composition of the canonical inclusion into the topological sum and the quotient map, namely,

$$j^i : X_i \hookrightarrow \coprod X_i \twoheadrightarrow \operatorname{colim} X_i .$$

It is an *exercise* to prove that this definition of colimit actually has the universal property. In the book [27] there is a general treatment of the topic of colimits of topological spaces, these being called (as by many other authors) *direct limits* (see further below).

In the algebraic case we have an analogous situation, namely, given a chain or *direct system* of abelian groups (or rings, vector spaces, etc.) and homomorphisms

$$A_1 \xrightarrow{h_2^1} A_2 \xrightarrow{h_3^2} A_3 \longrightarrow \cdots ,$$

we define its *colimit* as

$$\operatorname{colim} A_i = \left(\bigoplus_{i \geq 1} A_i\right) / A' ,$$

where A' is the subgroup of $\bigoplus A_i$ generated by the differences $h_k^i(a_i) - a_i \in A_i \oplus A_k \subset \bigoplus A_i$, $k > i$, where $h_k^i = h_k^{k-1} \circ h_{k-1}^{k-2} \circ \cdots \circ h_{i+1}^i$. In other words, we identify each group A_i with its image in A_k. For each i we have homomorphisms $h^i : A_i \longrightarrow \operatorname{colim} A_i$ given by the composition of the canonical inclusion in the direct sum and the epimorphism in the colimit

$$h^i : A_i \rightarrowtail \bigoplus A_i \twoheadrightarrow \operatorname{colim} A_i\,.$$

We have, as in the topological case, that

$$h^k \circ h_k^i = h^i : A_i \longrightarrow \operatorname{colim} A_i\,.$$

The algebraic colimit also has the following *universal property*. If $\{f^i : A_i \longrightarrow B \mid i \geq 1\}$ is a family of homomorphisms such that $f^{i+1} \circ h_{i+1}^i = f^i : A_i \longrightarrow B$ for all $i \geq 1$ or, equivalently, $f^k \circ h_k^i = f^i : A_i \longrightarrow B$ for all $k > i \geq 1$, then there exists a unique homomorphism $f : \operatorname{colim} A_i \longrightarrow B$ such that $f \circ h^i = f^i$. Diagrammatically we have the following:

$$\begin{tikzcd} A_i \arrow[rr, "h^i"] \arrow[dr, "f^i"'] & & \operatorname{colim} A_i \arrow[dl, dashed, "f"] \\ & B. & \end{tikzcd}$$

Dually, for an *inverse system* of abelian groups and homomorphisms

$$\cdots \xrightarrow{h_3^4} A^3 \xrightarrow{h_2^3} A^2 \xrightarrow{h_1^2} A^1$$

we have a homomorphism

$$d : \prod A^i \longrightarrow \prod A^i$$

such that

$$d(a_1, a_2, a_3, \dots) = \left(a_1 - h_1^2(a_2), a_2 - h_2^3(a_3), a_3 - h_3^4(a_4), \dots\right)\,.$$

We define its *limit* as the kernel of d,

$$\lim A^i = \ker(d)\,,$$

and its *derived limit* as the cokernel of d,

$$\lim{}^1 A^i = \operatorname{coker}(d) = \left(\prod A^i\right) / \operatorname{im}(d)\,.$$

In this way we obtain an exact sequence

$$0 \longrightarrow \lim A^i \longrightarrow \prod A^i \longrightarrow \prod A^i \longrightarrow \lim{}^1 A^i \longrightarrow 0\,.$$

Dually, in the case of the colimit, for each i we have homomorphisms $h_i : \lim A^i \longrightarrow A^i$ given by the composite

$$\lim A^i \longrightarrow \prod A^i \xrightarrow{\mathrm{proj}_i} A^i\,.$$

The limit also has a universal property dual to that of the colimit. It is the following.

If $\{f_i : B \longrightarrow A^i \mid i \geq 1\}$ is a family of maps such that $h_i^{i+1} \circ f_{i+1} = f_i : B \longrightarrow A^i$ for all $i \geq 1$ or, equivalently (defining $h_i^k = h_i^{i+1} \circ h_{i+1}^{i+2} \circ \cdots \circ h_{k-1}^k$), such that $h_i^k \circ f_k = f_i : B \longrightarrow A^i$ for all $k > i \geq 1$, then there exists a unique homomorphism $f : B \longrightarrow \lim A^i$ such that $h_i \circ f = f_i$. Diagrammatically, this is expressed as

$$\begin{tikzcd}
 & B \arrow[dl, dashed, "f"'] \arrow[dr, "f_i"] & \\
\lim A^i \arrow[rr, "h_i"'] & & A^i\,.
\end{tikzcd}$$

As we have already mentioned above, frequently one refers to the colimit as the *direct limit*, and one denotes it by the symbol $\lim_{\rightarrow}$ or dirlim. Likewise, one often says *inverse limit* instead of limit, and one denotes it by the symbol $\lim_{\leftarrow}$ or invlim. In order to avoid confusion between these, we prefer the nomenclature of colimit and limit, which is more in agreement with the dual categorical character of both concepts. A systematic treatment of colimits and limits can be found in the book by Mac Lane [46], which is, moreover, an excellent general reference for the categorical concepts (functors, natural transformations, etc.) that will be mentioned in this text and briefly described below.

Categories, Functors, and Natural Transformations

Throughout the text we use the concept of functor. This is inherent to the concept of a category, whose definition we now give.

A *category* $\mathcal{C}$ consists of a class of *objects* and, for each pair of objects A, B, a set of *morphisms* $\mathcal{C}(A, B)$ with *domain* A and *codomain* B. If $f \in \mathcal{C}(A, B)$, one usually writes $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. For every triple of objects A, B, C, there is a function

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

associating to a pair of morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ their *composite*

$$g \circ f : A \longrightarrow C\,.$$

Two axioms are satisfied:

Associativity. If $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$, then

$$h \circ (g \circ f) = (h \circ g) \circ f : A \longrightarrow C\,.$$

Identity. For every object B there is a morphism $1_B : B \longrightarrow B$ such that if $f : A \longrightarrow B$, then $1_B \circ f = f$, and if $g : B \longrightarrow C$, then $g \circ 1_B = g$.

Some examples of categories that will be useful in this text are the following:

1. The category $\mathcal{S}et$ of sets and functions, that is, the category whose objects are all sets, and for sets A, B, $\mathcal{S}et(A, B)$ is the set of functions from A to B.

2. The category $\mathcal{T}op$ of topological spaces and continuous maps.

3. The category $\mathcal{G}$ of groups and homomorphisms.

4. Given a partial order $\leq$ in a set X, there is a category $\mathcal{X}$ whose objects are the elements of X and such that the set $\mathcal{X}(x, y)$ is either the empty set or the set consisting of one element, according to whether $x \not\leq y$ or $x \leq y$.

There are many other examples, such as the category of pointed sets (i.e., nonempty sets each with a distinguished point called a *base point*) and pointed functions (i.e., functions preserving base points) $\mathcal{S}et_*$; of pointed topological spaces and pointed maps $\mathcal{T}op_*$; of abelian groups and homomorphisms $\mathcal{A}b$; of modules over a ring R and module homomorphisms $\mathcal{M}od_R$; of vector spaces and linear transformations $\mathcal{V}ect$; etc.

A morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is called an *isomorphism* if there is another morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. For example, isomorphisms in $\mathcal{S}et$ are *set equivalences*, in $\mathcal{T}op$ are *homeomorphisms*, and in $\mathcal{G}$ are *group isomorphisms*.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *covariant functor* (or *contravariant functor*) $T : \mathcal{C} \longrightarrow \mathcal{D}$ assigns to every object A of $\mathcal{C}$ an object $T(A)$ of $\mathcal{D}$ and to every morphism $f : A \longrightarrow B$ of $\mathcal{C}$ a morphism $f_* = T(f) : T(A) \longrightarrow T(B)$ (or $f^* = T(f) : T(B) \longrightarrow T(A)$) in such a way that

(a) $T(1_A) = 1_{T(A)}$,

(b) $T(g \circ f) = T(g) \circ T(f)$ (or $T(g \circ f) = T(f) \circ T(g)$).

Some examples are the following:

1. There is a covariant functor from the category of topological spaces and continuous maps to the category of sets and functions that assigns to every topological space its underlying set. This functor is usually called the *forgetful functor* because it "forgets" the structure of a topological space.

2. There is a covariant functor from the category of sets and functions to the category of topological spaces and continuous maps that assigns to every set the discrete topological space having it as an underlying set.

3. There is a covariant functor from the category of sets and functions to the category of (abelian) groups and homomorphisms that assigns to every set the free (abelian) group generated by the set.

4. There is a contravariant functor from the category of topological spaces and continuous maps to the category of rings and homomorphisms that assigns to every topological space the ring of its continuous real-valued functions.

5. A direct system (or inverse system) in a category $\mathcal{C}$ is a covariant functor (or contravariant functor) from the category $\mathbb{N}$ determined by the ordered set of the natural numbers (cf. example 4 on xxiv).

One can compare functors with each other. This is done by means of a suitable definition of a morphism between functors. Let $T_1, T_2 : \mathcal{C} \longrightarrow \mathcal{D}$ be functors of the same variance (either both covariant or both contravariant). A *natural transformation* φ from T_1 to T_2, in symbols $\varphi : T_1 \stackrel{\cdot}{\longrightarrow} T_2$, assigns to every object A of $\mathcal{C}$ a morphism $\varphi(A) : T_1(A) \longrightarrow T_2(A)$ of $\mathcal{D}$ in such a way that for every morphism $f : A \longrightarrow B$ of $\mathcal{C}$ the appropriate one of the following diagrams is commutative

$$
\begin{tikzcd}
T_1(A) \arrow[r, "T_1(f)"] \arrow[d, "\varphi(A)"'] & T_1(B) \arrow[d, "\varphi(B)"] \\
T_2(A) \arrow[r, "T_2(f)"'] & T_2(B)\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
T_1(A) \arrow[d, "\varphi(A)"'] & T_1(B) \arrow[l, "T_1(f)"'] \arrow[d, "\varphi(B)"] \\
T_2(A) & T_2(B)\,, \arrow[l, "T_2(f)"]
\end{tikzcd}
$$

according to whether T_1, T_2 are covariant or contravariant.

If $\varphi : T_1 \stackrel{\cdot}{\longrightarrow} T_2$ is a natural transformation such that $\varphi(A)$ is an isomorphism in $\mathcal{D}$ for each object A in $\mathcal{C}$, then φ is called a *natural equivalence*.

Smooth Approximation and Deformation of Maps

We shall need to approximate continuous maps with homotopic smooth maps, that is, maps with continuous derivatives of all orders. We present two results on this. First we approximate functions. This is done using the notion of a *smooth bump function*. Namely, given $A \subset V \subset \mathbb{R}^n$ where A is closed and V is open in $\mathbb{R}^n$, a *bump function* of A in V is a continuous function $h : \mathbb{R}^n \longrightarrow I$ such that $f|A = 1$ and $f|\mathbb{R}^n - V = 0$.

Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$\alpha(t) = \begin{cases} \mathrm{e}^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

This function is smooth and can be used to produce a second smooth function

$$\beta(t) = \frac{\alpha(1-t)}{\alpha(1-t) + \alpha(t)}\,,$$

which is such that

$$\begin{cases} \beta(t) = 1 & \text{if } t \leq 0, \\ 0 < \beta(t) < 1 & \text{if } 0 < t < 1, \\ \beta(t) = 0 & \text{if } t \geq 1. \end{cases}$$

Let $A = D_r(a)$ be the closed ball with center $a \in \mathbb{R}^n$ and radius $r > 0$, and let $V = \overset{\circ}{D}_s(a)$ be a larger open ball; that is, $s > r$. Then for $x \in \mathbb{R}^n$ the function

$$h(x) = \beta\left(\frac{|x-a|^2 - r^2}{s^2 - r^2}\right)$$

is a smooth bump function of A in V, as one may easily check.

Let now $U \subset \mathbb{R}^n$ be open and bounded, and let $V \subset \mathbb{R}^n$ be such that $V \supset \overline{U}$. Then there exists a smooth bump function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $\overline{U}$ in V defined as follows. Since $\overline{U}$ is compact, it can be covered with a finite number of open balls $\overset{\circ}{D}_1, \dots, \overset{\circ}{D}_k$ such that their closures $D_1, \dots, D_k$ are contained in V. Let $D'_1, \dots, D'_k$ be balls such that $D_i \subset \overset{\circ}{D}{}'_i \subset V$ and let h_i be a smooth bump function of D_i in $\overset{\circ}{D}{}'_i$. Define h by

$$h(x) = 1 - (1 - h_1(x)) \cdot \cdots \cdot (1 - h_k(x))\,.$$

We have now the desired *smooth approximation theorem*, which shows how one can smoothly approximate continuous functions.

Smooth approximation theorem. *Let $U \subset \mathbb{R}^n$ be open, and let $f : U \longrightarrow \mathbb{R}$ be a continuous map that is smooth in an open set $W \subset U$. Let moreover*

W', W'' be open sets such that $\overline{W}' \subset W''$ and W'' is bounded and contained in U. Finally, take $\varepsilon > 0$. Then there exists a function $g : U \longrightarrow \mathbb{R}$ that is smooth in $W \cup W'$ and satisfies

$$|g(x) - f(x)| < \varepsilon \textit{ for all } x \in U \textit{ and } g(x) = f(x) \textit{ for all } x \in W - \overline{W}''.$$

To obtain such a map g apply the Weierstrass approximation theorem (see [65]) to find a polynomial function $p(x)$ such that

$$|p(x) - f(x)| < \varepsilon \text{ for all } x \in U$$

and take a smooth bump function h of $\overline{W}'$ in W''. Then define

$$g(x) = h(x)p(x) + (1 - h(x))f(x) \text{ for } x \in U.$$

Then g is smooth in $W \cup W'$, $g|W' = p|W'$, $g|U - W'' = f|U - W''$, and $|g(x) - f(x)| < \varepsilon$ for all $x \in \overline{W}''$.

We now state the *smooth deformation theorem*, which shows how one can find smooth maps homotopic to given continuous maps.

Smooth deformation theorem. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be bounded open sets and let $\varphi : U \longrightarrow V$ be a continuous map. Take $W, W' \subset \mathbb{R}^m$ open such that $\overline{W} \subset W' \subset \overline{W}' \subset U$. Then there exists a map $\psi : U \longrightarrow V$ such that:*

(1) *$\psi|W : W \longrightarrow V$ is smooth.*

(2) *$\psi|U - W' = \varphi|U - W'$ and $\psi \simeq \varphi \operatorname{rel}(U - W')$.*

The *proof* is as follows. Cover the compact set $\varphi(\overline{W}')$ by a finite number of open balls contained in V, and let $\varepsilon > 0$ be smaller than one-half the smallest radius of the balls. Then use the smooth approximation theorem for each component of φ to obtain $\psi : U \longrightarrow \mathbb{R}^n$ such that it is smooth in W, $\psi|U - W' = \varphi|U - W'$, and $\|\psi(x) - \varphi(x)\| < \varepsilon$ for all $x \in U$. Then the linear deformation

$$H(x, t) = (1 - t)\varphi(x) - t\psi(x)$$

is a homotopy $H : U \times I \longrightarrow V$ from φ to ψ that coincides with φ on $U - W'$; i.e., it is relative to $U - W'$. In particular, $\psi(U) \subset V$.

Given a smooth map $\varphi : U \longrightarrow \mathbb{R}^n$, where $U \subset \mathbb{R}^m$ is open, we say that $x \in U$ is a *regular point* if the derivative $D\varphi(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is nonsingular.

In particular, if $m < n$, then no point $x \in U$ is regular. A point $y \in \mathbb{R}^n$ is a *regular value* if all points in $\varphi^{-1}(y)$ are regular.

The following result holds (see [57]).

Theorem 1. *If $y \in \mathbb{R}^n$ is a regular value of a smooth map $\varphi : U \longrightarrow \mathbb{R}^n$, where $U \subset \mathbb{R}^m$ is open, then $\varphi^{-1}(y) \subset U$ is a smooth manifold of dimension $m - n$. If, in particular, $m < n$, then $\varphi^{-1}(y) = \emptyset$.*

Another theorem that will be useful for us in this text is due to A. B. Brown, and in a sharper form to A. Sard. It states the following (see [57]).

Brown–Sard theorem. *Let $\varphi : U \longrightarrow \mathbb{R}^n$ be a smooth map, where $U \subset \mathbb{R}^m$ is open. Then the set of regular values of φ is dense in $\mathbb{R}^n$.*

Combining the smooth deformation theorem with the two previous results, one has the following theorem.

Theorem 2. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be bounded open sets and let $\varphi : U \longrightarrow V$ be a continuous map. Take $W, W' \subset \mathbb{R}^m$ open such that $\overline{W} \subset W' \subset \overline{W}' \subset U$. Then there exists a map $\psi : U \longrightarrow V$ such that:*

(1) *$\psi|W : W \longrightarrow V$ is smooth.*

(2) *$\psi|U - W' = \varphi|U - W'$ and $\psi \simeq \varphi \,\mathrm{rel}\,(U - W')$.*

(3) *There is a point $y \in V$ such that $\psi^{-1}(y)$ is a smooth $(m-n)$-manifold, and in particular, if $m < n$, then $\psi^{-1}(y) = \emptyset$.*

PARTITIONS OF UNITY

We shall now continue with a brief description of a notion that we will find useful, namely, the notion of a *partition of unity* subordinate to an open cover $\mathcal{U} = \{U_\lambda\}$ of a topological space X. This consists of a family of functions $\{\eta_\lambda : X \longrightarrow I\}$, indexed with the same index set that the cover $\mathcal{U}$ has, such that $\eta_\lambda|X - U_\lambda = 0$ for all λ, and moreover, each $x \in X$ has a neighborhood V such that $\eta_\lambda|V = 0$, except for a finite number of indices λ, and finally, $\sum_\lambda \eta_\lambda(x) = 1$ for all $x \in X$. (Note that the sum is always a finite sum.) A partition of unity subordinate to a given open cover is a useful tool, for example, for sets of functions or maps only partially defined and with values in $\mathbb{R}$, $\mathbb{C}$, or in some vector space. For example, it is an *exercise* to prove that if $\{f_\lambda : U_\lambda \longrightarrow \mathbb{R}\}$ is a family of continuous functions, then the function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = \sum_\lambda \eta_\lambda(x) f_\lambda(x)$ is well defined and is continuous.

A fundamental theorem concerning the topology of paracompact spaces is the following:

Theorem 3. *A topological space X is paracompact if and only if every open cover $\mathcal{U}$ of X admits a partition of unity subordinate to it.*

The books [60], [27], and [83] can be consulted in order to review this theorem and for general considerations about paracompact spaces.

For subspaces of $\mathbb{R}^n$ one can construct smooth partitions of unity making use of the smooth bump functions constructed in the previous paragraph.

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Chapter 1

Function Spaces

Function spaces will be the foundation of many of the constructions that will be made in this text. The aim of this chapter is to review the most important aspects of the topology of function spaces. We shall assume a knowledge of the concepts of point set topology such as those found in the texts [27, 34, 60, 83], for example.

1.1 Admissible Topologies

There are various ways to endow a set of maps with topologies that have different properties. In this section we shall study the most convenient topologies on the set of (continuous) maps between two topological spaces, namely those topologies that allow us to realize the necessary constructions and that have useful properties.

1.1.1 Definition. Let X, Y be sets. We denote by Y^X the set of functions $f : X \longrightarrow Y$.

We can interpret Y^X as the Cartesian product $\prod_{x \in X} Y_x$, where $Y_x = Y$ for all $x \in X$.

We now suppose that Y is a topological space. Then a canonical topology for Y^X is the *product topology* in $\prod Y_x$. A subbase for this topology is that formed by the family of sets $U^x = \{f \in Y^X \mid f(x) \in U\}$, where $x \in X$ and U is an open set in Y.

1.1.2 Exercise. Let $p_x : Y^X \longrightarrow Y$ be the projection defined by $p_x(f) = f(x)$. Show that the product topology is the smallest that makes all of the projections p_x, $x \in X$, continuous.

If we now also suppose that X is a topological space, we can consider the subset $M(X,Y)$ of Y^X that consists of all the continuous maps. In the following we shall introduce a canonical topology in $M(X,Y)$. We consider the *evaluation* map

$$e' : Y^X \times X \longrightarrow Y$$

such that $e'(f,x) = f(x)$, and its restriction

$$e : M(X,Y) \times X \longrightarrow Y.$$

1.1.3 DEFINITION. We say that a topology in $M(X,Y)$ is *admissible* if the evaluation e is continuous with respect to it.

It is possible that $M(X,Y)$ does not have any admissible topology.

1.2 COMPACT-OPEN TOPOLOGY

The compact-open topology is a topology on $M(X,Y)$ that takes into account both the topology of X and the topology of Y and that generalizes the product topology.

1.2.1 DEFINITION. The *compact-open* topology in $M(X,Y)$ has as subbase the family of sets

$$U^K = \{f \in M(X,Y) \mid f(K) \subset U\},$$

where $K \subset X$ is compact and U is an open set in Y.

If $\mathcal{T}$ is a topology in $M(X,Y)$, we shall denote by $M_{\mathcal{T}}(X,Y)$ the corresponding topological space. We shall denote it by $M_{\mathrm{co}}(X,Y)$ if $\mathcal{T} = \mathrm{co}$ is the compact-open topology.

1.2.2 **Proposition**. *The compact-open topology* (co) *is coarser than any admissible topology in $M(X,Y)$. (That is,* $\mathrm{co} \subset \mathcal{T}$ *for every admissible topology $\mathcal{T}$.)*

Proof: We have to show that every open set in $M_{\mathrm{co}}(X,Y)$ is open in $M_{\mathcal{T}}(X,Y)$ if $\mathcal{T}$ is admissible. For this it suffices to show that U^K is in $\mathcal{T}$. We have that

$$e : M_{\mathcal{T}}(X,Y) \times X \longrightarrow Y$$

is continuous. Take $k \in K$ and $f \in U^K$, that is, $f(K) \subset U$. Since e is continuous and $e(f,k) = f(k) \in U$, there exist neighborhoods V_k of f in $M_{\mathcal{T}}(X,Y)$, and W_k of k in X, such that $e(V_k \times W_k) \subset U$.

The family $\{W_k\}$ forms an open cover of K, which is compact, so that there exists a finite subfamily $W_1, \ldots, W_n$ such that $K \subset W_1 \cup \cdots \cup W_n$. Let $V_1, \ldots, V_n$ be the corresponding V_i such that $e(V_i \times W_i) \subset U$, $i = 1, \ldots, n$. Put $V = V_1 \cap \cdots \cap V_n$. Then $f \in V$ and $V \subset U^K$, since if $g \in V$ and $k \in K$, then $k \in W_i$ for some i. So, $g(k) = e(g,k) \in e(V \times W_i) \subset e(V_i \times W_i) \subset U$, which implies that $g(K) \subset U$. And this shows that U^K is open in $M_{\mathcal{T}}(X,Y)$. $\square$

From now on we shall denote $M_{\text{co}}(X,Y)$ simply by $M(X,Y)$.

1.2.3 **Proposition**. *If X is a locally compact Hausdorff space, then the compact-open topology* co *is admissible.*

Proof: We have to show that $e : M(X,Y) \times X \longrightarrow Y$ is continuous.

Let $U \subset Y$ be open and take $(f,x) \in e^{-1}(U)$. Since $e(f,x) = f(x) \in U$ and f is continuous, there exists a neighborhood W of x in X such that $f(W) \subset U$. Since X is locally compact and Hausdorff, there exists V open with compact closure $\overline{V}$ such that $x \in V \subset \overline{V} \subset W$.

Then $(f,x) \in U^{\overline{V}} \times V$, which is open in $M(X,Y) \times X$. It suffices to show that $U^{\overline{V}} \times V \subset e^{-1}(U)$. Indeed, if $f' \in U^{\overline{V}}$ and $x' \in V$, then $f'(x') \in U$, that is, $e(f',x') \in U$. $\square$

1.2.4 **Corollary**. *If X is a locally compact Hausdorff space, then the topology* co *is the smallest admissible in $M(X,Y)$.* $\square$

1.2.5 Exercise. Let X be a set endowed with the discrete topology and let Y be any topological space. Show that $M(X,Y)$ with the co topology is (homeomorphic to) the topological product $\prod_{x \in X} Y_x$, $Y_x = Y$, as described above.

1.3 The Exponential Law

If X, Y, Z are sets, the *exponential law* establishes an equivalence of sets

$$Z^{X \times Y} \cong (Z^Y)^X \,.$$

To realize this, it suffices to define

$$\varphi : Z^{X\times Y} \longrightarrow (Z^Y)^X \quad \text{by} \quad \varphi(f)(x)(y) = f(x,y)$$

and, as its inverse,

$$\psi : (Z^Y)^X \longrightarrow Z^{X\times Y} \quad \text{by} \quad \psi(g)(x,y) = g(x)(y)\,.$$

We now would like an analogous result for $M(X,Y)$.

1.3.1 **Proposition**. *Let X, Y, Z be topological spaces with Y Hausdorff and locally compact. Then we have an equivalence of sets*

$$\varphi : M(X\times Y, Z) \longrightarrow M(X, M(Y,Z)).$$

Proof: In order to define φ as above, we must show that if $f : X\times Y \longrightarrow Z$ is continuous, then $\varphi(f)(x) : Y \longrightarrow Z$ is continuous and $\varphi(f) : X \longrightarrow M(Y,Z)$ is continuous.

For the first statement, let us note that $\varphi(f)(x)$ is the composite

$$Y \xrightarrow{i_x} X\times Y \xrightarrow{f} Z\,,$$

where $i_x(y) = (x,y)$, which clearly is continuous. (Note that if $X = \emptyset$, the proposition is trivial.)

For the second assertion, let U^K be a subbasic open set in $M(Y,Z)$. It suffices to show that $\varphi(f)^{-1}(U^K)$ is open in X. So take $x \in \varphi(f)^{-1}(U^K)$. Then $f(x,k) \in U$ for all $k \in K$ and there exist neighborhoods W_k of x, V_k of k, with $f(W_k \times V_k) \subset U$. Since K is compact, the family $\{V_k\}$ contains a finite subfamily $V_1, \dots, V_m$ that covers K. Put $W = W_1 \cap \cdots \cap W_m$, where W_i is such that $f(W_i \times V_i) \subset U$. Then W is a neighborhood of x in X. We claim that $W \subset \varphi(f)^{-1}(U^K)$. Indeed, if $x' \in W$ and $k \in K$, then $\varphi(f)(x')(k) = f(x',k)$, but $k \in V_i$ for some i, and $x' \in W_i$, so $f(x',k) \in U$.

Thus we have proved that φ is well defined.

We claim now that with the above definition

$$\psi : M(X, M(Y,Z)) \longrightarrow M(X\times Y, Z)$$

is well defined. Let $g : X \longrightarrow M(Y,Z)$ be continuous. It suffices to show that $\psi(g)$ is continuous.

Let $U \subset Z$ be open. We claim that $\psi(g)^{-1}(U)$ is open. Take $(x,y) \in \psi(g)^{-1}(U)$, that is, $g(x)(y) \in U$. Since $g(x)$ is continuous, there exists a neighborhood W of y with $g(x)(W) \subset U$. Because Y is locally compact and Hausdorff, there exists an open set V with compact closure $\overline{V}$ such that $y \in V \subset \overline{V} \subset W$. Therefore, $g(x)(\overline{V}) \subset U$, and so $g(x) \in U^{\overline{V}}$, which is open in $M(Y,Z)$.

Since g is continuous, there exists a neighborhood T of x in X such that $g(T) \subset U^{\overline{V}}$. Take an element (x', y') in $T \times V$, which is a neighborhood of (x, y) in $X \times Y$. Then $\psi(g)(x', y') = g(x')(y') \in U$, and so $T \times V \subset \psi(g)^{-1}(U)$. $\square$

With an additional condition, the equivalence of sets in the previous proposition is a homeomorphism, namely, we have the next result.

1.3.2 **Theorem**. *If X, Y, Z are topological spaces such that X and Y are Hausdorff and Y is locally compact, then*

$$\varphi : M(X \times Y, Z) \longrightarrow M(X, M(Y, Z))$$

is a homeomorphism.

Proof: Let us show that φ and ψ are continuous.

First, it is an *exercise* to show that $(U^L)^K$ is a subbasic open set in $M(X, M(Y,Z))$ if U is open in Z, and K and L are compact in X and Y, respectively (cf. [27, XII.5] or 1.3.4 below). Then $K \times L$ is compact, and if $f \in U^{K\times L} \subset M(X \times Y, Z)$, then $\varphi(f)(K)(L) = f(K \times L) \in U$; that is, $\varphi(U^{K\times L}) \subset (U^L)^K$.

Now let U^J be a subbasic open set in $M(X \times Y, Z)$, with J compact in $X \times Y$. Put $K = \mathrm{proj}_X(J)$ and $L = \mathrm{proj}_Y(J)$. Then K and L are compact and $J \subset K \times L$. Let us show that $\psi((U^L)^K) \subset U^J$. Indeed, take $g \in (U^L)^K$ and $(x, y) \in J$. Then $\psi(g)(x, y) = g(x)(y) \in U$, provided that $x \in K$ and $y \in L$. $\square$

We have the function

$$T : M(X, Y) \times M(Y, Z) \longrightarrow M(X, Z) \tag{1.3.3}$$

given by composition.

1.3.4 Exercise. Prove that if X and Y are locally compact Hausdorff spaces, then the function T of (1.3.3) is continuous. In particular, if f :

$X \longrightarrow Y$ is continuous, then it induces (by restriction of T) a continuous map

$$f^{\#} : M(Y, Z) \longrightarrow M(X, Z)$$

such that $f^{\#}(g) = g \circ f$. Similarly, if $g : Y \longrightarrow Z$ is continuous, then it induces (again by restriction of T) a continuous map

$$g_{\#} : M(X, Y) \longrightarrow M(X, Z)$$

such that $g_{\#}(f) = g \circ f$. Prove, moreover, that for any general X and Y, $f^{\#}$ and $g_{\#}$ are, in fact, continuous.

1.3.5 DEFINITION. Let A be a subspace of X and let B be a subspace of Y. We denote by $M(X, A; Y, B)$ the subspace of $M(X, Y)$ that consists of the maps $f : X \longrightarrow Y$ such that $f(A) \subset B$. An important example of these subspaces is $M(X, x_0; Y, y_0)$, which consists of those maps $f : X \longrightarrow Y$ such that $f(x_0) = y_0$, with $x_0 \in X$ and $y_0 \in Y$ being specified points. Such maps are called *pointed* (or *based*) *maps*, since they send the *base point* x_0 of X to the base point y_0 of Y.

1.3.6 EXAMPLE. Let $I = [0, 1]$ be the unit interval and $\partial I = \{0, 1\}$ its boundary. We can consider then the spaces

$$M(I, X) \supset M(I, 0; X, x_0) \supset M(I, \partial I; X, x_0)$$

for a *pointed space* (X, x_0). These spaces are known as *the space of free paths in X*, *the space of paths in X based on x_0* (or *path space* of X), and *the space of loops in X based on x_0* (or *loop space* of X), respectively. We usually denote $M(I, \partial I; X, x_0)$ by $\Omega(X, x_0)$ or, if the base point is obvious from context, by ΩX (cf. 1.3.9 further on).

1.3.7 DEFINITION. Let us consider the *pairs* of spaces (X, A) and (Y, B). We define their *product* to be the pair

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

So $(I, \partial I) \times (I, \partial I) = (I^2, \partial I^2)$, where I^2 is the unit square in the plane and ∂I^2 its boundary, which is homeomorphic to the circle $\mathbb{S}^1$ (see Figure 1.1).

Inductively, $(I^n, \partial I^n) \times (I, \partial I) = (I^{n+1}, \partial I^{n+1})$, where I^{n+1} is the unit cube in $\mathbb{R}^{n+1}$ and ∂I^{n+1} is its boundary, which is homeomorphic to the sphere

$$\mathbb{S}^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\right\}.$$

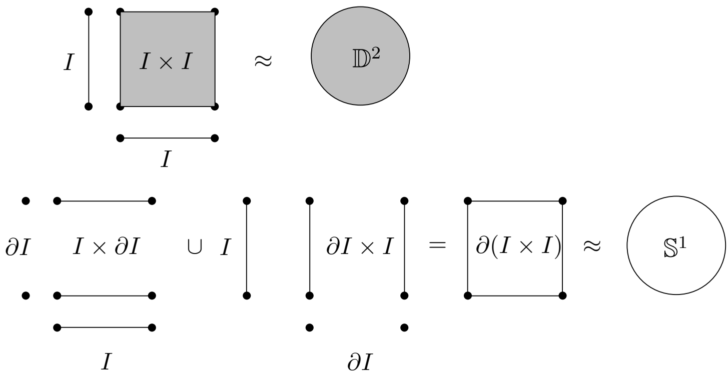

Figure 1.1

By the exponential law (which is also true for pairs: *exercise*) we have

$$M(I^{n+1}, \partial I^{n+1}; X, x_0) \approx M(I, \partial I; M(I^n, \partial I^n; X, x_0), \widetilde{x_0})\,, \tag{1.3.8}$$

where $\widetilde{x_0} \in M(I^n, \partial I^n; X, x_0)$ is such that $\widetilde{x_0}(I^n) = x_0$.

1.3.9 Definition. The space $M(I^n, \partial I^n; X, x_0)$ is called the *n-loop space* of X and is denoted by

$$\Omega^n(X, x_0).$$

If the base point is obvious from context, then we abuse notation and write $\Omega^n X$.

By (1.3.8) we have

$$\Omega(\Omega^n(X, x_0), \widetilde{x_0}) \approx \Omega^{n+1}(X, x_0).$$

1.3.10 Exercise. Let X be a pointed space. Prove that we have a homeomorphism

$$\Omega^n(X, x_0) \approx M(\mathbb{S}^n, *; X, x_0).$$

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Chapter 2

Connectedness and Algebraic Invariants

In this chapter we shall introduce the concepts of path connectedness and of homotopy of continuous maps between two spaces. We shall study the sets of homotopy classes of maps and relate this with path connectedness. Finally, we shall define the homotopy groups of a topological space, which are important algebraic invariants for such spaces.

2.1 Path Connectedness

Path connectedness is a stronger concept than topological connectedness and is better suited for studying homotopy properties. It is based on the concept of a path in a topological space X.

2.1.1 Definition. Let X be a topological space. We define the following relation on it: $x \simeq y$ in X if there exists $\alpha \in M(I, X)$ such that $\alpha(0) = x$ and $\alpha(1) = y$. We say that x is *connected* with y by the *path* α (see 2.5.1 below). The space X is *path connected* or, also, 0-*connected*, if $x \simeq y$ for each pair of points $x, y \in X$.

2.1.2 Exercise. Prove that $\simeq$ is an equivalence relation on X.

2.1.3 Definition. The equivalence classes, denoted by $[x]$, divide X into disjoint subsets called *path components* of X. Let $\pi_0(X)$ be the set of equivalence classes.

This is an important *topological invariant*, which we shall study later on. This invariant "measures" the "disjoint" pieces into which X can be decomposed, as the illustration in Figure 2.1 (where $|\cdot|$ denotes cardinality) shows for a space X in the plane. In particular, X is path connected if and only if X has only one path component.

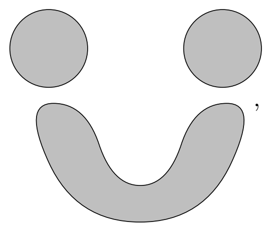
$X =$  , $|\pi_0(X)| = 3$

Figure 2.1

Let $f : X \longrightarrow Y$ be continuous. Then f induces a function

$$f_* : \pi_0(X) \longrightarrow \pi_0(Y)$$

such that $f_*[x] = [f(x)]$. This function is well defined (*exercise*).

The construction π_0 has the following *functorial* properties, whose proof is a simple *exercise* for the reader.

2.1.4 **Proposition**. *The construction π_0 is functorial, that is, the following assertions hold.*

(a) *If $f : X \longrightarrow X$ is the identity, then*

$$f_* : \pi_0(X) \longrightarrow \pi_0(X)$$

is also the identity.

(b) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then*

$$(g \circ f)_* = g_* \circ f_* : \pi_0(X) \longrightarrow \pi_0(Z).$$

In particular, if $f : X \longrightarrow Y$ is a homeomorphism, then $f_* : \pi_0(X) \longrightarrow \pi_0(Y)$ is an equivalence of sets (isomorphism). □

2.2 Homotopy Classes

The relation of homotopy of maps generalizes path connectedness of points. It is the fundamental concept of homotopy theory. In this section we give the basic ideas that underlie it.

2.2.1 Definition. Let $f, g : X \longrightarrow Y$ be continuous maps. We say that f is *homotopic* to g (in symbols $f \simeq g$) if there exists a *homotopy* of f to g, that is, a map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Analogously, we define the concept of homotopy between maps of pairs of spaces; namely, if $f, g : (X, A) \longrightarrow (Y, B)$ are maps of pairs, then $f \simeq g$ if there exists a *homotopy of pairs* of f to g, $H : (X, A) \times I \longrightarrow (Y, B)$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

2.2.2 Exercise. Prove that the relation $\simeq$ is an equivalence relation.

2.2.3 Exercise. Prove that $x, y \in X$ are connected by a path if and only if the maps $c_x, c_y : * \longrightarrow X$, such that $c_x(*) = x$ and $c_y(*) = y$, are homotopic. That is, $x \simeq y$ if and only if $c_x \simeq c_y$.

2.2.4 Definition. Given X, Y, we denote by $[X, Y]$ the set of *homotopy classes* of maps $X \longrightarrow Y$, that is, of equivalence classes of maps $X \longrightarrow Y$ modulo the relation $\simeq$. Analogously, we define the set $[X, A; Y, B]$. In particular, if $X = (X, x_0)$, $Y = (Y, y_0)$ are *pointed spaces*, then we denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps between X and Y.

2.2.5 Note. If the space X is Hausdorff and locally compact and if the space Y is Hausdorff, then $[X, Y] = \pi_0(M(X, Y))$. Analogously, $[X, A; Y, B] = \pi_0(M(X, A; Y, B))$.

2.2.6 **Proposition**. *Let X and Y be locally compact Hausdorff spaces. Then, identifying*

$$\pi_0(M(X, Y) \times M(Y, Z)) \quad \textit{with} \quad \pi_0(M(X, Y)) \times \pi_0(M(Y, Z)),$$

the function T in (1.3.3) determines a function

$$[X, Y] \times [Y, Z] \longrightarrow [X, Z]$$

(given by composition). In particular, $f : X \longrightarrow Y$ induces

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

and $g : Y \longrightarrow Z$ induces

$$g_* : [X, Y] \longrightarrow [X, Z].$$

(For these last two statements we do not need any assumptions on X and Y.) □

Obviously, an analogous result holds for pairs of spaces.

The concept of a homeomorphism of topological spaces can be generalized; namely, a map $f : X \longrightarrow Y$ is a *homotopy equivalence* if it has a *homotopy inverse*, that is, a map $g : Y \longrightarrow X$ such that the homotopy classes $[g \circ f] \in [X, X]$ and $[f \circ g] \in [Y, Y]$ coincide with $[\mathrm{id}_X]$ and $[\mathrm{id}_Y]$, respectively.

2.2.7 **Proposition**. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then f induces bijections (equivalences of sets)*

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

and

$$f_* : [Z, X] \longrightarrow [Z, Y]$$

for any space Z.

Proof: If g is the homotopy inverse of f, then g^* and g_* are the inverses of f^* and f_*, respectively. □

2.2.8 DEFINITION. Let $\{X_\alpha \mid \alpha \in \Lambda\}$ be a family of topological spaces. We denote their coproduct or topological sum by $\coprod_{\alpha\in\Lambda} X_\alpha$. If $\{(X_\alpha, A_\alpha) \mid \alpha \in \Lambda\}$ is a family of pairs of spaces, then we define its *coproduct* or *topological sum* as

$$\coprod_{\alpha\in\Lambda} (X_\alpha, A_\alpha) = \left(\coprod_{\alpha\in\Lambda} X_\alpha, \coprod_{\alpha\in\Lambda} A_\alpha \right) .$$

If $\{X_\alpha \mid \alpha \in \Lambda\}$ is a family of pointed spaces, we define its *coproduct* or *wedge sum* (shortly called *wedge*) as the quotient space

$$\bigvee_{\alpha\in\Lambda} X_\alpha = \coprod_{\alpha\in\Lambda} X_\alpha / \{x_\alpha \mid \alpha \in \Lambda\} ,$$

where for each α, $x_\alpha \in X_\alpha$ is the base point. One may check that there is an embedding $\bigvee_{\alpha\in\Lambda} X_\alpha \hookrightarrow \prod_{\alpha\in\Lambda} X_\alpha$ such that each component X_α maps into the "axis" $\widehat{X}_\alpha = \{(y_\beta) \in \prod_{\beta\in\Lambda} X_\beta \mid y_\beta = x_\beta \text{ if } \beta \neq \alpha\}$.

2.2.9 **Proposition**. *If $(X, A) = \coprod_{\alpha\in\Lambda} (X_\alpha, A_\alpha)$, then*

$$[X, A; Y, B] \cong \prod_{\alpha\in\Lambda} [X_\alpha, A_\alpha; Y, B] .$$

In particular, if X_α, $\alpha \in \Lambda$, are pointed spaces, then

$$\left[\bigvee_{\alpha\in\Lambda} X_\alpha, Y \right]_* \cong \prod_{\alpha\in\Lambda} [X_\alpha, Y]_* .$$

Proof: Given an element $[f] \in [X, A; Y, B]$, let $f_\alpha = f \circ i_\alpha$, where $i_\alpha : (X_\alpha, A_\alpha) \hookrightarrow (X, A)$ is the inclusion. Then $[f] \mapsto ([f_\alpha])$ determines

$$[X, A; Y, B] \longrightarrow \prod_{\alpha \in \Lambda} [X_\alpha, A_\alpha; Y, B] .$$

Now, given $([f_\alpha]) \in \prod_{\alpha \in \Lambda} [X_\alpha, A_\alpha; Y, B]$, the maps $f_\alpha : (X_\alpha, A_\alpha) \longrightarrow (Y, B)$ determine a map $f : (X, A) \longrightarrow (Y, B)$ such that $f \circ i_\alpha = f_\alpha$. So, $([f_\alpha]) \mapsto [f]$ is the desired inverse. □

2.3 Topological Groups

With the aim of introducing algebraic structures in $[X, Y]$ we have to recall the notion of a topological group as well as some other related notions.

2.3.1 Definition. A topological space G is a *topological group* if it is supplied with a continuous map

$$\mu : G \times G \longrightarrow G ,$$

called *multiplication*, that gives G the structure of a group in such a way that the map from G to G given by $x \mapsto x^{-1}$ is continuous. If we simply write $xy = \mu(x, y)$, then the conditions on μ and $x \mapsto x^{-1}$ are equivalent to requiring that the function

$$\widetilde{\mu} : G \times G \longrightarrow G$$

given by $\widetilde{\mu}(x, y) = xy^{-1}$ be continuous.

2.3.2 Examples. The following are examples of topological groups:

(i) $G = \mathbb{R}$, the real numbers with the usual topology and sum.

(ii) $G = \mathbb{R}^n$, the Euclidean space of dimension n with the usual topology and the usual sum of vectors.

(iii) $G = \mathbb{S}^1 = \{\mathrm{e}^{\mathrm{i}x} \in \mathbb{C} \mid x \in \mathbb{R}\}$, the complex numbers of norm 1 with the topology induced by that of $\mathbb{C}$ and multiplication of complex numbers, that is,

$$\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^{\mathrm{i}(x+y)} .$$

(iv) If $\mathrm{M}_{m\times n}(\mathbb{R})$ denotes the set of matrices that have m rows and n columns and have real entries, with the topology given by the bijection

$$\mathrm{M}_{m\times n}(\mathbb{R}) \equiv \mathbb{R}^{mn}$$

that places the rows "one after the other," we have a continuous map

$$\mathrm{M}_{m\times n}(\mathbb{R}) \times \mathrm{M}_{n\times l}(\mathbb{R}) \longrightarrow \mathrm{M}_{m\times l}(\mathbb{R})$$

given by matrix multiplication.

In particular, if $m = n$, then $\mathrm{M}_{n\times n}(\mathbb{R})$ has a multiplicative structure. Nonetheless, inverses do not always exist.

The determinant

$$\det : \mathrm{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

is a continuous function. Therefore, $\det^{-1}(\mathbb{R} - 0)$ is an open subset of $\mathrm{M}_{n\times n}(\mathbb{R})$, and this subset is indeed a group under matrix multiplication. We denote this subset by $\mathrm{GL}_n(\mathbb{R})$ and call it the *real general linear group of dimension n*. Note that det is a continuous homomorphism of this group to the multiplicative (topological) group $\mathbb{R} - 0$.

Let G be a topological group. It is an *exercise* to show (cf. 1.3.4) that $M(X, G)$ is a topological group with the following multiplication:

$$M(X,G) \times M(X,G) \longrightarrow M(X,G)\,,$$

$$(f, g) \mapsto \mu \circ (f, g) = fg\,;$$

that is, $(fg)(x) = f(x)g(x)$. Similarly, $\pi_0(G)$ also acquires a group structure, which is defined by

$$\mu : G \times G \longrightarrow G$$

as follows. Let

$$\overline{\mu} : \pi_0(G) \times \pi_0(G) \longrightarrow \pi_0(G)$$

be such that

$$\overline{\mu}([x], [y]) = [\mu(x, y)] = [xy].$$

In the same way, we obtain the following general statement.

2.3.3 **Proposition**. *Let G be a topological group. Then for every space X, the set $[X, G]$ has an induced group structure. If $f : X \longrightarrow Y$ is continuous, then*

$$f^* : [Y, G] \longrightarrow [X, G]$$

is a homomorphism of groups, and if, on the other hand, $g : G \longrightarrow H$ is a continuous homomorphism of topological groups, then

$$g_* : [X, G] \longrightarrow [X, H]$$

is a homomorphism. Finally, if G is abelian, then $[X, G]$ is also abelian. □

2.4 Homotopy of Mappings of the Circle into Itself

In this section we shall analyze from the homotopical viewpoint the maps of the circle into itself. These maps will provide us with an example of mappings that are not homotopically trivial, and furthermore, in a sense they will provide us with a fundamental example of these. We follow closely the very convenient approach of [71].

Recall that the points of the circle $\mathbb{S}^1 \subset \mathbb{C}$ have the form $\mathrm{e}^{2\pi \mathrm{i} t}$. Let $q : I \longrightarrow \mathbb{S}^1$ be the identification such that $q(t) = \mathrm{e}^{2\pi \mathrm{i} t}$.

Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous pointed function, that is, such that $\varphi(0) = 0$, that also satisfies $\varphi(1) = n \in \mathbb{Z}$. The map $I \longrightarrow \mathbb{S}^1$ such that $t \mapsto \mathrm{e}^{2\pi \mathrm{i} \varphi(t)}$ is compatible with the identification q. Hence it determines a pointed map

$$\widehat{\varphi} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 ,$$

that is, $\widehat{\varphi}(1) = 1$, such that $\widehat{\varphi}(\mathrm{e}^{2\pi \mathrm{i} t}) = \mathrm{e}^{2\pi \mathrm{i} \varphi(t)}$. Therefore, one has a commutative diagram

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d, "q"'] & \mathbb{R} \arrow[d] \\ \mathbb{S}^1 \arrow[r, dashed, "\widehat{\varphi}"'] & \mathbb{S}^1 . \end{tikzcd}$$

We might say, in plain words, that the values of the map φ run along the interval $[0, n]$ (since we start from 0 and arrive at n) in one time unit, that is, while letting the argument of the function run along the interval $[0, 1]$. Consequently, the map $\widehat{\varphi}$ is such that while its argument runs about $\mathbb{S}^1$ once, starting at 1 and returning to 1, its value runs around $\mathbb{S}^1$ n times, also starting at 1 and returning to 1. In other words, after one turn of the argument, there are n turns of the value of $\widehat{\varphi}$. More precisely, this number n counts n counterclockwise turns if $n > 0$, and $-n$ clockwise turns if $n < 0$. We shall prove in what follows that any mapping $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ coincides with $\widehat{\varphi}$ for some $\varphi : I \longrightarrow \mathbb{R}$, that is, that one can "unwind" the mapping.

2.4.1 **Proposition**. *Given any pointed map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, that is, such that $f(1) = 1$, there exists a unique pointed function $\varphi : I \longrightarrow \mathbb{R}$, that is, with $\varphi(0) = 0$, such that $f(\zeta) = \widehat{\varphi}(\zeta)$, $\zeta \in \mathbb{S}^1$.*

Proof: The function is unique, since if $\varphi, \psi : (I, 0) \longrightarrow (\mathbb{R}, 0)$ are such that $\widehat{\varphi} = \widehat{\psi} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, that is, if they are such that $\mathrm{e}^{2\pi\mathrm{i}\varphi(t)} = \mathrm{e}^{2\pi\mathrm{i}\psi(t)}$, then $\psi(t) - \varphi(t) \in \mathbb{Z}$ for all $t \in I$. Therefore, since the function $I \longrightarrow \mathbb{Z}$ given by $t \mapsto \psi(t) - \varphi(t)$ is continuous, and since I is connected and $\mathbb{Z}$ discrete, it follows that this function is constant. Moreover, since $\psi(0)-\varphi(0) = 0-0 = 0$, then $\psi = \varphi$.

Let us see now that φ exists. We need a mapping φ such that $\varphi(0) = 0$ and such that $f(\mathrm{e}^{2\pi\mathrm{i}t}) = \mathrm{e}^{2\pi\mathrm{i}\varphi(t)}$. To that end, let us take the main branch, log, of the complex logarithm; namely, if $z = r\mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{C}$, $r > 0$, $-\pi < \alpha < \pi$, then $\log(z) = \ln(r) + \mathrm{i}\alpha$, where ln is the natural logarithm function. Let $h : I \longrightarrow \mathbb{S}^1$ be such that $h(t) = f(\mathrm{e}^{2\pi\mathrm{i}t})$. Since I is compact, h is uniformly continuous, and so there exists a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of I such that

$$|h(t) - h(t_j)| < 2 \quad \text{if} \quad t \in [t_j, t_{j+1}] \quad \text{and} \quad j = 0, 1, \ldots, k-1\,.$$

Hence $h(t) \neq -h(t_j)$, that is, $h(t)\cdot h(t_j)^{-1} \neq -1$. Therefore, $\log(h(t)\cdot h(t_j)^{-1})$ is well defined. The desired function is thus the following. If $t \in [t_j, t_{j+1}]$, take

$$\varphi(t) = \frac{1}{2\pi\mathrm{i}}\left(\log\left(\frac{h(t_1)}{h(t_0)}\right) + \cdots + \log\left(\frac{h(t_j)}{h(t_{j-1})}\right) + \log\left(\frac{h(t)}{h(t_j)}\right)\right).$$

Then φ is well defined, continuous, and has real values. Using the exponential law $\mathrm{e}^{a+b} = \mathrm{e}^a\mathrm{e}^b$, and $\mathrm{e}^{\log(z)} = z$, since $h(t_0) = h(0) = 1$, one gets

$$\mathrm{e}^{2\pi\mathrm{i}\varphi(t)} = \frac{h(t)}{h(t_0)} = h(t) = f(\mathrm{e}^{2\pi\mathrm{i}t})\,.$$

□

As a consequence of this last proposition, we obtain the fundamental result of this section.

2.4.2 **Theorem**. *Given any mapping $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, there exists a unique pointed function $\varphi : I \longrightarrow \mathbb{R}$ such that $f(\zeta) = f(1)\cdot\widehat{\varphi}(\zeta)$, $\zeta \in \mathbb{S}^1$ (where the dot here means the complex product).*

Proof: Let $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be given by $g(\zeta) = f(1)^{-1} \cdot f(\zeta)$. Then $g(1) = 1$, and therefore by 2.4.1, there exists a unique pointed function $\varphi : I \longrightarrow \mathbb{R}$ such that $g(\zeta) = \widehat{\varphi}(\zeta)$. Therefore, $f(\zeta) = f(1) \cdot \widehat{\varphi}(\zeta)$. □

Given a function $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(0) = 0$ and $\varphi(1) = n \in \mathbb{Z}$, then $\varphi \simeq \varphi_n$ rel $\{0, 1\}$ for $\varphi_n : I \longrightarrow \mathbb{R}$ given by $\varphi_n(s) = ns$, since $H : I \times I \longrightarrow \mathbb{R}$ defined by

$$H(s, t) = (1 - t)\varphi(s) + nst$$

is a homotopy relative to $\{0, 1\}$. Applying the exponential mapping to both φ and φ_n we obtain the following result.

2.4.3 **Lemma**. *Let* $\varphi : I \longrightarrow \mathbb{R}$ *satisfy* $\varphi(0) = 0$ *and* $\varphi(1) = n \in \mathbb{Z}$. *Then* $\widehat{\varphi} \simeq \widehat{\varphi_n}$ rel $\{1\}$. □

Given a mapping $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we have by Theorem 2.4.2 that $f = f(1) \cdot \widehat{\varphi}$, that is, f is the result of composing a mapping of the type $\widehat{\varphi}$ with a rotation given by multiplying by a constant unit complex number. It is easy to verify (*exercise*) that any rotation is homotopic to the identity map $\mathrm{id}_{\mathbb{S}^1}$; therefore, $f \simeq \widehat{\varphi}$ for some $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(0) = 0$ and $\varphi(1) = n \in \mathbb{Z}$. By 2.4.3, we have the following.

2.4.4 **Proposition**. *Given* $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *there exists a unique* $n \in \mathbb{Z}$ *such that* $f \simeq \widehat{\varphi_n} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. □

We have the following definition.

2.4.5 DEFINITION. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be continuous and let $\varphi : I \longrightarrow \mathbb{R}$ be the unique function that by 2.4.2 exists and is such that $f(\zeta) = f(1) \cdot \widehat{\varphi}(\zeta)$. Since the integer $\varphi(1) = n$ is well defined, we define the *degree* of f as this integer n and denote it by $\deg(f)$.

It is geometrically clear what is meant by $\deg(f)$, since by 2.4.2 this integer indicates how many times $f(\zeta)$ turns around $\mathbb{S}^1$ when ζ turns once around $\mathbb{S}^1$. This motion of $f(\zeta)$ is counterclockwise if $n > 0$ and clockwise if $n < 0$, while if $n = 0$, it means that $f \simeq c_0$, that is, the total number of turns is 0.

We observe that $\deg(f)$ depends only on the homotopy class of f; namely, one has the following.

2.4.6 **Lemma**. *If* $f \simeq g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, *then* $\deg(f) = \deg(g)$.

Proof: Let $H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ be a homotopy such that $H(\zeta, 0) = f(\zeta)$, and $H(\zeta, 1) = g(\zeta)$, and let $f_s : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be given by $f_s(\zeta) = H(\zeta, s)$. By 2.4.2, there exists a unique continuous function $\varphi_s : I \longrightarrow \mathbb{R}$ such that $\varphi_s(0) = 0$, $\varphi_s(1) \in \mathbb{Z}$, and $f_s(\zeta) = f_s(1) \cdot \widehat{\varphi_s}(\zeta)$. We shall see that the mapping $I \times I \longrightarrow \mathbb{R}$ given by $(t, s) \mapsto \varphi_s(t)$ is a homotopy; that is, it is continuous. As in the proof of Proposition 2.4.1, the map $h : I \times I \longrightarrow \mathbb{S}^1$ given by $(s, t) \mapsto h(s, t) = f_s(\mathrm{e}^{2\pi \mathrm{i} t})$ is uniformly continuous, and hence one can choose the partition of I in the proof of that proposition in such a way that

$$|h(s, t) - h(s, t_j)| < 2$$

if $s \in I,\ t \in [t_j, t_{j+1}]$, and $j = 0, 1, \ldots, k - 1$. As before, one can now define φ_s with the same formula, but inserting in it the map $h_s : t \longrightarrow h(s, t)$ instead of h; that is, if $s \in I$ and $t \in [t_j, t_{j+1}]$, then

$$\varphi_s(t) = \frac{1}{2\pi \mathrm{i}} \left(\log \left(\frac{h(s, t_1)}{h(s, t_0)} \right) + \cdots + \log \left(\frac{h(s, t_j)}{h(s, t_{j-1})} \right) + \log \left(\frac{h(s, t)}{h(s, t_j)} \right) \right) .$$

Hence $\varphi_s(t)$ is continuous as a function of s and of t; in particular, the function $s \mapsto \varphi_s(1)$ is continuous, and since $\varphi_s(1) \in \mathbb{Z}$, it has to be constant. Since $f(\zeta) = f(1) \cdot \widehat{\varphi_0}(\zeta)$ and $g(\zeta) = g(1) \cdot \widehat{\varphi_1}(\zeta)$, we obtain that $\deg(f) = \varphi_0(1) = \varphi_1(1) = \deg(g)$. $\square$

Hence the degree determines a function $[\mathbb{S}^1, \mathbb{S}^1] \longrightarrow \mathbb{Z}$. The fundamental result in this section, which shows us how an invariant is used for classification problems, is the following.

2.4.7 **Theorem**. *The function*

$$[\mathbb{S}^1, \mathbb{S}^1] \longrightarrow \mathbb{Z} \quad \textit{given by} \quad [f] \mapsto \deg(f)$$

is well defined and bijective. More precisely, one has the following.

(a) *If $n \in \mathbb{Z}$, then the map $g_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ given by $g_n(\zeta) = \zeta^n$ is such that* $\deg(g_n) = n$.

(b) *Take $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. Then $f \simeq g$ if and only if* $\deg(f) = \deg(g)$.

Proof: (a) Since $g_n(\mathrm{e}^{2\pi \mathrm{i} t}) = \mathrm{e}^{2\pi \mathrm{i} n t}$, we have that $g_n = \widehat{\varphi_n}$; hence $\deg(f) = \varphi_n(1) = n$.

(b) By 2.4.6, if $f \simeq g$, then $\deg(f) = \deg(g)$.

Conversely, if $\deg(f) = \deg(g) = n$, then we have that $f(\zeta) = f(1) \cdot \widehat{\varphi}(\zeta)$ and $g(\zeta) = g(1) \cdot \widehat{\psi}(\zeta)$, where $\varphi(0) = \psi(0) = 0$ and $\varphi(1) = \psi(1) = n$.

Since multiplication by $f(1)$ and by $g(1)$ yields rotations, and so maps that are homotopic to $\mathrm{id}_{\mathbb{S}^1}$, and since by the considerations before 2.4.3 we have $\varphi \simeq \varphi_n \simeq \psi$, it follows that $f \simeq \widehat{\varphi} \simeq \widehat{\varphi_n} \simeq \widehat{\psi} \simeq g$. $\square$

2.4.8 EXAMPLES.

(a) The map $\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has degree 1, since $\mathrm{id}_{\mathbb{S}^1} = g_1$.

(b) If $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is *nullhomotopic*, i.e., if it is homotopic to the constant map, then $\deg(f) = 0$, since then $f \simeq g_0$.

(c) The reflection $\rho : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ on the x-axis, that is, the map ρ such that $\rho(\zeta) = \overline{\zeta}$, has degree -1, since $\rho = g_{-1}$.

2.4.9 **Proposition**. *Given $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, then*

$$\deg(f \cdot g) = \deg(f) + \deg(g)\,,$$

where $f \cdot g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ denotes the mapping $\zeta \mapsto f(\zeta)g(\zeta)$, using the complex multiplication in $\mathbb{S}^1$.

Proof: If $f \simeq g_m$ and $g \simeq g_n$, then $f \cdot g \simeq g_m \cdot g_n = g_{m+n}$. $\square$

2.4.10 **Proposition**. *Given $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, then*

$$\deg(f \circ g) = \deg(f)\deg(g)\,.$$

Proof: If $f \simeq g_m$ and $g \simeq g_n$, then $f \circ g \simeq g_m \circ g_n = g_{mn}$. $\square$

2.4.11 **Corollary**. *If $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism, then* $\deg(f) = \pm 1$. *Consequently, $f \simeq \mathrm{id}_{\mathbb{S}^1}$ or $f \simeq \rho$, where ρ is the reflection given by taking complex conjugates.*

Proof: Since $f \circ f^{-1} = \mathrm{id}$, then $\deg(f)\deg(f^{-1}) = 1$; this is possible only if $\deg(f) = \deg(f^{-1}) = \pm 1$. In particular, we have that $\deg(f) = \deg(f^{-1})$. $\square$

2.4.12 DEFINITION. We say that a map $f : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ is *odd* if for every $x \in \mathbb{S}^m$, $f(-x) = -f(x)$; we say that the map is *even* if for every $x \in \mathbb{S}^m$, $f(-x) = f(x)$.

2.4.13 **Theorem**.

(a) *If* $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *is odd, then* $\deg(f)$ *is odd.*

(b) *If* $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *is even, then* $\deg(f)$ *is even.*

Proof: (a) By 2.4.2, there is a map $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi(1) = \deg(f)$, and

$$f(\mathrm{e}^{2\pi \mathrm{i}t}) = f(1) \cdot \mathrm{e}^{2\pi \mathrm{i}\varphi(t)}\,.$$

From $-\mathrm{e}^{2\pi \mathrm{i}t} = \mathrm{e}^{2\pi \mathrm{i}(t+\frac{1}{2})}$ and $-f(\mathrm{e}^{2\pi \mathrm{i}t}) = f(-\mathrm{e}^{2\pi \mathrm{i}t}) = f(\mathrm{e}^{2\pi \mathrm{i}(t+\frac{1}{2})})$ it follows that

$$\mathrm{e}^{2\pi \mathrm{i}(\varphi(t)+\frac{1}{2})} = -\mathrm{e}^{2\pi \mathrm{i}\varphi(t)} = \mathrm{e}^{2\pi \mathrm{i}\varphi(t+\frac{1}{2})}\,,$$

and therefore

$$\varphi\left(t + \frac{1}{2}\right) = \varphi(t) + \frac{1}{2} + k\,,$$

where k is an integer that does not depend on t, since I is connected and φ is continuous. For $t = 0$ one has that $\varphi(\frac{1}{2}) = \varphi(0 + \frac{1}{2}) = \varphi(0) + \frac{1}{2} + k = \frac{1}{2} + k$. For $t = \frac{1}{2}$, one then has

$$\deg(f) = \varphi(1) = \varphi\left(\frac{1}{2} + \frac{1}{2}\right) = \varphi\left(\frac{1}{2}\right) + \frac{1}{2} + k = \frac{1}{2} + k + \frac{1}{2} + k = 1 + 2k\,,$$

and therefore $\deg(f)$ is odd.

The even case is proved analogously. □

2.4.14 EXERCISE. Prove (b) in the theorem above.

2.4.15 EXERCISE. The set $[\mathbb{S}^1, \mathbb{S}^1]$ has an *additive structure* (that is, of an abelian group), given by $[f] + [g] = [f \cdot g]$ (see 2.4.9) and a *multiplicative structure* given by $[f][g] = [f \circ g]$ (see 2.4.10). Prove that $[\mathbb{S}^1, \mathbb{S}^1]$ is a commutative ring with $0 = [g_0]$ ($g_0(\zeta) = 1$ for all $\zeta \in \mathbb{S}^1$) and $1 = [g_1]$ ($g_1(\zeta) = \zeta$ for all $\zeta \in \mathbb{S}^1$) with respect to these structures. Conclude that the function $[\mathbb{S}^1, \mathbb{S}^1] \longrightarrow \mathbb{Z}$ given by $[f] \mapsto \deg(f)$ is a ring isomorphism.

2.4.16 REMARK. For any space X, one may consider the set of homotopy classes $[X, \mathbb{S}^1]$. Using the (abelian) multiplicative structure of $\mathbb{S}^1 \subset \mathbb{C}$ given by complex multiplication, this set becomes an abelian group. Later on, we shall see that for a nice space X this group is the so-called *first cohomology group* of X and is denoted by $H^1(X)$. According to Exercise 2.4.15, we have now proved that $H^1(\mathbb{S}^1) \cong \mathbb{Z}$.

2.4.17 **Proposition**. *The inclusions* $i, j : \mathbb{S}^1 \hookrightarrow \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *given by* $i(z) = (z, 1)$, $j(z) = (1, z)$ *are not nullhomotopic and are not homotopic to each other; that is,* $0 \neq [i] \neq [j] \neq 0$.

Proof: If i and j were nullhomotopic, then the composites $\mathrm{proj}_1 \circ i = \mathrm{id}_{\mathbb{S}^1}$ and $\mathrm{proj}_2 \circ j = \mathrm{id}_{\mathbb{S}^1}$ would also be nullhomotopic, thus contradicting 2.4.8(a). Similarly, if i and j were homotopic, then the composites $\mathrm{proj}_1 \circ i = \mathrm{id}_{\mathbb{S}^1} = g_1$ and $\mathrm{proj}_1 \circ j = g_0$ would also be homotopic, a result that would contradict 2.4.8(a) and (b). □

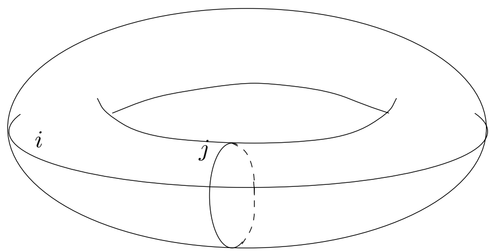


Figure 2.2

Proposition 2.4.17 above, showing that the maps i are j not homotopic, poses the idea that each of the two maps "surrounds" a certain "hole." In fact, i surrounds the "exterior hole" of the tube forming the torus, and j the "interior hole," and these two holes are essentially different (see Figure 2.2).

The next example is probably more eloquent. If we bore a hole into the complex plane $\mathbb{C}$, let us say, to obtain the complement of the origin $\mathbb{C} - 0$, then the inclusion $i : \mathbb{S}^1 \hookrightarrow \mathbb{C} - 0$ is not nullhomotopic, since if it were, then the map

$$\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \xrightarrow{i} \mathbb{C} - 0 \xrightarrow{r} \mathbb{S}^1$$

would also be nullhomotopic, where $r(z) = z/|z|$. What this shows is that the map $i : \mathbb{S}^1 \longrightarrow \mathbb{C} - 0$ detects the hole. It is in this sense that we shall systematize in the next section the study of maps $\mathbb{S}^1 \longrightarrow X$ for any topological space X in order to detect holes or, in other words, to measure certain kinds of complications in the structure of the space X.

2.4.18 REMARK. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be continuous and $z_0 \notin f(\mathbb{S}^1)$. A reasonable question is the following: How many times does the curve described by f turn around z_0? The answer is not always intuitively clear, as is shown in Figure 2.3.

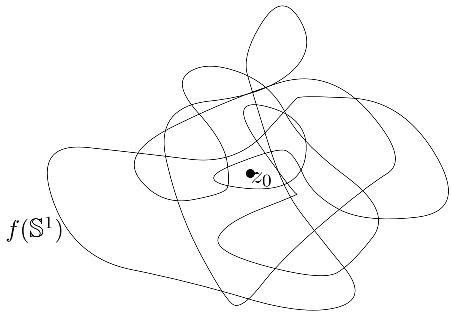


Figure 2.3

The answer is as follows. First, if $r : \mathbb{C} - 0 \longrightarrow \mathbb{S}^1$ is the retraction given by $r(z) = z/|z|$, then the map

$$f_{z_0} : \mathbb{S}^1 \xrightarrow{f} \mathbb{C} - z_0 \xrightarrow{t_{z_0}} \mathbb{C} - 0 \xrightarrow{r} \mathbb{S}^1 ,$$

where $t_{z_0}(z) = z - z_0$, is well defined. Then the answer to the question posed is that the curve described by f surrounds the point z_0 precisely $\deg(f_{z_0})$ times. This number is called the *winding number* of the curve $f(\mathbb{S}^1)$, and we denote it by $W(f, z_0)$. In other words,

(2.4.19) $$W(f, z_0) = \deg(f_{z_0}) , \quad \text{where} \quad f_{z_0}(\zeta) = \frac{f(\zeta) - z_0}{|f(\zeta) - z_0|} .$$

For example, see [25] for a systematic and more general study of the degree, the winding number, and other related concepts.

As a matter of fact, when f is differentiable, then the winding number around z_0 corresponds to the number obtained by the Cauchy formula; that is,

$$W(f, z_0) = \deg(f_{z_0}) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} \frac{f'(\zeta)}{f(\zeta) - z_0} \mathrm{d}\zeta .$$

(See [30] or [8].)

2.4.20 Definition. A topological space X is *contractible* if there exists a homotopy equivalence between it and a one-point space, or equivalently, if there exists a homotopy $F : X \times I \longrightarrow X$ that starts with the identity and ends with the constant map $c(x) = x_0$, namely, if id_X is nullhomotopic. We call such a homotopy F a *contraction*.

Having been able to classify maps $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ up to homotopy brings many nice consequences. From the fact that $\deg(\mathrm{id}_{\mathbb{S}^1}) = 1$ one has that $\mathrm{id}_{\mathbb{S}^1}$ is not nullhomotopic, and from this we obtain the following.

2.4.21 **Theorem**. *The circle $\mathbb{S}^1$ is not contractible.*

Proof: If it were contractible, then $\mathrm{id}_{\mathbb{S}^1}$ would be nullhomotopic. $\square$

In the example of $i : \mathbb{S}^1 \longrightarrow \mathbb{C} - 0$, we saw that $r : \mathbb{C} - 0 \longrightarrow \mathbb{S}^1$ is a *retraction* of the punctured plane $\mathbb{C} - 0$ to the subspace $\mathbb{S}^1$; this way of thinking allows us to prove an interesting fact, which is the following.

2.4.22 **Proposition**. *There is no retraction $r : \mathbb{D}^2 \longrightarrow \mathbb{S}^1$, that is, there is no map r such that $r|_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$*

Proof: Since $\mathbb{D}^2$ is contractible, any map defined on $\mathbb{D}^2$ is nullhomotopic, and in particular r would be so too. But this would be a contradiction, since the composition of r with the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$, which is $\mathrm{id}_{\mathbb{S}^1}$, would also be nullhomotopic. Such an r cannot exist. $\square$

The proposition above allows us to prove a very important result in topology with many applications. It is known as Brouwer's fixed point theorem.

2.4.23 **Theorem**. *Every map $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ has a* fixed point*, that is, a point $x_0 \in \mathbb{D}^2$ such that $f(x_0) = x_0$.*

Proof: If there were no such x_0, then we would have $f(x) \neq x$ for all $x \in \mathbb{D}^2$. Hence the points x and $f(x)$ would determine a ray that starts at $f(x)$ and intersects $\mathbb{S}^1$ in exactly one point $r(x)$. (See Figure 2.4.) The map $r\colon \mathbb{D}^2 \to \mathbb{S}^1$ is well defined, continuous, and is also a retraction. However, the existence of such a retraction contradicts Proposition 2.4.22. $\square$

2.4.24 Exercise. For a given map f, find an explicit formula for the retraction $r : \mathbb{D}^2 \longrightarrow \mathbb{S}^1$ described in the proof of Brouwer's fixed point theorem 2.4.23.

2.4.25 Exercise. Take $X = \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1,\ |y| \leq 2, |z| \leq 3\}$, and consider the map $f : X \longrightarrow \mathbb{R}^3$ given by

$$f(x, y, z) = \left(x - \frac{y^2 + z^2 + 1}{14}, y - \frac{x^2 + z^2 + 4}{14}, z - \frac{x^2 + y^2 + 9}{14}\right) .$$

Prove that the equation $f(x, y, z) = 0$ has a solution (in X). (Hint: Use Brouwer's fixed point theorem 2.4.23.)

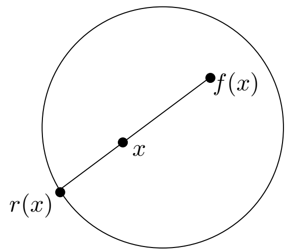


Figure 2.4

The concept of degree is so useful that it has applications outside of topology. A nice example of this is the following proof of the fundamental theorem of algebra.

2.4.26 **Theorem**. (Fundamental theorem of algebra) *Every nonconstant polynomial with complex coefficients has a root. That is, if*

$$f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n ,$$

$n > 0$, $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$, *then there exists* $z_0 \in \mathbb{C}$ *such that* $f(z_0) = 0$.

Proof: Assuming that f does not have a root, the mapping $z \mapsto f(z)$ would determine a map $f : \mathbb{C} \longrightarrow \mathbb{C} - 0$. If we take $\mu = |a_0| + |a_1| + \cdots + |a_{n-1}| + 1$ and $z \in \mathbb{S}^1$, then

$$\begin{aligned} |f(\mu z) - \mu^n z^n| &\leq |a_0| + \mu|a_1| + \cdots + \mu^{n-1}|a_{n-1}| \\ &\leq \mu^{n-1}(|a_0| + |a_1| + \cdots + |a_{n-1}|) \quad (\mu \geq 1) \\ &< \mu^n = |\mu^n z^n| \quad (\mu > |a_0| + |a_1| + \cdots + |a_{n-1}|). \end{aligned}$$

Therefore, $f(\mu z)$ lies in the interior of a circle with center at $\mu^n z^n$ and radius $|\mu^n z^n|$, and so the line segment connecting $f(\mu z)$ with $\mu^n z^n$ does not contain the origin. Hence $H(z,t) = (1-t)f(\mu z) + t\mu^n z^n$ determines a homotopy $H : \mathbb{S}^1 \times I \longrightarrow \mathbb{C} - 0$, starting with the map $z \mapsto f(\mu z)$ and ending with the map $z \mapsto \mu^n z^n$. Since the first map is nullhomotopic using the nullhomotopy $(z,t) \mapsto f((1-t)\mu z)$, so also is the second map. Therefore, by composing it with the known retraction $r : \mathbb{C} - 0 \longrightarrow \mathbb{S}^1$ given by $r(z) = z/|z|$, we obtain that the map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ given by $z \mapsto z^n$ would be nullhomotopic. But this last map is g_n, and so we have contradicted 2.4.7. $\square$

Another application of the degree, or more precisely of the winding number $W(f,z)$ defined above in 2.4.18, is to prove a version of the Jordan curve theorem. This assertion will be based on the following proposition.

2.4.27 Proposition. *Let $f : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be continuous, and let z_0 and z_1 be points in the same path component of $\mathbb{C} - f(\mathbb{S}^1)$. Then $W(f, z_0) = W(f, z_1)$.*

Proof: If $\lambda : z_0 \simeq z_1$ is a path, then $f_{\lambda(t)}$ given by

$$f_{\lambda(t)}(\zeta) = \frac{f(\zeta) - \lambda(t)}{|f(\zeta) - \lambda(t)|}$$

(see 2.4.19) is a homotopy from f_{z_0} to f_{z_1}; consequently,

$$W(f, z_0) = \deg(f_{z_0}) = \deg(f_{z_1}) = W(f, z_1)\,.$$

□

The following is a weak version of the famous Jordan curve theorem.

2.4.28 Theorem. *Given any map $f : \mathbb{S}^1 \longrightarrow \mathbb{C}$, the complement of its image $\mathbb{C} - f(\mathbb{S}^1)$ contains only one unbounded path component. For z inside this component, one has that $W(f, z) = 0$.*

Proof: Since $f(\mathbb{S}^1)$ is compact, being the continuous image of a compact set, the Heine–Borel theorem guarantees that it is bounded. So its complement $\mathbb{C} - f(\mathbb{S}^1)$ contains an unbounded component V. If $\mu > 0$ is large enough, then $f(\mathbb{S}^1) \subset D = \{z \in \mathbb{C} \mid |z| \leq \mu\}$, $\mathbb{C} - D \subset \mathbb{C} - f(\mathbb{S}^1)$, and, since D is bounded, $(\mathbb{C} - D) \cap V \neq \emptyset$. Hence, since $\mathbb{C} - D$ is path connected, $\mathbb{C} - D \subset V$ and V is the only unbounded component of $\mathbb{C} - f(\mathbb{S}^1)$. If $z \in V$ and $z' \in \mathbb{C} - D$, then by 2.4.27, $W(f, z) = W(f, z')$. Moreover, the homotopy

$$H(\zeta, t) = \frac{(1-t)f(\zeta) - z'}{|(1-t)f(\zeta) - z'|}$$

starts with $f_{z'}$ and ends with a constant map, and so one has that $W(f, z') = \deg(f_{z'}) = 0$. □

The classical Jordan curve theorem states that *given an embedding $e : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$, then the complement $\mathbb{R}^2 - e(\mathbb{S}^1)$ has exactly two components, one bounded and one unbounded.* The latter is the one given by 2.4.28. One can prove that $W(e, z) = \pm 1$ if z lies in the bounded component.

Another beautiful result in algebraic topology is the Borsuk–Ulam theorem, which we shall now prove only in its two-dimensional version. This result implies the nonexistence of an embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^2$.

2.4.29 **Theorem**. (Borsuk–Ulam) *Given a continuous map*

$$f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2 ,$$

there is a point $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.

Proof: If we assume that $f(x) \neq f(-x)$ for every point $x \in \mathbb{S}^2$, then one can define two maps, namely,

$$f_1 : \mathbb{S}^2 \longrightarrow \mathbb{S}^1 \quad \text{given by} \quad f_1(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} ,$$
$$f_2 : \mathbb{D}^2 \longrightarrow \mathbb{S}^1 \quad \text{given by} \quad f_2(x_1, x_2) = f_1\left(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right) .$$

If we define $g = f_2|_{\mathbb{S}^1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, then we have, on the one hand, that g is nullhomotopic, since the homotopy

$$H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1 , \qquad H(\zeta, t) = f_2((1-t)\zeta) ,$$

is a nullhomotopy. On the other hand, g is odd, that is, $g(-\zeta) = -g(\zeta)$, since f_1 is odd. By 2.4.13(a) one has that $\deg(g)$ is odd, thus contradicting that g is nullhomotopic. □

2.4.30 Note. The Borsuk–Ulam theorem is often described in meteorological terms as follows. If we assume that temperature T and atmospheric pressure P are continuous functions of location on the surface of the Earth, then both determine a map $f = (T, P) : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$. The theorem asserts that in this case there exists a pair of antipodal points with the same temperature and atmospheric pressure.

If $g : \mathbb{S}^2 \longrightarrow \mathbb{S}^1$ is continuous, then it cannot be odd; that is, it cannot happen that $g(-x) = -g(x)$, since the composite

$$\mathbb{S}^2 \xrightarrow{g} \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$$

would be a counterexample to the Borsuk–Ulam theorem 2.4.29. In the proof of this theorem, by assuming the contrary of its assertion, that is, the existence of $f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ such that for every $x \in \mathbb{S}^2$, $f(x) \neq f(-x)$, we could construct an odd map $g : \mathbb{S}^2 \longrightarrow \mathbb{S}^1$. We have hence that the Borsuk–Ulam theorem is equivalent to the following.

2.4.31 **Theorem**. *There are no continuous odd maps $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^1$.* □

2.4.32 REMARK. There is a general version of the Borsuk–Ulam theorem stating that *given a continuous map* $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$, *there is a point* $x \in \mathbb{S}^2$ *such that* $f(x) = f(-x)$. As before, this assertion is equivalent to saying that *there are no continuous odd maps* $f : \mathbb{S}^n \longrightarrow \mathbb{S}^{n-1}$. In order to prove these facts, more sophisticated machinery is needed. One possibility is given by using the cohomology groups of the projective spaces, as will be seen later on in Chapter 11 (see 11.8.28 and 11.8.29).

2.4.33 EXERCISE. Let $f : \mathbb{D}^2 \longrightarrow \mathbb{R}^2$ be an odd map on the boundary, that is, such that if $x \in \mathbb{S}^1$, then $f(-x) = -f(x)$. Prove that there exists $x_0 \in \mathbb{D}^2$ such that $f(x_0) = 0$.

2.4.34 EXERCISE. Consider the following system of equations:

$$\begin{aligned} x \cos y &= x^2 + y^2 - 1\,, \\ y \cos x &= \tan 2\pi(x^3 + y^3)\,. \end{aligned}$$

Using the last exercise, prove that the system has a solution (x_0, y_0) such that $x_0^2 + y_0^2 \leq 1$.

One last result in this section, whose proof is an application of the Borsuk–Ulam theorem, is the so-called ham sandwich theorem. In order to state it, we need the following preparatory considerations. For each point $a = (a_1, a_2, a_3) \in \mathbb{S}^2$ and each element $d \in \mathbb{R}$, let $E(a, d) \subset \mathbb{R}^3$ be the plane given by the equation

$$\gamma_a(x) = a_1x_1 + a_2x_2 + a_3x_3 - d = 0\,,$$

and let $E^+(a, d)$ and $E^-(a, d)$ be the half-spaces of $\mathbb{R}^3$ such that $\gamma_a(x) \geq 0$ and $\gamma_a(x) \leq 0$, respectively. Obviously, $E^+(-a, -d) = E^-(a, d)$. Let $A_1, A_2, A_3 \subset \mathbb{R}^3$ be subsets such that the maps $f_\nu^\pm : \mathbb{S}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$, where $f_\nu^\pm(a, d)$ is the volume of $A_\nu \cap E^\pm(a, d)$ for $\nu = 1, 2, 3$, are well defined and continuous. Moreover, for each $a \in \mathbb{S}^2$ there exists a unique $d_a \in \mathbb{R}$ depending continuously on a and such that $f_1^+(a, d_a) = f_1^-(a, d_a)$. This last condition means that given any family of parallel planes, there exists only one that divides the set A_1 in two portions of equal volume. Clearly, $d_{-a} = -d_a$. Under these conditions, one has the following result.

2.4.35 **Theorem**. (Ham sandwich theorem) *There exists a plane in* $\mathbb{R}^3$ *dividing each of the subsets* A_1, A_2, A_3 *in portions of equal volume.*

Proof: If $f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ is the map given by

$$f(a) = (f_2^+(a, d_a), f_3^+(a, d_a)),$$

then, by the assumptions, f is well defined and continuous. By the Borsuk–Ulam theorem 2.4.29 there exists $b \in \mathbb{S}^2$ such that $f(b) = f(-b)$. By the properties of d_a and $E^\pm(a, d)$, one has for this b that $f_\nu^+(b, d_b) = f_\nu^+(-b, d_{-b}) = f_\nu^+(-b, -d_b) = f_\nu^-(b, d_b)$, as was required. □

2.4.36 Note. As indicated by its name, a gastronomic interpretation of the ham sandwich theorem can be given if we assume that A_1 is the bread, A_2 the butter and A_3 the ham that will be used to prepare a sandwich. The theorem guarantees that it is possible to cut the sandwich with a flat knife, independent of the distribution of the ingredients, in such a way that each of the two pieces contains exactly the same amount of bread, butter, and ham.

2.4.37 Exercise. Prove the Borsuk–Ulam theorem in dimension 1; that is, prove that given a map $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$, there exists $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$. (Hint: Apply the intermediate value theorem to the map $g : \mathbb{S}^1 \longrightarrow \mathbb{R}$ given by $g(x) = f(x) - f(-x)$.)

2.4.38 Exercise. State the ham sandwich theorem in $\mathbb{R}^2$ and apply the former exercise to prove it.

2.4.39 Exercise. Indicate which of the following maps f are nullhomotopic and which are not.

(a) $f : \mathbb{S}^n \longrightarrow \mathbb{S}^{n+1}$, $f(x) = (x, 0)$.

(b) $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$, $f(\zeta) = (\zeta^2, \zeta^3)$.

(c) $f : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$, $f(\xi, \eta) = (\xi\eta, 1)$.

(d) $f : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$, $f(x, y) = (x^2 - y^2, 2xy)$.

(e) $f : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$, $f(x, y) = (x^2, y)$.

2.5 The Fundamental Group

Historically, the first important concept of algebraic topology was the fundamental group. This is also the first properly algebraic invariant of a topological space to be studied in this book. We shall associate to a topological space this group, which in general is not abelian and whose structure provides us with valuable information about the space.

We shall start by giving the definition of the fundamental group, which in the beginning depends on the basic concept of a path inside a topological space. Although we have already given the definition of path in 2.1.1 and have used the concept in the preceding chapter, for the sake of completeness of this chapter we shall recall it.

2.5.1 Definition. Let X be a topological space and take points $x_0, x_1 \in X$. A *path* from x_0 to x_1 is a map $\omega : I \longrightarrow X$ such that $\omega(0) = x_0$ and $\omega(1) = x_1$ (see Figure 2.5). As before, we denote it by $\omega : x_0 \simeq x_1$. The point x_0 will be called the *origin* (or *beginning*) of ω, and x_1 the *destination* (or *end point*) of ω, and both will be called *extreme points* of the path. If both extreme points coincide, that is, if $x_0 = x_1$, we say that the path is *closed* or simply that it is a *loop* based at x_0.

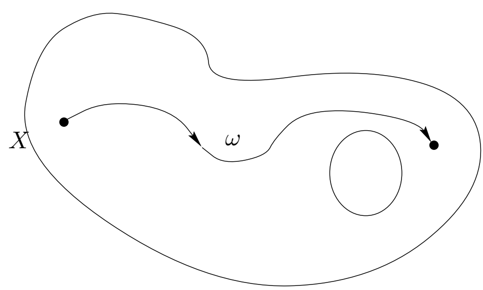


Figure 2.5

2.5.2 Examples.

(a) If $x \in X$, then $c_x : I \longrightarrow X$ given by $c_x(t) = x$ for every $t \in I$ is the *constant path* or *constant loop*.

(b) Take $n \in \mathbb{Z}$. The path $\omega_n : I \longrightarrow \mathbb{S}^1$ given by $\omega_n(t) = \mathrm{e}^{2\pi i n t}$ is the loop of *degree* n in the circle. It has the effect of wrapping around $\mathbb{S}^1$ n times (counterclockwise if $n > 0$, clockwise if $n < 0$, and if $n = 0$, it does not wrap around) as t runs along I; ω_n is the associated loop of the map $g_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ defined in 2.4.7(a).

(c) In the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the paths $\omega_1^1, \omega_1^2 : I \longrightarrow T^2$ given by $\omega_1^1(t) = (\mathrm{e}^{2\pi \mathrm{i}t}, 1) = (\omega_1(t), 1)$ and $\omega_1^2(t) = (1, \mathrm{e}^{2\pi \mathrm{i}t}) = (1, \omega_1(t))$ are loops, which will be called the *unitary equatorial loop* and the *unitary meridional loop*. (See 2.4.17.) More generally, we have in T^2 the loops $\omega_m^1, \omega_n^2 : I \longrightarrow T^2$ given by $\omega_m^1(t) = (\omega_m(t), 1)$ and $\omega_n^2(t) = (1, \omega_n(t))$.

Figure 2.6 shows the generators ω_1^1 and ω_1^2 in the torus.

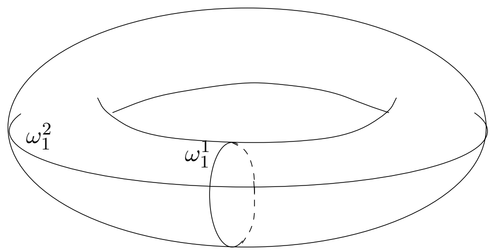


Figure 2.6

In general, as one can see in the preceding examples, as well as in Figure 2.5, as the parameter t varies from 0 to 1, the point $\omega(t)$ describes a curve or path in X connecting the points x_0 and x_1. Two paths $\omega, \sigma : I \longrightarrow X$ are *equal* if as maps they are equal, that is, if for every $t \in I$, $\omega(t) = \sigma(t)$. It is not enough that they have the same images. For instance, the loops ω_n in $\mathbb{S}^1$ defined in 2.5.2(b) are all different from each other. Given any numbers $a < b \in \mathbb{R}$ and any map $\gamma : [a, b] \longrightarrow X$, the canonical homeomorphism $I \longrightarrow [a, b]$ given by $t \mapsto (1-t)a + tb$ transforms γ into a new path $\widehat{\gamma} : I \longrightarrow X$ such that $\widehat{\gamma}(t) = \gamma((1-t)a + tb)$, so that in principle, any such map γ is canonically a path. For technical reasons, it is convenient always to assume $a = 0$ and $b = 1$.

2.5.3 EXERCISE. Prove that giving a path $\sigma : x_o \simeq x_1$ in X is equivalent to giving a homotopy $H : c_{x_0} \simeq c_{x_1} : * \longrightarrow X$, where c_x represents the map from the one-point space $*$ into X with value x.

As in the case of loops, as we saw in the last chapter, it is sometimes possible to multiply paths by each other as well as to define inverses, as we shall now see.

2.5.4 DEFINITION. Given a path $\omega : I \longrightarrow X$, we define the *inverse path* as $\overline{\omega} : I \longrightarrow X$, where $\overline{\omega}(t) = \omega(1-t)$. If $\omega : x_0 \simeq x_1$, then $\overline{\omega} : x_1 \simeq x_0$. Two paths $\omega, \sigma : I \longrightarrow X$ are *connectable* if $\omega(1) = \sigma(0)$; in this case one can define the *product* of ω and σ as the path $\omega\sigma : I \longrightarrow X$ given by

$$(\omega\sigma)(t) = \begin{cases} \omega(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \sigma(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

If $\omega : x_0 \simeq x_1$ and $\sigma : x_1 \simeq x_2$, then $\omega\sigma : x_0 \simeq x_2$. In particular, the pairs $\omega, \overline{\omega}$; $\overline{\omega}, \omega$; c_{x_0}, ω; and ω, c_{x_1} are always connectable, and their products $\omega\overline{\omega}$, $\overline{\omega}\omega$, $c_{x_0}\omega$, and ωc_{x_1} are defined. Nonetheless, in general, $\omega\overline{\omega} \neq c_{x_0}$, $c_{x_0}\omega \neq \omega$, etc. This bad behavior is corrected with the following definition.

2.5.5 Definition. Two paths $\omega_0, \omega_1 : I \longrightarrow X$ are said to be *homotopic* if they have the same extreme points x_0 and x_1 and there exists a *homotopy* $H : I \times I \longrightarrow X$ such that $H(s,0) = \omega_0(s)$, $H(s,1) = \omega_1(s)$, $H(0,t) = x_0$, $H(1,t) = x_1$, for every $s,t \in I$; that is, H is a homotopy relative to $\{0,1\}$. This we denote, as usual, by $H : \omega_0 \simeq \omega_1 \text{ rel } \partial I$; if it is not necessary to emphasize the homotopy, then the fact that ω_0 and ω_1 are homotopic is simply denoted by $\omega_0 \simeq \omega_1$. Figure 2.7 illustrates this concept. If a loop ω is homotopic to the constant loop c_{x_0}, that is, $\omega \simeq c_{x_0}$, one says that it is *nullhomotopic* or *contractible*.

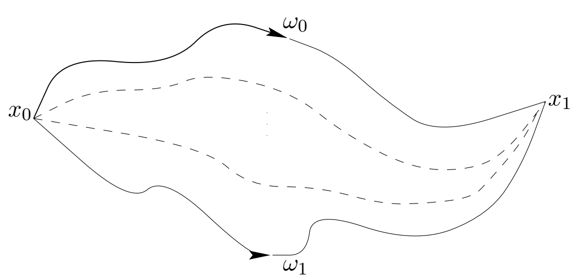


Figure 2.7

In relation to the comments following Definition 2.5.4, we have the following lemma.

2.5.6 **Lemma**. *Let $\omega : x_0 \simeq x_1$, $\sigma : x_1 \simeq x_2$ and $\gamma : x_2 \simeq x_3$ be paths in X. Then one has the following facts.*

(a) $\omega(\sigma\gamma) \simeq (\omega\sigma)\gamma$.

(b) $c_{x_0}\omega \simeq \omega\,, \quad \omega c_{x_1} \simeq \omega$.

(c) $\omega\overline{\omega} \simeq c_{x_0}\,, \quad \overline{\omega}\omega \simeq c_{x_1}\,.$

Proof:
(a) The homotopy $H : I \times I \longrightarrow X$ given by

$$H(s,t) = \begin{cases} \omega(\frac{4s}{2-t}) & \text{if } 0 \le s \le \frac{2-t}{4}, \\ \sigma(4s+t-2) & \text{if } \frac{2-t}{4} \le s \le \frac{3-t}{4}, \\ \gamma(\frac{4s+t-3}{t+1}) & \text{if } \frac{3-t}{4} \le s \le 1, \end{cases}$$

is well defined and is such that $H : \omega(\sigma\gamma) \simeq (\omega\sigma)\gamma$.

(b) The homotopies $H, K : I \times I \longrightarrow X$ given by

$$H(s,t) = \begin{cases} x_0 & \text{if } 0 \le s \le \frac{1-t}{2}, \\ \omega(\frac{2s+t-1}{t+1}) & \text{if } \frac{1-t}{2} \le s \le 1, \end{cases}$$

$$K(s,t) = \begin{cases} \omega(\frac{2s}{t+1}) & \text{if } 0 \le s \le \frac{1+t}{2}, \\ x_1 & \text{if } \frac{1+t}{2} \le s \le 1, \end{cases}$$

are well defined and are such that $H : c_{x_0}\omega \simeq \omega$ and $K : \omega c_{x_1} \simeq \omega$.

(c) The homotopies $H, K : I \times I \longrightarrow X$ given by

$$H(s,t) = \begin{cases} \omega(2s(1-t)) & \text{if } 0 \le s \le \frac{1}{2}, \\ \omega(2(1-s)(1-t)) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

$$K(s,t) = \begin{cases} \omega(2(1-s)(1-t)) & \text{if } 0 \le s \le \frac{1}{2}, \\ \omega(2s(1-t)) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

are well defined and are such that $H : \omega\overline{\omega} \simeq c_{x_0}$ and $K : \overline{\omega}\omega \simeq c_{x_1}$. □

In what follows, we shall frequently write the expresion

$$\omega_1\omega_2\cdots\omega_k\,,$$

without parentheses, which, if it is not stated otherwise, means the path

$$\omega_1\omega_2\cdots\omega_k(t) = \begin{cases} \omega_1(kt) & \text{if } 0 \le t \le \frac{1}{k}, \\ \omega_2(kt-1) & \text{if } \frac{1}{k} \le t \le \frac{2}{k}, \\ \quad\vdots & \qquad\vdots \\ \omega_k(kt-k+1) & \text{if } \frac{k-1}{k} \le t \le 1, \end{cases}$$

that is, all paths in the product are *uniformly* traveled.

One has the following.

2.5.7 **Lemma**. *The relation $\omega \simeq \sigma$ is an equivalence relation.*

Proof: The homotopy $H(s,t) = \omega(s)$ proves that $\omega \simeq \omega$.

If $H : \omega \simeq \sigma$, then $\overline{H} : I \times I \longrightarrow X$, given by $\overline{H}(s,t) = H(s, 1-t)$, is such that $\overline{H} : \sigma \simeq \omega$.

Finally, if $H : \omega \simeq \sigma$ and $K : \sigma \simeq \gamma$, then the homotopy $L : I \times I \longrightarrow X$ defined by

$$L(s,t) = \begin{cases} H(s,2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ K(s,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

is a homotopy relative to $\{0,1\}$, is well defined, and satisfies $L : \omega \simeq \gamma$. □

In what follows we shall denote the equivalence class of ω by $[\omega]$ and we shall call it the *homotopy class* of ω. We are especially interested in homotopy classes of loops based at a specific point x and in particular, in the class $[c_x]$, which will be denoted by 1_x or, if there is no danger of confusion, by 1.

If $H : \omega_0 \simeq \omega_1$ and $K : \sigma_0 \simeq \sigma_1$, then the homotopy $HK : I \longrightarrow X$ given by

$$HK(s,t) = \begin{cases} H(2s,t) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ K(2s-1,t) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

is well defined and is such that $HK : \omega_0\sigma_0 \simeq \omega_1\sigma_1$. Hence we may define the *product* of the homotopy classes of two connectable paths ω and σ by the formula

$$[\omega][\sigma] = [\omega\sigma]\,.$$

Using this and 2.5.6 we have the following result.

2.5.8 **Proposition**. *Let $\omega : w \simeq x$, $\sigma : x \simeq y$, and $\gamma : y \simeq z$ be paths in X. Then the following identities hold:*

(a) $[\omega]([\sigma][\gamma]) = ([\omega][\sigma])[\gamma]$.

(b) $1_w[\omega] = [\omega] = [\omega]1_x$.

(c) $[\omega][\overline{\omega}] = 1_w$, $[\overline{\omega}][\omega] = 1_x$.
(For this reason, $[\overline{\omega}]$ is denoted by $[\omega]^{-1}$.) □

Thanks to (a), we have that the product of homotopy classes of paths is associative. Hence there shall not be any confusion if one writes simply $[\omega][\sigma][\gamma]$.

2.5.9 EXERCISE. Prove that if $\omega_n : I \longrightarrow \mathbb{S}^1$, $n \in \mathbb{Z}$, is as in 2.5.2(b), then $[\omega_n] = [\omega_1]^n$. (Hint: $\omega_1^2 = \omega_2$; proceed by induction over n.)

The concept of fundamental group depends on a base point $x_0 \in X$.

If we restrict 2.5.8 to loops (closed paths), we have the following result.

2.5.10 **Theorem** and DEFINITION. *Let (X, x_0) be a pointed space. Then the set*

$$\pi_1(X, x_0) = \{[\lambda] \mid \lambda \text{ is a loop based at } x_0\}$$

is a group with respect to the multiplication $[\lambda][\mu] = [\lambda\mu]$ with neutral element $1 = 1_{x_0} = [c_{x_0}]$ and with $[\lambda]^{-1}$ as the inverse of each $[\lambda]$. This group is called the *fundamental group* of X based at the point x_0. □

2.5.11 EXERCISE. Prove that the definition of the fundamental group $\pi_1(X, x_0)$ is consistent with the definition of the first homotopy group ($n = 1$) given in 2.10.9. (Hint: A loop $\lambda : I \longrightarrow X$ based at x_0 determines a pointed map $\mathbb{S}^1 \longrightarrow X$, and conversely.)

Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map. If $\lambda : I \longrightarrow X$ is a loop based at x_0, then the composite $f \circ \lambda : I \longrightarrow Y$ is a loop based at y_0. Besides, if c_{x_0} is the constant loop in X, then $f \circ c_{x_0} = c_{y_0}$ is the constant loop in Y, and given loops λ and μ in X, one has

$$f \circ (\lambda\mu) = (f \circ \lambda)(f \circ \mu)\,.$$

2.5.12 EXERCISE. Prove the last assertion in its general form, that is, if $f : X \longrightarrow Y$ is continuous and λ and μ are connectable paths in X, then $f \circ \lambda$ and $f \circ \mu$ are connectable in Y and $f \circ (\lambda\mu) = (f \circ \lambda)(f \circ \mu)$.

2.5.13 **Theorem**. *A pointed map $f : (X, x_0) \longrightarrow (Y, y_0)$ induces a group homomorphism*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)\,,$$

given by $f_*([\lambda]) = [f \circ \lambda]$.

Proof: If $H : \lambda_0 \simeq \lambda_1$ rel ∂I is a homotopy of loops in X based at x_0, that is, $H(s, 0) = \lambda_0(s)$, $H(s, 1) = \lambda_1(s)$, $H(0, t) = x_0 = H(1, t)$, then clearly $f \circ H : f \circ \lambda_0 \simeq f \circ \lambda_1$ rel ∂I, so that the function $f_*([\lambda]) = [f \circ \lambda]$ is well defined.

The remarks before the statement of the theorem prove that $f_*([\lambda\mu]) = [f \circ (\lambda\mu)] = [(f \circ \lambda)(f \circ \mu)] = f_*([\lambda])f_*([\mu])$, which shows that f_* is a group homomorphism. □

The construction of the fundamental group is *functorial*; that is, it behaves well with respect to maps, as the following immediate result shows.

2.5.14 **Theorem**. *Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed spaces and let $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ be pointed maps. Then one has the following properties:*

(a) $\mathrm{id}_{X*} = 1_{\pi_1(X,x_0)} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$.

(b) $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$. □

Because of the conditions (a) and (b) above, the correspondence

$$\begin{array}{ccc} X & & \pi_1(X, x_0) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle f_*} \\ Y & & \pi_1(Y, y_0) \end{array}$$

is said to be a *functor*.

2.5.15 EXAMPLES.

(a) If $\lambda : I \longrightarrow \mathbb{R}^n$ is a loop based at 0, then the homotopy $H(s,t) = (1-t)\lambda(s)$ is a nullhomotopy. Hence $[\lambda] = 1 \in \pi_1(\mathbb{R}^n, 0)$. Therefore, $\pi_1(\mathbb{R}^n, 0) = 1$; that is, the fundamental group of $\mathbb{R}^n$ is the trivial group.

(b) As in the previous example, one can prove that $\pi_1(\mathbb{D}^n, 0) = 1$.

(c) Recall that a subset $X \subset \mathbb{R}^n$ is convex if given two points $x, y \in X$, then for every $t \in I$, $(1-t)x + ty \in X$; that is, the straight line segment joining x and y lies inside X. Given any point $x_0 \in X$ and any loop $\lambda : I \longrightarrow X$ based at x_0, the homotopy $H(s,t) = (1-t)\lambda(s) + tx_0$ is a nullhomotopy relative to ∂I. Therefore, $[\lambda] = 1 \in \pi_1(X, x_0)$. Hence the fundamental group of any convex set is trivial.

(d) Recall that a topological space X is contractible to $x_0 \in X$ if the identity map id_X is *nullhomotopic*, that is, if there exists a contraction $D : X \times I \longrightarrow X$ given by $D(x, 0) = x$, $D(x, 1) = x_0$, $t \in I$; X is *strongly contractible* if moreover, the homotopy D satisfies $D(x_0, t) = x_0$ for all $t \in I$. (Cf. 2.4.20.) If X is (strongly) contractible to $x_0 \in X$, then every loop $\lambda : I \longrightarrow X$ based at x_0 is nullhomotopic, as the nullhomotopy $H(s,t) = D(\lambda(s), t)$ shows, where $D : X \times I \longrightarrow X$ is a contraction, that is, $D(x, 0) = x$, $D(x, 1) = x_0 = D(x_0, t)$, $t \in I$. Therefore, $\pi_1(X, x_0) = 1$; that is, *the fundamental group of every contractible space is trivial.*

2.5.16 **Proposition**. *Let (X, x_0) and (Y, y_0) be pointed spaces. Then the function*

$$\varphi = (\mathrm{proj}_{X*}, \mathrm{proj}_{Y*}) : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is a group isomorphism.

Proof: The function is clearly a homomorphism. If $\lambda : I \longrightarrow X \times Y$ is a loop satisfying $\varphi([\lambda]) = (1, 1)$, then the loops $\lambda_1 = \mathrm{proj}_X \circ \lambda : I \longrightarrow X$ and $\lambda_2 = \mathrm{proj}_Y \circ \lambda : I \longrightarrow Y$ are nullhomotopic, say through the nullhomotopies $H_1 : I \times I \longrightarrow X$ and $H_2 : I \times I \longrightarrow Y$. Therefore, $H = (H_1, H_2) : I \longrightarrow X \times Y$ is a nullhomotopy of the loop $(\lambda_1, \lambda_2) = \lambda : I \longrightarrow X \times Y$. Consequently, $[\lambda] = 1$, and φ is a monomorphism.

On the other hand, if $([\lambda_1], [\lambda_2]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is an arbitrary element, then the loop $\lambda = (\lambda_1, \lambda_2) : I \longrightarrow X \times Y$ is such that $\varphi([\lambda]) = ([\lambda_1], [\lambda_2])$. So φ is an epimorphism. □

Up to now, we have only had explicit examples of trivial fundamental groups. In the next section we shall see examples of nontrivial fundamental groups.

In what follows we shall analyze the relationship between the fundamental groups of a space X with respect to two different base points x_0 and x_1.

If $x_0 \in X$ lies in the path component X_0 of X and λ is a loop in X based at x_0, then, since I is path connected, the image of λ lies in X_0. Moreover, if $H : \lambda \simeq \mu$ is a homotopy in X, then the image of the homotopy also lies inside X_0. These remarks establish the truth of the following statement.

2.5.17 **Proposition**. *Let X be a pointed space with base point x_0. If X_0 is the path component of X containing $x_0 \in X$, then the inclusion map $i : X_0 \hookrightarrow X$ induces an isomorphism $i_* : \pi_1(X_0, x_0) \longrightarrow \pi_1(X, x_0)$.* □

Proposition 2.5.17 allows us to restrict the analysis of the fundamental group to path-connected spaces. Indeed for such spaces the fundamental group is well defined, up to isomorphism, independent of the base point. More precisely, we have the following result.

2.5.18 **Theorem**. *Let $\omega : x_0 \simeq x_1$ be a path in X. There is an isomorphism*

$$\varphi_\omega : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$

given by $\varphi_\omega([\lambda]) = [\omega][\lambda][\omega]^{-1}$.

Proof: Since λ is a loop based at x_1, ω and λ are connectable, and so also are $\omega\lambda$ and $\overline{\omega}$; therefore, the function φ_ω is well defined, and indeed it depends only on the class $[\omega]$.

To see that it is a homomorphism, we have by 2.5.8 that

$$\varphi_\omega([\lambda][\mu]) = [\omega][\lambda][\mu][\overline{\omega}] = [\omega][\lambda][\overline{\omega}][\omega][\mu][\overline{\omega}] = \varphi_\omega([\lambda])\varphi_\omega([\mu])\,.$$

Hence φ_ω is a homomorphism.

Moreover, the homomorphism $\varphi_{\overline{\omega}} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is clearly the inverse of φ_ω. □

2.5.19 EXERCISE. Check that in fact, $\varphi_\omega \circ \varphi_{\overline{\omega}} = 1_{\pi_1(X,x_0)}$ and $\varphi_{\overline{\omega}} \circ \varphi_\omega = 1_{\pi_1(X,x_1)}$.

If in Theorem 2.5.18 we take in particular ω to be a loop based at x_0, that is, such that $[\omega] \in \pi_1(X, x_0)$, then φ_ω is precisely the *inner automorphism* of $\pi_1(X, x_0)$ given by conjugation with the element $[\omega]$.

2.5.20 REMARK. Theorem 2.5.18 allows us to write $\pi_1(X)$ for a path connected space X without reference to the base point. Notice, however, that in general there is no canonical isomorphism between the fundamental group at two different base points. Therefore, $\pi_1(X)$ is really a family of isomorphic groups.

The concept introduced in what follows will be an important concept in this textbook, as it also is in general.

2.5.21 DEFINITION. A topological space X is said to be *simply connected* if it is path connected (0-connected) and for some base point $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is trivial. Frequently, a simply connected space is also called 1-*connected*.

The spaces given in 2.5.15 are all simply connected spaces. We have the following characterization of this concept.

2.5.22 **Proposition**. *Let X be a path-connected space. The following are equivalent.*

(a) *X is simply connected.*

(b) $\pi_1(X,x) = 1$ *for every point* $x \in X$.

(c) *Every loop* $\lambda : I \longrightarrow X$ *is nullhomotopic.*

(d) $\omega \simeq \sigma$ rel ∂I *for any two paths with the same extreme points* x *and* y.

Proof: (a) $\Leftrightarrow$ (b) follows from Theorem 2.5.18, since, because X is path connected, there is always a path $\omega : x_0 \simeq x$ in X.

(b) $\Rightarrow$ (c), for if $\lambda : I \longrightarrow X$ is a loop based at x, then $[\lambda] \in \pi_1(X,x) = 1$. Hence $[\lambda] = 1$; that is, λ is nullhomotopic.

(c) $\Rightarrow$ (d), since $\omega\overline{\sigma}$ is a loop based at x and so is nullhomotopic; that is, $\omega\overline{\sigma} \simeq c_x$. Therefore, by Lemma 2.5.6,

$$(\omega\overline{\sigma})\sigma \simeq c_x\sigma\,.$$

But by the same lemma the left-hand side is homotopic to $\omega(\overline{\sigma}\sigma) \simeq \omega$, while the right-hand side is homotopic to σ. Hence, since $\simeq$ is an equivalence relation, $\omega \simeq \sigma$.

(d) $\Rightarrow$ (a), for if $[\lambda] \in \pi_1(X,x_0)$, then since λ and c_{x_0} have the same extreme points, $\lambda \simeq c_{x_0}$; that is, $[\lambda] = 1$. Hence $\pi_1(X,x_0) = 1$, and so X is simply connected. $\square$

Let $f,g : (X,x_0) \longrightarrow (Y,y_0)$ be homotopic maps between pointed spaces and let $H : X \times I \longrightarrow Y$ be a homotopy relative to $\{x_0\}$. If $\lambda : I \longrightarrow X$ is a loop in X based at x_0, then as we saw above, $f \circ \lambda$ and $g \circ \lambda$ are loops in Y based at y_0; moreover, the homotopy $(s,t) \mapsto H(\lambda(s),t)$ is a homotopy between the loops $f \circ \lambda$ and $g \circ \lambda$ relative to $\{0,1\}$, i.e., $[f \circ \lambda]$ and $[g \circ \lambda]$ are the same element in $\pi_1(Y,y_0)$. Thus, we have shown the following.

2.5.23 **Proposition**. *Let* $f,g : (X,x_0) \longrightarrow (Y,y_0)$ *be homotopic maps of pointed spaces. Then* $f_* = g_* : \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$. $\square$

Indeed, the result above has a stronger version; one has the following theorem.

2.5.24 **Theorem**. *Let* $f,g : X \longrightarrow Y$ *be homotopic maps and, if* $H : f \simeq g$ *is a homotopy, let* $\gamma : I \longrightarrow Y$ *be the path given by* $\gamma(t) = H(x_0,t)$*, for some point* $x_0 \in X$*. Then* $f_* = \varphi_\gamma \circ g_* : \pi_1(X,x_0) \longrightarrow \pi_1(Y,f(x_0))$*, where* φ_γ *is as in* 2.5.18.

Proof: Take $[\lambda] \in \pi_1(X, x_0)$ and let $F : I \times I \longrightarrow Y$ be given by

$$F(s,t) = \begin{cases} H(\lambda(2(1-t)s), 2st) & \text{if } 0 \le s \le \frac{1}{2}, \\ H(\lambda(1+2t(s-1)), t+(1-t)(2s-1)) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

It is straightforward to check that F is a homotopy relative to $\{0,1\}$ of the path product $(f \circ \lambda)\gamma$ to $\gamma(g \circ \lambda)$. Therefore, $[f \circ \lambda][\gamma] = [\gamma][g \circ \lambda]$, that is, $f_*([\lambda]) = \varphi_\gamma g_*([\lambda])$. □

By the theorem above, we have that the fundamental group is a *homotopy invariant*; i.e., it depends only on the homotopy type of the space. The following holds.

2.5.25 **Theorem**. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then the induced homomorphism $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism for every point $x_0 \in X$.*

Proof: Let $g : Y \longrightarrow X$ be a homotopy inverse of f; hence $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. By 2.5.24, we have

$$\begin{aligned} (g \circ f)_* &= \varphi_\gamma : \pi_1(X, x_0) \longrightarrow \pi_1(X, gf(x_0)), \\ (f \circ g)_* &= \varphi_\mu : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(Y, fgf(x_0)), \end{aligned}$$

for certain paths γ in X and μ in Y. That is, $g_* \circ f_*$ and $f_* \circ g_*$ are group isomorphisms with the inverse of the first being α, say. So, $g_* \circ (f_* \circ \alpha) = 1$ and $((f_* \circ \alpha) \circ g_*) \circ f_* = f_*$, but since f_* is an epimorphism, $(f_* \circ \alpha) \circ g_* = 1$; that is, g_* is an isomorphism. Therefore, since $(\alpha \circ g_*) \circ f_* = 1$ and $\alpha \circ g_*$ is an isomorphism, so is f_*. □

2.5.26 NOTE. Let $A \subset X$ and take $x_0 \in A$. Then, the inclusion $i : A \hookrightarrow X$ induces a homomorphism $i_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$, which, as shown by the case $A = \mathbb{S}^1 \subset \mathbb{D}^2 = X$, it is not in general a monomorphism. However, if λ is a loop in A representing an element in $\pi_1(A, x_0)$, then $i_*([\lambda])$ is represented by the loop $i \circ \lambda$, which is essentially the same loop λ, but now thought of as a loop in X. As is shown by the special case mentioned above, the fact that λ is a loop in A that is contractible in X does not mean that it is contractible in A; that is, if $i_*([\lambda]) = 0$, then it does not necessarily follow that $[\lambda] = 0$.

If $\lambda : I \longrightarrow X$ is a loop based at x_0, then λ determines a pointed map $\widetilde{\lambda} : (\mathbb{S}^1, 1) \longrightarrow (X, x_0)$ given by $\widetilde{\lambda}(\mathrm{e}^{2\pi \mathrm{i} t}) = \lambda(t)$. Conversely, a pointed map $f : (\mathbb{S}^1, 1) \longrightarrow (X, x_0)$ determines a loop λ_f based at x_0 given by $\lambda_f(t) = f(\mathrm{e}^{2\pi \mathrm{i} t})$. In other words, we have the next statement.

2.5.27 Proposition. *The function* $\pi_1(X,x_0) \longrightarrow [\mathbb{S}^1,1;X,x_0]$ *given by* $[\lambda] \mapsto [\widetilde{\lambda}]$ *is bijective.* □

More generally, we have the following.

2.5.28 Theorem. *Let* X *be path connected, and let*

$$\Phi : \pi_1(X,x_0) \longrightarrow [\mathbb{S}^1, X]$$

be given by $\Phi([\lambda]) = [\widetilde{\lambda}]$ *by ignoring the base points. Then* Φ *is surjective. Moreover, if* $\alpha, \beta \in \pi_1(X,x_0)$*, then* $\Phi(\alpha) = \Phi(\beta)$ *if and only if there exists* $\gamma \in \pi_1(X,x_0)$ *such that* $\alpha = \gamma\beta\gamma^{-1}$*; that is,* α *and* β *are conjugates.*

Proof: Every map $f : \mathbb{S}^1 \longrightarrow X$ is homotopic to a map $g : \mathbb{S}^1 \longrightarrow X$ such that $g(1) = x_0$, since if $\sigma : f(1) \simeq x_0$ is some path, then the homotopy

$$H(s,t) = \begin{cases} \sigma(t-3s) & \text{if } 0 \le s \le \frac{t}{3}, \\ f(\mathrm{e}^{2\pi \mathrm{i}(\frac{3s-t}{3-2t})}) & \text{if } \frac{t}{3} \le s \le \frac{3-t}{3}, \\ \sigma(3s+t-3) & \text{if } \frac{3-t}{3} \le s \le 1, \end{cases}$$

is such that $H(s,0) = f(\mathrm{e}^{2\pi \mathrm{i}s})$ and $H(s,1)$ is the product loop $\overline{\sigma}\lambda_f\sigma$; in other words, the homotopy $K : \mathbb{S}^1 \times I \longrightarrow X$ given by $K(\mathrm{e}^{2\pi\mathrm{i}s},t) = H(s,t)$ starts at f and ends at a map g such that $g(1) = \sigma(1) = x_0$. This shows that Φ is surjective.

Let us now assume that $\Phi([\lambda]) = \Phi([\mu])$; then we have a homotopy $L : \mathbb{S}^1 \times I \longrightarrow X$ such that $L(\mathrm{e}^{2\pi\mathrm{i}s},0) = \lambda(s)$ and $L(\mathrm{e}^{2\pi\mathrm{i}s},1) = \mu(s)$. Thus, the path $\sigma : I \longrightarrow X$ given by $\sigma(t) = L(1,t)$ is a loop representing an element $\gamma = [\sigma] \in \pi_1(X,x_0)$. Thanks to the homotopy

$$F(s,t) = \begin{cases} H(2(1-t)s, 2st) & \text{if } 0 \le s \le \frac{1}{2}, \\ H(1+2t(s-1), t+(1-t)(2s-1)) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

which is analogous to the one in the proof of 2.5.24, where $H(s,t) = L(\mathrm{e}^{2\pi\mathrm{i}s},t)$, one has $\lambda\sigma \simeq \sigma\mu$.

Conversely, if $\lambda\sigma \simeq \sigma\mu$, then there exists a homotopy $H : \lambda \simeq \sigma\mu\overline{\sigma}$ rel ∂I. So $K(\mathrm{e}^{2\pi\mathrm{i}s},t) = H(s,t)$ is a well-defined homotopy from $\widetilde{\lambda}$ to $\widetilde{\sigma\mu\overline{\sigma}}$. On the other hand, the homotopy

$$G(s,t) = \begin{cases} \sigma(3s+t) & \text{if } 0 \le s \le \frac{1-t}{3}, \\ \mu(\frac{3s+t-1}{1+2t}) & \text{if } \frac{1-t}{3} \le s \le \frac{2+t}{3}, \\ \sigma(3-3s+t) & \text{if } \frac{2+t}{3} \le s \le 1, \end{cases}$$

is such that $G : \sigma\mu\overline{\sigma} \simeq \mu$ and $G(0,t) = \sigma(t) = G(1,t)$; therefore, it defines a homotopy $M : \mathbb{S}^1 \times I \longrightarrow X$ such that $M(\mathrm{e}^{2\pi \mathrm{i} s}, t) = G(s,t)$, starting at $\widetilde{\sigma\mu\overline{\sigma}}$ and ending at $\widetilde{\mu}$. Thus the homotopies K and M may be composed to yield one from $\widetilde{\lambda}$ to $\widetilde{\mu}$; that is, $\Phi([\lambda]) = \Phi([\mu])$. □

2.6 THE FUNDAMENTAL GROUP OF THE CIRCLE

The circle $\mathbb{S}^1$ is path connected, and thus its fundamental group is independent of the choice of base point. The natural base point is $1 \in \mathbb{S}^1$. In Section 2.4 we did all the necessary computations to understand this group. We shall use the results of that section, and as there, we keep close to the approach of [71]. The following lemma will be very useful.

2.6.1 **Lemma**. *The loop product of two loops in $\mathbb{S}^1$ is homotopic to the product of the loops realized as maps with complex values.*

Proof: Let λ, $\mu : I \longrightarrow \mathbb{S}^1$ be two loops. Take the homotopy

$$H(s,t) = \begin{cases} \lambda(2s) & \text{if } 0 \leq s \leq \frac{1-t}{2}, \\ \lambda(\frac{2s-t+1}{2}) \cdot \mu(\frac{2s+t-1}{2}) & \text{if } \frac{1-t}{2} \leq s \leq \frac{1+t}{2}, \\ \mu(2s-1) & \text{if } \frac{1+t}{2} \leq s \leq 1, \end{cases}$$

where $\zeta \cdot \eta$ represents the product in $\mathbb{S}^1$ of the unit complex numbers ζ and η. This homotopy starts with the loop product $\lambda\mu$ and ends with the complex product of complex maps $\lambda \cdot \mu$. □

By the previous lemma, we have that if $[\lambda], [\mu] \in \pi_1(\mathbb{S}^1, 1)$, then $[\lambda][\mu] = [\lambda \cdot \mu]$, and therefore, since the complex product is commutative, we have that $[\lambda][\mu] = [\mu][\lambda]$; that is, we have the following consequence of the previous lemma.

2.6.2 **Lemma**. *The fundamental group of the circle $\pi_1(\mathbb{S}^1, 1)$ is abelian.* □

2.6.3 NOTE. One can give a direct proof of the fact that the fundamental group of the circle is abelian. To start, let $\lambda, \mu : I \longrightarrow \mathbb{S}^1$ be loops. The homotopy $H : I \times I \longrightarrow \mathbb{S}^1$ given by

$$H(s,t) = \begin{cases} \mu(2st) \cdot \lambda(2(1-t)s) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \mu(t + (1-t)(2s-1)) \cdot \lambda(1 + 2t(s-1)) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

where $\zeta \cdot \eta$ is the product of the complex numbers ζ and η in $\mathbb{S}^1$, is such that $H : \lambda\mu \simeq \mu\lambda$; that is, $[\lambda][\mu] = [\mu][\lambda]$.

The homotopy above is indeed the composite of two maps, namely of the map $f : I \times I \longrightarrow I \times I$ given by

$$f(s,t) = \begin{cases} (2(1-t)s, 2st) & \text{if } 0 \le s \le \frac{1}{2}, \\ (1+2t(s-1), t+(1-t)(2s-1)) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

and the map $g : I \times I \longrightarrow \mathbb{S}^1$ given by $g(s,t) = \mu(t) \cdot \lambda(s)$. The map f takes the sides $\{0\} \times I$ and $\{1\} \times I$ of the square onto the vertices $(0,0)$ and $(1,1)$, respectively, and the sides $I \times \{0\}$ and $I \times \{1\}$ to $I \times \{0\} \cup \{1\} \times I$ and $\{0\} \times I \cup I \times \{1\}$, respectively. On the other hand, the map g "translates" the loop λ in $\mathbb{S}^1$ along the loop μ. What this looks like is shown in Figure 2.8.

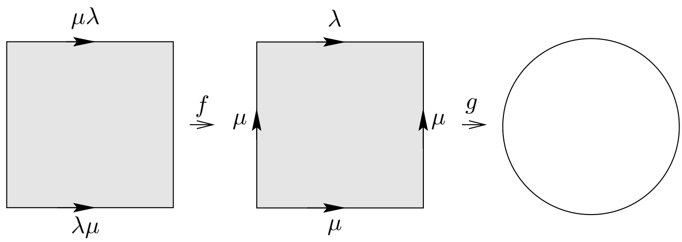


Figure 2.8

2.6.4 EXERCISE. Prove that the fundamental group of every (path-connected) topological group G based at 1, that is, $\pi_1(G,1)$, is abelian. (Hint: One may use the same proof as given for 2.6.1.)

2.6.5 EXERCISE. Let G be a topological group (or an H-space; see next section). Prove that if λ, $\mu : I \longrightarrow G$ are loops, then $[\lambda][\mu] = [\lambda \cdot \mu]$, where $\cdot$ represents the group multiplication. Use this to show that $\pi_1(G,1)$ is abelian. (Hint: Use 2.10.10 below.)

Let us recall the function $\deg : [\mathbb{S}^1, \mathbb{S}^1] \longrightarrow \mathbb{Z}$ defined in 2.4.5, and the function $\Phi : \pi_1(\mathbb{S}^1, 1) \longrightarrow [\mathbb{S}^1, \mathbb{S}^1]$ of the previous section. Let $\Psi = \deg \circ \Phi : \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}$. We summarize what we did in Section 2.4 in the following result.

2.6.6 **Theorem**. $\Psi : \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}$ *is a group isomorphism.*

Proof: By 2.4.7 and by 2.5.24, since in this case φ_γ is the identity, Ψ is bijective. Thus it is enough to check that it is a group homomorphism. Take $\alpha = [\lambda], \beta = [\mu] \in \pi_1(\mathbb{S}^1, 1)$; by 2.6.1, $\alpha\beta = [\lambda \cdot \mu]$. If $\widetilde{\lambda}, \widetilde{\mu} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ are representatives of $\Phi(\alpha), \Phi(\mu)$, respectively, then $\Psi(\alpha\beta) = \Psi([\lambda \cdot \mu]) = \deg(\widetilde{\lambda \cdot \mu}) = \deg(\widetilde{\lambda} \cdot \widetilde{\mu}) = \deg(\widetilde{\lambda}) + \deg(\widetilde{\mu}) = \Psi(\alpha) + \Psi(\beta)$, where the next to the last equality comes from 2.4.9. □

Let $\gamma_n : I \longrightarrow \mathbb{S}^1$ be given by $\gamma_n(t) = \mathrm{e}^{2\pi int} = g_n(\mathrm{e}^{2\pi\mathrm{i}t})$. Then $\Phi([\gamma_n]) = [g_n]$, and thus $\Psi([\gamma_n]) = \deg(g_n) = n$. Hence in particular, $\Psi([\gamma_1]) = 1$ is a generator of $\mathbb{Z}$ as an infinite cyclic group. We have thus the following result.

2.6.7 **Theorem**. *$\pi_1(\mathbb{S}^1, 1)$ is an infinite cyclic group generated by $[\gamma_1]$, that is, by the homotopy class of the loop $t \mapsto \mathrm{e}^{2\pi\mathrm{i}t}$.* □

2.6.8 DEFINITION. The class $[\gamma_1]$ is called the *canonical generator* of the infinite cyclic group $\pi_1(\mathbb{S}^1, 1)$.

If one works with a path-connected space, then as we already proved in 2.5.18, its fundamental group is essentially independent of the base point. In what follows, whenever the base point either is clear or irrelevant, we shall denote the fundamental group of a path-connected space X simply by $\pi_1(X)$.

2.6.9 EXAMPLES. If a space X has the same homotopy type of $\mathbb{S}^1$, then $\pi_1(X) \cong \mathbb{Z}$; we have the following:

(a) $\pi_1(\mathbb{C} - 0) \cong \mathbb{Z}$. The isomorphism is defined by $[\lambda] \mapsto W(f_\lambda, 0)$, the winding number around 0 of the map $f_\lambda : \mathbb{S}^1 \longrightarrow \mathbb{C}$ given by $f_\lambda(\mathrm{e}^{2\pi\mathrm{i}t}) = \lambda(t)$.

(b) If Y is contractible and $X = Y \times \mathbb{S}^1$, then, by 2.5.16 and 2.5.15(d), $\pi_1(X) \cong \pi_1(Y) \times \pi_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. In particular, if $X = \mathbb{D}^2 \times \mathbb{S}^1$ is a *solid torus*, $\pi_1(X) \cong \mathbb{Z}$.

(c) If M is the Moebius band, then $\pi_1(M) \cong \mathbb{Z}$. In fact, the *equatorial loop* $\lambda_e : I \longrightarrow M$ such that $\lambda_e(t) = q(t, \frac{1}{2})$, where $q : I \times I \longrightarrow M$ is the canonical identification, represents a generator of $\pi_1(M)$.

The following example, in particular, is very important. It is an immediate consequence of 2.5.16 and 2.6.7.

2.6.10 Example. If $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is the torus and $x_0 = (1,1) \in \mathbb{T}^2$, then

$$\pi_1(\mathbb{T}^2, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}\,. \tag{2.6.11}$$

Moreover, if $\gamma_1^1, \gamma_1^2 : I \longrightarrow \mathbb{T}^2$ are the *canonical loops* $\gamma_1^1(t) = (\gamma_1(t), 1)$, $\gamma_1^2(t) = (1, \gamma_1(t))$, then we may reformulate (2.6.11) by saying that $\pi_1(\mathbb{T}^2, x_0)$ is the free abelian group generated by the classes $\alpha_1 = [\gamma_1^1]$ and $\alpha_2 = [\gamma_1^2]$.

As a generalization of the previous example, we may prove immediately by induction the following.

2.6.12 **Proposition**. *Let*

$$\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}\,.$$

Then $\pi_1(\mathbb{T}^n)$ is the free abelian group generated by the classes $[\gamma_1^1], \dots, [\gamma_1^n]$ defined by

$$\gamma_1^i(t) = (1, \dots, \underbrace{\gamma_1(t)}_{i}, \dots, 1) \in \mathbb{T}^n\,.$$

□

Let $g_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the map of degree n given by $g_n(\zeta) = \zeta^n$. For the canonical loop $\gamma_1 : I \longrightarrow \mathbb{S}^1$, such that $[\gamma_1]$ is the canonical generator of $\pi_1(\mathbb{S}^1)$, one has that $g_n \circ \gamma_1 = \gamma_n$, so that $(g_n)_*([\gamma_1]) = [\gamma_n] = [\gamma_1]^n$ (since by the considerations prior to 2.6.7, $\Psi(\gamma_n) = n$). Hence $g_{n*} : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1)$ is $g_{n*}(\alpha) = \alpha^n$. Since $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has degree n implies $f \simeq g_n$, we therefore have the following theorem.

2.6.13 **Theorem**. *Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfy* $\deg(f) = n$. *Then the homomorphism $f_* : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1)$ is given by $f_*(\alpha) = \alpha^n$.* □

2.6.14 Note. Strictly speaking, in the previous theorem one has the homomorphism $f_* : \pi_1(\mathbb{S}^1, 1) \longrightarrow \pi_1(\mathbb{S}^1, f(1))$; thus the statement of the theorem can be more precisely applied to the composite

$$\pi_1(\mathbb{S}^1, 1) \xrightarrow{f_*} \pi_1(\mathbb{S}^1, f(1)) \xrightarrow{(r_{f(1)}^{-1})_*} \pi_1(\mathbb{S}^1, 1)\,,$$

where $r_{f(1)}^{-1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is the rotation in $\mathbb{S}^1$ given by multiplying by $f(1)^{-1}$, which is homotopic to the identity.

Another interesting and useful example is the following.

2.6.15 EXAMPLE. Let $f_{bd}^{ac} : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ be given by $f_{bd}^{ac}(\zeta, \eta) = (\zeta^a \cdot \eta^b, \zeta^c \cdot \eta^d)$, $a, b, c, d \in \mathbb{Z}$. Then, by 2.6.13 and 2.6.10, $(f_{bd}^{ac})_* : \pi_1(\mathbb{T}^2) \longrightarrow \pi_1(\mathbb{T}^2)$ is such that $(f_{bd}^{ac})_*(\alpha_1) = \alpha_1^a \alpha_2^c$ and $(f_{bd}^{ac})_*(\alpha_2) = \alpha_1^b \alpha_2^d$, if $\alpha_1, \alpha_2 \in \pi_1(\mathbb{T}^2)$ are as in 2.6.10.

2.6.16 EXERCISE. Check all details of the assertions in the example above and characterize the values of a, b, c, d for which $(f_{bd}^{ac})_*$ is an isomorphism. What can be said about the map f_{bd}^{ac} for these values?

2.6.17 EXERCISE. Let $\varphi : \pi_1(\mathbb{T}^2) \longrightarrow \pi_1(\mathbb{T}^2)$ be any homomorphism. Prove that there exists $f : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such that $f_* = \varphi$. Moreover, show that if φ is an isomorphism, then f can be chosen to be a homeomorphism. (Hint: Use Example 2.6.15.)

2.6.18 EXERCISE. Prove that $\mathbb{T}^m \approx \mathbb{T}^n$ if and only if $m = n$.

2.6.19 EXERCISE. Prove that a loop $\lambda : I \longrightarrow \mathbb{S}^1$ is such that $[\lambda] \in \pi_1(\mathbb{S}^1)$ is a generator if and only if $W(f_\lambda, 0) = \pm 1$, where $f_\lambda : \mathbb{S}^1 \longrightarrow \mathbb{C}$ is given by $f_\lambda(\mathrm{e}^{2\pi \mathrm{i} t}) = \lambda(t)$ and W is the winding number function.

2.6.20 EXERCISE. If M is the Moebius band and $f : \mathbb{S}^1 \longrightarrow \partial M$ is a homeomorphism, prove that the loop $\lambda_f : I \longrightarrow M$ given by $\lambda_f(t) = f(\mathrm{e}^{2\pi \mathrm{i} t}) \in M$ satisfies $[\lambda_f] = \alpha^2$ for α one of the generators of $\pi_1(M) \cong \mathbb{Z}$ (see 2.6.9(c)). Conclude that the boundary ∂M is not a retract of M.

2.7 H-SPACES

In Sections 2.2 and 2.3 above we have seen that the fact that a topological space Y has a compatible group structure, namely, that it is a topological group, implies that the homotopy set $[X, Y]$ inherits a group structure. We can impose even weaker conditions on Y than that of being a group and still have that $[X, Y]$ is a group for every X. These conditions are those that define the concept of an H-space, which we shall study in this section.

2.7.1 CONVENTION. From here on, we shall be concerned mainly with **pointed spaces** and **pointed maps**. We shall use the notation $M_*(X, Y)$ for the set of pointed maps from X to Y endowed with the compact-open topology. Analogously, we shall use the notation $[X, Y]_*$ for the set of pointed homotopy classes of pointed maps from X to Y, namely for the set $[X, x_0; Y, y_0]$.

2.7.2 DEFINITION. A topological space W is an *H-space* if it is a pointed space equipped with a continuous map

$$\mu : W \times W \longrightarrow W\,,$$

called the *H-multiplication*, such that if $e : W \longrightarrow W$ is the constant map whose value is the base point $e(W) = w_0$, then it is an *identity up to homotopy*, or an *H-identity*; that is, the composites

$$W \xrightarrow{(e,\mathrm{id})} W \times W \xrightarrow{\mu} W, \quad W \xrightarrow{(\mathrm{id},e)} W \times W \xrightarrow{\mu} W$$

are homotopic to the identity map of W.

We say that W is *homotopy associative* or *H-associative* if the composites $\mu \circ (\mu \times \mathrm{id}), \mu \circ (\mathrm{id} \times \mu) : W \times W \times W \longrightarrow W$ are homotopic, that is, if the following diagram commutes up to homotopy:

$$\begin{tikzcd}
W \times W \times W \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"'] & W \times W \arrow[d, "\mu"] \\
W \times W \arrow[r, "\mu"'] & W
\end{tikzcd}$$

Note that in the algebraic case of a group, strict commutativity of this diagram is equivalent to associativity of the multiplication.

A map $j : W \longrightarrow W$ determines *inverses up to homotopy*, or *H-inverses*, if the composites

$$W \xrightarrow{\ (\mathrm{id},j)\ } W \times W \xrightarrow{\mu} W, \quad W \xrightarrow{\ (j,\mathrm{id})\ } W \times W \xrightarrow{\mu} W$$

are each homotopic to $e : W \longrightarrow W$, that is, if they are nullhomotopic.

These properties coincide with the axioms of a group, with the reservation that they hold only up to homotopy. We now have the following concept.

2.7.3 DEFINITION. An H-associative H-space equipped with a map that determines H-inverses is called an *H-group*. An H-space, or an H-group, W is *homotopy abelian* or *H-abelian* if the maps $\mu, \mu \circ T : W \times W \longrightarrow W$ are homotopic, where $T(x, y) = (y, x)$.

2.7.4 DEFINITION. If W and W' are H-spaces and $h : W \longrightarrow W'$ is continuous, we say that h is an *H-homomorphism* if the composites

$$W \times W \xrightarrow{\mu} W \xrightarrow{h} W', \quad W \times W \xrightarrow{h\times h} W' \times W' \xrightarrow{\mu'} W'$$

are homotopic, that is, if the diagram

$$\begin{tikzcd} W \times W \arrow[r, "\mu"] \arrow[d, "h\times h"'] & W \arrow[d, "h"] \\ W' \times W' \arrow[r, "\mu'"'] & W' \end{tikzcd}$$

commutes up to homotopy.

2.7.5 Definition. Let W be a pointed space. We say that $[X,W]_*$ has a *natural group structure in* X if

(a) for every pointed space X, $[X,W]_*$ has a group structure such that the class $[e]$ of the constant map $e : X \longrightarrow W$ is the unit of the group, and if

(b) for every pointed map $f : X \longrightarrow Y$, the induced function

$$f^* : [Y,W]_* \longrightarrow [X,W]_*$$

is a homomorphism of groups.

In the same way as with groups, the multiplication μ of an H-space W induces a multiplication in $M_*(X,W)$. We have, in fact, the following general result.

2.7.6 **Theorem**. *Let W be a pointed space. Then $[X,W]_*$ has a natural group structure in X if and only if W is an H-group.*

Proof: If W is an H-group, it is completely straightforward that $[X,W]_*$ acquires a natural group structure in X. Conversely, let us suppose that $[X,W]_*$ has a natural group structure in X. Let $p_1, p_2 : W \times W \longrightarrow W$ be the projections onto the first factor and onto the second factor. Let $\mu : W \times W \longrightarrow W$ be a map that represents the product $[p_1][p_2]$ in the group structure in $[W \times W, W]_*$. It is easy to show that in fact, this map μ is a multiplication that gives W the structure of an associative H-space. On the other hand, there exists a map $j : W \longrightarrow W$ that represents in the group structure in $[W,W]_*$ the inverse of the class of $\mathrm{id} : W \longrightarrow W$, that is, such that $[j] = [\mathrm{id}]^{-1}$. The map j determines H-inverses, and so W has the structure of an H-group. □

2.7.7 Exercise. Reexamine all the details of the proof of Theorem 2.7.6.

2.7.8 EXERCISE. Prove that if W is an H-abelian H-group, then $[X, W]_*$ is an abelian group.

2.7.9 **Proposition**. *If $h : W \longrightarrow W'$ is an H-homomorphism of H-spaces, then for every space X,*

$$h_* : [X, W]_* \longrightarrow [X, W']_*$$

is a homomorphism. □

2.8 LOOP SPACES

A fundamental example of an H-group is the loop space of a pointed topological space, as defined in 1.3.9.

2.8.1 DEFINITION. If Y is a pointed space with base point y_0, then its loop space ΩY has the structure of an H-group, as follows. Let

$$\mu : \Omega Y \times \Omega Y \longrightarrow \Omega Y$$

be such that for loops $\alpha, \beta \in \Omega Y$,

$$\mu(\alpha, \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

2.8.2 EXERCISE. Verify that μ is continuous.

2.8.3 **Lemma**. *μ is an H-multiplication.*

Proof: If $e : \Omega Y \longrightarrow \Omega Y$ is the constant map whose value is the constant loop $\alpha_0 : I \longrightarrow Y, \alpha_0(t) = y_0$, we see that this is an H-unit, that is $\mu(\beta, \alpha_0) \simeq \beta$ and $\mu(\alpha_0, \beta) \simeq \beta$, for every loop β.

The first homotopy is given by

$$F : \Omega Y \times I \longrightarrow \Omega Y\,,$$

where

$$F(\beta, s)(t) = \begin{cases} \beta(\frac{2t}{1+s}) & \text{if } 0 \leq t \leq \frac{1+s}{2}, \\ y_0 & \text{if } \frac{1+s}{2} \leq t \leq 1. \end{cases}$$

The second homotopy is analogous; it is an *exercise* to write it. □

2.8.4 **Lemma**. *μ is H-associative.*

Proof: The homotopy

$$G : \Omega Y \times \Omega Y \times \Omega Y \times I \longrightarrow \Omega Y$$

between $\mu \circ (\mu \times \mathrm{id})$ and $\mu \circ (\mathrm{id} \times \mu)$ is as follows:

$$G(\alpha, \beta, \gamma, s)(t) = \begin{cases} \alpha(\frac{4t}{1+s}) & \text{if } 0 \le t \le \frac{1+s}{4}, \\ \beta(4t - 1 - s) & \text{if } \frac{1+s}{4} \le t \le \frac{2+s}{4}, \\ \gamma(\frac{4t-2-s}{2-s}) & \text{if } \frac{2+s}{4} \le t \le 1. \end{cases}$$

□

2.8.5 **Lemma**. *Let $j : \Omega Y \longrightarrow \Omega Y$ be such that $j(\alpha)(t) = \alpha(1 - t)$. Then j determines H-inverses.*

Proof: The homotopy

$$H : \Omega Y \times I \longrightarrow \Omega Y\,,$$

where

$$H(\alpha, s)(t) = \begin{cases} \alpha(2(1 - s)t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \alpha(2(1 - s)(1 - t)) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

begins with $\mu(\alpha, j(\alpha))$ and ends with α_0. The second homotopy is given in an analogous manner. It is an *exercise* to write it. □

As a consequence of the three previous lemmas we have the following result.

2.8.6 **Theorem**. *For every pointed space Y, ΩY is an H-group, and so for every space X, $[X, \Omega Y]_*$ is a group. If $f : X \longrightarrow X'$ is continuous, then*

$$f^* : [X', \Omega Y]_* \longrightarrow [X, \Omega Y]_*$$

is a homomorphism. Finally, if $g : Y \longrightarrow Y'$ is a pointed map ($g(y_0) = y_0'$), then $\Omega g : \Omega Y \longrightarrow \Omega Y'$ defined as the restriction of $g_\#$ (cf. (1.3.3)), is an H-homomorphism. Therefore,

$$(\Omega g)_* : [X, \Omega W]_* \longrightarrow [X, \Omega W']_*$$

is a group homomorphism. □

2.9 H-COSPACES

There is a "dual" exposition to that which we have just done in the two previous sections. The idea is to define spaces Q in such a way that $[Q, Y]_*$ is a group for arbitrary Y and such that whenever $g : Y \longrightarrow Y'$ is continuous, then $g_* : [Q, Y]_* \longrightarrow [Q, Y']_*$ is a homomorphism.

In the same way as the topological product $\times$ is needed to define the notion of H-space, we now need the concept dual to the topological product, but in the pointed case.

2.9.1 DEFINITION. Let X and Y be pointed spaces. Their topological product $X \times Y$ is also pointed with base point (x_0, y_0) if $x_0 \in X$ and $y_0 \in Y$ are the base points of X and Y, respectively. Notice that the *reduced coproduct* or the *wedge sum* of X and Y can be considered as a subspace of $X \times Y$,

$$X \vee Y = \{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}\,;$$

that is, $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$. (See Figure 2.9.)

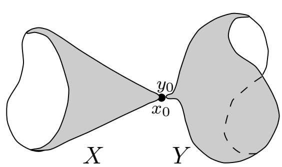


Figure 2.9

In a dual manner to the product, the wedge has the following property: For given pointed maps $f : X \longrightarrow Z$, $\quad g : Y \longrightarrow Z$, there then is defined a pointed map

$$\langle f, g\rangle : X \vee Y \longrightarrow Z$$

given by

$$\langle f, g\rangle(x, y) = \begin{cases} f(x) & \text{if } y = y_0, \\ g(y) & \text{if } x = x_0. \end{cases}$$

On the other hand, if now $f : X \longrightarrow X'$, $\quad g : Y \longrightarrow Y'$ are pointed maps, these define a pointed map

$$f \vee g : X \vee Y \longrightarrow X' \vee Y'$$

given by

$$(f \vee g)(x, y) = \big(f(x), g(y)\big).$$

2.9.2 NOTE. Given a finite number of pointed spaces $X_1, \dots, X_k$, their wedge $X_1 \vee \cdots \vee X_k$ can be seen as the subspace $\{(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k \mid x_i \text{ is the base point for all but at most one value of } i\}$. However, observe that for an infinite number of pointed spaces this is not the case.

2.9.3 DEFINITION. A topological space Q is an *H-cospace* if it is a pointed space equipped with a continuous map

$$\nu : Q \longrightarrow Q \vee Q,$$

called *H-comultiplication*, such that if $e : Q \longrightarrow Q$ is the constant map whose value is the base point, $e(Q) = q_0$, then it is a *counit up to homotopy*, or an *H-counit*; that is, the composites

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{\langle \mathrm{id}, e \rangle} Q, \quad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{\langle e, \mathrm{id} \rangle} Q$$

are homotopic to the identity of Q.

We say that Q is *homotopy coassociative*, or *H-coassociative*, if the composites $(\nu \vee \mathrm{id}) \circ \nu$, $(\mathrm{id} \vee \nu) \circ \nu : Q \longrightarrow Q \vee Q \vee Q$ are homotopic, that is, if the following diagram commutes up to homotopy:

$$\begin{tikzcd}
Q \arrow[r, "\nu"] \arrow[d, "\nu"'] & Q \vee Q \arrow[d, "\nu \vee \mathrm{id}"] \\
Q \vee Q \arrow[r, "\mathrm{id} \vee \nu"'] & Q \vee Q \vee Q
\end{tikzcd}$$

A map $j : Q \longrightarrow Q$ *determines coinverses up to homotopy*, or *H-coinverses*, if the composites

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{\langle j, \mathrm{id} \rangle} Q, \quad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{\langle \mathrm{id}, j \rangle} Q$$

are each homotopic to $e : Q \longrightarrow Q$.

2.9.4 DEFINITION. An H-coassociative H-cospace equipped with a map that determines H-coinverses is called an *H-cogroup*. An H-cospace, or an H-cogroup, is *homotopy coabelian*, or *H-coabelian*, if the maps $\nu, S \circ \nu : Q \longrightarrow Q \vee Q$ are homotopic, where $S : Q \vee Q \longrightarrow Q \vee Q$ is the restriction of $T : Q \times Q \longrightarrow Q \times Q \quad (T(x,y) = (y,x))$.

2.9.5 DEFINITION. If Q and Q' are H-cospaces and $k : Q' \longrightarrow Q$ is continuous, we say that k is an *H-cohomomorphism* if the composites

$$Q' \xrightarrow{k} Q \xrightarrow{\nu} Q \vee Q, \quad Q' \xrightarrow{\nu'} Q' \vee Q' \xrightarrow{k \vee k} Q' \vee Q'$$

are homotopic; that is, if the diagram

$$\begin{tikzcd} Q' \arrow[r, "\nu'"] \arrow[d, "k"'] & Q' \vee Q' \arrow[d, "k \vee k"] \\ Q \arrow[r, "\nu"'] & Q \vee Q \end{tikzcd}$$

commutes up to homotopy.

An H-cogroup satisfies, up to homotopy, the axioms of a "cogroup," that is, the dual of the axioms of a group. These are obtained from the group axioms by reversing arrows and substituting Cartesian products $\times$ with coproducts $\vee$. If Y is an arbitrary pointed space, the assignment $Q \mapsto [Q, Y]_*$ reverses arrows in such a way that the relations that a cogroup Q satisfies up to homotopy are now satisfied by $[Q, Y]_*$, except with the arrows reversed. More precisely, we have a function

$$\overline{\nu} : [Q, Y]_* \times [Q, Y]_* \longrightarrow [Q, Y]_* \,,$$

given by

$$\overline{\nu}([f], [g]) = [\langle f, g \rangle \circ \nu] \,;$$

that is, the pair $([f], [g])$ is sent to the homotopy class of the composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{\langle f,g \rangle} Y \,.$$

It is an *exercise* to verify that $\overline{\nu}$ is well defined, that is, that it does not depend on the choice of the representatives f, g in the classes $[f]$ and $[g]$.

2.9.6 DEFINITION. Let Q be a pointed space. We say that $[Q, Y]_*$ has a *natural group structure in* Y if

(a) for every pointed space Y, $[Q, Y]_*$ has a group structure such that the class $[e]$ of the constant map $e : Q \longrightarrow Y$ is the unit of the group, and if

(b) for every pointed map $f : Y \longrightarrow X$, the induced function

$$f_* : [Q, Y]_* \longrightarrow [Q, X]_*$$

is a homomorphism of groups.

We have the following general result, dual to 2.7.6.

2.9.7 **Theorem**. *Let Q be a pointed space. Then $[Q,Y]_*$ has a natural group structure in Y if and only if Q is an H-cogroup.*

Proof: If Q is an H-cogroup, then as we have already indicated earlier, $[Q,Y]_*$ acquires a multiplication $\nu : [Q,Y]_* \times [Q,Y]_* \longrightarrow [Q,Y]_*$, and it is easy to prove that with it we obtain a natural group structure in Y. Conversely, let us suppose that $[Q,Y]_*$ has a natural group structure in Y. Let $i_1, i_2 : Q \longrightarrow Q \vee Q$ be the inclusions into the first and the second cofactors. Let $\nu : Q \longrightarrow Q \vee Q$ be a map that represents the product $[i_1][i_2]$ in the group structure in $[Q, Q \vee Q]_*$. It is easy to prove that in fact, this map ν is a comultiplication that gives Q the structure of an H-coassociative H-cospace. On the other hand, there exists a map $j : Q \longrightarrow Q$ that represents in the group structure in $[Q,Q]_*$ the inverse of the class of $\mathrm{id} : Q \longrightarrow Q$, that is, such that $[j] = [\mathrm{id}]^{-1}$. The map j determines H-coinverses, so that Q has the structure of an H-cogroup. □

2.9.8 EXERCISE. Reexamine all of the details of the proof of Theorem 2.9.7.

2.9.9 EXERCISE. Prove that if Q is an H-coabelian H-cogroup, then $[Q,Y]_*$ is an abelian group.

2.9.10 **Proposition**. *If $k : Q' \longrightarrow Q$ is an H-cohomomorphism of H-cogroups, then for every space Y,*

$$k^* : [Q,Y]_* \longrightarrow [Q',Y]_*$$

is a homomorphism of groups. □

2.10 SUSPENSIONS

The typical example of an H-cogroup is provided by the reduced suspension of a pointed space. This construction is, in a certain sense, the dual to the construction of the loop space that we have studied earlier.

2.10.1 DEFINITION. If X is a pointed space, we define its *reduced suspension* ΣX as the quotient

$$\Sigma X = X \times I / X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I,$$

which again is a pointed space whose base point is the image of $X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I$, after it has been collapsed to a point in the above quotient.

We denote by $x \wedge t$ the class of $(x,t) \in X \times I$. Thus the base point is $\overline{x}_0 = x_0 \wedge t = x \wedge 0$. If $f : X \longrightarrow Y$ is a pointed map, then $f \times \mathrm{id}_I$ induces a pointed map

$$\Sigma f : \Sigma X \longrightarrow \Sigma Y\,,$$

which satisfies $\Sigma f(x \wedge t) = f(x) \wedge t$.

2.10.2 Definition. We define a comultiplication

$$\nu : \Sigma X \longrightarrow \Sigma X \vee \Sigma X$$

by

$$\nu(x \wedge t) = \begin{cases} (x \wedge 2t, \overline{x}_0) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (\overline{x}_0, x \wedge (2t-1)) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

which has the effect of pinching the "equator" of ΣX (see Figure 2.10).

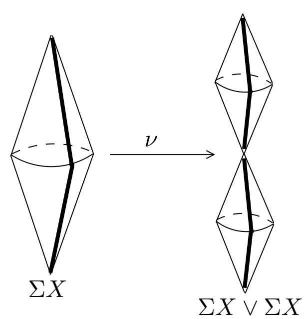


Figure 2.10

2.10.3 Exercise. Verify that ν gives ΣX the structure of an H-cogroup. In particular, let $\tau : \Sigma X \longrightarrow \Sigma X$ be given by $\tau(x \wedge t) = x \wedge (1-t)$; then prove that τ determines H-coinverses. (Hint: There is a way to use the homotopies of 2.8.1 in order to obtain the homotopies needed here.)

We have therefore the following result.

2.10.4 **Theorem**. *For every pointed space X, ΣX is an H-cogroup, and consequently, for every space Y, $[\Sigma X, Y]_*$ is a group. If $g : Y' \longrightarrow Y$ is continuous, then*

$$g_* : [\Sigma X, Y']_* \longrightarrow [\Sigma X, Y]_*$$

is a homomorphism of groups. Finally, if $f : X \longrightarrow X'$ *is a pointed map, then* $\Sigma f : \Sigma X \longrightarrow \Sigma X'$ *is an* H*-cohomomorphism, and so*

$$(\Sigma f)^* : [\Sigma X', Y]_* \longrightarrow [\Sigma X, Y]_*$$

is a homomorphism of groups. □

This theorem is not surprising if we observe the following proposition.

2.10.5 **Proposition**. *There is a homeomorphism*

$$M_*(\Sigma X, Y) \approx M_*(X, \Omega Y)$$

such that the induced bijection

$$[\Sigma X, Y]_* \equiv [X, \Omega Y]_*$$

is an isomorphism of groups.

Proof: To $g : \Sigma X \longrightarrow Y$ we assign $\widehat{g} : X \longrightarrow \Omega Y$ by defining $\widehat{g}(x)(t) = g(x \wedge t)$. Dually, to $f : X \longrightarrow \Omega Y$ we assign $\check{f} : \Sigma X \longrightarrow Y$ by $\check{f}(x \wedge t) = f(x)(t)$. These correspondences induce the desired homeomorphism and its inverse, as we show easily. This homeomorphism establishes a bijection of the path components, that is, the bijection that we seek. It is an easy *exercise* to prove that this bijection is an isomorphism of groups. □

2.10.6 Exercise. Show that the bijection $[\Sigma X, Y]_* \equiv [X, \Omega Y]_*$ is natural in X and in Y; namely, show that if $f : X' \longrightarrow X$ and $g : Y \longrightarrow Y'$ are pointed maps, then the diagrams

$$\begin{array}{ccc} [\Sigma X, Y]_* & \longrightarrow & [X, \Omega Y]_* \\ {\scriptstyle (\Sigma f)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ [\Sigma X', Y]_* & \longrightarrow & [X', \Omega Y]_* \end{array}$$

and

$$\begin{array}{ccc} [\Sigma X, Y]_* & \longrightarrow & [X, \Omega Y]_* \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (\Omega g)_*} \\ [\Sigma X, Y']_* & \longrightarrow & [X, \Omega Y']_* \end{array}$$

commute, where the horizontal arrows represent the corresponding isomorphisms.

2.10.7 REMARK. Let $\tau : \Sigma X \longrightarrow \Sigma X$ be as in 2.10.3. The function $\tau^* : [\Sigma X, Z]_* \longrightarrow [\Sigma X, Z]_*$ is in general not a homomorphism; it sends an element to its inverse. If $[\Sigma X, Z]_*$ is abelian (written additively), then it is the isomorphism given by multiplication by -1, i.e., by changing sign.

2.10.8 EXERCISE. If $n > 0$, prove that the n-sphere $\mathbb{S}^n$ is the suspension of the $(n-1)$-sphere, that is, $\mathbb{S}^n \approx \Sigma\mathbb{S}^{n-1}$.

Using the previous exercise, we can define the following.

2.10.9 DEFINITION. The set of pointed homotopy classes

$$\pi_n(X) = [\mathbb{S}^n, X]_*$$

is a group, called the *nth homotopy group* of X.

By proposition 2.10.5, for $n \geq 1$ we have

$$\pi_n(X) \cong \pi_{n-1}(\Omega X).$$

If we consider the bijection $[\Sigma^2 X, Y]_* \equiv [\Sigma X, \Omega Y]_*$, we obtain two group structures in the set on the right, since ΣX gives one group structure and ΩY gives another. But actually, these two structures coincide, and even more holds. Namely, we have the following general algebraic result, which relates two group multiplications in a set.

2.10.10 **Lemma**. *Let G be a set equipped with two multiplications $*, \bullet$ such that*

(a) $*, \bullet$ *have a common bilateral unit, and*

(b) $*, \bullet$ *are mutually distributive.*

Then $*$ and $\bullet$ coincide, as well as being both commutative and associative.

Proof: Take $w, x, y, z \in G$. Also let $e \in G$ be the unit.

(a) means that $e * x = x * e = x = e \bullet x = x \bullet e$.

(b) means that $(w * x) \bullet (y * z) = (w \bullet y) * (x \bullet z)$.

Therefore,

$$x * y = (x \bullet e) * (e \bullet y) = (x * e) \bullet (e * y) = x \bullet y\,,$$

and so $*$ and $\bullet$ coincide. Moreover,

$$x * y = (e \bullet x) * (y \bullet e) = (e * y) \bullet (x * e) = y \bullet x = y * x,$$

and so the structure is commutative. Finally,

$$x * (y * z) = (x \bullet e) * (y \bullet z) = (x * y) \bullet (e * z) = (x * y) * z,$$

and so the structure is associative. □

2.10.11 EXERCISE. Prove that if Q is an H-cogroup and W is an H-group, then the two multiplicative structures induced in $[Q, W]_*$ satisfy the hypotheses of the previous lemma.

Consequently, we have the following statements.

2.10.12 **Corollary**. *If Q is an H-cogroup and W is an H-group, then the set $[Q, W]_*$ has the structure of an abelian group.* □

2.10.13 **Corollary**. *For $n \geq 2$, the isomorphic groups*

$$[\Sigma^n X, Y]_* \cong [X, \Omega^n Y]_*$$

are abelian. □

And we have in particular the following consequence:

2.10.14 **Corollary**. *The homotopy groups of X, namely $\pi_n(X)$, are abelian if $n \geq 2$.* □

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CHAPTER 3

HOMOTOPY GROUPS

The last chapter ended with the definition of the homotopy groups of a pointed space. In this chapter, after a short section on some particularly interesting attaching spaces, we shall start with an analysis of the fundamental group of a pointed space, by proving the Seifert–van Kampen theorem; then we shall compute the fundamental groups of some spaces. Further on, the notion of homotopy group will be generalized to a definition for pairs of spaces, and we shall study these groups systematically. All of the spaces that we consider in this chapter, as well as all of the maps, are pointed.

3.1 ATTACHING SPACES; CYLINDERS AND CONES

A very useful construction in homotopy theory, as well as in other areas, is the attaching space of a continuous map. This construction allows us to obtain new spaces from given spaces. In this section we shall introduce the concept and examine the important particular cases of mapping cylinder and mapping cone.

3.1.1 DEFINITION. Let X and Y be (pointed) topological spaces, $A \subset X$ a closed subset (containing the base point), and $f : A \longrightarrow Y$ a continuous (pointed) map. The *attaching space* $Y \cup_f X$ is defined as the following quotient

$$Y \cup_f X = X \sqcup Y/\!\sim ,$$

where the relation $\sim$ is given as follows: $a \in A$ is identified with $f(a) \in Y$. Clearly, the composite $k : Y \overset{j}{\hookrightarrow} X \sqcup Y \overset{q}{\to} Y \cup_f X$ is an inclusion (as a closed subspace), where q is the quotient map. (See Figure 3.1.) (In a natural way, in the pointed case, $Y \cup_f X$ has a base point.)

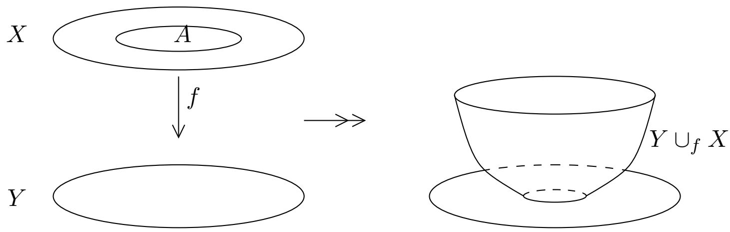


Figure 3.1

3.1.2 EXAMPLES. Let $f : X \longrightarrow Y$ be continuous. Then X can be represented as a subspace of the *cylinder* over X, $X \times I$, identifying it with the bottom of the cylinder, $X \approx X \times \{0\}$. We define the *mapping cylinder* of f as

$$M_f = Y \cup_f (X \times I) .$$

(See Figure 3.2.)

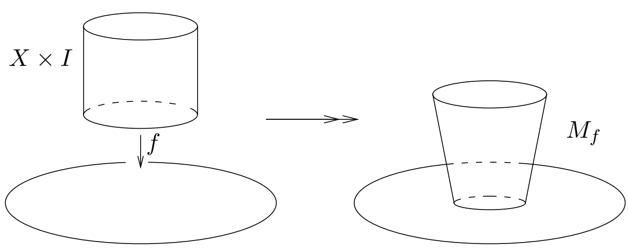


Figure 3.2

In the same way, X is a subspace of the *cone* over X, CX, which is defined as the quotient of the cylinder,

$$CX = X \times I/\{x_0\} \times I \cup X \times \{1\} ,$$

where once more we identify X with the bottom of the cone, $X \approx X \times \{0\} \subset CX$.

We define the *mapping cone* or *homotopy cofiber* of f as

$$C_f = Y \cup_f CX .$$

(See Figure 3.3.)

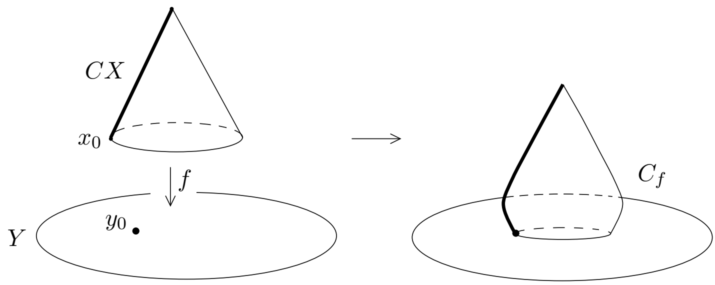


Figure 3.3

3.1.3 Exercise. By 3.1.1, $Y \hookrightarrow CX \sqcup Y \longrightarrow C_f$ is an inclusion. Prove that $C_f/Y \approx \Sigma X$.

3.1.4 Definition. Let $f : X \longrightarrow Y$ be continuous. We say that f is *nullhomotopic* if $f \simeq c$, in other words, if f is homotopic to the map $c : X \longrightarrow Y$, where $c(X) = \{y_0\}$ and where the *nullhomotopy* $H : f \simeq c$ is *pointed*; that is, it satisfies $H(x_0, t) = y_0$ for all $t \in I$.

3.1.5 **Lemma**. *$f : X \longrightarrow Y$ is nullhomotopic if and only if it admits an extension $F : CX \longrightarrow Y$.*

Proof: Let $H : X \times I \longrightarrow Y$ be a nullhomotopy. Then $H(\{x_0\} \times I \cup X \times \{1\}) = \{y_0\}$, and so it determines a map

$$F : CX \longrightarrow Y$$

given by $F(x, 0) = H(x, 0) = f(x)$. Therefore, F extends f.

Conversely, if $F : CX \longrightarrow Y$ extends f, then the composite

$$H : X \times I \longrightarrow CX \xrightarrow{F} Y$$

is a nullhomotopy of f. □

We have the following lemma, which shows that the inclusion $X \hookrightarrow CX$ has a homotopy extension property.

3.1.6 **Lemma**. *Let $F : CX \longrightarrow Y$ be continuous and let $H : X \times I \longrightarrow Y$ be a homotopy that starts with $f = F|X$. Then we can extend H to a homotopy*

$G : CX \times I \longrightarrow Y$ that starts with F. That is, in the commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "j_0"] \arrow[d, hook] & X \times I \arrow[d, hook] \arrow[ddr, bend left, "H"] & \\ CX \arrow[r, hook, "j_0"] \arrow[drr, bend right, "F"'] & CX \times I \arrow[dr, dashed, "G"] & \\ & & Y \end{tikzcd}$$

there exists $G : CX \times I \longrightarrow Y$ that makes both triangles commute.

Proof: Let us define

$$G(\overline{(x,t)}, t') = \begin{cases} F\overline{(x, 1-(1-t)(1+t'))} & \text{if } (1-t)(1+t') \leq 1, \\ H(x, (1-t)(1+t')-1) & \text{if } (1-t)(1+t') \geq 1, \end{cases}$$

where $\overline{(x,t)}$ denotes the image of $(x,t) \in X \times I$ in CX. Then G extends H. So the diagram commutes as desired. □

Because of this, we say that the pair (CX, X) has the *homotopy extension property*, HEP, which will be studied systematically in the next chapter (see 4.1.1).

Before concluding this section we shall study some homotopy properties related to the constructions of the mapping cone and the mapping cylinder. These will be useful for us in subsequent chapters. The next one generalizes 3.1.5.

3.1.7 **Proposition**. *Let us consider the maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $g \circ f$ is nullhomotopic if and only if there exists $G : C_f \longrightarrow Z$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[rr, hook] \arrow[dr, "g"'] & & C_f \arrow[dl, dashed, "G"] \\ & & Z & \end{tikzcd}$$

commutes, that is, if and only if g has an extension G to the mapping cone of f.

Proof: By 3.1.5, $g \circ f : X \longrightarrow Z$ is nullhomotopic if and only if $g \circ f$ has an extension $H : CX \longrightarrow Z$. Clearly, $(H, g) : CX \sqcup Y \longrightarrow Z$ determines the map G that we seek.

Conversely, if there exists $G : C_f \longrightarrow Z$, then the composite $CX \hookrightarrow Y \cup_f CX = C_f \xrightarrow{G} Z$ is an extension of $g \circ f$, so that by 3.1.5, $g \circ f$ is nullhomotopic. □

3.1.8 **Proposition**. *Let $g : Y \longrightarrow Z$ be continuous and Z be path connected. Suppose, furthermore, that $\pi_{n-1}(Z) = 0$. Then, given $f : \mathbb{S}^{n-1} \longrightarrow Y$, g admits an extension $G : Y \cup_f \mathbb{D}^n \longrightarrow Z$.*

Proof: Since $\pi_{n-1}(Z) = 0$, the composite $g \circ f : \mathbb{S}^{n-1} \longrightarrow Z$ is nullhomotopic. By 3.1.7, g admits an extension $G : C_f \longrightarrow Z$, but clearly, $C_f \approx Y \cup_f \mathbb{D}^n$. □

3.2 The Seifert–van Kampen Theorem

After having given some constructions of new spaces out of old, in this section we come back to the fundamental group. A very useful tool is a formula that in some cases allows us to compute the fundamental group of certain spaces in terms of the fundamental groups of parts of them. Before going to the general formula, as an example of it, let us first analyze a special case.

3.2.1 **Proposition**. *Let $X = X_1 \cup X_2$ with X_1, X_2 open subsets. If X_1 and X_2 are simply connected and $X_1 \cap X_2$ is path connected, then X is simply connected.*

Proof: Let $\lambda : I \longrightarrow X$ be a loop based at $x_0 \in X_1 \cap X_2$. We have that $\{\lambda^{-1}(X_1), \lambda^{-1}(X_2)\}$ is an open cover of I. There exists a number $\delta > 0$, called the *Lebesgue number* of this cover, such that if $0 \leq t - s < \delta$, then $[s,t] \subset \lambda^{-1}(X_1)$ or $[s,t] \subset \lambda^{-1}(X_2)$. Hence, one has a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval I such that

$$\lambda([t_0,t_1]) \subset X_1, \quad \lambda([t_1,t_2]) \subset X_2, \ \ldots, \lambda([t_{k-1},t_k]) \subset X_2 \,.$$

Since $\lambda(t_i) \in X_1 \cap X_2$, there exist paths $\omega_i : x_0 \simeq \lambda(t_i)$ in $X_1 \cap X_2$, $i = 1, 2, \ldots, k-1$; let moreover ω_0 as well as ω_k denote the constant path at $x_0 = \lambda(t_0) = \lambda(0) = \lambda(1) = \lambda(t_k)$. The loops

$$\mu_i(t) = \begin{cases} \omega_{i-1}(3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \lambda_i(3t-1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \omega_i(3-3t) & \text{if } \frac{2}{3} \leq t \leq 1, \end{cases}$$

where $\lambda_i(t) = \lambda((1-t)t_{i-1} + tt_i)$ is the portion of λ in the interval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, k$, lie in X_1 or in X_2, and therefore they are contractible in X_1 or in X_2 and hence in X; that is, $\mu_i \simeq 0$ in X. Since $\lambda \simeq \mu_1\mu_2 \cdots \mu_k$, we have that λ is contractible, that is, $\lambda \simeq 0$. Figure 3.4 shows the proof graphically. □

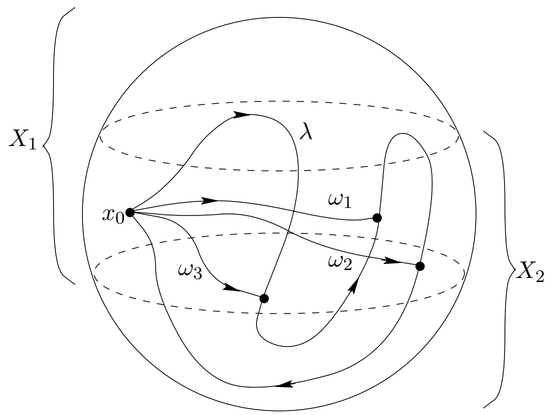


Figure 3.4

An important application is given in the next example.

3.2.2 Example. If $n > 1$, then the sphere $\mathbb{S}^n$ is simply connected. For if $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ are the poles of the sphere and $X_1 = \mathbb{S}^n - S$, $X_2 = \mathbb{S}^n - N$, then the hypotheses of 3.2.1 hold, since X_1 and X_2, being homeomorphic to $\mathbb{R}^n$, are contractible, and $X_1 \cap X_2$ is path connected, since $X_1 \cap X_2 \approx \mathbb{S}^{n-1} \times (-1, 1) \simeq \mathbb{S}^{n-1}$.

3.2.3 Exercise. Prove that *if X is path connected, then its (reduced) suspension ΣX is simply connected.*

3.2.4 Note. The previous exercise is quite straightforward if instead of ΣX, one takes the *unreduced suspension*, defined by $SX = X \times I/\simeq$, where $(x, s) \simeq (y, t)$ if and only if $x = y$ and $s = t$ or $s = t = 0$ or 1, since in it one has a "north pole" and a "south pole" as in the case of the spheres. There is a canonical quotient map $SX \longrightarrow \Sigma X$, which collapses the meridian $\{(x_0, t) \mid t \in I\}$ in SX onto the base point. One may prove that if the space X is well pointed (see Chapter 5), then the quotient map is a homotopy equivalence. This fact is a consequence then of Lemma 3.3.2 below.

The Seifert–van Kampen theorem is a generalization of 3.2.1, because it allows one to compute the fundamental group of a union of open subspaces if one knows the fundamental groups of each of them and the way that the fundamental group of the intersection relates to these.

We shall use the concept of a free product $G_1 * G_2$ of two groups, which, in brief, consists of finite words $x_1x_2 \cdots x_{2n}$, where $x_1 \in G_1$, $x_2 \in G_2$, $x_3 \in G_1$, $x_4 \in G_2$, and so on, and no term x_i is the trivial element, with the possible exceptions of x_1 and x_{2n}, and the product of two such words is obtained by juxtaposition, then omitting trivial elements, and finally grouping together consecutive elements in the same group (see [48]).

Before stating the Seifert–van Kampen theorem in its general form, let us prove a generalization of 3.2.1. Take a topological space $X = X_1 \cup X_2$ with $X_1 \cap X_2 \neq \emptyset$ and $x_0 \in X_1 \cap X_2$. Then, by the functoriality of the fundamental group, the commutative diagram of inclusions of topological spaces

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & X_1 \arrow[d, hook, "j_1"] \\ X_2 \arrow[r, hook, "j_2"'] & X \end{tikzcd}$$

induces a commutative diagram of group homomorphisms

$$\begin{tikzcd} \pi_1(X_1 \cap X_2, x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(X_1, x_0) \arrow[d, "j_{1*}"] \\ \pi_1(X_2, x_0) \arrow[r, "j_{2*}"'] & \pi_1(X, x_0)\,. \end{tikzcd}$$

3.2.5 **Lemma**. *If X_1 and X_2 are open sets in X and are such that they as well as $X_1 \cap X_2$ are 0-connected, then $\pi_1(X, x_0)$ is generated by the images of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ under j_{1*} and j_{2*}, respectively. Therefore, the homomorphism*

$$\varphi : \pi_1(X_1, x_0) * \pi_1(X_2, x_0) \longrightarrow \pi_1(X, x_0)$$

induced by j_{1*} and j_{2*} is an epimorphism.

The *proof* of this result is essentially the same as the one given for 3.2.1. We leave it as an *exercise* to the reader. □

According to the previous lemma, if we want to compute $\pi_1(X)$ in terms of $\pi_1(X_1)$, $\pi_1(X_2)$, and $\pi_1(X_1 \cap X_2)$, the only thing left to do is to compute the subgroup $N = \ker(\varphi)$. Using similar techniques (though more complicated), one can show that N is the normal subgroup of $\pi_1(X_1, x_0) * \pi_1(X_2, x_0)$ generated by the set

$$\{i_{1*}(\alpha) i_{2*}(\alpha)^{-1} \mid \alpha \in \pi_1(X_1 \cap X_2, x_0)\}$$

(see [60], [9], or [50]). We thus have the main theorem.

3.2.6 Theorem. (Seifert–van Kampen) *Let $X = X_1 \cup X_2$, with X_1, X_2 open. If X_1, X_2, and $X_1 \cap X_2$ are nonempty and path connected, then, for $x_0 \in X_1 \cap X_2$,*

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0)/N,$$

where N is the normal subgroup generated by the set

$$\{i_{1*}(\alpha)i_{2*}(\alpha)^{-1} \mid \alpha \in \pi_1(X_1 \cap X_2, x_0)\}.$$

$\square$

We have some very nice applications of this theorem, which allow us to compute a number of fundamental groups for several spaces. The first computation is the following.

3.2.7 Corollary. *Under the assumptions of* 3.2.6 *one has the following.*

(a) *If X_2 is simply connected, then*

$$j_{1*} : \pi_1(X_1, x_0) \longrightarrow \pi_1(X, x_0)$$

is an epimorphism and $\ker j_{1*}$ *is the normalizer of the subgroup*

$$i_{1*}(\pi_1(X_1 \cap X_2, x_0)).$$

(b) *If $X_1 \cap X_2$ is simply connected, then*

$$j_{1*} \cdot j_{2*} : \pi_1(X_1, x_0) * \pi_1(X_2, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism.

(c) *If X_2 and $X_1 \cap X_2$ are simply connected, then*

$$j_{1*} : \pi_1(X_1, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism. $\square$

3.2.8 Proposition. *The fundamental group of a wedge of k copies of the circle, $\mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_k$, is freely generated by the elements*

$$\alpha_1, \dots, \alpha_k \in \pi_1(\mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_k, x_0),$$

where x_0 is the base point of the wedge obtained from all of the elements $1 \in \mathbb{S}^1_i$, and the class α_i is represented by the canonical loop $\lambda_i : I \longrightarrow \mathbb{S}^1_i \hookrightarrow \mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_k$ given by $\lambda_i(t) = \mathrm{e}^{2\pi \mathrm{i}t} \in \mathbb{S}^1_i$. Therefore,

$$\pi_1(\underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{k}) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k}.$$

Proof: By induction on k. For a wedge of two circles, $X = \mathbb{S}^1_1 \vee \mathbb{S}^1_2$, take $X_1 = \mathbb{S}^1_1 \vee (\mathbb{S}^1_2 - \{-1\})$ and $X_2 = (\mathbb{S}^1_1 - \{-1\}) \vee \mathbb{S}^1_2$. Then X, X_1, and X_2 satisfy the hypotheses of the Seifert–van Kampen theorem, and since $X_1 \cap X_2$ is homeomorphic to the open cross $\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$, it is contractible. Thus using 3.2.7(b) and the fact that the inclusions $\mathbb{S}^1_1 \hookrightarrow X_1$ and $\mathbb{S}^1_2 \hookrightarrow X_2$ induce isomorphisms in the fundamental groups, one has that $\pi_1(\mathbb{S}^1_1, 1) * \pi_1(\mathbb{S}^1_2, 1) \longrightarrow \pi_1(\mathbb{S}^1_1 \vee \mathbb{S}^1_2, x_0)$ is an isomorphism. Moreover, since the classes α_1 and α_2 come from the canonical generators of $\pi_1(\mathbb{S}^1_1, 1)$ and $\pi_1(\mathbb{S}^1_2, 1)$, they are the generators of $\pi_1(\mathbb{S}^1_1 \vee \mathbb{S}^1_2, x_0)$ as a free group. Therefore, the group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1, x_0)$ is isomorphic to $\mathbb{Z} * \mathbb{Z}$.

If for a wedge of $k-1$ copies of $\mathbb{S}^1$ the result is true, then take

$$X_1 = \mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_{k-1} \vee (\mathbb{S}^1_k - \{-1\}),$$

which has the same homotopy type via the inclusion of $\mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_{k-1}$, and take

$$X_2 = (\mathbb{S}^1_1 - \{-1\}) \vee \cdots \vee (\mathbb{S}^1_{k-1} - \{-1\}) \vee \mathbb{S}^1_k,$$

which also via the inclusion has the same homotopy type of $\mathbb{S}^1_k$. Since $X_1 \cap X_2$ is homeomorphic to a "star" with $2k$ rays, it is contractible, and, again by 3.2.7(b),

$$\pi_1(\mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_{k-1}, x_0) * \pi_1(\mathbb{S}^1_k, 1) \longrightarrow \pi_1(\mathbb{S}^1_1 \vee \cdots \vee \mathbb{S}^1_k, x_0)$$

is an isomorphism. And as was the case for $k = 2$, we have that $\alpha_1, \ldots, \alpha_k$ are its generators as a free group, as we wanted to prove. □

The Seifert–van Kampen theorem can be used to study the fundamental group of a space with a cell attached.

3.2.9 **Proposition**. *For Y path connected, let $f : \mathbb{S}^{n-1} \longrightarrow Y$ be continuous, $n \geq 3$. If $y_0 \in Y$, then the canonical inclusion $i : Y \hookrightarrow Y \cup_f e^n$ induces an isomorphism*

$$i_* : \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(Y \cup_f e^n, y_0).$$

Proof: Let $X = Y \cup_f e^n$ and let $q : \mathbb{D}^n \sqcup Y \longrightarrow X$ be the identification. The subspaces $X_1 = q((\mathbb{D}^n - \{0\}) \sqcup Y)$ and $X_2 = q(\overset{\circ}{\mathbb{D}}{}^n)$ are open. Notice that the canonical inclusion $Y \hookrightarrow X_1$ is a homotopy equivalence and that X_2 is contractible. Moreover, the intersection $X_1 \cap X_2 \approx \overset{\circ}{\mathbb{D}}{}^n - \{0\}$, which has the same homotopy type of the sphere $\mathbb{S}^{n-1}$, is simply connected, since $n \geq 3$. Therefore, by 3.2.7(c), if $x_0 \in X_1 \cap X_2$, then the inclusion $X_1 \hookrightarrow X$ induces an isomorphism $\pi_1(X_1, x_0) \longrightarrow \pi_1(X, x_0)$.

Take now a path $\omega : x_0 \simeq y_0$ in X_1. Then the homomorphism induced by the inclusion $i_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, y_0)$ factors as indicated in the commutative diagram

$$\begin{tikzcd}
\pi_1(Y, y_0) \arrow[r, "i_*"] \arrow[d, "\cong"] & \pi_1(X, y_0) \arrow[dd, "\cong"', "\varphi_\omega"] \\
\pi_1(X_1, y_0) \arrow[d, "\varphi_\omega"', "\cong"] & \\
\pi_1(X_1, x_0) \arrow[r, "\cong"] & \pi_1(X, x_0) \, ,
\end{tikzcd}$$

where the unnamed isomorphisms are induced by inclusions and the φ_ω are the isomorphisms of 2.5.18 in X_1 and in X, respectively. Therefore, i_* is an isomorphism, as desired. □

Let us now see what happens in the case of the attachment of a 2-cell.

3.2.10 **Proposition**. *Let $f : \mathbb{S}^1 \longrightarrow Y$ be continuous. If $\lambda_f : I \longrightarrow Y$ is the loop given by $\lambda_f(t) = f(\mathrm{e}^{2\pi \mathrm{i} t})$ and $\omega : y_0 \simeq f(1)$ is a path in Y, then the inclusion $i : Y \hookrightarrow Y \cup_f e^2$ induces an epimorphism $i_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Y \cup_f e^2, y_0)$, and its kernel is the normal subgroup N_{α_f} generated by the element $\alpha_f = [\omega \lambda_f \overline{\omega}] \in \pi_1(Y, y_0)$. Therefore,*

$$\pi_1(Y \cup_f e^2, y_0) \cong \pi_1(Y, y_0)/N_{\alpha_f} \, .$$

The group N_{α_f} does not depend on the path ω, since the loop $\mu_f = \omega \lambda_f \overline{\omega}$ that surrounds the cell is contractible in $Y \cup_f e^2$, because it can be contracted over the cell, as shown in Figure 3.5. Before attaching the cell one has $\mu_f \not\simeq 0$, but after doing it, $\mu_f \simeq 0$. Therefore, $i_*(\alpha_f) = [\mu_f] = 1$ in $\pi_1(Y \cup_f e^2, y_0)$. One says that the element $\alpha_f \in \pi_1(Y, y_0)$ *is killed by attaching the 2-cell* using the map f.

Proof: Using the same notation as in the previous proof, we have that the canonical inclusion $Y \hookrightarrow X_1$ is a homotopy equivalence and that X_2 is contractible. Moreover, the intersection $X_1 \cap X_2 \approx \mathring{\mathbb{D}}^2 - \{0\}$ has the same homotopy type of the circle $\mathbb{S}^1$ and so is not simply connected. By 3.2.7(a) the inclusion $X_1 \hookrightarrow X$ induces an epimorphism on the fundamental group, and so $i_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, y_0)$ is an epimorphism.

On the other hand, if $z_0 = q(0) = q(1)$, then the loop $\lambda'_f : I \longrightarrow X$ given by $\lambda'_f(t) = q(\frac{1}{2}\mathrm{e}^{2\pi \mathrm{i} t})$, which indeed lies inside $X_1 \cap X_2$, generates $\pi_1(X_1 \cap X_2, z_0) \cong \mathbb{Z}$. Also, the deformation retraction of X_2 into Y deforms λ'_f

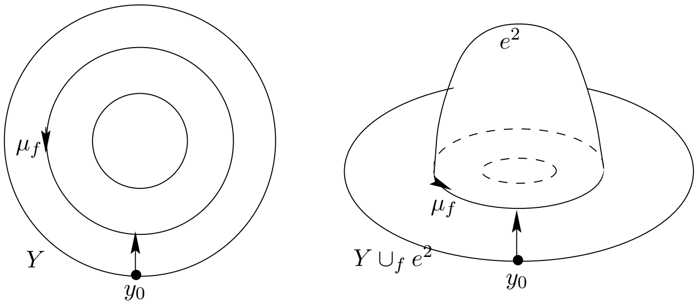


Figure 3.5

in λ_f. Letting $j : X_1 \hookrightarrow X$ denote the inclusion, we know from 3.2.7(a) that $\ker(j_*)$ is generated as a normal subgroup by the element $[\lambda'_f]$, and so $\ker(i_* : \pi_1(Y, f(1)) \longrightarrow \pi_1(X, f(1)))$ is generated by $[\lambda_f]$, and, as in the previous proof, $\ker(i_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, y_0))$ is generated by α_f. □

Inductively, it is possible to prove the following result.

3.2.11 **Corollary**. *If the 2-cells $e_1^2, e_2^2, \ldots, e_k^2$ are attached to Y using the maps $f_1, f_2, \ldots, f_k : \mathbb{S}^1 \longrightarrow Y$, then*

$$\pi_1(Y \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_k^2, y_0) \cong \pi_1(Y, y_0)/N_{\{\alpha_{f_1}, \alpha_{f_2}, \ldots, \alpha_{f_k}\}} \, .$$

□

3.2.12 Examples.

(a) For any integer $k \geq 1$, let $X_k = \mathbb{S}^1 \cup e^2$, where the cell is attached using the map $g_k : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree k, $g_k(\zeta) = \zeta^k$. If $[\alpha] \in \pi_1(\mathbb{S}^1, 1)$ is the canonical generator, then $\pi_1(X_k, 1) \cong \pi_1(\mathbb{S}^1, 1)/N_{\{\alpha_k\}}$, where $\alpha_k = [\lambda_{g_k}] \in \pi_1(\mathbb{S}^1, 1)$. By 2.6.13, $\alpha_k = \alpha_1^k \in \pi_1(\mathbb{S}^1, 1)$; that is, α_k is the kth power of the canonical generator. Therefore,

$$\pi_1(X_k, 1) \cong \mathbb{Z}/k,$$

that is, this fundamental group is cyclic of order k.

(b) The construction of (a) for $k = 2$ produces $X_2 \approx \mathbb{RP}^2$, that is, the projective plane. Therefore,

$$\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2.$$

There are several ways of grasping this fact. If, for instance, we realize $\mathbb{RP}^2$ by identifying antipodal points in the boundary of $\mathbb{D}^2$, then the map $\lambda_1 : I \longrightarrow \mathbb{D}^2$ given by $\lambda_1(t) = \mathrm{e}^{\pi i t}$ determines a loop λ' in $\mathbb{RP}^2$ (see Figure 3.6(a)). Since by 3.2.7(a), $\pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{RP}^2)$ is an epimorphism, the class $[\lambda']$ generates $\pi_1(\mathbb{RP}^2)$; that is, this group is cyclic. Defining $\lambda_2(t) = \mathrm{e}^{\pi i(t+1)}$ and $\lambda = \lambda_1\lambda_2$, we have that λ surrounds $\mathbb{D}^2$ once and therefore is contractible. Since λ_1, λ_2, and λ' all determine the same homotopy class in $\pi_1(\mathbb{RP}^2)$, $[\lambda']^2 = 1 \in \pi_1(\mathbb{RP}^2)$, that is, this group is cyclic of order 2.

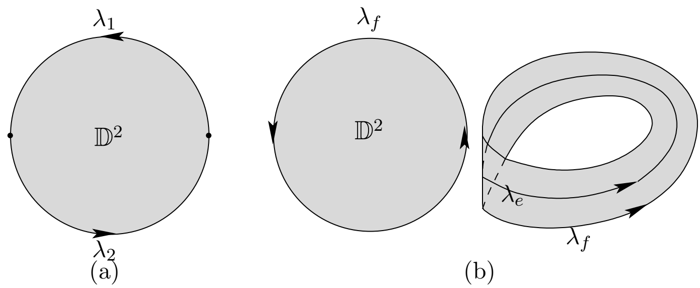


Figure 3.6

Another way of looking at this is the following. $\mathbb{RP}^2$ is obtained by attaching a 2-cell to the Moebius band M along its boundary, which is homeomorphic to $\mathbb{S}^1$. Since M has the same homotopy type of $\mathbb{S}^1$, the equatorial loop λ_e that surrounds the equator of M once (see 2.6.9(c)) generates $\pi_1(M)$ as an infinite cyclic group. If $f : \mathbb{S}^1 \longrightarrow \partial M \hookrightarrow M$ is a homeomorphism onto the boundary of M, the loop λ_f in $\mathbb{RP}^2 = M \cup_f e^2$ is such that it deforms inside M to the equator to become λ_e^2 (see Figure 3.6(b)). Consequently, $[\lambda_e]^2 = 1 \in \pi_1(\mathbb{RP}^2)$, so we again see that this group is cyclic of order 2.

Considering $\mathbb{RP}^2$ as a quotient of $\mathbb{S}^2$ by identifying antipodal points, we may repeat the construction above. A path $\lambda : I \longrightarrow \mathbb{S}^2$ that uniformly travels along one-half of the equator of the sphere determines in $\mathbb{RP}^2$ a loop μ, generating $\pi_1(\mathbb{RP}^2)$ and whose square comes from λ^2. Since it travels along the whole equator of $\mathbb{S}^2$, the loop λ_2 can be deformed into a constant loop, and so $[\mu]^2 = 1$ in $\pi_1(\mathbb{RP}^2)$.

(c) The *orientable surface of genus* g, S_g, is obtained by attaching a 2-cell to the wedge of $2g$ circles $S^1_{2g} = \mathbb{S}^1_{a_1} \vee \mathbb{S}^1_{b_1} \vee \cdots \vee \mathbb{S}^1_{a_g} \vee \mathbb{S}^1_{b_g}$ with the map $f_g : \mathbb{S}^1 \longrightarrow S^1_{2g}$, such that as the argument travels around the circle counterclockwise, the values of the map first go around $\mathbb{S}^1_{a_1}$ counterclockwise, then $\mathbb{S}^1_{b_1}$ also counterclockwise, then again $\mathbb{S}^1_{a_1}$ but now clockwise, and then $\mathbb{S}^1_{b_1}$ clockwise, and so on, and finishing by going around $\mathbb{S}^1_{b_g}$ clockwise. (See [50], [9], or [60].) Then the associated loop $\lambda_g = \lambda_{f_g} : I \longrightarrow S^1_{2g}$ is the loop product $\lambda_{a_1}\lambda_{b_1}\overline{\lambda}_{a_1}\overline{\lambda}_{b_1}\lambda_{a_2}\cdots\overline{\lambda}_{a_g}\overline{\lambda}_{b_g}$, where λ_{a_i} and λ_{b_i} are the canonical loops in $\mathbb{S}^1_{a_i} = \mathbb{S}^1$ and $\mathbb{S}^1_{b_i} = \mathbb{S}^1$, and $\overline{\lambda}_{a_i}$ and $\overline{\lambda}_{b_g}$ are their inverses, $i = 1, \dots, g$. By 3.2.8, $\pi_1(S^1_{2g})$ is freely generated by the classes $\alpha_i = [\lambda_{a_i}]$, $\beta_i = [\lambda_{b_i}]$.

By 3.2.10, $\pi_1(S_g) \cong \pi_1(S^1_{2g})/N_{\alpha_{f_g}}$. That is,

$$\pi_1(S_g) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g} / N_{\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}}\,,$$

where α_i is the generator of the $(2i-1)$th copy of $\mathbb{Z}$ and β_i of the $2i$th, $i = 1, \dots, g$. In terms of generators and relations, this fact is usually written as

$$\pi_1(S_g) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} \rangle\,,$$

and one says that this group *has as generators* the elements $\alpha_1, \beta_1, \dots,$ α_g, β_g subject only to the *relation*

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1\,.$$

(d) Analogously to (c) we can compute the fundamental group of a *nonorientable surface N_g of genus g* defined as the result of attaching a 2-cell to a wedge of g circles $S^1_g = \mathbb{S}^1_{a_1} \vee \cdots \vee \mathbb{S}^1_{a_g}$ but now with the map $f_g : \mathbb{S}^1 \longrightarrow S^1_g$ such that as the argument travels around the circle counterclockwise, the values of the map first go around $\mathbb{S}^1_{a_1}$ counterclockwise, then $\mathbb{S}^1_{a_2}$ also counterclockwise, and so on, and finishing by going around $\mathbb{S}^1_{a_g}$ counterclockwise. Therefore, we now have

$$\pi_1(N_g) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{g} / N_{\alpha_1^2\cdots\alpha_g^2}\,,$$

where α_i is the generator of the ith copy $\mathbb{Z}$. In terms of generators and relations, one has

$$\pi_1(N_g) = \langle \alpha_1, \dots, \alpha_g \mid \alpha_1^2\cdots\alpha_g^2 \rangle\,;$$

that is, this group has as generators the elements $\alpha_1, \dots, \alpha_g$ subject to the one relation $\alpha_1^2\cdots\alpha_g^2 = 1$.

Using examples (c) and (d) above, we can distinguish surfaces of different genus.

3.2.13 **Corollary**. *No two of the surfaces*

$$S_0, S_1, S_2, \ldots, N_1, N_2, \ldots$$

have the same homotopy type, and in particular, they are not homeomorphic.

Proof: If the fundamental groups of these surfaces are abelianized, we have

$$\pi_1(S_g)^{\mathrm{ab}} \cong \mathbb{Z}^{2g}, \qquad \pi_1(N_h)^{\mathrm{ab}} \cong \mathbb{Z}^{h-1} \times (\mathbb{Z}/2).$$

Here $\mathbb{Z}^0$ denotes 0. Since no two of these groups are isomorphic, we have that no two of these surfaces have the same homotopy type. This implies that no two of them are homeomorphic. □

3.2.14 EXERCISE. Compute the fundamental groups of the following spaces:

(a) $\mathbb{S}^1 \vee \mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{RP}^2$, $\mathbb{RP}^2 \vee \mathbb{RP}^2$, $\mathbb{RP}^2 \times \mathbb{RP}^2$.

(b) $\mathbb{R}^3 - C$, where C is the circle $x^2 + y^2 = 1$, $z = 0$.

(c) $(\mathbb{S}^1 \times \mathbb{S}^1) \cup e^2$, where the 2-cell is attached using the map $f(\zeta) = (\zeta^2, \zeta^3)$.

3.3 HOMOTOPY SEQUENCES I

In this section we shall introduce a sequence of spaces constructed out of mapping cones and maps between them. This sequence has the property that when we apply the functor $[-, W]_*$, namely, the functor of pointed homotopy classes of maps into a pointed space W, we get an exact sequence. We shall also introduce, dually, a sequence of spaces constructed out of the so-called homotopy fibers and maps between them. This sequence has the dual property that when we apply the functor $[W, -]_*$, namely, the functor of pointed homotopy classes of maps from a pointed space W, we get an exact sequence.

Let $f : X \longrightarrow Y$ be continuous. Then using the mapping cone construction we can define the following sequence:

$$X \xrightarrow{f} Y \xrightarrow{i_1} C_f \xrightarrow{i_2} C_{i_1} \xrightarrow{i_3} C_{i_2} \longrightarrow \cdots, \tag{3.3.1}$$

where i_1 is the canonical inclusion of Y into $C_f = Y \cup_f CX$ and, analogously, i_k is the canonical inclusion of $C_{i_{k-2}}$ into the mapping cone of i_{k-1}, $C_{i_{k-1}} = C_{i_{k-2}} \cup_{i_{k-1}} CC_{i_{k-3}}$.

It is possible to identify, up to homotopy, the spaces C_{i_k} in terms of X and Y. To do this, let us consider the following lemma.

3.3.2 Lemma. *Take $Y' \subset Y$ and suppose that there exists a homotopy $H : Y \times I \longrightarrow Y$ such that*

(a) $H(y, 0) = y$,

(b) $H(Y' \times I) \subset Y'$,

(c) $H(Y' \times \{1\}) = \{y_0\}$.

Then the identification $q : Y \longrightarrow Y/Y'$ is a homotopy equivalence.

Proof: Using (c), we can define $s : Y/Y' \longrightarrow Y$ given by

$$s(q(y)) = H(y, 1).$$

Then H is a homotopy between id_Y and $s \circ q$.

On the other hand, using (b), H determines a homotopy $\overline{H} : (Y/Y') \times I \longrightarrow Y/Y'$ such that

$$\overline{H}(q(y), t) = q\,H(y, t).$$

So $\overline{H}$ begins with $\mathrm{id}_{Y/Y'}$ and ends with $q \circ s$. □

3.3.3 Corollary. *Let $f : X \longrightarrow Y$ be continuous, $i : Y \longrightarrow C_f$ the canonical inclusion, and C_i its mapping cone. Then $CY \subset C_i$ and the identification*

$$C_i \longrightarrow C_i/CY$$

is a homotopy equivalence. Moreover, $C_i/CY \approx C_f/Y$.

Proof: Using 3.3.2 it is enough to construct a homotopy

$$H : C_i \times I \longrightarrow C_i$$

that sends CY into itself and that begins with the identity and ends with the constant map. First let

$$F : CY \times I \longrightarrow CY$$

be the contraction $F(\overline{(y, t)}, s) = \overline{(y, 1 - (1 - s)(1 - t))}$. It is easy to see that $C_i = C_f \cup_i CY = (Y \cup_f CX) \cup_i CY$ is the quotient of $CX \sqcup CY$ that identifies $\overline{(x, 0)} \in CX$ with $\overline{(f(x), 0)} \in CY$. Let $q : CX \sqcup CY \longrightarrow C_i$ be that identification. So the canonical inclusion $j : CY \longrightarrow C_i$ is clearly the restriction of q to CY.

Let G be given by

$$G : X \times I \xrightarrow{f\times \mathrm{id}} Y \times I \hookrightarrow CY \times I \xrightarrow{F} CY \overset{j}{\hookrightarrow} C_i\,.$$

Then G is a homotopy satisfying

$$G(x,0) = j\overline{(f(x),0)} = q\overline{(f(x),0)} = q\overline{(x,0)}\,;$$

that is, $G(x,0) = g\overline{(x,0)} \in C_i$, where g denotes the composite $CX \hookrightarrow CX \sqcup CY \longrightarrow C_i$. So $G : X \times I \longrightarrow C_i$ is a homotopy that begins with $g|X$. Using 3.1.6 we can extend G to a homotopy $F' : CX \times I \longrightarrow C_i$ such that $F'(\overline{(x,0)},s) = G(x,s) = j\overline{(f(x),s)}$. So we can define $H : C_i \times I \longrightarrow C_i$ by

$$H(\overline{(z,t)},s) = \begin{cases} F'(\overline{(z,t)},s) & \text{if } \overline{(z,t)} \in CX,\\ F(\overline{(z,t)},s) & \text{if } \overline{(z,t)} \in CY, \end{cases}$$

which is well defined, since if $x \in X$, then q identifies $\overline{(x,0)}$ with $\overline{(f(x),0)}$ in C_i, and we have that

$$F'(\overline{(x,0)},s) = G(x,s) = \overline{(f(x),s)} = F(\overline{(f(x),0)},s)\,.$$

Finally, it is clear that $C_i/CY \approx C_f/Y$ holds, as one can see in Figure 3.7. □

Using the above and Exercise 3.1.3, we have in the sequence (3.3.1) the following homotopy equivalences:

(3.3.4) $$C_{i_1} \simeq C_{i_1}/CY \approx C_f/Y \approx \Sigma X\,,$$

and in a similar way,

(3.3.5) $$C_{i_2} \simeq C_{i_2}/C(C_f) \approx C_{i_1}/C_f \approx \Sigma Y\,.$$

Actually, we have the following property.

3.3.6 EXERCISE. Let $q_1 : C_{i_1} \longrightarrow \Sigma X$ and $q_2 : C_{i_2} \longrightarrow \Sigma Y$ be the homotopy equivalences (3.3.4) and (3.3.5), and let τ be as in 2.10.3. Prove that the diagram

$$\begin{tikzcd} C_{i_1} \arrow[r, "i_3"] \arrow[d, "\tau\circ q_1"', "\simeq"] & C_{i_2} \arrow[d, "q_2", "\simeq"'] \\ \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \end{tikzcd}$$

commutes up to homotopy.

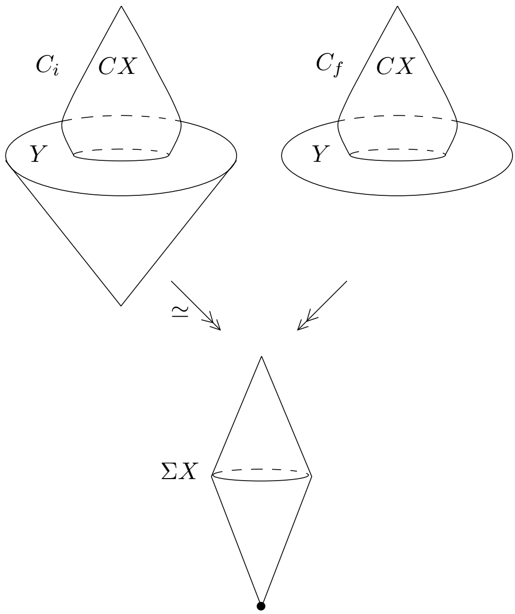


Figure 3.7

In this way, (3.3.1) is transformed into

$$\begin{aligned} X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow C_{\Sigma f} \\ \longrightarrow \Sigma^2 X \longrightarrow \Sigma^2 Y \longrightarrow \cdots . \end{aligned} \tag{3.3.7}$$

This sequence is frequently known as the *Barratt–Puppe sequence* of the map $f : X \longrightarrow Y$.

Let us now observe that the sequence

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad i \quad} C_f$$

is *h-coexact*; that is, we have the following assertion.

3.3.8 **Proposition**. *Let W be an arbitrary pointed space. Then the sequence*

$$[C_f, W]_* \xrightarrow{\quad i^* \quad} [Y, W]_* \xrightarrow{\quad f^* \quad} [X, W]_*$$

is exact; that is,

$$\operatorname{im}(i^*) = \ker(f^*) = \{[\varphi] \in [Y,W]_* \mid f^*[\varphi] = [\varphi \circ f] = [e_0]\},$$

where $e_0 : X \longrightarrow W$ is the constant map.

Proof: First observe that $i \circ f : X \longrightarrow C_f$ is nullhomotopic. This is so, since we have the homotopy

$$H(x,t) = \overline{(x, 1-t)} \in Y \cup_f CX,$$

which for $t = 0$ is constant and for $t = 1$ gives the point $\overline{(x,0)}$, which is identified with $f(x)$ in C_f. Thus $f^*i^*([\psi]) = [\psi \circ i \circ f] = [e_0]$ for all $[\psi] \in [C_f, W]_*$, and so $\operatorname{im}(i^*) \subset \ker(f^*)$.

If we now suppose that $f^*[\varphi] = [\varphi \circ f] = 0 = [e_0]$, then $\varphi \circ f$ is nullhomotopic. Let

$$H : X \times I \longrightarrow W$$

be a nullhomotopy of $\varphi \circ f : X \longrightarrow W$. Then $H(x_0, t) = H(x, 1) = w_0$, and therefore $H(\{x_0\} \times I \cup X \times \{1\}) = \{w_0\}$. So H defines a map

$$\psi' : CX \longrightarrow W$$

such that $\psi'\overline{(x,t)} = H(x,t)$. Since $\psi'\overline{(x,0)} = H(x,0) = \varphi f(x)$, we then can define

$$\psi : C_f = Y \cup_f CX \longrightarrow W$$

such that $\psi\overline{(x,t)} = \psi'\overline{(x,t)}$ and $\psi(y) = \varphi(y)$. Consequently, $[\psi] \in [C_f, W]_*$ satisfies $i^*([\psi]) = [\psi \circ i] = [\varphi]$, and so $\ker(f^*) \subset \operatorname{im}(i^*)$. □

3.3.9 **Corollary**. *The sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact, where $j : C_f \longrightarrow \Sigma X = C_f/Y$ is the quotient map; that is, the sequence of sets (and groups)

$$[\Sigma Y, W]_* \xrightarrow{(\Sigma f)^*} [\Sigma X, W]_* \xrightarrow{j^*} [C_f, W]_* \xrightarrow{i^*} [Y,W]_* \xrightarrow{f^*} [X,W]_*$$

is exact.

Proof: From the sequence (3.3.1) we have that the portions $Y \longrightarrow C_f \longrightarrow C_{i_1}$ and $C_f \longrightarrow C_{i_1} \longrightarrow C_{i_2}$ are as in 3.3.8; therefore, h-coexact. By Exercise 3.3.6, and interchanging C_{i_1} by ΣX and C_{i_2} by ΣY, we obtain the desired h-coexact sequence, since the effect of τ does not change kernels and images (only signs). □

Therefore, we have the following consequence.

3.3.10 **Corollary**. *Given a pointed map $f : X \longrightarrow Y$, we have an exact sequence*

$$\begin{aligned} \cdots \longrightarrow [\Sigma^k C_f, W]_* \longrightarrow [\Sigma^k Y, W]_* \xrightarrow{(\Sigma^k f)^*} [\Sigma^k X, W]_* \longrightarrow \\ [\Sigma^{k-1} C_f, W]_* \longrightarrow \cdots \longrightarrow [\Sigma X, W]_* \longrightarrow [C_f, W]_* \longrightarrow \\ \longrightarrow [Y, W]_* \longrightarrow [X, W]_* \end{aligned} \tag{3.3.11}$$

for every pointed space W.

Proof: Since $[\Sigma Z, W]_* \cong [Z, \Omega W]_*$ in a natural way (see 2.10.5 and 2.10.6), we may reduce each 5-term portion of the sequence to the first 5 and then apply Corollary 3.3.9 above. □

3.3.12 EXERCISE. Show by induction that there is a homeomorphism

$$\varphi^k : C_{\Sigma^k f} \approx \Sigma^k C_f$$

such that $\varphi^k \circ i^k = \Sigma^k i$, where $i^k : \Sigma^k Y \longrightarrow C_{\Sigma^k f}$ and $i : Y \longrightarrow C_f$ are the canonical inclusions. Therefore, the exact sequence (3.3.11) is equivalent to the following exact sequence for the Barrat-Puppe sequence.

$$\begin{aligned} \cdots \longrightarrow [C_{\Sigma^k f}, W]_* \longrightarrow [\Sigma^k Y, W]_* \xrightarrow{(\Sigma^k f)^*} [\Sigma^k X, W]_* \longrightarrow \\ [C_{\Sigma^{k-1} f}, W]_* \longrightarrow \cdots \longrightarrow [\Sigma X, W]_* \longrightarrow [C_f, W]_* \longrightarrow \\ \longrightarrow [Y, W]_* \longrightarrow [X, W]_*\,, \end{aligned} \tag{3.3.13}$$

for every pointed space W.

Everything done above has a dual version. We shall sketch the results, and the reader should figure out all the proofs.

First of all, there are dual versions of the mapping cylinder and the mapping cone. Accepting the space $M(I, Y)$ of free paths of Y as the dual of the cylinder, if $f : X \longrightarrow Y$ is continuous, then we have the following definition.

3.3.14 DEFINITION. Define the *mapping path space* of f as

$$E_f = \{(x, \alpha) \in X \times M(I, Y) \mid \alpha(1) = f(x)\}\,.$$

There is also a dual concept of the mapping cone, namely, we define the *homotopy fiber* of a pointed map f as

$$P_f = \{(x, \alpha) \in X \times M(I, Y) \mid \alpha(0) = y_0,\ \alpha(1) = f(x)\}\,.$$

3.3.15 Exercise. Take $x_0 \in A \subset X$, and let $i : A \hookrightarrow X$ be the inclusion map. Prove that the mapping path space of i, E_i, is homeomorphic to

$$\{\alpha \in M(I,X) \mid \alpha(1) \in A\},$$

and that the homotopy fiber of i, P_i, is homeomorphic to

$$\{\alpha \in M(I,X) \mid \alpha(0) = x_0 \text{ and } \alpha(1) \in A\}.$$

There are canonical maps for the mapping path space and for the homotopy fiber. One is $p : E_f \longrightarrow Y$, such that $p(x,\alpha) = \alpha(0)$, whose fiber $p^{-1}(y_0) = P_f$; another map is $q : P_f \longrightarrow X$, such that $q(x,\alpha) = x$, whose fiber $q^{-1}(x_0) \approx \Omega Y$. This assertion is dual to the statement of Exercise 3.1.3.

Dual to the cone is the *path space* $PY = \{\alpha \in M(I,Y) \mid \alpha(0) = y_0\}$, which has a canonical projection $p : PY \longrightarrow Y$ given by $p(\alpha) = \alpha(1)$. Then there is the following dual to Lemma 3.1.5.

3.3.16 **Lemma**. *$f : X \longrightarrow Y$ is nullhomotopic if and only if it admits a lifting $F : X \longrightarrow PY$, that is, such that $p \circ F = f$.* □

Dual to 3.1.6, the projection $p : PY \longrightarrow Y$ has a homotopy lifting property.

3.3.17 **Lemma**. *Let $F : X \longrightarrow PY$ be a continuous map and let $H : X \times I \longrightarrow Y$ be a homotopy that starts with $p \circ F$. Then we can lift H to a homotopy $G : X \times I \longrightarrow PY$ that starts with F. That is, in the commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, hook, "j_0"'] & PY \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "G"] & Y
\end{tikzcd}$$

there exists $G : X \times I \longrightarrow PY$ that makes both triangles commute.

A nice *exercise* stressing the duality of the concept of the homotopy fiber of a map with that of mapping cone is the proof of the following fact, which is dual to Proposition 3.1.7.

3.3.18 **Proposition**. *Let us consider the maps $W \xrightarrow{g} X \xrightarrow{f} Y$. Then $f \circ g$ is nullhomotopic if and only if there exists $G : W \longrightarrow P_f$ such that the diagram*

$$\begin{tikzcd}
P_f \arrow[rr, two heads, "q"] & & X \arrow[r, "f"] & Y \\
& W \arrow[ul, dashed, "G"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

commutes; that is, if and only if g has a lifting G to the homotopy fiber of f. $\square$

Let $f : X \longrightarrow Y$ be continuous; using the homotopy fiber construction, we may define the sequence

$$\cdots \longrightarrow P_{q_2} \xrightarrow{q_3} P_{q_1} \xrightarrow{q_2} P_f \xrightarrow{q_1} X \xrightarrow{f} Y\,, \tag{3.3.19}$$

where q_1 is the canonical projection of P_f onto X, and, analogously, q_k is the canonical projection of the homotopy fiber of q_{k-1}, $P_{q_{k-1}}$, onto $P_{q_{k-2}}$.

As before, we may identify, up to homotopy, the spaces P_{q_k}.

Dual to 3.3.3, we have the following.

3.3.20 **Proposition**. *Let $f : X \longrightarrow Y$ be continuous, $q : P_f \longrightarrow Y$ the canonical projection, and P_q its homotopy fiber. Then the inclusion $\Omega Y \hookrightarrow P_q$ is a homotopy equivalence.* $\square$

Dual to Exercise 3.3.6 one may solve the following.

3.3.21 Exercise. Let $j_1 : \Omega Y \hookrightarrow P_{q_1}$ and $j_2 : \Omega X \hookrightarrow P_{q_2}$ be the homotopy equivalences (3.3.20), and let $\sigma : \Omega Y \longrightarrow \Omega Y$ be dual to the map $\tau : \Sigma X \longrightarrow \Sigma X$ in 2.10.3. Prove that the diagram

$$\begin{tikzcd}
P_{q_2} \arrow[r, "j_3"] & P_{q_1} \\
\Omega X \arrow[u, "j_2", "\simeq"'] \arrow[r, "\Omega f"'] & \Omega Y \arrow[u, "\simeq", "j_1\circ\sigma"']
\end{tikzcd}$$

commutes up to homotopy.

We may transform (3.3.19) into

$$\begin{array}{c} \cdots \longrightarrow \Omega^2 X \longrightarrow \Omega^2 Y \longrightarrow \\ \longrightarrow P_{\Omega f} \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow P_f \longrightarrow X \xrightarrow{f} Y\,. \end{array} \tag{3.3.22}$$

This sequence is the *dual Barratt–Puppe sequence* of the map $f : X \longrightarrow Y$.

Dual to the previous case, the sequence

$$P_f \xrightarrow{\quad q \quad} X \xrightarrow{\quad f \quad} Y$$

is *h-exact*, that is, we have the following assertion.

3.3.23 **Proposition**. *Let W be an arbitrary pointed space. Then the sequence*
$$[W,P_f]_* \xrightarrow{q_*} [W,X]_* \xrightarrow{f_*} [W,Y]_*$$
is exact; that is,
$$\operatorname{im}(q_*) = \ker(f_*) = \{[\varphi] \in [W,X]_* \mid f_*[\varphi] = [f\circ\varphi] = [e_0]\}\,,$$
where $e_0 : W \longrightarrow Y$ is the constant map. □

As a corollary, we obtain the dual of 3.3.9.

3.3.24 **Corollary**. *The sequence*
$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{p} P_f \xrightarrow{j} X \xrightarrow{f} Y$$
is h-exact, where $p : \Omega Y \hookrightarrow P_q \longrightarrow P_f$ is the canonical embedding; that is, the sequence of sets (and groups)
$$[W,\Omega X]_* \xrightarrow{(\Omega f)_*} [W,\Omega Y]_* \xrightarrow{p_*} [W,P_f]_* \xrightarrow{q_*} [W,X]_* \xrightarrow{f_*} [W,Y]_*$$
is exact. □

Hence, we have the following consequence.

3.3.25 **Corollary**. *Given a pointed map $f : X \longrightarrow Y$, we have an exact sequence*
$$\begin{aligned} &\cdots \longrightarrow [W,\Omega^k P_f]_* \longrightarrow [W,\Omega^k X]_* \xrightarrow{(\Omega^k f)_*} [W,\Omega^k Y]_* \longrightarrow \\ &[W,\Omega^{k-1}P_f]_* \longrightarrow \cdots \longrightarrow [W,\Omega Y]_* \longrightarrow [W,P_f]_* \longrightarrow \\ &\longrightarrow [W,X]_* \longrightarrow [W,Y]_* \end{aligned} \tag{3.3.26}$$
for every pointed space W. □

There is also a dual version of Exercise 3.3.12.

3.4 Homotopy Groups

In what follows we shall study the relations between the homotopy groups of a pair of spaces and those of the individual spaces of the pair.

Let X be a pointed space with base point $x_0 \in X$. If I is the interval $[0,1]$, then $\partial I = \{0,1\} \subset I$ is its boundary, and $0 \in \partial I \subset I$ will be considered as the base point of both spaces. For $n \geq 1$, $\pi_n(X)$ is the group $[\Sigma^n(\partial I), X]_*$, since we have $\partial I \approx \mathbb{S}^0$ and $\Sigma^n(\partial I) \approx \mathbb{S}^n$ according to 2.10.8. For $n = 0$ the sets $[\partial I, X]_*$ and $\pi_0(X)$ coincide.

3.4.1 DEFINITION. Let (X, A) be a pair of pointed spaces with base point $x_0 \in A \subset X$. For $n \geq 1$ we define

$$\pi_n(X, A) = [\Sigma^{n-1}(I, \partial I); X, A]_*\,,$$

where $\Sigma^k(I, \partial I) = (\Sigma^k I, \Sigma^k(\partial I)) \approx (\mathbb{D}^{k+1}, \mathbb{S}^k)$, where $\mathbb{D}^{k+1} \subset \mathbb{R}^{k+1}$ is the unit disk and where $[-;-]_*$ represents the set of pointed homotopy classes of pairs.

3.4.2 **Proposition**. *The construction π_n has the following properties.*

(a) *$\pi_n(X, A)$ is a group if $n \geq 2$ and is abelian if $n \geq 3$.*

(b) *A map $f : (\mathbb{D}^n, \mathbb{S}^{n-1}, *) \longrightarrow (X, A, x_0)$ represents the neutral element of $\pi_n(X, A)$ if and only if f is homotopic as a map of pointed pairs to a map g such that $g(\mathbb{D}^n) \subset A$.*

The group $\pi_n(X, A)$, $n \geq 2$, is called the *n-homotopy group of the pair* (X, A).

Proof: (a) This is shown essentially in the same way as 2.10.4 and 2.10.13, since the structure is given by the H-comultiplication in $\mathbb{D}^n \approx \Sigma^{n-1} I$, $n \geq 2$.

(b) Suppose that $f \simeq g$ and that $g(\mathbb{D}^n) \subset A$ and let H be a homotopy of pointed pairs between f and g. We define

$$G : (\mathbb{D}^n \times I, \mathbb{S}^{n-1} \times I, \{*\} \times I) \longrightarrow (X, A, x_0)$$

by

$$G(x,t) = \begin{cases} H(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g((2-2t)x + (2t-1)*) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Therefore, G is a nullhomotopy of f.

If we suppose, conversely, that we have a nullhomotopy

$$G : (\mathbb{D}^n \times I, \mathbb{S}^{n-1} \times I, \{*\} \times I) \longrightarrow (X, A, x_0)$$

of f, we define $H : \mathbb{D}^n \times I \longrightarrow X$ by

$$H(x,t) = \begin{cases} G\left(\frac{2x}{2-t}, t\right) & \text{if } 0 \leq \|x\| \leq \frac{1-t}{2}, \\ G\left(\frac{x}{\|x\|}, 2 - 2\|x\|\right) & \text{if } 1 - \frac{t}{2} \leq \|x\| \leq 1. \end{cases}$$

Thus g given by $g(x) = H(x, 1)$ satisfies $g(\mathbb{D}^n) \subset A$. □

3.4.3 Remark. Let $J^n = I^n \times \{0\} \cup \partial I^n \times I \subset \partial I^{n+1}$. Then it is an easy *exercise* (see 3.4.9) to verify that there is a bijection

$$[I^{n+1}, \partial I^{n+1}, J^n; X, A, x_0] \cong [I^{n+1}/J^n, \partial I^{n+1}/J^n, *; X, A, x_0]\,.$$

Since $I^{n+1}/J^n \approx \mathbb{D}^n$ and $\partial I^{n+1}/J^n \approx \mathbb{S}^{n-1}$, one can also consider (as some authors do)

$$\pi_n(X, A) = [I^{n+1}, \partial I^{n+1}, J^n; X, A, x_0]\,,$$

in which case the group operation is given by taking $[F] \cdot [G] = [H]$, where $H : I^{n+1} = I^n \times I \longrightarrow X$ is given by

$$H(s_1, \ldots, s_n, t) = \begin{cases} F(s_1, \ldots, 2s_i, \ldots, s_n, t) & \text{if } 0 \leq s_i \leq \frac{1}{2}, \\ G(s_1, \ldots, 2s_i - 1, \ldots, s_n, t) & \text{if } \frac{1}{2} \leq s_i \leq 1; \end{cases}$$

for $i = 1, \ldots, n$.

If, in particular, we consider the pair (X, x_0), then the map of pairs

$$\varphi : \Sigma^{n-1}(I, \partial I) \longrightarrow (X, \{x_0\})$$

maps $\Sigma^{n-1}(\partial I) \approx \mathbb{S}^{n-1}$ to x_0. In this way, φ determines a pointed map

$$\psi : \Sigma^{n-1} I/\Sigma^{n-1}(\partial I) \longrightarrow X\,.$$

Now we have as well

$$\Sigma^{n-1} I/\Sigma^{n-1}(\partial I) \approx \mathbb{D}^n/\mathbb{S}^{n-1} \approx \mathbb{S}^n\,,$$

from which we obtain a bijection between $[\Sigma^{n-1}(I, \partial I); X, \{x_0\}]_*$ and $[\mathbb{S}^n, X]_*$. This proves that

$$\pi_n(X, \{x_0\}) \cong \pi_n(X)$$

if $n \geq 1$, and so we will identify these two sets.

So the inclusion $j : (X, \{x_0\}) \longrightarrow (X, A)$ induces

$$j_* : \pi_n(X) \longrightarrow \pi_n(X, A)\,, \tag{3.4.4}$$

which is a homomorphism if $n \geq 2$. Since $\Sigma^n(\partial I)$ is path connected if $n \geq 1$, if X' is the path component of X that contains x_0, then $X' \subset X$ induces isomorphisms

$$\pi_n(X') \longrightarrow \pi_n(X)$$

if $n \geq 1$.

By restricting to the second term of the pair we obtain the homomorphism ∂ in the following diagram:

(3.4.5)
$$\begin{tikzcd} \pi_n(X,A) \arrow[rr, dashed, "\partial"] \arrow[d, equal] & & \pi_{n-1}(A) \arrow[d, leftrightarrow, "\cong"] \\ {[\Sigma^{n-1}(I,\partial I);X,A]_*} \arrow[rr] & & {[\Sigma^{n-1}(\partial I),A]_*}\,. \end{tikzcd}$$

The homomorphism ∂ of (3.4.5) is called the *connecting homomorphism of the homotopy groups* of the pair (X,A). Combining (3.4.4) and (3.4.5) we obtain a sequence

(3.4.6)
$$\begin{aligned} \cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X,A) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots \\ \longrightarrow \pi_1(X,A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)\,, \end{aligned}$$

which is called the *homotopy sequence* of the pair (X,A).

In the following section we shall prove that (3.4.6) is exact. For this purpose we need a generalization of 3.3.8 for pairs of spaces.

3.4.7 **Proposition**. *Let $f:(X,A)\longrightarrow(Y,B)$ be a pointed map of (pointed) pairs and let $f':X\longrightarrow Y$ and $f'':A\longrightarrow B$ be its restrictions. Then the sequence*

$$(X,A) \xrightarrow{\quad f \quad} (Y,B) \xrightarrow{\quad i \quad} C_f$$

is h-coexact, where $C_f=(C_{f'},C_{f''})$. This means that for any pair of pointed spaces (Z,C), the sequence

$$[C_{f'},C_{f''};Z,C]_* \xrightarrow{i^*} [Y,B;Z,C]_* \xrightarrow{f^*} [X,A;Z,C]_*$$

is exact.

Proof: Just as in 3.3.8, $i\circ f:(X,A)\longrightarrow(C_{f'},C_{f''})$ is nullhomotopic as a map of pairs. So $\operatorname{im}(i^*)\subset\ker(f^*)$. If now $\varphi:(Y,B)\longrightarrow(Z,C)$ is such that $\varphi\circ f$ is nullhomotopic as a map of pairs, then any nullhomotopy

$$H:(X\times I,A\times I)\longrightarrow(Z,C)$$

defines a map of pairs

$$\eta:(CX,CA)\longrightarrow(Z,C)\,,$$

which (as in the proof of 3.3.8) extends to $\varphi:(Y,B)\longrightarrow(Z,C)$. Here we are considering the domain as a subpair of $(C_{f'},C_{f''})$. Having shown all this, η and φ define

$$\psi:(C_{f'},C_{f''})\longrightarrow(Z,C)$$

such that $\psi\circ i=\varphi$. Therefore, $\ker(f^*)\subset\operatorname{im}(i^*)$. $\square$

3.4.8 REMARK. There is an approach to the homotopy groups using homotopy fibers instead of mapping cones, namely, using 2.10.5, and since $\mathbb{S}^n = \Sigma^n\mathbb{S}^0$ for a pointed space X, we have that $\pi_n(X) = [\mathbb{S}^n, X]_* = [\mathbb{S}^0, \Omega^n X]_*$, and analogously for pairs, namely $\pi_n(X, A) = [I, \partial I; \Omega^{n-1}(X, A)]_*$, where $\Omega^{n-1}(X, A) = (\Omega^{n-1}X,\ \Omega^{n-1}A)$. It is an exercise to reconstruct the homotopy sequence of a pair (3.4.6).

3.4.9 EXERCISE. Let $J^{q-1} = (\partial I^{q-1} \times I) \cup (I^{q-1} \times \{0\})$ and let $x_0 \in A \subset X$ (cf. 3.4.3). Prove that

$$\pi_q(X, A) = [I^q, \partial I^q, J^{q-1}; X, A, x_0]$$

and that

$$\pi_{q-1}(A) = [\partial I^q, J^{q-1}; A, x_0]\,,$$

so that $\partial : \pi_q(X, A) \longrightarrow \pi_{q-1}(A)$ is given by $\partial([\alpha]) = [\alpha|\partial I^q]$.

3.4.10 EXERCISE. Take $x_0 \in A \subset X$ and let $P(X; x_0, A)$ be the homotopy fiber of the inclusion $A \hookrightarrow X$ (see 3.3.15). Prove that

$$\pi_q(X, A) \cong \pi_{q-1}(P(X; x_0, A))\,.$$

(Hint: Let $\alpha : (I^q, \partial I^q, J^{q-1}) \longrightarrow (X, A, x_0)$ correspond with $\widehat{\alpha} : \partial I^q \longrightarrow P(X; x_0, A)$ by $\widehat{\alpha}(s)(t) = \alpha(s, t)$, $(s, t) \in I^{q-1} \times I = I^q$.)

3.5 HOMOTOPY SEQUENCES II

In the same way as we obtained the sequence (3.3.7) we can obtain the sequence

$$\begin{aligned}(X, A) \xrightarrow{f} (Y, B) \longrightarrow (C_{f'}, C_{f''}) \longrightarrow (\Sigma X, \Sigma A) \xrightarrow{\Sigma f} (\Sigma Y, \Sigma B) \longrightarrow \\ \longrightarrow (C_{\Sigma f'}, C_{\Sigma f''}) \longrightarrow (\Sigma^2 X, \Sigma^2 A) \longrightarrow \cdots\,.\end{aligned}$$

Combining 3.4.7 with this sequence we obtain the following:

3.5.1 **Corollary**. *Given $f : (X, A) \longrightarrow (Y, B)$ a map of pointed pairs, we have an exact sequence*

$$\begin{aligned}\cdots \longrightarrow [C_{\Sigma^k f'}, C_{\Sigma^k f''}; Z, C]_* \longrightarrow [\Sigma^k(Y, B); Z, C]_* \longrightarrow \\ \longrightarrow [\Sigma^k(X, A); Z, C]_* \longrightarrow [C_{\Sigma^{k-1} f'}, C_{\Sigma^{k-1} f''}; Z, C]_* \longrightarrow \\ \cdots \longrightarrow [\Sigma(X, A); Z, C]_* \longrightarrow [C_{f'}, C_{f''}; Z, C]_* \longrightarrow \\ \longrightarrow [Y, B; Z, C]_* \longrightarrow [X, A; Z, C]_*\,,\end{aligned} \tag{3.5.2}$$

for each pointed pair (Z, C). □

3.5.3 Remark. In a way similar to 3.3.11 we obtain an exact sequence

$$\begin{aligned}(3.5.4)\qquad &\cdots \longrightarrow [\Sigma^k(C_{f'}, C_{f''}); Z, C]_* \longrightarrow [\Sigma^k(Y, B); Z, C]_* \longrightarrow \\ &[\Sigma^k(X, A);\ Z, C]_* \longrightarrow [\Sigma^{k-1}(C_{f'}, C_{f''}); Z, C]_* \longrightarrow \\ &\cdots \longrightarrow [\Sigma(X, A);\ Z, C]_* \longrightarrow [C_{f'}, C_{f''};\ Z, C]_* \longrightarrow \\ &\longrightarrow [Y, B; Z, C]_* \longrightarrow [X, A;\ Z, C]_*\,,\end{aligned}$$

for each pointed pair (Z, C).

3.5.5 **Theorem**. *The homotopy sequence (3.4.6) of a pair* (X, A) *is exact.*

Proof: Let $i : (\partial I, 0) \longrightarrow (\partial I, \partial I)$ be the inclusion. Let us consider the following sequence of pairs for i:

$$\begin{aligned}(3.5.6)\qquad &(\partial I, 0) \xrightarrow{i} (\partial I, \partial I) \xrightarrow{j} (C_{i'}, C_{i''}) \xrightarrow{k} (\Sigma\partial I, 0) \longrightarrow \\ &\xrightarrow{\Sigma} (\Sigma\partial I, \Sigma\partial I) \longrightarrow (C_{\Sigma i'}, C_{\Sigma i''}) \longrightarrow \cdots\,.\end{aligned}$$

We have a homeomorphism

$$g : (C_{i'}, C_{i''}) \longrightarrow (I, \partial I)$$

given by

$$g'\overline{(0, t)} = 0, \qquad\qquad g'\overline{(1, t)} = 1 - t$$

(since the mapping cone is reduced; see Figure 3.8).

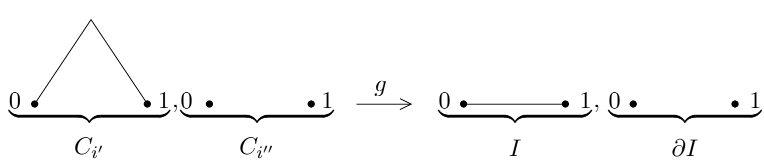


Figure 3.8

Then $\widetilde{j} = g \circ j : (\partial I, \partial I) \longrightarrow (I, \partial I)$ is the inclusion.

On the other hand, $k \circ g^{-1} : (I, \partial I) \longrightarrow (\Sigma\partial I, 0)$ is the composite

$$\widetilde{k} : (I, \partial I) \longrightarrow (I/\partial I, \overline{0}) \longrightarrow \Sigma(\partial I, 0),$$

so that (3.5.6) is transformed into

$$(\partial I, 0) \xrightarrow{i} (\partial I, \partial I) \xrightarrow{\widetilde{j}} (I, \partial I) \xrightarrow{\widetilde{k}} \Sigma(\partial I, 0) \xrightarrow{\Sigma i} \cdots\,,$$

which, using (3.5.2), gives rise to the exact sequence

$$\begin{aligned} &[\Sigma^k(I,\partial I);X,A]_* \longrightarrow [\Sigma^k(\partial I,\partial I);X,A]_* \longrightarrow \\ &\longrightarrow [\Sigma^k(\partial I,0);X,A]_* \longrightarrow [\Sigma^{k-1}(I,\partial I);X,A]_* \\ &\cdots \longrightarrow [\Sigma(\partial I,0);X,A]_* \xrightarrow{\widetilde{k^*}} [I,\partial I;X,A]_* \xrightarrow{\widetilde{j^*}} \\ &\longrightarrow [\partial I,\partial I;X,A]_* \longrightarrow [\partial I,0;X,A]_*\,. \end{aligned} \tag{3.5.7}$$

Since we clearly have

$$\begin{aligned} [\Sigma^k(I,\partial I);X,A]_* &= \pi_{k+1}(X,A) \qquad \text{(by definition)},\\ [\Sigma^k(\partial I,\partial I);X,A]_* &= [\Sigma^k(\partial I),A]_* = \pi_k(A)\,,\\ [\Sigma^k(\partial I,0);X,A]_* &= [\Sigma^k(\partial I),0;X,x_0]_* = \pi_k(X)\,, \end{aligned}$$

we see that (3.5.7) is the desired sequence (3.4.6). □

We can summarize the most important results of this chapter in the following theorem.

3.5.8 **Theorem**. *Let (X,A) be a pair of pointed spaces, with base point $x_0 \in A \subset X$. For every $n \geq 1$ we associate to it the sets*

$$\pi_n(X,A), \quad \pi_{n-1}(A), \quad \pi_{n-1}(X)$$

and the functions

$$\begin{aligned} \partial\ : \pi_n(X,A) &\longrightarrow \pi_{n-1}(A)\,,\\ i_*\ : \pi_{n-1}(A) &\longrightarrow \pi_{n-1}(X)\,,\\ j_*\ : \pi_{n-1}(X) &\longrightarrow \pi_{n-1}(X,A)\,. \end{aligned}$$

Moreover, if $f:(X,A)\longrightarrow(Y,B)$ is a map of pointed pairs with restrictions $f':X\longrightarrow Y$ and $f'':A\longrightarrow B$, we associate to f the functions

$$\begin{aligned} f_* : \pi_n(X,A) &\longrightarrow \pi_n(Y,B)\,,\\ f''_* : \pi_{n-1}(A) &\longrightarrow \pi_{n-1}(B)\,,\\ f'_* : \pi_{n-1}(X) &\longrightarrow \pi_{n-1}(Y)\,. \end{aligned}$$

These have the following properties:

(a) *The sets $\pi_n(X,A)$, $\pi_{n-1}(A)$, and $\pi_{n-1}(X)$ are groups if $n \geq 2$ and they are abelian if $n \geq 3$. Also f_*, f''_*, and f'_* are homomorphisms of groups in these cases. Moreover, $\pi_0(A)$ and $\pi_0(X)$ are the sets of path components of A and X, respectively.*

(b) *If* $f = \mathrm{id} : (X, A) \longrightarrow (X, A)$, *then* $f_* = 1_{\pi_n(X,A)}$, $f''_* = 1_{\pi_{n-1}(A)}$ *and* $f'_* = 1_{\pi_{n-1}(X)}$.

(c) *If* $f : (X, A) \longrightarrow (Y, B)$ *and* $g : (Y, B) \longrightarrow (Z, C)$ *are maps of pointed pairs, then* $(g \circ f)_* = g_* \circ f_*$, $(g'' \circ f'')_* = g''_* \circ f''_*$, *and* $(g' \circ f')_* = g'_* \circ f'_*$.

(d) *For* $f : (X, A) \longrightarrow (Y, B)$, *the diagrams*

$$\begin{CD} \pi_n(X,A) @>{\partial}>> \pi_{n-1}(A) \\ @V{f_*}VV @VV{f''_*}V \\ \pi_n(Y,B) @>>{\partial}> \pi_{n-1}(B) \end{CD}$$

$$\begin{CD} \pi_{n-1}(A) @>{i_*}>> \pi_{n-1}(X) \\ @V{f''_*}VV @VV{f'_*}V \\ \pi_{n-1}(B) @>>{i_*}> \pi_{n-1}(Y) \end{CD}$$

if $n \geq 1$, *and*

$$\begin{CD} \pi_{n-1}(X) @>{j_*}>> \pi_{n-1}(X,A) \\ @V{f'_*}VV @VV{f_*}V \\ \pi_{n-1}(Y) @>>{j_*}> \pi_{n-1}(Y,B) \end{CD}$$

if $n \geq 2$, *are commutative.*

(e) *For every pointed pair* (X, A) *the sequence*

$$\begin{aligned} \cdots \longrightarrow \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \xrightarrow{j_*} \\ \longrightarrow \pi_{n-1}(X,A) \xrightarrow{\partial} \cdots \longrightarrow \pi_1(X,A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X) \end{aligned}$$

is exact. In particular, if X *satisfies* $\pi_n(X) = 0$ *for all* $n \geq 0$, *then*

$$\partial : \pi_n(X, A) \longrightarrow \pi_{n-1}(A)$$

is a bijection for $n \geq 1$.

(f) *If the maps of pointed pairs* $f, g : (X, A) \longrightarrow (Y, B)$ *are homotopic, then*

$$\begin{aligned} f_* &= g_* : \pi_n(X,A) \longrightarrow \pi_n(Y,B)\,, \\ f''_* &= g''_* : \pi_{n-1}(A) \longrightarrow \pi_{n-1}(B)\,, \\ f'_* &= g'_* : \pi_{n-1}(X) \longrightarrow \pi_{n-1}(Y)\,. \end{aligned}$$

(g) *If X is contractible, that is, if the map* id_X *is homotopic to the constant map* $e_0 : X \longrightarrow X$*, then*

$$\pi_n(X) = 0, \qquad n \geq 0. \qquad \square$$

3.5.9 REMARK. From part (f) we obtain, in particular, that if $f : (X, A) \longrightarrow (Y, B)$ is a homotopy equivalence, that is, if there exists $g : (Y, B) \longrightarrow (X, A)$ such that $g \circ f \simeq \mathrm{id}_{(X,A)}$ and $f \circ g \simeq \mathrm{id}_{(Y,B)}$, then $f_* : \pi_*(X, A) \longrightarrow \pi_*(Y, B)$ is an isomorphism. (We are assuming that the homotopies are of pointed pairs. Nevertheless, it is possible to prove that if the homotopies are only of pairs, without preserving the base point, then f_* is still an isomorphism for every $x \in A$; cf. 4.4.8.)

3.5.10 EXERCISE. Prove that given a pointed space X and pointed subspaces $B \subset A \subset X$, where $x_0 \in B$ is the common base point of the three spaces, we have a long exact sequence

$$\cdots \longrightarrow \pi_n(A, B) \longrightarrow \pi_n(X, B) \longrightarrow \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A, B) \longrightarrow \cdots,$$

called the *(exact) homotopy sequence* of the triple (X, A, B). (Hint: Define the connecting homomorphism ∂ as the composite

$$\pi_n(X, A) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(A, B)$$

of the homomorphism defined in (3.4.5) and that induced by the inclusion. Then put together the exact sequences of homotopy groups of the pairs (X, A), (X, B), and (A, B).)

Clearly, the exact homotopy sequence of a pointed pair (X, A) is the same as that of the triple (X, A, x_0).

3.5.11 REMARK. As before, there is an approach to the homotopy sequence of a pair using loop spaces instead of suspensions. The details are left to the reader as an *exercise.*

Chapter 4

Homotopy Extension and Lifting Properties

We already saw in the previous chapter that the inclusion $X \hookrightarrow CX$ of a space X into its (reduced) cone has a homotopy extension property (see 3.1.6); we also saw that the projection $PY \twoheadrightarrow Y$ of the (pointed) path space of a space onto the space Y has, dually, a homotopy lifting property (see 3.3.17). In this chapter we shall study systematically these two properties. More precisely, we analyze families of maps that have one of the two essentially dual properties, generally known as the homotopy extension and homotopy lifting properties. These topics are of great importance in algebraic topology and will be used in subsequent chapters.

4.1 Cofibrations

In this section we analyze maps having the homotopy extension property (HEP) in various aspects and prove some basic results.

4.1.1 Definition. Assume that $A \subset X$ and that $\mathcal{C}$ is a class of topological spaces. We say that the pair (X, A) has the *homotopy extension property with respect to* $\mathcal{C}$, abbreviated $\mathcal{C}$-HEP, if for every $Y \in \mathcal{C}$ and for every map $f : X \longrightarrow Y$ and every homotopy $H : A \times I \longrightarrow Y$ that starts with $f|A$, we can extend H to a homotopy $\widehat{H} : X \times I \longrightarrow Y$ that starts with f.

Putting this definition into diagrammatical form, we have that (X, A) has the $\mathcal{C}$-HEP if and only if, given the commutative diagram

(4.1.2)
$$\begin{tikzcd} & X \arrow[dr, "j_0"] \arrow[rrr, bend left, "f"] & & & \\ A \arrow[ur, hook, "i"] \arrow[dr, "j_0"'] & & X \times I \arrow[rr, dashed, "\widehat{H}"] & & Y \\ & A \times I \arrow[ur, hook, "i\times\mathrm{id}"'] \arrow[rrru, bend right, "H"'] & & & \end{tikzcd}$$

with $Y \in \mathcal{C}$, where $i : A \hookrightarrow X$ is the inclusion and $j_0 : X \longrightarrow X \times I$ (respectively, $j_0 : A \longrightarrow A \times I$) is the inclusion into the lower face, $j_0(x) = (x, 0)$ (respectively, $j_0(a) = (a, 0)$), there exists a map $\widehat{H}$, as indicated by the dashed arrow, that makes the two triangles commute.

In other words, this definition says that for any $Y \in \mathcal{C}$ the commutative diagram of function spaces

(4.1.3)
$$\begin{tikzcd} M(X \times I, Y) \arrow[r, "(i\times\mathrm{id})^{\#}"] \arrow[d, "j_0^{\#}"'] & M(A \times I, Y) \arrow[d, "j_0^{\#}"] \\ M(X, Y) \arrow[r, "i^{\#}"'] & M(A, Y) \end{tikzcd}$$

has the property that whenever $f \in M(X, Y)$ and $H \in M(A \times I, Y)$ satisfy $j_0^{\#}(H) = i^{\#}(f) = f|A$, then there exists $\widehat{H} \in M(X \times I, Y)$ satisfying $j_0^{\#}(\widehat{H}) = f$ and $(i \times \mathrm{id})^{\#}(\widehat{H}) = \widehat{H}|A \times I = H$.

4.1.4 Definition. If $\mathcal{C}$ is the class of all spaces and (X, A) has the $\mathcal{C}$-HEP, then we simply say that (X, A) has the *homotopy extension property* (HEP).

The following is a concept that is apparently more general, but that turns out to coincide essentially with the above definition when all is said and done.

4.1.5 Definition. A continuous map $j : A \longrightarrow X$ is a *cofibration* if for every topological space Y and every map $f : X \longrightarrow Y$ and every homotopy $H : A \times I \longrightarrow Y$ satisfying $H(a, 0) = fj(a)$ for $a \in A$ there exists a homotopy $\widehat{H} : X \times I \longrightarrow Y$ such that $\widehat{H}(j(a), t) = H(a, t)$ for $a \in A$ and $t \in I$ and such that $\widehat{H}(x, 0) = f(x)$ for $x \in X$. In other words, given diagram (4.1.2) with the change that we have substituted the inclusion i with the map j, there exists $\widehat{H}$ as before.

Actually, this definition is not more general than the previous one, as we shall see.

4.1.6 **Proposition**. *If $j : A \longrightarrow X$ is a cofibration, then j is an embedding; that is, it defines a homeomorphism $A \longrightarrow j(A)$. In the latter case, j is a cofibration if and only if the pair $(X, j(A))$ has the* HEP.

Proof: Let $Z_j = X \cup_j A \times I$ be the mapping cylinder of j and let $q : X \sqcup A \times I \longrightarrow Z_j$ be the quotient map. The map $X \longrightarrow X \times I$ given by $x \mapsto (x, 0)$ and the map $A \times I \longrightarrow X \times I$ given by $(a, t) \mapsto (j(a), t)$ together determine a map $i : Z_j \longrightarrow X \times I$ in the quotient.

Because j is a cofibration, the map $f : X \longrightarrow Z_j$ given by $f(x) = q(x)$ and the map $H : A \times I \longrightarrow Z_j$ given by $H(a, t) = q(a, t)$ together determine a map $\widehat{H} : X \times I \longrightarrow Z_j$ such that $\widehat{H} \circ i : Z_j \longrightarrow Z_j$ is the identity. So i defines a homeomorphism $Z_j \approx i(Z_j) = X \times 0 \cup j(A) \times I \subset X \times I$.

Since $q|A \times 1$ is a homeomorphism $A \times 1 \approx q(A \times 1)$, we have a homeomorphism $i \circ q|A \times 1 : A \times 1 \longrightarrow j(A) \times 1$. But since $iq(a, 1) = (j(a), 1)$, we have that j is a homeomorphism onto its image. □

We can assume from now on that any given cofibration $j : A \longrightarrow X$ is always an inclusion $j : A \hookrightarrow X$, and we shall say without any further distinction either that an inclusion is a cofibration or that the corresponding pair has the HEP.

We shall prove in the following some fundamental properties of cofibrations. To simplify notation, we write 0 for the set $\{0\} \subset I$.

4.1.7 **Theorem**. *Let $A \subset X$ be closed. Then the inclusion $j : A \hookrightarrow X$ is a cofibration if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.*

Proof: If j is a cofibration, then the map $f : X \longrightarrow X \times 0 \cup A \times I$ given by $f(x) = (x, 0)$ and the map $H : A \times I \longrightarrow X \times 0 \cup A \times I$ given by $H(a, t) = (a, t)$ together determine a map $r = \widehat{H} : X \times I \longrightarrow X \times 0 \cup A \times I$, which obviously is a retraction.

Conversely, if we have a retraction $r : X \times I \longrightarrow X \times 0 \cup A \times I$, then for any space Y, any map $f : X \longrightarrow Y$, and any homotopy $H : A \times I \longrightarrow Y$ satisfying $H(a, 0) = fj(a)$ for $a \in A$ we can define a homotopy $\widehat{H} : X \times I \longrightarrow Y$ by

$$\widehat{H}(x, t) = \begin{cases} f \circ \mathrm{proj}_X \circ r(x, t) & \text{if } (x, t) \in r^{-1}(X \times 0), \\ H \circ r(x, t) & \text{if } (x, t) \in r^{-1}(A \times I). \end{cases}$$

Then $\widehat{H}$ is continuous, since $X \times 0$ and $A \times I$ are closed in $X \times I$. □

4.1.8 Note. Note that the first part of the previous proof does not require that A be closed in X. As a matter of fact, it is possible to prove the second part without using that hypothesis (see [74]). Moreover, if X is Hausdorff and $A \hookrightarrow X$ is a cofibration, then A is closed in X. To prove this, note that

$X \times I$ also is Hausdorff, and so $X \times 0 \cup A \times I$ is closed, because it is a retract of $X \times I$. Consequently, $A \times 1$ is also closed in $X \times 1$, or equivalently, A is closed in X.

If the space X is sufficiently separable, the property of an inclusion $j : A \hookrightarrow X$ being a cofibration is a local property. We have, in fact, the next assertion.

4.1.9 **Proposition**. *Let X be a normal space. Then the inclusion $j : A \hookrightarrow X$ is a cofibration if and only if the inclusion $j : A \hookrightarrow V$ is a cofibration for some open neighborhood V of A in X.*

Proof: Let V be a neighborhood of A in X such that the inclusion $j : A \hookrightarrow V$ is a cofibration. By the previous proposition, there exists a retraction $r' : V \times I \longrightarrow V \times 0 \cup A \times I$. Because X is normal, there exists a *contraction* W of V, that is, a neighborhood W of A such that $A \subset W \subset \overline{W} \subset V$. By Urysohn's lemma ([83, 15.6]), there exists a function $\alpha : X \longrightarrow I$ such that $\alpha|A = 1$ and $\alpha|X - W = 0$. In order to apply again the previous proposition, we define a retraction $r : X \times I \longrightarrow X \times 0 \cup A \times I$ by

$$r(x,t) = \begin{cases} r'(x, t\alpha(x)) & \text{if } x \in \overline{W}, \\ (x, 0) & \text{if } x \in X - W. \end{cases}$$

This is obviously a well-defined retraction. □

4.1.10 NOTE. In the first part of the previous proof instead of a retraction r' it is sufficient to assume the existence of a map $r' : V \times I \longrightarrow X \times 0 \cup A \times I$ such that its restriction $r'|V \times 0 \cup A \times I$ is the inclusion. Such a map is called a *weak retraction*. Given this modification of one of the hypotheses, the proof remains the same.

4.1.11 DEFINITION. Suppose that $A \subset X$. We say that A is a *strong deformation retract of a neighborhood* V if there exists a homotopy $H : V \times I \longrightarrow X$ such that

(i) $H(x, 0) = x, \qquad x \in V,$

(ii) $H(a, t) = a, \qquad a \in A,\ t \in I,$

(iii) $H(x, 1) \in A, \qquad x \in V.$

We shall see that this condition is almost sufficient to guarantee that the inclusion $A \hookrightarrow X$ is a cofibration.

4.1.12 **Theorem**. *Assume that X is normal and that $A \subset X$ is closed and is a strong deformation retract of a neighborhood V. If there exists a function $\psi : X \longrightarrow I$ such that $A = \psi^{-1}(0)$ and $\psi(X - V) = 1$, then the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof: According to Proposition 4.1.9 it is enough to prove that the inclusion $A \hookrightarrow V$ is a cofibration, and by Theorem 4.1.7 (or actually by Note 4.1.10) it is enough to construct a weak retraction

$$r : V \times I \longrightarrow X \times 0 \cup A \times I\,.$$

Since A is a strong deformation retract of V, there exists a homotopy $H : V \times I \longrightarrow X$ as in Definition 4.1.11. Put $W = \psi^{-1}(\frac{1}{2}, 1]$ and put $\varphi = \min(2\psi, 1)$. Then W is a neighborhood of $X - V$ satisfying $\varphi(W) = 1$. We define r by the formula

$$r(x,t) = \begin{cases} \left(H\left(x, \frac{t}{\varphi(x)}\right), 0\right) & \text{if } t \leq \varphi(x), \\ (H(x,1), t - \varphi(x)) & \text{if } t \geq \varphi(x), \end{cases}$$

This is well defined if $\varphi(x) > 0$, since the sets $\{(x,t) \in V \times I \mid t \leq \varphi(x)\}$ and $\{(x,t) \in V \times I \mid t \geq \varphi(x)\}$ are closed and the two functions that define r coincide on their common domain where $t = \varphi(x)$. We have to prove that we can extend the map r continuously when $\varphi(x) = 0$, that is, for $x \in A$. But for $a \in A$ we have $(H(a,t), t) = (a,t)$, and so we extend r by putting $r(a, 0) = (a, 0)$. In these points $(a, 0)$ the function r so defined is continuous. And this in turn follows from the fact that H is continuous and I is compact, so that given any neighborhood D of a in X, there exists another neighborhood $D' \subset D$ such that $H(D' \times I) \subset D$, and consequently, for any $\varepsilon > 0$ we have that $r(D' \times [0, \varepsilon)) \subset D \times [0, \varepsilon)$. $\square$

4.1.13 Definition. A Hausdorff space X is *perfectly normal* if for every pair of closed disjoint sets A and B in X there exists a continuous function $\varphi : X \longrightarrow I$ such that $A = \varphi^{-1}(0)$ and $B = \varphi^{-1}(1)$.

The class of perfectly normal spaces evidently includes metric spaces, but it also includes CW-complexes (which will be introduced later on). Consequently, we have the following theorem, which turns out to be important for a large class of spaces.

4.1.14 **Theorem**. *Let X be perfectly normal and let $A \subset X$ be closed. If A is a strong deformation retract of a neigborhood in X, then the inclusion $A \hookrightarrow X$ is a cofibration.* □

Alternatively, it is sufficient to require that X be normal, and that A be a G_δ in X, that is, that A be closed and that it be the intersection of a countable family of open sets in X.

4.1.15 Exercise. Prove that if X is normal, A is a G_δ, and A is a strong deformation retract of a neighborhood V in X, then the inclusion $A \hookrightarrow X$ is a cofibration. (Hint: Put $A = \bigcap V_n$, where each $V_n \subset V$ is an open neighborhood of A in X. Using Urysohn's lemma, there exists a function $f_n : X \longrightarrow I$ for each n such that $f_n|A = 0$ and $f_n|X - V_n = 1$, and there exists a function $g : X \longrightarrow I$ such that $g|X - V = 0$ and $g|A = 1$. If we define $f_A(x) = \sum(f_n(x)/2^n)$, then the function

$$\psi(x) = \frac{f_A(x)}{f_A(x) + g(x)}$$

satisfies the conditions of Theorem 4.1.13.)

We conclude this section with the next theorem, which gives us various ways to recognize cofibrations.

4.1.16 **Theorem**. *Let X be normal and let $A \subset X$ be closed. Then the following are equivalent:*

(a) *The inclusion $A \hookrightarrow X$ is a cofibration.*

(b) *There exists a homotopy $D : X \times I \longrightarrow X$ and a function $\varphi : X \longrightarrow I$ such that $A \subset \varphi^{-1}(0)$ and*

$$\begin{aligned} D(x,0) &= x\,, \quad x \in X\,, \\ D(a,t) &= a\,, \quad a \in A\,,\ t \in I\,, \\ D(x,t) &\in A\,, \quad x \in X\,,\ t > \varphi(x)\,. \end{aligned}$$

(c) *The subset A is a strong deformation retract of a neighborhood V in X, and there exists $\psi : X \longrightarrow I$ such that $A = \psi^{-1}(0)$ and $\psi|X - V = 1$.*

Proof: For property (a) we shall use the characterization given in Theorem 4.1.7, namely, that there exists a retraction $r : X \times I \longrightarrow X \times 0 \cup A \times I$.

(a) $\Rightarrow$ (b) Given r we define φ and D as follows:

$$\begin{aligned} \varphi(x) &= \sup_{t \in I} |t - \mathrm{proj}_I r(x,t)|\,, \quad x \in X\,, \\ D(x,t) &= \mathrm{proj}_X r(x,t)\,, \quad x \in X\,,\ t \in I\,. \end{aligned}$$

(b) $\Rightarrow$ (c) Given D and φ we define $V = \varphi^{-1}[0,1)$. Then V is a neighborhood of A in X. Moreover, A is a strong deformation retract of V, since if we define $H : V \times I \longrightarrow X$ as $D|V \times I$, then H satisfies the conditions of Definition 4.1.11. We then define $\psi : X \longrightarrow I$ by

$$\psi(x) = \inf\{t \in I | D(x,t) \in A\}\,.$$

(c) $\Rightarrow$ (a) This follows from Theorem 4.1.12. □

4.1.17 Exercise. Prove that in the previous proof the map ψ is indeed continuous.

The following statement can be proved in various ways, for example by applying 4.1.7. Nevertheless, we shall prove it using 4.1.16.

4.1.18 **Proposition**. *The inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n$ is a cofibration.*

Proof: Since $\mathbb{D}^n$ is normal, using 4.1.16(c) it is enough to prove that there exist a neighborhood V of $\mathbb{S}^{n-1}$ in $\mathbb{D}^n$ as well as a function $\psi : \mathbb{D}^n \longrightarrow I$ and, finally, a strong deformation retraction $D : V \times I \longrightarrow \mathbb{D}^n$.

Put $V = \mathbb{D}^n - 0$ and $\psi(x) = 1 - |x|$ and $D(x,t) = (1-t)x + t(x/|x|)$. Then we have $\psi^{-1}(0) = \mathbb{S}^{n-1}$ and $\psi|\mathbb{D}^n - V = \psi(0) = 1$ and furthermore $D(x,0) = x$ and $D(x,1) = x/|x| \in \mathbb{S}^{n-1}$. Also, if $x \in \mathbb{S}^{n-1}$, then we have $|x| = 1$ and $D(x,t) = x$. □

4.1.19 Exercise. Prove 4.1.18 using 4.1.7; that is, prove that $\mathbb{D}^n \times 0 \cup \mathbb{S}^{n-1} \times I$ is a retract of $\mathbb{D}^n \times I$.

4.2 Some Results on Cofibrations

There are various rather useful properties of cofibrations, as we shall see in this section.

4.2.1 **Theorem**. *If $j : A \hookrightarrow X$ is a cofibration and A is contractible, then the quotient map $q : X \longrightarrow X/A$ is a homotopy equivalence.*

Proof: Let $H : A \times I \longrightarrow A$ be a contraction, that is, a homotopy such that $H(a, 0) = a$ and $H(a, 1) = *$, where $*$ represents some point in A. Because j is a cofibration, there exists $F : X \times I \longrightarrow X$ satisfying $F(x, 0) = x$ and $F(a, t) = H(a, t)$. Let $F_t : X \longrightarrow X$ be the map given by $F_t(x) = F(x, t)$. In particular, we have that $F_0 = \mathrm{id}_X$ and the restriction $F_1|A$ is constant. Therefore, the map F_1 determines a map $q' : X/A \longrightarrow X$ such that $q' \circ q = F_1$. So F determines a homotopy $\mathrm{id}_X \simeq q' \circ q$.

Conversely, since $F_t(A) \subset A$, the composition $q \circ F_t$ determines a homotopy $G : (X/A) \times I \longrightarrow X/A$ satisfying $G(q(x), t) = qF_t(x)$. We have that $G(q(x), 0) = qF_0(x) = q(x)$ and that $G(q(x), 1) = qF_1(x) = qq'(q(x))$, and so it follows that G is a homotopy $\mathrm{id}_{X/A} \simeq q \circ q'$. Then q and q' are homotopy inverses. □

4.2.2 **Lemma**. *If $A \hookrightarrow X$ is a cofibration, then the canonical inclusion $CA \hookrightarrow X \cup CA$ is also a cofibration.*

Proof: According to Theorem 4.1.7, it is enough to construct a retraction $r' : (X \cup CA) \times I \longrightarrow (X \cup CA) \times 0 \cup (CA) \times I$. Since $A \hookrightarrow X$ is a cofibration, again using Theorem 4.1.7, there exists a retraction $r : X \times I \longrightarrow X \times 0 \cup A \times I$. This retraction and the identity $\mathrm{id}_{(CA)\times I}$ define a map $(X \times I) \sqcup (CA) \times I \longrightarrow (X \times 0 \cup A \times I) \sqcup (CA) \times I$ that determines the desired retraction r' after taking the obvious quotients. It merely suffices to observe that these quotients are well defined, since I is compact. □

Since the cone CA over any space A is contratible, we have the following consequence of Theorem 4.2.1 and the previous lemma.

4.2.3 **Corollary**. *If $A \hookrightarrow X$ is a cofibration, then the quotient map $X \cup CA \longrightarrow X \cup CA/CA \approx X/A$ is a homotopy equivalence.* □

4.2.4 DEFINITION. A commutative square of topological spaces and maps

$$\begin{tikzcd}
 & X \arrow[dr, "h"] & \\
A \arrow[ur, "g"] \arrow[dr, "f"'] & & Z \\
 & Y \arrow[ur, "k"'] &
\end{tikzcd}$$

is called a *pushout* if given maps $h' : X \longrightarrow W$ and $k' : Y \longrightarrow W$ such that $h' \circ g = k' \circ f$, then there exists a unique map $\varphi : Z \longrightarrow W$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dr, "h"] \arrow[drr, bend left, "h'"] & & \\ A \arrow[ur, "g"] \arrow[dr, "f"'] & & Z \arrow[r, dashed, "\varphi"] & W\,. \\ & Y \arrow[ur, "k"'] \arrow[urr, bend right, "k'"'] & & \end{tikzcd}$$

4.2.5 Example. A typical example of a pushout is given by an attaching space (see 3.1.1); namely, let $A \subset X$ be closed and take a map $g : A \longrightarrow Y$. Then the following is a pushout diagram.

$$\begin{tikzcd} & X \arrow[dr, "h"] & \\ A \arrow[ur, hook, "i"] \arrow[dr, "f"'] & & Y \cup_f X\,, \\ & Y \arrow[ur, hook, "k"'] & \end{tikzcd}$$

where $h : X \hookrightarrow X \sqcup Y \xrightarrow{q} Y \cup_f X$.

4.2.6 Exercise. Given a pushout diagram

$$\begin{tikzcd} & X \arrow[dr, "h"] & \\ A \arrow[ur, hook, "i"] \arrow[dr, "f"'] & & Z\,, \\ & Y \arrow[ur, hook, "k"'] & \end{tikzcd}$$

prove that if i is a cofibration, then k is also a cofibration.

There is a convenient way to convert, up to homotopy equivalence, any closed inclusion into a cofibration. Explicitly, we have the next result.

4.2.7 **Proposition**. *Let $A \hookrightarrow X$ be an inclusion of a closed subset into a topological space. Then the embedding $A \hookrightarrow X \times 0 \cup A \times I$ of A into the upper face of the cylinder, given by the inclusion map $a \mapsto (a, 1)$, is a cofibration.*

Proof: Put $\widetilde{X} = X \times 0 \cup A \times I$ and put $\widetilde{A} = A \times 1 \subset \widetilde{X}$. We shall prove that $\widetilde{X} \times 0 \cup \widetilde{A} \times I$ is a retract of $\widetilde{X} \times I$. To do this, let $\widetilde{r} : \widetilde{X} \times I \longrightarrow \widetilde{X} \times 0 \cup \widetilde{A} \times I$ be defined by

$$\widetilde{r}(x, 0, s) = (x, 0, 0)$$

if $(x,0) \in X \times 0 \subset \widetilde{X}$, $s \in I$, and by

$$\widetilde{r}(a,t,s) = \begin{cases} (a, 1, s - \frac{(1-s)(1-t)}{t}) & \text{if } t \geq 1-s, \\ (a, t + \frac{st}{1-s}, 0) & \text{if } t \leq 1-s, \end{cases}$$

if $(a,t) \in A \times I \subset \widetilde{X}$, $s \in I$. It can be immediately verified that $\widetilde{r}$ is continuous and is a retraction. So by 4.1.7, the inclusion $\widetilde{A} \hookrightarrow \widetilde{X}$ is a cofibration. $\square$

In the previous proposition the inclusion $j : X \longrightarrow \widetilde{X}$ given by $x \mapsto (x,0)$ is a homotopy equivalence with inverse $p : \widetilde{X} \longrightarrow X$ defined to be the projection $(x,0) \mapsto x$ and $(a,t) \mapsto a$. The composition $i \circ p$ is homotopic to $\mathrm{id}_{\widetilde{X}}$ by the homotopy defined by $(x,0,s) \mapsto (x,0)$ and $(a,t,s) \mapsto (a,st)$. Furthermore, the restriction of p to $\widetilde{A}$ is a homeomorphism. In this way, we obtain a commutative triangle

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ A \arrow[ur, hook] \arrow[r, hook] & X, \end{tikzcd}$$

where the vertical arrow is a homotopy equivalence and the diagonal arrow (which is an inclusion) is a cofibration.

The previous proposition is a particular case of a more general result, which states that any map can be replaced up to homotopy by a cofibration. The proof is essentially the same as that of 4.2.7.

4.2.8 **Theorem**. *Let $f : A \longrightarrow X$ be continuous and let M_f be the mapping cylinder of f* (see 3.1.2). *Let $j : A \longrightarrow M_f$ be defined by $j(a) = (a,1) \in M_f$. Then the following assertions hold:*

(a) *The map j is a cofibration.*

(b) *If $p : M_f \longrightarrow X$ is defined by $p(a,t) = f(a)$ and $p(x) = x$ for $(a,t) \in A \times I$ and for $x \in X$, then p is a homotopy equivalence satisfying $p \circ j = f$. So we have a commutative triangle*

$$\begin{tikzcd} & M_f \arrow[d, "p"] \\ A \arrow[ur, hook] \arrow[r, "f"'] & X, \end{tikzcd}$$

where the vertical arrow is a homotopy equivalence and the diagonal arrow (which is an inclusion) is a cofibration. $\square$

4.2.9 Exercise. Give the details of the proof of 4.2.8.

The class of cofibrations is a large class that contains the inclusions into a CW-complex of any subcomplex (see the following chapter) and the inclusions into an ANR of any closed subset that is also an ANR. Both of these are very important classes of spaces. We shall now study a bit of the latter class. We refer the reader to [31] for some additional results about this subject.

4.2.10 Definition. Let X be a metric space. Then X is called an *absolute neighborhood retract*, or in abbreviated form an ANR, if every time that we have an embedding $X \hookrightarrow Y$ of X as a closed subspace into a normal space Y, then the image of X in Y is a retract of an open neighborhood. Equivalently this condition says that whenever we have a closed subset A in a normal space Y as well as a map $f : A \longrightarrow X$, then we can extend f to an open neighborhood of A in Y.

4.2.11 Exercise. Prove the equivalence just mentioned above in Definition 4.2.10.

The class of ANRs is a large class that includes manifolds of finite dimension as well as paracompact manifolds modeled on Banach spaces. More generally, we can prove that any ANR can be embedded as a retract of an open subset of a normed topological vector space. This large class of spaces has interesting properties related to the HEP. For example, we have the following assertion, due to Borsuk.

4.2.12 **Proposition**. *Let A be a closed subspace of a metric space X. Then the pair (X, A) has the $\mathcal{A}$-*HEP*, where $\mathcal{A}$ is the class of all* ANR*s.*

Proof: Let Y be an ANR. It is enough to prove that any map $f : X \times 0 \cup A \times I \longrightarrow Y$ admits an extension to $X \times I$. Since Y is an ANR, we have by (the equivalent) definition that there exists an extension $H : U \longrightarrow Y$, where U is a neighborhood of $X \times 0 \cup A \times I$ in $X \times I$. Because I is compact, there exists a neighborhood V of A in X such that $V \times I \subset U$. Since X is metric, there exists $\psi : X \longrightarrow I$ satisfying $\psi|A = 1$ and $\psi|X - V = 0$. Then we extend f to the map $F : X \times I \longrightarrow Y$ defined by $F(x, t) = H(x, \psi(x)t)$. $\square$

4.2.13 **Theorem**. *If X is an* ANR *and $A \subset X$ is closed and is also itself an* ANR*, then the pair (X, A) has the* HEP.

Proof: It is enough to construct a retraction $r : X \times I \longrightarrow X \times 0 \cup A \times I$. To do this, we observe that since $X \times 0$, $A \times I$, and their intersection are all closed ANRs inside of their union, then their union is also an ANR. (See [31].) So, according to the proof of the previous theorem, it suffices to use here $Y = X \times 0 \cup A \times I$ and $f = \mathrm{id}$. □

In fact, the converse of Theorem 4.2.13 also is true; namely, we have the following assertion.

4.2.14 EXERCISE. Prove that if X is an ANR and $A \subset X$ is closed and the pair (X, A) has the HEP, then A is an ANR. (Hint: Because (X, A) has the HEP, it follows that A is a retract of a neighborhood U in X. So given any closed subset B of a normal space Y and any map $f : B \longrightarrow A$, we can extend f to $g : W \longrightarrow X$, where W is a neighborhood of B in Y. Then use a retraction $r : U \longrightarrow A$ to restrict g to a suitable neighborhood in such a way that its image lies in A.)

The results 4.2.13 and 4.2.14 show the relevance of the class of ANRs within the framework of the theory of cofibrations. They assert that in order for a closed subset of an ANR to be ANR, a necessary and sufficient condition is that the inclusion map be a cofibration. That is to say, we have the following extension of 4.2.13.

4.2.15 **Theorem**. *Suppose that X is an* ANR *and that $A \subset X$ is closed. Then A is an* ANR *if and only if the inclusion $A \hookrightarrow X$ is a cofibration.* □

The statements made in the following exercises are obtained directly from the definition of cofibration.

4.2.16 EXERCISE. Prove that if the inclusion $A \hookrightarrow X$ is a cofibration, then the inclusion $A \times Z \hookrightarrow X \times Z$ also is a cofibration for every space Z.

4.2.17 EXERCISE. Prove that the composition of cofibrations is a cofibration. Specifically, show that if $j' : A' \hookrightarrow A$ and $j : A \hookrightarrow X$ are cofibrations, then so also is the composite $j \circ j' : A' \hookrightarrow X$.

4.2.18 EXERCISE. Prove that if the inclusion $A \hookrightarrow X$ is a cofibration, then so also is the inclusion $A \hookrightarrow CX$ given by the composite $A \hookrightarrow X \hookrightarrow CX$.

4.3 Fibrations

In this section we shall study a class of maps, namely fibrations, with a property dual to that of cofibrations. In analogy to Section 4.1 we shall analyze homotopy lifting properties (HLPs). We are going to place these maps into classes according to the type of HLP they have.

A dual property to homotopy extension is homotopy lifting. With the idea of emphasizing this duality, we shall indicate throughout this section which extension property is dual to each lifting property when the latter is introduced.

4.3.1 Definition. Assume that $p : E \longrightarrow B$ is continuous and that $\mathcal{C}$ is a class of topological spaces. We say that p has the *homotopy lifting property with respect* to $\mathcal{C}$, denoted by $\mathcal{C}$-HLP, if for every $X \in \mathcal{C}$, every map $f : X \longrightarrow E$, and every homotopy $H : X \times I \longrightarrow B$ that begins with $p \circ f$ we can then *lift* H to a homotopy $\widetilde{H} : X \times I \longrightarrow E$ that begins with f, that is, such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(x, 0) = f(x)$. If a map $p : E \longrightarrow B$ has the $\mathcal{C}$-HLP, then we shall also say that it is a $\mathcal{C}$-*fibration.*

Putting this definition into diagrammatic form, we have that p has the $\mathcal{C}$-HLP if and only if for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B, \end{tikzcd} \tag{4.3.2}$$

where $X \in \mathcal{C}$ and where $j_0 : X \longrightarrow X \times I$ is the inclusion $j_0(x) = (x, 0)$, there exists a map $\widetilde{H}$, as indicated by the dashed arrow, that makes the two triangles commute.

In other words, this definition says that if $X \in \mathcal{C}$, then in the commutative diagram

$$\begin{tikzcd} M(X \times I, E) \arrow[r, "j_0^{\#}"] \arrow[d, "p_{\#}"'] & M(X, E) \arrow[d, "p_{\#}"] \\ M(X \times I, B) \arrow[r, "j_0^{\#}"'] & M(X, B) \end{tikzcd} \tag{4.3.3}$$

we have that whenever $f \in M(X, E)$ and $H \in M(X \times I, B)$ satisfy $j_0^{\#}(H) = p_{\#}(f)$, then there exists $\widetilde{H} \in M(X \times I, E)$ such that $p_{\#}(\widetilde{H}) = H$ y $j_0^{\#}(\widetilde{H}) = f$.

The dual character of Definition 4.3.1, when put face to face with Definition 4.1.1, is apparent when we modify diagram (4.3.2) as

$$
\begin{tikzcd}
 & & M(I,B) \arrow[dr, "e_0"] & \\
X \arrow[urr, bend left, "H'"] \arrow[r, dashed, "\widetilde{H}'"] \arrow[drr, bend right, "f"'] & M(I,E) \arrow[ur, "p\#"] \arrow[dr, "e_0"'] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}
\tag{4.3.4}
$$

and compare it with (4.1.2). Here $e_0 : M(I,E) \longrightarrow E$, (respectively, $e_0 : M(I,B) \longrightarrow B$) is evaluation at 0, namely $e_0(\alpha) = \alpha(0)$ for $\alpha \in M(I,E)$ (respectively, for $\alpha \in M(I,B)$). So p has the $\mathcal{C}$-HLP if and only if for every commutative diagram (4.3.4) with $X \in \mathcal{C}$ there exists a map $\widetilde{H}'$, as indicated by the dashed arrow, that makes the two triangles on the left commute.

The relations between H' and H, and between $\widetilde{H}'$ and $\widetilde{H}$, are given by the identities

$$H'(x)(t) = H(x,t) \qquad \text{and} \qquad \widetilde{H}'(x)(t) = \widetilde{H}(x,t)\,.$$

4.3.5 Exercise. Prove the equivalence of the definitions based on the diagrams (4.3.2) and (4.3.4).

4.3.6 Exercise. Suppose that $p : E \longrightarrow B$ has the $\mathcal{C}$-HLP and that $U \subset B$. Prove that the restriction $p_U = p|p^{-1}U : E_U = p^{-1}U \longrightarrow U$ also has the $\mathcal{C}$-HLP. This $\mathcal{C}$-fibration is called the *induced* $\mathcal{C}$*-fibration* (or simply the *pullback* $\mathcal{C}$*-fibration*).

4.3.7 Definition. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be continuous. A map $\widetilde{f} : E' \longrightarrow E$ is called *fiber-preserving* if it sends "fibers into fibers," that is, if there exists a continuous $f : B' \longrightarrow B$ such that the square

$$
\begin{tikzcd}
E' \arrow[r, "\widetilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}
$$

commutes, and therefore the fiber $(p')^{-1}(b')$ goes under $\widetilde{f}$ into the fiber $F_b = p^{-1}(f(b))$ for every $b' \in B'$.

Dually to Definition 4.2.4, we have the next concept.

4.3.8 DEFINITION. A commutative square of topological spaces and maps

$$\begin{tikzcd}
& E \arrow[dr, "p"] & \\
E' \arrow[ur, "g"] \arrow[dr, "q"'] & & B \\
& B' \arrow[ur, "f"'] &
\end{tikzcd}$$

is called a *pullback* if given maps $g' : W \longrightarrow E$ and $q' : W \longrightarrow B'$ such that $p \circ g' = f \circ q'$, then there exists a unique map $\psi : W \longrightarrow E'$ such that the following diagram commutes:

$$\begin{tikzcd}
& & E \arrow[dr, "p"] & \\
W \arrow[urr, bend left, "g'"] \arrow[r, dashed, "\psi"] \arrow[drr, bend right, "@'"'] & E' \arrow[ur, "g"] \arrow[dr, "q"'] & & B \\
& & B' \arrow[ur, "f"'] &
\end{tikzcd}$$

4.3.9 EXAMPLE. Suppose that $p : E \longrightarrow B$ and $f : B' \longrightarrow B$ are continuous. Put $E' = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$, and define $p' : E' \longrightarrow B'$ by $p'(b', e) = b'$ and $\widetilde{f} : E' \longrightarrow E$ by $\widetilde{f}(b', e) = e$. Then the corresponding commutative square is a pullback diagram. We say that p' *is induced from* p *by* f. It is denoted by $E' = f^*E$ (this space is also called the *pullback space*).

Notice that $\widetilde{f} : f^*E \longrightarrow E$ is a fiber-preserving map.

4.3.10 EXERCISE. Prove the following functorial properties of the construction defined in 4.3.9.

(a) If $f = \mathrm{id}_B$, then $f^*E \approx E$, where the homeomorphism is given by the associated map $\widetilde{f}$.

(b) If we also have $g : B'' \longrightarrow B'$, then $(fg)^*E \approx g^*f^*E$.

The next result generalizes the statement of Exercise 4.3.6.

4.3.11 **Proposition**. *If $p : E \longrightarrow B$ is a $\mathcal{C}$-fibration and $g : B' \longrightarrow B$ is continuous, then the map induced from p by g, namely $p' : E' \longrightarrow B'$, is a $\mathcal{C}$-fibration and is called* the induced *$\mathcal{C}$-fibration.*

Proof: Assume that $X \in \mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[d, "j_0"'] & E' \arrow[r, "\widetilde{g}"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H'"'] \arrow[ur, dashed, "\widetilde{H}'"] & B' \arrow[r, "g"'] & B.
\end{tikzcd}$$

Then we want to construct $\widetilde{H}' : X \times I \longrightarrow E'$ satisfying $\widetilde{H}' j_0 = g'$ and $p'\widetilde{H}' = H'$.

Put $f = \widetilde{g} \circ g'$ and $H = g \circ H'$. Since p is a $\mathcal{C}$-fibration, there exists $\widetilde{H} : X \times I \longrightarrow E$ such that $\widetilde{H} \circ j_0 = f$ and $p \circ \widetilde{H} = H$. Let $\widetilde{H}'$ be defined as $\widetilde{H}'(x,t) = (H'(x,t), \widetilde{H}(x,t)) \in E'$. Then we have $\widetilde{H}'(x,0) = (H'(x,0), \widetilde{H}(x,0)) = (p'g'(x), \widetilde{g}g'(x)) = g'(x)$, and so $\widetilde{H}' \circ j_0 = g'$. Also, we clearly have that $p' \circ \widetilde{H}' = H'$. □

4.3.12 Definition. Let $p : E \longrightarrow B$ be a $\mathcal{C}$-fibration. If $\mathcal{C}$ is the class of hypercubes I^n (or equivalently, as can be proved, the class of CW-complexes), then we say that $p : E \longrightarrow B$ is a *Serre fibration*. Moreover, if $\mathcal{C}$ is the class of all spaces, then we say that p is a *Hurewicz fibration*, or simply a *fibration* if this will not cause confusion.

4.3.13 Exercise. Let $p : E \longrightarrow B$ be a Hurewicz fibration. Prove that there exists a map

$$\Gamma : E \times_B M(I,B) = \{(e,\alpha) \in E \times M(I,B) \mid p(e) = \alpha(0)\} \longrightarrow M(I,E)$$

such that $\Gamma(e,\alpha)(0) = e$ and $p\Gamma(e,\alpha)(t) = \alpha(t)$ for $(e,\alpha) \in E \times_B M(I,B)$ and for $t \in I$.

Suppose furthermore that $p : E \longrightarrow B$ is given. Prove that if there exists Γ as above, then p is a Hurewicz fibration.

This map $\Gamma : E \times_B M(I,B) \longrightarrow M(I,E)$, whose existence characterizes the Hurewicz fibrations, is called *path-lifting map* (PLM). (Hint: Apply 4.3.4 where $X = E \times_B M(I,B)$ and where the maps $f : E \times_B M(I,B) \longrightarrow E$ and $H' : E \times_B M(I,B) \longrightarrow M(I,B)$ are defined to be the projection maps.)

4.3.14 Note. Observe that this PLM is the dual concept of the retraction predicted by Theorem 4.1.7. Namely, in this case the space $E \times_B M(I,B)$ is the *pullback* (limit) of the diagram

$$M(I,B) \xrightarrow{q_0} B \xleftarrow{p} E\,,$$

where $q_0(\alpha) = \alpha(0)$, whereas if $i : A \hookrightarrow X$ is a closed inclusion, then the space $X \times 0 \cup A \times I$ is the *pushout* (colimit) of the diagram

$$A \times I \xleftarrow{j_0} A \xhookrightarrow{i} X\,,$$

where $j_0(a) = (a,0)$.

Dually to Exercise 4.2.6, one can solve the following.

4.3.15 EXERCISE. Given a pullback diagram

$$\begin{tikzcd} & E \arrow[dr, "p"] & \\ E' \arrow[ur, "g"] \arrow[dr, "q"'] & & B\,, \\ & B' \arrow[ur, "f"'] & \end{tikzcd}$$

prove that if p is a Hurewicz fibration, then q is also a Hurewicz fibration.

4.3.16 EXERCISE. Let B be a topological space with base point $x_0 \in B$ and let $PB = \{\omega : I \longrightarrow B \mid \omega(0) = x_0\}$ be the *path space* of B with the compact-open topology. Then the map $q : PB \longrightarrow B$ defined by $q(\omega) = \omega(1)$ is a Hurewicz fibration whose fiber is the loop space ΩB and whose total space is the contractible space PB. (Hint: The map $\Gamma : PB \times_B M(I,B) \longrightarrow M(I,PB)$ defined by

$$\Gamma(\omega,\sigma)(t)(s) = \begin{cases} x_0 & \text{if } 4s \leq t, \\ \omega(\frac{4s-t}{4-2t}) & \text{if } t \leq 4s \leq 4-t, \\ \sigma(\frac{4s+t-4}{2s-1}) & \text{if } 4-t \leq 4s, \end{cases}$$

is a PLM. Also, the homotopy $H : PB \times I \longrightarrow PB$ given by $H(\omega,s) = \omega_s$, where $\omega_s(t) = \omega((1-s)t)$, is a contraction of PB to a point.

4.3.17 EXERCISE. Let $p : E \longrightarrow B$ be a Hurewicz fibration where B is path connected, and let b_0 and b_1 be points in B. Prove that the *fibers* $F_0 = p^{-1}(b_0)$ and $F_1 = p^{-1}(b_1)$ have the same homotopy type. (Hint: If $\alpha : b_0 \simeq b_1$ is a path, then for every point $e \in F_0$ there exists a path $\widetilde{\alpha} : I \longrightarrow E$ such that $\widetilde{\alpha} = \Gamma(e,\alpha)$, where Γ is a PLM (see 4.3.13). Then the map $F_0 \longrightarrow F_1$ given by $e \mapsto \widetilde{\alpha}(1)$ is a homotopy equivalence.)

4.3.18 EXERCISE. Let $p : E \longrightarrow B$ be a Hurewicz fibration.

(a) If B is contractible to point b_0 and $F = p^{-1}(b_0)$ is the fiber at b_0, prove that there exists a homotopy equivalence $\varphi : E \longrightarrow B \times F$ such that the triangle

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

commutes, where $\pi : B \times F \longrightarrow B$ is the projection. (Hint: Let $H : B \times I \longrightarrow B$ be a contraction, that is, a homotopy such that $H(b,0) = b$ and $H(b,1) = b_0$, and let $\widehat{H} : B \longrightarrow M(I,B)$ be given by $\widehat{H}(b)(t) = H(b,t)$. If

$$\Gamma : E \times_B M(I,B) \longrightarrow M(I,E)$$

is a PLM (4.3.13), then

$$\varphi(e) = (p(e), \Gamma(e, \widehat{H}(p(e)))(1))$$

is the desired homotopy equivalence.)

In this case we say that the fibration $p : E \longrightarrow B$ is *homotopically trivial.*

(b) Assume that B has a cover $\mathcal{U}$ formed by open contractible sets. Conclude that for every $U \in \mathcal{U}$, the induced fibration $p_U : E_U = p^{-1}U \longrightarrow U$ (4.3.6) is homotopically trivial. (Hint: Compare this property with Definition 4.5.1.)

4.3.19 Exercise. Let $p : E \longrightarrow B$ be a homotopically trivial Hurewicz fibration; i.e., there exists a homotopy equivalence $\varphi : E \longrightarrow B \times F$ such that the triangle

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

commutes, where $\pi : B \times F \longrightarrow B$ is the projection, and assume that (B,A) has the HEP. Prove that the induced fibration $p_A : E_A = p^{-1}A \longrightarrow A$ is also homotopically trivial. Conclude that there is a homotopy equivalence of pairs $(E, E_A) \longrightarrow (B,A) \times F$ over the identity of (B,A).

In some sense the next proposition shows the dual character of the HLP and the HEP.

4.3.20 **Proposition**. *Let $A \subset X$ be closed. Suppose that (X,A) has the $\mathcal{C}$-HEP and that X is locally compact and Hausdorff. If B is locally compact and Y satisfies $M(B,Y) \in \mathcal{C}$, then $i^{\#} : M(X,Y) \longrightarrow M(A,Y)$, where $i : A \hookrightarrow X$, has the $\{B\}$-HLP.*

Proof: Let us consider the commutative square

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, hook, "j_0"'] & M(X,Y) \arrow[d, "i^\#"] \\ B \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & M(A,Y). \end{tikzcd}$$

Then $i^\#$ will have the $\{B\}$-HLP if there exists an $\widetilde{H}$ that makes the two triangles commute. To do this, consider the commutative diagram

$$\begin{tikzcd} & X \arrow[dr] \arrow[rr, bend left, "f'"] & & \\ A \arrow[ur, hook] \arrow[dr] & & X \times I \arrow[r, dashed, "\widetilde{H}'"] & M(B,Y)\,, \\ & A \times I \arrow[ur, hook] \arrow[rr, bend right, "H'"'] & & \end{tikzcd}$$

where f' and H' correspond to f and H, respectively, under the exponential bijection (applied two times); that is, $f'(x)(b) = f(b)(x)$ and $H'(a,t)(b) = H(b,t)(a)$. Then f' and H' are continuous, since B and X are locally compact and Hausdorff. By hypothesis $\widetilde{H}'$ exists. So, defining $\widetilde{H}$ by $\widetilde{H}(b,t)(x) = \widetilde{H}'(x,t)(b)$, it turns out that $\widetilde{H}$ is continuous (once again because B is locally compact; see [27, Chapter XII]) and has the desired properties. □

In a dual way to Theorem 4.2.8, we have the following.

4.3.21 **Theorem**. *Let $f : Y \longrightarrow B$ be continuous, Y path connected, and let E_f be the mapping path space of f* (see 3.3.14). *Let $p : E_f \longrightarrow B$ be defined by $p(y,\beta) = \beta(1)$, $(y,\beta) \in E_f$. Then the following assertions hold:*

(a) *The map p is a fibration.*

(b) *If $i : Y \longrightarrow E_f$ is defined by $i(y) = (y, c_{f(y_0)})$ for $y \in Y$ and $c_{f(y_0)} \in M(I,B)$ is the constant path, then i is a homotopy equivalence satisfying $p \circ i = f$. So we have a commutative triangle*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i"'] & B \\ E_f\,, \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

where the vertical arrow is a homotopy equivalence and the diagonal arrow (which is surjective) is a Hurewicz fibration. □

The *proof* is somehow dual to the proof of 4.2.8. To prove that p is a Hurewicz fibration, one may construct an adequate PLM. Details are left to the reader as an *exercise*.

4.3.22 NOTE. In a beautiful article, N. Steenrod [70] (see also [22]) describes how, by working in the category of compactly generated spaces already studied systematically by Kelley [39], the hypothesis of local compactness can be made superfluous in the previous proof.

We say that a Hausdorff topological space X is *compactly generated* if it satisfies the following condition:

(CG) A subset A of X is closed if and only if $A \cap C$ is closed for every compact subset C of X.

That is to say, a space is compactly generated if its topology is the *weak topology* generated by all of its compact subsets or, to put it in other words, if the space has the *union topology* associated to its compact subsets.

Assume that X is any given Hausdorff space. By using (CG) we can define in X a new topology that turns X into a compactly generated space. We denote by kX the space X with this new topology. Evidently, we have that $\mathrm{id} : kX \longrightarrow X$ is continuous. In fact, it is a homeomorphism if and only if X is compactly generated. Furthermore, X and kX have exactly the same compact subsets. It is also clear that X and kX have the same homotopy groups, since the continuous image of any sphere always lies in a compact subset.

In the category of compactly generated spaces (also called k-spaces) we apply the k construction to the traditional constructions of new spaces from given spaces in order to guarantee that these new spaces belong to the same category. In particular, the product of two compactly generated spaces X and Y is given by $k(X \times Y)$ in this category. Analogously, $kM(X, Y)$ is a good definition for the topology of the function space, since the exponential laws turn out to hold.

The category of compactly generated spaces is very large. In fact, it contains all locally compact Hausdorff spaces as well as all spaces that satisfy the first countability axiom, such as metric spaces [83, 27]. By construction CW-complexes also are in this category. These topics are also treated in detail by B. Gray in his book [31].

In light of the previous note, the duality between cofibration and Hurewicz fibration is clarified further in the following consequence of 4.3.20.

4.3.23 **Corollary**. *Let $i : A \hookrightarrow X$ be a cofibration in the category of compactly generated spaces. Then for every compactly generated space B, the induced map $i^{\#} : kM(X,B) \longrightarrow kM(A,B)$ is a Hurewicz fibration.* $\square$

At this point it is worthwhile to mention some other results that connect the concepts of fibration and cofibration. These results are proved in [74], and we refer the reader to that article for their proofs.

4.3.24 **Theorem**. *Suppose that $j(A)$ is closed in X, where $j : A \longrightarrow X$ is a map. Then these two statements are equivalent:*

(a) *Given a Hurewicz fibration $p : E \longrightarrow B$ and a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B, \end{tikzcd}$$

there exists a lifting $h : X \longrightarrow E$ such that $p \circ h = f$ and $h \circ j = g$.

(b) *The map j is a cofibration and a homotopy equivalence.*

If (a) *and* (b) *hold, then the lifting h of f is unique up to a homotopy relative to $j(A)$.* $\square$

By taking X instead of A, $X \times I$ instead of X, and a homotopy $H : X \times I \longrightarrow B$ in part (a) of the previous theorem, since in this case $j = j_0 : x \mapsto (x, 0)$ is a cofibration and a homotopy equivalence, there always exists a lifting $G : X \times I \longrightarrow E$ with the desired properties. So we recover the definition of a fibration. Moreover, since the previous theorem states that the lifting is unique up to homotopy relative to $j_0(X)$, we obtain the following.

4.3.25 **Corollary**. *Let $p : E \longrightarrow B$ be a Hurewicz fibration. Given a homotopy $H : X \times I \longrightarrow B$ and a map $f : X \longrightarrow E$ such that $H(x,0) = pf(x)$ for all $x \in X$, there exists a homotopy $\widetilde{H} : X \times I \longrightarrow E$, which is unique up to homotopy relative to $X \times \{0\}$, such that $\widetilde{H}(x,0) = f(x)$ and $p\widetilde{H}(x,t) = H(x,t)$ for all $x \in X$ and all $t \in I$.* $\square$

The next result is dual to 4.3.24.

4.3.26 **Theorem**. *Assume that $p : E \longrightarrow B$. Then these two statements are equivalent:*

(a) *Given a (closed) cofibration $j : A \longrightarrow X$ and a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B, \end{tikzcd}$$

there exists a lifting $h : X \longrightarrow E$ such that $p \circ h = f$ and $h \circ j = g$.

(b) *The map p is a Hurewicz fibration and a homotopy equivalence.*

If (a) *and* (b) *hold, then the lifting h of f is unique up to a homotopy that is vertical with respect to p (that is, a homotopy that preserves fibers).* □

By taking $M(I, Y)$ instead of E, Y instead of B, and a map $\widehat{H} : A \longrightarrow M(I, Y)$ or, equivalently, a homotopy $H : A \times I \longrightarrow Y$, in part (a) of the previous theorem, since in this case $p = q_0 : \beta \mapsto \beta(0)$ is a Hurewicz fibration and a homotopy equivalence, there always exists an extension $\widehat{\widetilde{H}} : X \longrightarrow M(I, Y)$ of $\widetilde{H}$ or, equivalently, an extension of H, $\widetilde{H} : X \times I \longrightarrow E$, with the desired properties (observe that the lifting of f predicted in the theorem is also an extension of g). So we recover the definition of a cofibration. Moreover, since the previous theorem states that the extension is unique up to vertical homotopy, we obtain the following.

4.3.27 **Corollary**. *Let $j : A \longrightarrow X$ be a closed cofibration. Given a homotopy $H : A \times I \longrightarrow Y$ and a map $f : X \longrightarrow Y$ such that $H(a, 0) = fj(a)$ for all $a \in A$, there exists a homotopy $\widetilde{H} : X \times I \longrightarrow Y$, which is unique up to homotopy relative to $X \times \{0\}$, such that $\widetilde{H}(x, 0) = f(x)$ and $\widetilde{H}(a, t) = H(a, t)$ for all $x \in X$ and all $t \in I$.* □

Corollary 4.3.25 follows from a more general result than 4.3.24, which we state and prove now.

4.3.28 **Proposition**. *Let $p : E \longrightarrow B$ be a Hurewicz fibration and let $H_0, H_1 : X \times I \longrightarrow E$ be homotopies such that*

(i) *there is a homotopy $\mathcal{H} : p \circ H_0 \simeq p \circ H_1$;*

(ii) *there is a homotopy* $G : H_0|X \times \{0\} \simeq H_1|X \times \{0\}$*;*

(iii) $pG(x,0,t) = \mathcal{H}(x,0,t)$ *for all* $x \in X$*,* $t \in I$*.*

Then there exists a homotopy $\widetilde{\mathcal{H}} : H_0 \simeq H_1$ *such that* $\widetilde{\mathcal{H}}(x,0,t) = G(x,0,t)$ *for all* $x \in X$*,* $t \in I$*, and* $p \circ \widetilde{\mathcal{H}} = \mathcal{H}$*.*

Proof: Let $C = I \times \{0\} \cup I \times \{1\} \cup \{0\} \times I \subset I \times I$. We define a map $\varphi : X \times C \longrightarrow E$ by

$$\varphi(x,s,t) = \begin{cases} H_0(x,s) & \text{if } t = 0, \\ H_1(x,s) & \text{if } t = 1, \\ G(x,0,t) & \text{if } s = 0. \end{cases}$$

There is a homeomorphism of pairs $\alpha : (I \times I, C) \longrightarrow (I \times I, I \times \{0\})$. If $i : C \hookrightarrow I \times I$ is the inclusion, then $p \circ \varphi = \mathcal{H} \circ (\mathrm{id} \times i)$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
X \times I \times \{0\} \arrow[d, hook] & X \times C \arrow[l, "\mathrm{id}_X \times (\alpha|C)"', "\approx"] \arrow[r, "\varphi"] \arrow[d, hook, "\mathrm{id}_X \times i"] & E \arrow[d, "p"] \\
X \times I \times I & X \times I \times I \arrow[l, "\approx", "\mathrm{id}_X \times (\alpha|C)"'] \arrow[r, "\mathcal{H}"'] & B\,.
\end{tikzcd}$$

By the HLP, there exists a homotopy $\widehat{\mathcal{H}} : X \times I \times I \longrightarrow E$ such that $p \circ \widehat{\mathcal{H}} = \mathcal{H} \circ (\mathrm{id}_X \times \alpha^{-1})$ and $\widehat{\mathcal{H}}|X \times I \times \{0\} = \varphi \circ (\mathrm{id}_X \times (\alpha|C)^{-1})$. Therefore, the desired homotopy $\widetilde{\mathcal{H}}$ is given by $\widetilde{\mathcal{H}} = \widehat{\mathcal{H}} \circ (\mathrm{id}_X \times \alpha)$. □

Let $p : E \longrightarrow B$ be a Hurewicz fibration and assume that there is a map $f : X \longrightarrow E$ and a homotopy $H : X \times I \longrightarrow B$ such that $pf(x) = H(x,0)$ for all $x \in X$. Assume, moreover, that H_0 and H_1 are two liftings of H such that $H_0(x,0) = f(x) = H_1(x,0)$ for all $x \in X$. Then, taking $\mathcal{H}$ and G as constant homotopies, we can apply the previous proposition and conclude that $H_0 \simeq H_1$. Thus we have given an alternative proof of 4.3.25.

There is a corollary of 4.3.28, which, as a matter of fact, is equivalent to 4.3.25, as follows.

4.3.29 **Corollary**. *Let* $p : E \longrightarrow B$ *be a Hurewicz fibration. Then its path-lifting map is unique up to homotopy.*

Proof: Let $\Gamma_0, \Gamma_1 : E \times_B M(I,B) \longrightarrow M(I,E)$ be two lifting maps for p, and consider their associated maps

$$\widehat{\Gamma}_0, \widehat{\Gamma}_1 : (E \times_B M(I,B)) \times I \longrightarrow E\,,$$

under the exponential law. It is clear that these maps satisfy the conditions for H_0 and H_1 of 4.3.28. Hence, there is a homotopy $\widehat{\mathcal{H}} : \widehat{\Gamma}_0 \simeq \widehat{\Gamma}_1$, whose map associated under the exponential law gives a homotopy $\mathcal{H} : \Gamma_0 \simeq \Gamma_1$ such that for each t the map $\Gamma_t : (\beta, e) \mapsto \mathcal{H}(\beta, e, t)$ is also a lifting map for p. $\square$

Dually to what we did above, we can deduce Corollary 4.3.27 from a more general result, dual to 4.3.28, which we state and prove now.

4.3.30 **Proposition**. *Let $j : A \hookrightarrow X$ be a closed cofibration and let $H_0, H_1 : X \times I \longrightarrow Y$ be homotopies such that*

(i) *there is a homotopy $\mathcal{H} : H_0|A \times I \simeq H_1|A \times I$;*

(ii) *there is a homotopy $G : H_0|X \times \{0\} \simeq H_1|X \times \{0\}$;*

(iii) *$G(a, 0, t) = \mathcal{H}(a, 0, t)$ for all $a \in A$, $t \in I$.*

Then there exists a homotopy $\widetilde{\mathcal{H}} : H_0 \simeq H_1$ such that $\widetilde{\mathcal{H}}(x, 0, t) = G(x, 0, t)$ for all $x \in X$, $t \in I$, $\widetilde{\mathcal{H}}(a, s, t) = G(a, s, t)$, and $\widetilde{\mathcal{H}}|A \times I \times I = \mathcal{H}$.

Proof: As in the proof of 4.3.28, take the subset $C \subset I \times I$ and the homeomorphism of pairs $\alpha : (I \times I, C) \longrightarrow (I \times I, I \times \{0\})$. Let $D = (X \times I \times \{0\}) \cup (X \times I \times \{1\}) \cup (A \times I \times I) \cup (X \times \{0\} \times I) \subset X \times I \times I$ and define a homeomorphism $\beta : (X \times \{0\} \cup A \times I) \times I \longrightarrow D$ by

$$\beta(x, 0, t) = (x, \alpha^{-1}(x, 0)), \quad \beta(a, s, t) = (a, \alpha^{-1}(s, t)).$$

Let now $\varphi : D \longrightarrow Y$ be defined by

$$\varphi(x, s, t) = \begin{cases} H_0(x, s) & \text{if } t = 0, \\ H_1(x, s) & \text{if } t = 1, \\ \mathcal{H}(a, s, t) & \text{if } a \in A, \\ G(x, 0, t) & \text{if } s = 0. \end{cases}$$

Since $A \hookrightarrow X$ is a closed cofibration, by 4.1.7 there exists a retraction $r' : X \times I \longrightarrow X \times \{0\} \cup A \times I$. Define $\widetilde{\mathcal{H}}' : X \times I \times I \longrightarrow Y$ as the composite

$$\widetilde{\mathcal{H}}' : X \times I \times I \xrightarrow{r' \times \mathrm{id}_I} (X \times \{0\} \cup A \times I) \times I \xrightarrow{\beta} D \xrightarrow{\varphi} Y.$$

Therefore, the desired homotopy $\widetilde{\mathcal{H}}$ is given by $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}' \circ (\mathrm{id}_X \times \alpha)$. $\square$

Let $j : A \hookrightarrow X$ be a closed cofibration and assume that there is a map $f : X \longrightarrow Y$ and a homotopy $H : A \times I \longrightarrow Y$ such that $H(a, 0) = fj(a)$ for all $a \in A$. Assume, moreover, that H_0 and H_1 are two extensions of H such that $H_0(x, 0) = f(x) = H_1(x, 0)$ for all $x \in X$. Then, taking $\mathcal{H}$ and G as constant homotopies, we can apply the previous proposition and conclude that $H_0 \simeq H_1$. Thus we have given an alternative proof of 4.3.27.

There is also a corollary of 4.3.30, which, as a matter of fact, is an equivalent result to 4.3.27, as follows.

4.3.31 **Corollary**. *Let $j : A \hookrightarrow X$ be a closed cofibration. Then its retraction $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$ is unique up to homotopy.*

Proof: Let $r_0, r_1 : X \times I \longrightarrow X \times \{0\} \cup A \times I$ be two retractions. Taking $Y = X \times \{0\} \cup A \times I$, it is clear that these maps satisfy the conditions for H_0 and H_1 of 4.3.30. Hence, there is a homotopy $\widetilde{\mathcal{H}} : r_0 \simeq r_1$ such that for any t the map $r_t : (x, s) \mapsto \mathcal{H}(x, s, t)$ is also a retraction for j. □

Finally, here is another interesting result, which links fibrations and cofibrations. It also is proved in [74].

4.3.32 **Theorem**. *Let B be normal. If the pair (B, A) has the* HEP *with A closed in B and if $p : E \longrightarrow B$ is a Hurewicz fibration, then the pair (E, E_A) has the* HEP.

Proof: Using Theorem 4.1.16, we can take $\varphi : B \longrightarrow I$ and $D : B \times I \longrightarrow B$ as in part (b) of that theorem. Since p is a Hurewicz fibration, there exists a lifting $\widetilde{H} : E \times I \longrightarrow E$ of the homotopy $D \circ (p \times \mathrm{id}_I)$ that makes the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}_E"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ E \times I \arrow[r, "D\circ(p\times \mathrm{id}_I)"'] \arrow[ur, dashed, "\widetilde{H}"] & B. \end{tikzcd}$$

We then define $D' : E \times I \longrightarrow E$ by

$$D'(e, t) = \widetilde{H}(e, \min\{t, \varphi p(e)\}) .$$

Then D' and $\varphi' = \varphi \circ p$ satisfy the hypotheses of 4.1.16 again. □

Let us now analyze Serre fibrations.

4.3.33 Theorem. *If $p : E \longrightarrow B$ is a Serre fibration, then for $q \geq 1$,*

$$p_* : \pi_q(E, F_b) \longrightarrow \pi_q(B)$$

is an isomorphism, where $b \in B$ and $e \in F_b = p^{-1}(b)$ are arbitrary base points with respect to which we take the homotopy groups.

Proof: Let $F : (I^q, \partial I^q) \longrightarrow (B, b)$ be a representative of a class in $\pi_q(B)$. In particular, looking at F as

$$F : I^{q-1} \times I \longrightarrow B\,,$$

we then have that $F(x, 0) = b$, since $F(\partial I^q) = \{b\}$. So if we take $f : I^{q-1} \longrightarrow E$ to be constant, specifically $f(I^{q-1}) = \{e\}$, then the diagram

$$\begin{tikzcd} I^{q-1} \arrow[r, "f"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ I^{q-1} \times I \arrow[r, "F"'] & B \end{tikzcd}$$

is commutative. By hypothesis, there exists $\widetilde{F} : I^{q-1} \times I \longrightarrow E$ such that $\widetilde{F}(x, 0) = f(x) = e$ and $p\widetilde{F}(x, t) = F(x, t)$. Since $p\widetilde{F}(\partial I^q) = F(\partial I^q) = \{b\}$, we have that $\widetilde{F}(\partial I^q) \subset p^{-1}(b)$, and so $\widetilde{F} : I^q \longrightarrow E$ determines an element in $\pi_q(E, p^{-1}(b))$ such that $p_*[\widetilde{F}] = [F]$. Therefore, p_* is an epimorphism.

Let us now show that p_* is a monomorphism. To do this we note that if we have a pointed pair (X, A, x_0), then we can set up a bijection

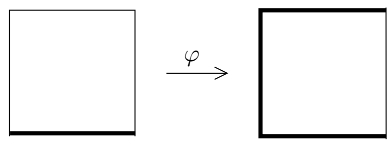
$$[(\mathbb{D}^q, \mathbb{S}^{q-1}, *), (X, A, x_0)] \leftrightarrow [(I^q, \partial I^q, J^{q-1}), (X, A, x_0)]\,,$$

where $J^{q-1} = \partial I^{q-1} \times I \cup I^{q-1} \times \{0\}$. So we have that

$$\pi_q(E, p^{-1}(b), e) = [(I^q, \partial I^q, J^{q-1}), (E, p^{-1}(b), e)]\,.$$

Let $\widetilde{F} : (I^q, \partial I^q, J^{q-1}) \longrightarrow (E, p^{-1}(b), e)$ be a representative of an element in $\pi_q(E, p^{-1}(b), e)$ such that $p_*[\widetilde{F}] = 1$, that is, $p \circ \widetilde{F} \simeq 0$. Also let $H : (I^q, \partial I^q) \times I \longrightarrow (B, b)$ be a homotopy such that $H(y, 0) = p\widetilde{F}(y)$ and $H(y, 1) = b$. Then we have the commutative diagram

$$\begin{tikzcd} I^q \arrow[r, "\varphi_0", "\approx"'] \arrow[d, "j_0"'] & I^q \times \{0\} \cup I^q \times \{1\} \cup J^{q-1} \times I \arrow[r, "h"] \arrow[d] & E \arrow[d, "p,"] \\ I^q \times I \arrow[r, "\approx", "\varphi"'] & I^q \times I \arrow[r, "H"'] & B, \end{tikzcd}$$



$(I^q \times I, I^q \times \{0\}) \longrightarrow (I^q \times I, I^q \times \{0\} \cup I^q \times \{1\} \cup J^{q-1} \times I)$

Figure 4.1

where $\varphi : (I^q \times I, I^q \times \{0\}) \longrightarrow (I^q \times I, I^q \times \{0\} \cup I^q \times \{1\} \cup J^{q-1} \times I)$ is a homeomorphism of pairs and φ_0 is the restriction to the lower face, as Figure 4.1 shows.

Then we have that $h|I^q \times \{0\} = \widetilde{F}$ and that $h|I^q \times \{1\} \cup J^{q-1} \times I$ is the constant map whose value is e. Since p is a Serre fibration, there exists $K' : I^q \times I \longrightarrow E$ such that

$$K'(y,0) = h\varphi_0(y) \quad \text{and} \quad p \circ K' = H \circ \varphi\,.$$

Then $K = K' \circ \varphi^{-1} : I^q \times I \longrightarrow E$ is a homotopy such that

$$\begin{aligned} K(y,0) &= K'\varphi^{-1}(y,0) = h(y,0) = \widetilde{F}(y)\,, \\ K(y,1) &= K'\varphi^{-1}(y,1) = h(y,1) = e\,, \end{aligned}$$

and $K(J^{q-1} \times I) = \{e\}$. Moreover, since $pK(\partial I^q \times I) = H(\partial I^q \times I) = \{b\}$, we have that $K(\partial I^q \times I) \subset p^{-1}(b)$. So K is a nullhomotopy for $\widetilde{F}$, implying that $[\widetilde{F}] = 1$. Therefore p_* is a monomorphism. □

4.3.34 **Corollary**. *If $p : E \longrightarrow B$ is a Serre fibration, then for $b \in B$ and $F = p^{-1}(b)$ we have that*

$$\cdots \longrightarrow \pi_q(F) \longrightarrow \pi_q(E) \longrightarrow \pi_q(B) \longrightarrow \pi_{q-1}(F) \longrightarrow \cdots$$

is an exact sequence.

Proof: Consider the homotopy sequence (3.4.6) of the pair (E, F). According to 4.3.23 each term $\pi_q(E, F)$ in it can be substituted with $\pi_q(B)$. So, defining a new connecting homomorphism ∂ as the composition of the isomorphism $(p_*)^{-1} : \pi_q(B) \longrightarrow \pi_q(E, F)$ followed by the connecting homomorphism ∂ of the pair (E, F) (3.4.6), we obtain the exact sequence that we were looking for. □

This sequence is known as the *exact homotopy sequence of the Serre fibration $p : E \longrightarrow B$.*

Let us now examine an interesting property relating fibrations to strong deformation retracts. First let us recall that $A \subset X$ is a *strong deformation retract* if there exists a homotopy

$$H : X \times I \longrightarrow X$$

such that

$$\begin{aligned} H(x,0) &= x\,, \quad x \in X\,, \\ H(a,t) &= a\,, \quad a \in A\,,\ t \in I\,, \\ H(x,1) &\in A\,, \quad x \in X\,. \end{aligned}$$

So $r : X \longrightarrow A$ defined by $r(x) = H(x,1)$ is a retraction.

4.3.35 **Proposition**. *Assume that $p : E \longrightarrow B$ is a $\mathcal{C}$-fibration and that $A \subset X$ is a strong deformation retract with $A, X \in \mathcal{C}$. If the square*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "h"] & B, \end{tikzcd}$$

commutes, that is, if $p \circ g = f|A$, then there exists $h : X \longrightarrow E$ such that $p \circ h = f$ and $h|A \simeq g$.

Proof: Suppose that $H : X \times I \longrightarrow X$ is a deformation of X that retracts X to A and that $r : X \longrightarrow A$ is the corresponding retraction. Let $F : X \times I \longrightarrow B$ be defined by $F = f \circ H$, and let $g' : X \longrightarrow E$ be defined by $g' = g \circ r$. We then obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "j_1"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B, \end{tikzcd}$$

where $j_1(x) = (x,1)$. Since $X \in \mathcal{C}$ and p has the $\mathcal{C}$-HLP, there exists $\widetilde{F} : X \times I \longrightarrow E$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(x,1) = g'(x)$. If we define $h : X \longrightarrow E$ by $h(x) = \widetilde{F}(x,0)$, then we have that $ph(x) = p\widetilde{F}(x,0) = F(x,0) = fH(x,0) = f(x)$ and $\widetilde{F}(a,0) = h(a)$. Also, for $a \in A$ we have $\widetilde{F}(a,1) = g'(a) = gr(a) = g(a)$, and so $h|A \simeq g$ follows. $\square$

4.3.36 EXERCISE. Prove that if in 4.3.35 the inclusion is also a cofibration, then we can prove that there exists h such that $h|A = g$.

4.3.37 EXERCISE. We say that a map $f : B' \longrightarrow B$ is a *weak homotopy equivalence* if for every $q \geq 0$ the induced map $f_* : \pi_q(B') \longrightarrow \pi_q(B)$ is an isomorphism (see 5.1.17). Let $p : E \longrightarrow B$ be a Serre fibration, and let $p' : E' \longrightarrow B'$ be the fibration induced from p by f. So we have the commutative diagram:

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B\,. \end{tikzcd}$$

Prove that if f is a weak homotopy equivalence, then so also is $\widetilde{f}$. (Hint: Let F be the common fiber of p and p'. We then have the commutative diagram

$$\begin{tikzcd} \pi_q(E',F) \arrow[r, "\widetilde{f}_*"] \arrow[d, "p'_*"'] & \pi_q(E,F) \arrow[d, "p_*"] \\ \pi_q(B') \arrow[r, "f_*"'] & \pi_q(B)\,, \end{tikzcd}$$

where the vertical arrows are isomorphisms by 4.3.33. By hypothesis, the lower horizontal arrow is also an isomorphism. Therefore, the upper horizontal arrow is an isomorphism as well. Now apply the exact homotopy sequence for each of the pairs (E',F) and (E,F) (3.5.8(e)) and then the five lemma in order to prove that $\widetilde{f}_* : \pi_q(E') \longrightarrow \pi_q(E)$ is also an isomorphism.)

The next theorem generalizes 4.3.33.

4.3.38 **Theorem**. *Let $p : E \longrightarrow B$ be a Serre fibration. Then for every $A \subset B$, $b \in A$, and $e \in p^{-1}(b)$ we have an isomorphism*

$$p_* : \pi_q(E, E_A, e) \cong \pi_q(B, A, b)\,.$$

Proof: Assume that $f : (I^q, \partial I^q, J^{q-1}) \longrightarrow (B, A, b)$ represents an arbitrary element of $\pi_q(B,A,b)$. Then we have the commutative diagram

$$\begin{tikzcd} J^{q-1} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^q \arrow[r, "f"'] \arrow[ur, dashed, "h"] & B, \end{tikzcd}$$

where $g(J^{q-1}) = \{e\}$. Since there is a homeomorphism of pairs

$$(I^{q-1} \times I,\ I^{q-1} \times \{0\}) \approx (I^q,\ J^{q-1})\,,$$

there exists $h : I^q \longrightarrow E$ such that $p \circ h = f$ and $h|J^{q-1} = g$, just as in the proof of 4.3.33. Since $ph(\partial I^q) = f(\partial I^q) \subset A, h(\partial I^q) \subset E_A$, and $h(J^{q-1}) = \{e\}$, we have that $h : (I^q, \partial I^q, J^{q-1}) \longrightarrow (E, E_A, e)$ represents a preimage of $[f]$. Consequently, p_* is an epimorphism.

Suppose now that $g : (I^q, \partial I^q, J^{q-1}) \longrightarrow (E, E_A, e)$ satisfies $p \circ g \simeq 0$ and that $F : (I^q, \partial I^q, J^{q-1}) \times I \longrightarrow (B, A, b)$ is a nullhomotopy; that is, $F(s, 0) = pg(s)$ and $F(s, 1) = b$. Then we have the commutative diagram

$$\begin{tikzcd}
I^q \times \{0\} \cup I^q \times \{1\} \cup J^{q-1} \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^q \times I \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B,
\end{tikzcd}$$

where $f(s, 0) = g(s)$ for $s \in I^q$, $f(s, 1) = e$ for $s \in I^q$, and $f(s, t) = e$ for $s \in J^{q-1}$ and $t \in I$. Once again, as in the proof of 4.3.33, there exists $\widetilde{F} : I^q \times I \longrightarrow E$ such that $p \circ \widetilde{F} = F$, $\widetilde{F}(s, 0) = g(s)$, and $\widetilde{F}(s, 1) = e$. Moreover, since $F(\partial I^q \times I) \subset A$, we have that $\widetilde{F}(\partial I^q \times I) \subset E_A$, and therefore $\widetilde{F} : (I^q, \partial I^q, J^{q-1}) \longrightarrow (E, E_A, e)$ is a nullhomotopy of g, implying that $[g] = 1$. So p_* is a monomorphism. □

The concept of quasifibration, introduced by Dold and Thom [26] and presented next, is made exactly in order to obtain the exact homotopy sequence that we have for the Serre fibrations. Specifically, Theorem 4.3.33 inspires us to make the next definition.

4.3.39 Definition. (Dold–Thom) A map $p : E \longrightarrow B$ is called a *quasifibration* if for every point $b \in B$ and for every $e \in p^{-1}(b)$ we have that

$$p_* : \pi_q(E, p^{-1}(b)) \longrightarrow \pi_q(B)$$

is an isomorphism for all $q \geq 0$, where these groups (or possibly sets) are based on e and b, respectively.

We can prove the next result in the same way as we proved 4.3.34.

4.3.40 **Proposition**. *Assume that $p : E \longrightarrow B$ is a quasifibration and that $b \in B$ and $e \in p^{-1}(b) = F$. Then there exists a long exact sequence*

$$\cdots \longrightarrow \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{p_*} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \longrightarrow \cdots . \tag{4.3.41}$$

□

This is called the *exact homotopy sequence* of the quasifibration $p : E \longrightarrow B$.

In Appendix A we gather a series of criteria for determining when a map is a quasifibration. These are results that appear in [26]. Because their proofs are technically more complicated than those that we typically include here in the main text, we prefer not to treat them now.

4.3.42 NOTE. The articles [73] and [74] of Strøm systematically treat cofibrations and their relations with fibrations. Reading them would be an excellent complement to the material treated in the first three sections of this chapter.

4.4 POINTED AND UNPOINTED HOMOTOPY CLASSES

Let X and Y be pointed spaces with base points x_0 and y_0, respectively. In this section, using results of the previous sections, we analyze the differences and the relationship between the set of pointed homotopy classes of maps $[X, x_0; Y, y_0]$ and the set of unpointed homotopy classes $[X, Y]$. In order to do this we shall assume that the space X is *well pointed*, namely, that the inclusion $i : \{x_0\} \hookrightarrow X$ is a closed cofibration. This condition will enable us to define an action of the fundamental group $\pi_1(Y, y_0)$ on $[X, x_0; Y, y_0]$.

4.4.1 **Proposition**. *Let X be a well-pointed space and let Y be a pointed space. Then there is a right action of the fundamental group $\pi_1(Y, y_0)$ on the homotopy set $[X, x_0; Y, y_0]$, namely a function*

$$\begin{aligned} [X, x_0; Y, y_0] \times \pi_1(Y, y_0) &\longrightarrow [X, x_0; Y, y_0]\,, \\ ([f], [\alpha]) &\longmapsto [f] \cdot [\alpha]\,, \end{aligned}$$

such that if $[\alpha], [\beta] \in \pi_1(Y, y_0)$ and $[f] \in [X, x_0; Y, y_0]$, then

$$[f] \cdot 1 = [f] \quad \textit{and} \quad ([f] \cdot [\alpha]) \cdot [\beta] = [f] \cdot ([\alpha] \cdot [\beta])\,,$$

where $1 \in \pi_1(Y, y_0)$ is the the identity element.

Proof: Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map and $\alpha : (I, \partial I) \longrightarrow (Y, y_0)$ a loop based at y_0. Since $\{x_0\} \hookrightarrow X$ is a cofibration, there exists a homotopy $F : X \times I \longrightarrow Y$ that completes the following diagram:

$$\begin{tikzcd}
 & X \arrow[dr, "j_0"] \arrow[drr, bend left, "f"] & & \\
\{x_0\} \arrow[ur, hook, "i"] \arrow[dr, "j_0"'] & & X \times I \arrow[r, dashed, "F"] & Y . \\
 & \{x_0\} \times I \arrow[ur, hook, "i \times \mathrm{id}"'] \arrow[urr, bend right, "\alpha"'] & &
\end{tikzcd}$$

Define the action by setting $[f] \cdot [\alpha] = [F_1]$, where $F_1 : (X, x_0) \longrightarrow (Y, y_0)$ is defined by $F_1(x) = F(x, 1)$. In order to see that this homotopy class is well defined, consider another homotopy extension $F' : X \times I \longrightarrow Y$ of α starting with f. Then by 4.3.27, there is a homotopy $\widetilde{\mathcal{H}} : F \simeq F'$. Let $h : X \times I \longrightarrow Y$ be given by $h(x, t) = \widetilde{\mathcal{H}}(x, 1, t)$. Then $h(x, 0) = F(x, 1) = F_1(x)$ and $h(x, 1) = F'(x, 1) = F'_1(x)$. Therefore, we can associate $[F_1]$ to the pair (f, α).

In order to see that $[F_1]$ depends only on the homotopy classes of f and α, assume that F'_1 is associated to f' and α' and that there are homotopies $G : f \simeq f'$ and $\mathcal{H} : \alpha \simeq \alpha'$. Since $\mathcal{H}(x_0, 0, t) = y_0 = G(x_0, 0, t)$ for all $t \in I$, the conditions of Proposition 4.3.30 are fulfilled, and hence there is a homotopy $\widetilde{\mathcal{H}} : F \simeq F'$. If we define $h : X \times I \longrightarrow Y$ by $h(x, t) = \widetilde{\mathcal{H}}(x, 1, t)$, then $h : F_1 \simeq F'_1$. Therefore, the function $[f] \cdot [\alpha] = [F_1]$ is well defined.

To show that this is a group action, consider first the neutral element $1 = [c_{y_0}] \in \pi_1(Y, y_0)$, where $c_{y_0} : I \longrightarrow Y$ is the constant loop. Also take $f : (X, x_0) \longrightarrow (Y, y_0)$. Define $F : X \times I \longrightarrow Y$ by $F(x, t) = f(x)$, so that $[f] \cdot 1 = [f] \cdot [c_{y_0}] = [F_1] = [f]$. Finally, let α, β be loops in Y based at y_0 and let $F, G : X \times I \longrightarrow Y$ be homotopies such that $F(x, 0) = f(x)$, $F(x_0, t) = \alpha(t)$, $G(x, 0) = F_1(x) = F(x, 1)$, and $G(x_0, t) = \beta(t)$. Then $[f] \cdot [\alpha] = [F_1]$ and $([f] \cdot [\alpha]) \cdot [\beta] = [G_1]$. Defining $H : X \times I \longrightarrow Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

we have $H(x, 0) = f(x)$ and $H(x_0, t) = (\alpha \cdot \beta)(t)$. Hence

$$[f] \cdot ([\alpha] \cdot [\beta]) = [H_1] = [G_1] = ([f] \cdot [\alpha]) \cdot [\beta] .$$

□

In what follows we analyze the relationship between pointed and unpointed homotopy classes. We shall show that the latter are obtained by dividing out in the former the action of the fundamental group.

4.4.2 **Theorem**. *Let X be a well-pointed space and let Y be any path-connected pointed space. Let $\Phi : [X, x_0; Y, y_0] \longrightarrow [X, Y]$ be the function that associates to each pointed homotopy class $[f]$ the unpointed homotopy class $\Phi([f])$. Then Φ induces a bijection*

$$\overline{\Phi} : [X, x_0; Y, y_0]/\pi_1(Y, y_0) \longrightarrow [X, Y],$$

where the set on the left-hand side is the orbit *set; that is, it is the quotient that identifies $[f]$ with $[f]\cdot[\alpha]$ for every $[f] \in [X, x_0; Y, y_0]$ and $[\alpha] \in \pi_1(Y, y_0)$.*

Proof: By Proposition 4.4.1, the action is given by $[f] \cdot [\alpha] = [F_1]$, where $F : X \times I \longrightarrow Y$ is a homotopy such that $F(x, 0) = f(x)$ and $F(x_0, t) = \alpha(t)$. Thus f is freely (not pointed) homotopic to F_1, and so $\Phi([f]) = \Phi([f] \cdot [\alpha])$, which says that $\overline{\Phi}$ is well defined.

Let now $f, g : (X, x_0) \longrightarrow (Y, y_0)$ be pointed maps such that $\Phi([f]) = \Phi([g])$. Then there is a free homotopy $F : f \simeq g$ that defines a loop α in Y by $\alpha(t) = F(x_0, t)$. But $[f] \cdot [\alpha] = [F_1]$ and $F_1 = g$ imply that $[f] \cdot [\alpha] = [g]$. Hence $\overline{\Phi}$ is injective.

Now let $g : X \longrightarrow Y$ be any unpointed map. Since Y is path connected, there is a path $\alpha : g(x_0) \simeq y_0$. Since $\{x_0\} \longrightarrow X$ is a cofibration, there is a homotopy $H : X \times I \longrightarrow Y$ such that $H(x, 0) = g(x)$ and $H(x_0, t) = \alpha(t)$. In particular, $H(x_0, 1) = y_0$. Therefore, the map $H_1 : x \mapsto H(x, 1)$ is such that $[H_1] \in [X, x_0; Y, y_0]$ and $\Phi([H_1]) = [g]$. Hence $\overline{\Phi}$ is surjective. □

4.4.3 NOTE. Let X be well pointed and Y be pointed. From the proof of the theorem above, we have that the quotient

$$[X, x_0; Y, y_0]/\pi_1(Y, y_0)$$

is in bijective correspondence with the set of free homotopy classes of pointed maps from (X, x_0) to (Y, y_0). When Y is path connected, this set coincides with $[X, Y]$, the set of free homotopy classes of unpointed maps from X to Y.

4.4.4 EXERCISE. Let $A \hookrightarrow X$ and $B \hookrightarrow Y$ be closed cofibrations. Show that $X \times B \cup A \times Y \hookrightarrow X \times Y$ is also a closed cofibration.

4.4.5 **Proposition**. *Let (W, w_0) be a well-pointed H-space with H-multiplication $\mu : W \times W \longrightarrow W$. Then μ is homotopic to another H-multiplication μ' such that $\mu'(w_0, w) = w = \mu'(w, w_0)$. Explicitly, if $e : W \longrightarrow W$ is the* *constant map whose value is the base point,* $e(W) = w_0$*, then it is a* strict identity*; that is, the composites*

$$W \xrightarrow{(e,\mathrm{id})} W \times W \xrightarrow{\mu} W, \quad W \xrightarrow{(\mathrm{id},e)} W \times W \xrightarrow{\mu} W$$

are the identity map of W*.*

Proof: Let $L, R : W \times I \longrightarrow W$ be homotopies such that $L : \mu \circ (e, \mathrm{id}) \simeq \mathrm{id}$ and $R : \mu \circ (\mathrm{id}, e) \simeq \mathrm{id}$; that is, $L(w, 0) = \mu(w_0, w)$, $L(w, 1) = w$, $R(w, 0) = \mu(w, w_0)$, $R(w, 1) = w$ for all $w \in W$. Define a homotopy $H : (W \vee W) \times I \longrightarrow W$ by $H(w, w_0, t) = R(w, t)$ and $H(w_0, w, t) = L(w, t)$, and consider the following commutative diagram:

$$\begin{tikzcd}
& W \times W \arrow[dr, "j_0"] \arrow[rr, bend left, "\mu"] & & \\
W \vee W \arrow[ur, hook] \arrow[dr, "j_0"'] & & (W \times W) \times I \arrow[r, dashed, "\widetilde{H}"] & W . \\
& (W \vee W) \times I \arrow[ur, hook] \arrow[urr, bend right, "H"'] & &
\end{tikzcd}$$

By Exercise 4.4.4, $W \vee W \hookrightarrow W \times W$ is a cofibration, and so there exists $\widetilde{H} : (W \times W) \times I \longrightarrow W$ such that $\widetilde{H}_0 = \mu$. Setting $\mu' = \widetilde{H}_1$ gives us the desired H-multiplication. □

4.4.6 **Proposition**. *Let* (X, x_0) *be a well-pointed space and let* (W, w_0) *be a well-pointed* H*-space. Then* $\pi_1(W, w_0)$ *acts trivially on* $[X, x_0; W, w_0]$*.*

Proof: Let $f : (X, x_0) \longrightarrow (W, w_0)$ be a pointed map and $\alpha : (I, \partial I) \longrightarrow (W, w_0)$ a loop based at w_0. By Proposition 4.4.5, the H-space W has a product μ' for which e is a strict identity. Define a homotopy $H : X \times I \longrightarrow W$ by $H(x, t) = \mu'(f(x), \alpha(t))$. Then we have a commutative diagram

$$\begin{tikzcd}
& X \arrow[dr, "j_0"] \arrow[rr, bend left, "f"] & & \\
\{x_0\} \arrow[ur, hook] \arrow[dr, "j_0"'] & & X \times I \arrow[r, "H"] & W . \\
& \{x_0\} \times I \arrow[ur, hook] \arrow[urr, bend right, "\alpha"'] & &
\end{tikzcd}$$

Therefore, $[f] \cdot [\alpha] = [H_1] = [f]$. □

As an immediate consequence of the previous proposition and of Theorem 4.4.2, we have the following result.

4.4.7 **Corollary**. *Let (X, x_0) be a well-pointed space and Y a path-connected H-space with identity element y_0. Then the function Φ that forgets base points determines an isomorphism*

$$\Phi : [X, x_0; Y, y_0] \cong [X, Y]. \qquad \square$$

Corresponding to Theorem 2.5.18 about the invariance of the fundamental group when base points are changed, we have the following.

4.4.8 **Theorem**. *Let X be a space and $\omega : x_0 \simeq x_1$ a path in X. There is an isomorphism*

$$\psi_\omega : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$$

with the following properties:

(i) *If $\omega \simeq \sigma$ rel ∂I, then $\psi_\omega = \psi_\sigma$.*

(ii) $\psi_{c_{x_0}} = 1_{\pi_n(X,x_0)}$.

(iii) *If $\sigma : x_1 \simeq x_2$, then $\psi_{\omega\sigma} = \psi_\sigma \circ \psi_\omega$.*

(iv) *If $f : X \longrightarrow Y$ is a map such that $f(x_0) = y_0$ and $f(x_1) = y_1$, then the following is a commutative diagram:*

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "\psi_\omega"] \arrow[d, "f_*"'] & \pi_n(X, x_1) \arrow[d, "f_*"] \\
\pi_n(Y, y_0) \arrow[r, "\psi_{f\circ\omega}"'] & \pi_n(Y, y_1).
\end{tikzcd}$$

Proof: Let the map $F : (I^n, \partial I^n) \longrightarrow (X, x_0)$ represent an element in the group $\pi_n(X, x_0)$. Define $\overline{\omega} : \partial I^n \times I \longrightarrow X$ by $\overline{\omega}(s, t) = \omega(t)$. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
 & I^n \arrow[dr, "j_0"] \arrow[drr, bend left, "F"] & & \\
\partial I^n \arrow[ur, hook] \arrow[dr, "j_0"'] & & I^n \times I \arrow[r, dashed, "\widetilde{F}"] & X. \\
 & \partial I^n \times I \arrow[ur, hook] \arrow[urr, bend right, "\overline{\omega}"'] & &
\end{tikzcd}$$

Since $\partial I^n \hookrightarrow I^n$ is a cofibration, there is a homotopy $\widetilde{F} : I^n \times I \longrightarrow X$ making the two triangles in the diagram commute.

We define $\psi_\omega : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$ by $\psi_\omega([F]) = [\widetilde{F}_1]$. Using Proposition 4.4.1 as we did in the proof of Theorem 4.4.2, one can show that this function is well defined and satisfies (i).

Properties (ii), (iii), and (iv) are an easy *exercise* to verify.

To show that the function ψ_ω is a homomorphism, consider maps $F, G : (I^n, \partial I^n) \longrightarrow (X, x_0)$ and let $\widetilde{F}, \widetilde{G}$ be homotopies such that $\widetilde{F}(s,0) = F(s)$, $\widetilde{G}(s,0) = G(s)$, for $s \in I^n$, and $\widetilde{F}(s,t) = \overline{\omega}(s,t) = \widetilde{G}(s,t)$ for all $s \in \partial I^n$ and all $t \in I$. Define $H : I^n \times I \longrightarrow X$ by

$$H(s_1, \ldots, s_n, t) = \begin{cases} F(s_1, \ldots, s_{n-1}, 2s_n, t) & \text{if } 0 \leq s_n \leq \frac{1}{2}, \\ G(s_1, \ldots, s_{n-1}, 2s_n - 1, t) & \text{if } \frac{1}{2} \leq s_n \leq 1. \end{cases}$$

It follows that $H_0 = F \cdot G$ and $H_1 = \widetilde{F}_1 \cdot \widetilde{G}_1$. Therefore,

$$\psi_\omega([F] \cdot [G]) = [H_1] = [\widetilde{F}_1 \cdot \widetilde{G}_1] = [\widetilde{F}_1] \cdot [\widetilde{G}_1] = \psi_\omega([F]) \cdot \psi_\omega([G]) \,.$$

By properties (i), (ii), and (iii), ψ_ω is a bijection; hence, it is an isomorphism. $\square$

4.4.9 EXERCISE. Let X be a space and $\omega : x_0 \simeq x_1$ a path in X. Prove that if $n = 1$, then $\psi_\omega = \varphi_{\omega^{-1}} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$, where $\varphi_{\omega^{-1}}$ is the isomorphism corresponding to ω^{-1} according to Theorem 2.5.18. Here $\omega^{-1}(t) = \omega(1-t)$.

Generalizing Remark 3.5.9, we have the following.

4.4.10 **Theorem**. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then for every $x_0 \in X$ and $n \geq 1$,*

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an isomorphism.

Proof: Let $g : Y \longrightarrow X$ be a homotopy inverse to f and let $H : X \times I \longrightarrow X$ be a homotopy from id_X to $g \circ f$. Consider the homomorphism $(g \circ f)_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, gf(x_0))$. Recall that if $[F] \in \pi_n(X, x_0)$, then $(g \circ f)_*([F]) = [g \circ f \circ F]$. Define the homotopy $\widetilde{H} : I^n \times I \longrightarrow X$ to be the composite $\widetilde{H} = H \circ (f \times \mathrm{id}_I)$ and let $\omega : I \longrightarrow X$ be the path between x_0 and $gf(x_0)$ given by $\omega(t) = H(x_0, t)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
& I^n \arrow[dr, "j_0"] \arrow[drr, bend left, "F"] & & \\
\partial I^n \arrow[ur, hook] \arrow[dr, "j_0"'] & & I^n \times I \arrow[r, "\widetilde{H}"] & X \,. \\
& \partial I^n \times I \arrow[ur, hook] \arrow[urr, bend right, "\overline{\omega}"'] & &
\end{tikzcd}$$

By Theorem 4.4.8, $\psi_\omega([F]) = [\widetilde{H}_1] = [g \circ f \circ F] = (g \circ f)_*([F])$. Since ψ_ω is an isomorphism, $(g \circ f)_* = g_* \circ f_*$ is also an isomorphism. Similarly, $f_* \circ g_*$ is an isomorphism too; hence, f_* and g_* are isomorphisms. $\square$

4.4.11 Exercise. Reformulate and prove 4.4.8 and 4.4.10 for pairs of pointed spaces.

As an immediate corollary of 4.4.10, we have the following (cf. 3.5.8(f)).

4.4.12 **Corollary**. *If X is a contractible space, then $\pi_n(X, x_0) = 0$ for every $x_0 \in X$ and $n \geq 0$.* □

4.4.13 Remark. It is not true that every contractible space is strongly contractible, that is, can be contracted to a point keeping the point fixed. Consider, for instance, the subset of $\mathbb{R}^2$ consisting of all points of the straight line segments joining the point $(0,1)$ to the point $(\frac{1}{n}, 0)$ for each positive integer n, as well as to the point $(0,0)$. This space can be contracted, but not strongly contracted, to the point $(0,0)$. However, we have the following result.

4.4.14 **Proposition**. *If (X, x_0) is a well-pointed contractible space, then X is strongly contractible to x_0.*

Proof: By Theorem 4.4.2, we have a bijection

$$[X, x_0; X, x_0]/\pi_1(X, x_0) \longrightarrow [X, X]\,.$$

By Corollary 4.4.12, $\pi_1(X, x_0) = 0$; hence, there is a bijection between $[X, x_0; X, x_0]$ and $[X, X]$. Since X is contractible, $[X, X] = 0$, so that $[X, x_0; X, x_0] = 0$ and therefore $\mathrm{id}_X \simeq c_{x_0}$ rel x_0. Here 0 denotes the one-point set. □

4.5 Locally Trivial Bundles

In this section we shall review the concept of a locally trivial bundle, which is a special case of the more general concept of a "fiber bundle." This latter concept can be studied in detail in various books. In particular, we refer the reader to the classic book of Steenrod [69] as well as to Husemöller [37].

In the same way as Serre fibrations are not in general (Hurewicz) fibrations, locally trivial bundles also are not in general (Hurewicz) fibrations, although they indeed are Serre fibrations. Some authors call them "locally trivial fiber spaces."

4.5.1 DEFINITION. A map $p : E \longrightarrow B$ is a *locally trivial bundle* with *fiber* F if every point $b \in B$ has a neighborhood $U \subset B$ such that there exists a homeomorphism $\varphi_U : U \times F \longrightarrow p^{-1}U$ making the triangle

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & p^{-1}U \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

commute, where $p_U = p|p^{-1}U : p^{-1}U \longrightarrow U$ and where π is the projection onto U. From this commutative diagram we get that φ_U can be restricted to a homeomorphism of $\pi^{-1}(b) = \{b\} \times F \approx F$ onto $p^{-1}(b)$ for all $b \in U$. Because of this we say that the fiber is F (cf. 4.3.17(b)). The open cover of such sets U is called a *trivializing cover* of the bundle, and the maps φ_U trivializing maps.

4.5.2 EXAMPLE. If we can take $U = B$, that is, if $E \approx B \times F$, then we have a *trivial bundle*. In particular, if $E = B \times F$, then $p = \mathrm{proj}_B$ is a trivial bundle.

4.5.3 EXAMPLE. A locally trivial bundle $p : E \longrightarrow B$ whose fiber F is a discrete space is called a *covering map*. In particular, a covering map always is a local homeomorphism. Figure 4.2 shows what a covering map looks like locally.

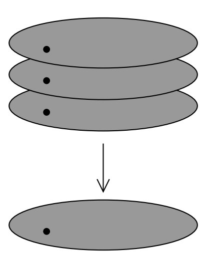


Figure 4.2

4.5.4 **Lemma**. *Every trivial bundle is a Hurewicz fibration.*

Proof: It is enough to assume that the given trivial bundle is of the form

$p = \mathrm{proj}_B : B \times F \longrightarrow B$. Let us consider the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B. \end{tikzcd}$$

If $f : X \longrightarrow E = B \times F$ is given as $f(x) = (f'(x), f''(x))$, we then define $\widetilde{H} : X \times I \longrightarrow E = B \times F$ by $\widetilde{H}(x,t) = (H(x,t), f''(x))$. $\square$

4.5.5 Example. Lemma 4.5.4 is not true if the fibration is assumed to be only homotopically trivial, that is, $E \simeq B \times F$, as we can show by considering the map $p : E \longrightarrow I$, where

$$E = \{0\} \times I \cup I \times \{0\} \quad \text{and} \quad F = \{*\}$$

and p is the projection onto the first factor (see Figure 4.3), since the path $\alpha = \mathrm{id}_I : I \longrightarrow I$ does not have a lifting to $\widetilde{\alpha} : I \longrightarrow E$ such that $\widetilde{\alpha}(0) = (0,1)$. (Cf. 4.3.13.)

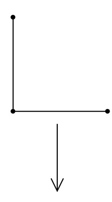


Figure 4.3

4.5.6 **Theorem**. *Every locally trivial bundle is a Serre fibration.*

Proof: Let $p : E \longrightarrow B$ be a locally trivial bundle. We have to prove that for every commutative square

$$\begin{tikzcd} I^q \arrow[r, "f"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ I^q \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

there exists $\widetilde{H} : I^q \times I \longrightarrow E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H} \circ j_0 = f$. For each point $b \in H(I^q \times I)$ there exists a neighborhood $U(b)$ of b such that

$p_{U(b)}$ is trivial, and so there exists a homeomorphism $\varphi_{U(b)} : F \times U(b) \longrightarrow E_{U(b)} = p^{-1}U(b)$. Since $H(I^q \times I)$ is compact, we can cover it with a finite number of such neighborhoods $U(b)$, say $U_1, \dots, U_k$. Since $I^q \times I$ is a compact metric space, there exists a number $\varepsilon > 0$, called the *Lebesgue number* of the cover $\{H^{-1}(U_i)\}$, such that every subset of diameter less than ε is contained in some $H^{-1}(U_i)$. Therefore, we can subdivide I^q into subcubes and take numbers $0 = t_0 < t_1 < \cdots < t_m = 1$ in such a way that if c is an n-face, then the image of $c \times [t_j, t_{j+1}]$ under H lies in some U_i. (Note that the 0-faces are vertices, the 1-faces are edges, etc.) Suppose that we have constructed $\widetilde{H}$ on $I^q \times [t_0, t_j]$. Then we shall construct $\widetilde{H}$ on $I^q \times [t_j, t_{j+1}]$ by defining it on each n-subface, using induction on n.

If c is a 0-face, then we pick some U_i such that $H(c \times [t_j, t_{j+1}]) \subset U_i$. Since $p\widetilde{H}(c, t_j) = H(c, t_j)$, we then have $\widetilde{H}(c, t_j) \subset E_{U_i}$. We define $\widetilde{H}(c, t) = \varphi_{U_i}(H(c, t), \mathrm{proj}_F \varphi_{U_i}^{-1}(\widetilde{H}(c, t_j)))$ for $t \in [t_j, t_{j+1}]$. This is well defined and continuous.

Assume that we have already constructed $\widetilde{H}$ on $\widetilde{c} \times [t_j, t_{j+1}]$ for every face $\widetilde{c}$ of dimension less than n and let c be an n-face. Let us then pick some U_i such that $H(c \times [t_j, t_{j+1}]) \subset U_i$. By hypothesis $\widetilde{H}$ is defined on $c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}]$. Clearly, there exists a homeomorphism of $c \times [t_j, t_{j+1}]$ to itself that sends $c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}]$ onto $c \times \{t_j\}$, and so using 4.5.4 we can complete the diagram

$$\begin{tikzcd}
c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}] \arrow[r, "\varphi_i^{-1} \circ \widetilde{H}|"] \arrow[d, hook] & U_i \times F \arrow[d, "\mathrm{proj}"] \\
c \times [t_j, t_{j+1}] \arrow[r, "H|"'] \arrow[ur, dashed, "\widetilde{K}"] & U_i.
\end{tikzcd}$$

Composing this lifting $\widetilde{K}$ with φ_i, we define $\widetilde{H}$ on $c \times [t_j, t_{j+1}]$. In this way we complete the induction step and obtain $\widetilde{H}|I^q \times [0, t_{j+1}]$. Finally, by induction on j, we define $\widetilde{H}$ on $I^q \times I$. □

4.5.7 EXERCISE. Using the same method of proof as in 4.5.6, prove the following statement:

4.5.8 **Proposition**. *Suppose that $p : E \longrightarrow B$ is continuous and that there exists an open cover $\{U\}$ of B such that for each open set U in the cover the restriction p_U is a Serre fibration. Then p is a Serre fibration.* □

4.5.9 EXERCISE. Assume that $p : E \longrightarrow B$ is a covering map. Prove that p has the *unique path-lifting property*; that is, p is such that for any given path $\alpha : I \longrightarrow B$ and any given point $y \in p^{-1}(\alpha(0))$ there exists a unique path $\widetilde{\alpha} : I \longrightarrow E$ satisfying $\widetilde{\alpha}(0) = y$ and $p \circ \widetilde{\alpha} = \alpha$. (Hint: Since p is a Serre fibration, the lifting always exists. To prove that it is unique, show that any two liftings with the same initial point y have to be homotopic fiber by fiber, using again the fact that p is a Serre fibration, and notice that this is possible only if both coincide, since the fiber is discrete.)

The following is a very important example.

4.5.10 EXAMPLE. Let $\mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C}$ be defined as

$$\mathbb{S}^3 = \{(z, z') \in \mathbb{C} \times \mathbb{C} \mid z\overline{z} + z'\overline{z}' = 1\}.$$

Also let us identify the *Riemann sphere*, defined by $\mathbb{C} \cup \{\infty\}$, with $\mathbb{S}^2$ by means of the stereographic projection $e : \mathbb{S}^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ defined by $e(\zeta) = (1/1 - z)(x + \mathrm{i}y)$ for $\zeta = (x, y, z)$ and $z < 1$ and by $e(0, 0, 1) = \infty$. This is shown in Figure 4.4.

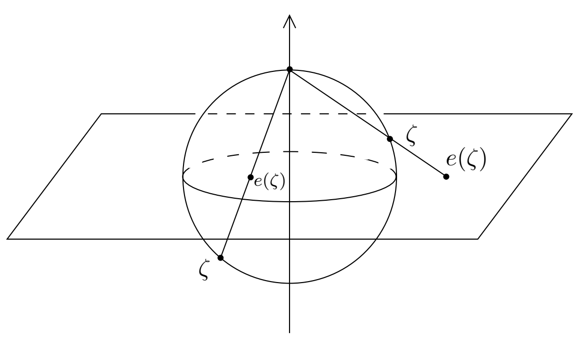


Figure 4.4

We have a map

$$p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

defined by $p(z, z') = z/z'$ if $z' \neq 0$ and by $p(z, z') = \infty$ if $z' = 0$. Then p is a locally trivial bundle with fiber $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid \zeta\overline{\zeta} = 1\}$, as we shall soon see.

Put $U = \mathbb{S}^2 - \{\infty\}(= \mathbb{C})$ and $V = \mathbb{S}^2 - \{0\}$. We define a homeomorphism

$$\varphi_U : U \times \mathbb{S}^1 \longrightarrow p^{-1}U$$

by $\varphi_U(z,\zeta) = \left(\zeta z/\sqrt{z\overline{z}+1}, \zeta/\sqrt{z\overline{z}+1}\right)$. It then has an inverse

$$\psi_U : p^{-1}U \longrightarrow U \times \mathbb{S}^1$$

given by $\psi_U(z,z') = (z/z', z'/|z'|)$.

We define another homeomorphism

$$\varphi_V : V \times \mathbb{S}^1 \longrightarrow p^{-1}V$$

by

$$\varphi_V(z,\zeta) = \left(|z|\zeta\sqrt{z\overline{z}+1}, |z|\zeta z\sqrt{z\overline{z}+1}\right)$$

if $z \in \mathbb{C} - \{0\}$ and by $\varphi_V(\infty,\zeta) = (\zeta, 0)$. Then its inverse

$$\psi_V : p^{-1}V \longrightarrow V \times \mathbb{S}^1$$

is given by $\psi_V(z,z') = (z/z', z/|z|)$ if $z' \neq 0$ and by $\psi_V(z,0) = (\infty, z)$.

So we have that $p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is locally trivial. This locally trivial bundle is called the *Hopf fibration.*

4.5.11 **Proposition**. *If $p : E \longrightarrow B$ is a locally trivial bundle and $f : B' \longrightarrow B$ is continuous, then the map $p' : E' \longrightarrow B'$ induced from p by f is a locally trivial bundle having the same fiber F as p has.*

Proof: Suppose that $b' \in B'$ and that U is a neighborhood of $f(b')$ in B such that there exists a homeomorphism φ_U that makes the triangle

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi U"] \arrow[dr] & & p^{-1}U \arrow[dl] \\ & U & \end{tikzcd}$$

commute. Put $U' = f^{-1}U$. Then U' is a neighborhood of b', and the map $\varphi_{U'} : U' \times F \longrightarrow (p')^{-1}U'$ given by $\varphi_{U'}(x',y) = (x', \varphi_U(f(x'),y))$ is a homeomorphism that makes the triangle

$$\begin{tikzcd} U' \times F \arrow[rr, "\varphi_{U'}"] \arrow[dr] & & (p')^{-1}U' \arrow[dl] \\ & U' & \end{tikzcd}$$

commute. $\square$

4.5.12 Example. Assume that $\mathbb{R}$ is the space of real numbers and consider the *exponential map*

$$p : \mathbb{R} \longrightarrow \mathbb{S}^1$$

defined by $p(t) = \mathrm{e}^{2\pi i t} \in \mathbb{S}^1 \subset \mathbb{C}$. Clearly, we have that $p(t) = p(t')$ if and only if $t' - t \in \mathbb{Z}$. So we have that $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ as abelian groups and as topological spaces. Let us show that it is a locally trivial bundle with fiber $\mathbb{Z}$ (see Figure 4.5). Put $U = \mathbb{S}^1 - \{1\}$, so that we have $p^{-1}U = \mathbb{R} - \mathbb{Z}$. Then there is a homeomorphism ψ_U that makes the triangle

$$\begin{tikzcd} p^{-1}U \arrow[rr, "\psi_U"] \arrow[dr] & & U \times \mathbb{Z} \arrow[dl] \\ & U & \end{tikzcd}$$

commute. It is given by $\psi_U(t) = (\mathrm{e}^{2\pi i t}, [t])$, where $[t] \in \mathbb{Z}$ satisfies $t = [t] + t'$ with $0 < t' < 1$. And its inverse $\varphi_U : U \times \mathbb{Z} \longrightarrow p^{-1}U$ is given by $\varphi_U(\zeta, n) = n + t$, where $\zeta = \mathrm{e}^{2\pi i t} \in U$ with $0 < t < 1$.

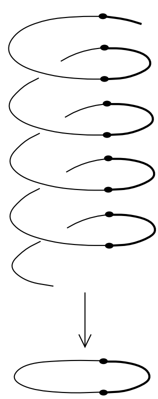


Figure 4.5

Analogously, if we put $V = \mathbb{S}^1 - \{-1\}$, so that

$$p^{-1}V = \mathbb{R} - \left(\mathbb{Z} + \frac{1}{2}\right) = \left\{ t \in \mathbb{R} \mid t \neq n + \frac{1}{2};\ n \in \mathbb{Z} \right\},$$

then we define $\psi_V : p^{-1}V \longrightarrow V \times \mathbb{Z}$ by $\psi_V(t) = \left(\mathrm{e}^{2\pi i t}, \left[t + \frac{1}{2}\right]\right)$. Then its inverse $\varphi_V : V \times \mathbb{Z} \longrightarrow p^{-1}V$ is given by $\varphi_V(\zeta, n) = n + t$ for $\zeta = \mathrm{e}^{2\pi i t} \in V$ with $-\frac{1}{2} < t < \frac{1}{2}$.

Also in this example, by using 4.3.34 and 4.5.6, we get an exact sequence

$$(4.5.12)\qquad \begin{array}{c} \cdots \longrightarrow \pi_q(\mathbb{Z}) \longrightarrow \pi_q(\mathbb{R}) \longrightarrow \pi_q(\mathbb{S}^1) \longrightarrow \pi_{q-1}(\mathbb{Z}) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(\mathbb{R}) \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_0(\mathbb{Z}) \longrightarrow \pi_0(\mathbb{R}). \end{array}$$

Since we have

$$\pi_q(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

and

$$\pi_q(\mathbb{R}) = 0 \quad \text{if} \ \ q \geq 0\,,$$

we obtain the next result.

4.5.13 **Theorem**. *The homotopy groups of* $\mathbb{S}^1$ *are given by*

$$\pi_q(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

□

That is to say, we have proved that $\mathbb{S}^1$ *is an Eilenberg–Mac Lane space of type* $K(\mathbb{Z},1)$ (see Chapter 6).

4.5.14 Exercise. Let $p : E \longrightarrow B$ be a covering map where B is path connected and *locally path connected.* To say that B is a locally path-connected space means that for each point $b \in B$ and each neighborhood U of b in B there is a neighborhood $V \subset U$ of b that is path connected. Let X be path connected. Prove that for every map $f : X \longrightarrow B$ and for all points $x_0 \in X$ and $y_0 \in p^{-1}(f(x))$, there exists a unique lifting $\widetilde{f} : X \longrightarrow E$ such that $\widetilde{f}(x_0) = y_0$ if and only if $f_*\pi_1(X, x_0) \subset p_*\pi_1(E, y_0)$. (Hint: For each point $x \in X$ let $\alpha : I \longrightarrow X$ be a path such that $\alpha(0) = x_0$ and $\alpha(1) = x$. Using 4.5.9, there exists a unique path $\widetilde{\alpha} : I \longrightarrow E$ such that $\widetilde{\alpha}(0) = y_0$ and $p \circ \widetilde{\alpha} = \alpha$. We then define $\widetilde{f} : X \longrightarrow E$ by $\widetilde{f}(x) = \widetilde{\alpha}(1)$. Using the hypotheses, prove that $\widetilde{f}$ is well defined and continuous.)

4.5.15 Exercise. Let $p : E \longrightarrow B$ be a covering map such that E is path connected. (This last condition is included by many authors in the definition of covering map.)

(a) Prove that we have a *transitive action* of the fundamental group of the base $\pi_1(B, b_0)$ on the fiber $F = p^{-1}b_0$ such that if $[\alpha] \in \pi_1(B, b_0)$ and $y \in F$, then $y \cdot [\alpha] = \widetilde{\alpha}(1)$, where $\widetilde{\alpha} : I \longrightarrow E$ is the lifting of α satisfying $\widetilde{\alpha}(0) = y$ (see 4.5.9). In other words, prove that $y \cdot 1 = y$

and that $y \cdot ([\alpha][\beta]) = (y \cdot [\alpha]) \cdot [\beta]$, where $1, [\alpha], [\beta] \in \pi_1(B, b_0)$ (that is, $\pi_1(B, b_0)$ *acts on* F). Moreover, prove that for every $y_1,\ y_2 \in F$ there exists $[\alpha] \in \pi_1(B, b_0)$ such that $y_1 \cdot [\alpha] = y_2$ (that is, the action is *transitive*). (Hint: The action is defined by using the unique path-lifting property 4.5.9. In order to prove that it is transitive, for any given y_1 and y_2 take a path $\widetilde{\alpha}$ from y_1 to y_2 and define $\alpha = p \circ \widetilde{\alpha}$.)

(b) Prove that the homomorphism $p_* : \pi_1(E, y_0) \longrightarrow \pi_1(B, b_0)$ is a monomorphism. (Hint: If $\widetilde{\alpha} : I \longrightarrow E$ is a closed path in E such that $\widetilde{\alpha}(0) = \widetilde{\alpha}(1) = y_0$ and such that $\alpha = p \circ \widetilde{\alpha} \simeq 0$ in B, then there is a lifting of every nullhomotopy of α, which in turn defines a nullhomotopy of $\widetilde{\alpha}$.)

(c) Assume that $y_0 \in F$. Prove that the function $[\alpha] \mapsto y_0 \cdot [\alpha]$ defines an isomorphism (as sets) between F and the set of (right) cosets of $p_*\pi_1(E, y_0)$ in $\pi_1(B, b_0)$. (Hint: One has $y_0 \cdot [\alpha] = y_0 \cdot [\beta]$ if and only if $p_*\pi_1(E, y_0)[\alpha] = p_*\pi_1(E, y_0)[\beta]$.)

(d) Suppose that E is simply connected, that is, $\pi_1(E) = 1$. Conclude that $\pi_1(B, b_0) \cong F$ as sets. A covering map $p : E \longrightarrow B$ such that $\pi_1(E) = 1$ is called a *universal covering map.*

4.5.16 EXERCISE. Let $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map, namely, $p(t) = \exp(2\pi i t)$. Prove that p is a universal covering map, so that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ at least as sets. (See Figure 4.6, and compare this with 4.5.12.)

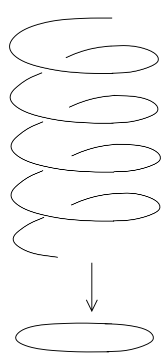


Figure 4.6

4.5.17 EXERCISE. Let $p : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$ for $n > 1$ be the canonical projection. Prove that p is a universal covering map whose fiber F consists of two points. Conclude that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$.

4.5.18 Note. The results stated in Exercises 4.5.15(b) and (c) can be obtained from the long exact homotopy sequence of a Serre fibration (see 4.3.34).

4.5.19 Exercise. Let B be a path-connected space that is also locally path connected and *semilocally 1-connected*. This means that B has the property that for every point $b \in B$ there exists a neighborhood $V \subset B$ of b such that the inclusion $i : V \hookrightarrow B$ satifies $i_*\pi_1(V, b) = 1$. Prove that there exists a universal covering map $p : E \longrightarrow B$, and in particular that E is path connected and simply connected ($\pi_1(E) = 1$). (Hint: Suppose that $b_0 \in B$. Take a cover V_j with $j \in J$ of B consisting of sets that are open, nonempty, and path connected just like the open set V above. Then for each j take a path α_j in B such that $\alpha(0) = b_0$ and $\alpha(1) \in V_j$, and moreover, such that α_j is the constant path whose value is b_0 if $b_0 \in V_j$. Next, for each $b \in V_i \cap V_j$ put $g_{ij}(b) = [\alpha_i\beta_i\beta_j^{-1}\alpha_j^{-1}] \in \pi_1(B, b_0)$, where β_k is a path in V_k from $\alpha_k(1)$ to b for $k = i, j$ (see Figure 4.7). Form the disjoint union

$$\coprod_j V_j \times \{\gamma\} \times \{j\} \subset B \times \pi_1(B, b_0) \times J\,,$$

where $\pi_1(B, b_0)$ and J are discrete, and identify (b, γ, j) and (b', γ', i) if $b = b'$ and $\gamma' = g_{ij}(b)\gamma$, thereby obtaining a topological space E and a map $p : E \longrightarrow B$. This is the desired covering map. Compare this with the construction of a vector bundle using cocycles in 8.1.1.)

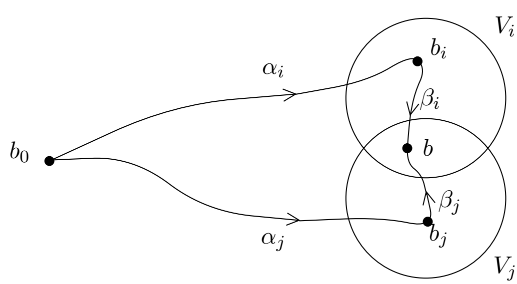


Figure 4.7

4.5.20 Exercise. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be locally trivial bundles with compact Hausdorff fibers over the same base space B. Prove that $\varphi : E \longrightarrow E'$ is a *bundle isomorphism* (that is, for each $x \in B$, the restriction to the fiber $\varphi_x : p^{-1}(x) \longrightarrow p'^{-1}(x)$ is a homeomorphism and φ

covers the identity map of B) if and only if φ itself is a homeomorphism. (Hint: Prove that the first condition implies that φ is a continuous, bijective, and open map using the fact that the group of homeomorphisms of the fiber $\mathrm{Homeo}(F, F)$ with the compact-open topology is then a topological group.)

4.5.21 Exercise. Assume that $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are locally trivial bundles with compact Hausdorff fibers. Prove that if $\widetilde{f} : E \longrightarrow E'$ is a *bundle morphism*, that is, there exists a continuous $f : B \longrightarrow B'$ such that $f \circ p = p' \circ \widetilde{f}$, and for each $x \in B$, the restriction to the fiber $\widetilde{f}_x : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$, then $E \approx f^*E'$. (Hint: Apply the previous exercise to E and f^*E'.)

4.5.22 Remark. The assertions of the two previous exercises are equally true if the fiber is discrete instead of compact. They also hold for *vector bundles*, that is, for locally trivial bundles $p : E \longrightarrow B$ such that their fiber F is a finite-dimensional vector space and given two trivializations $\varphi_U : U \times F \longrightarrow p^{-1}U$, $\varphi_V : V \times F \longrightarrow p^{-1}V$ and a point $x \in U \cap V$, the homeomorphism restricted to the fiber $\varphi_V^{-1} \circ \varphi_U : F \longrightarrow F$ is in fact a linear isomorphism (see 8.1.1 and compare with 8.1.13).

For more general locally trivial bundles $p : E \longrightarrow B$, the problem is that the group of homeomorphisms of the fiber $\mathrm{Homeo}(F, F)$ is not necessarily a topological group; that is, the function sending a homeomorphism to its inverse need not be continuous. Therefore, one might instead assume that for each trivializing U and V, the map $U \cap V \longrightarrow \mathrm{Homeo}(F, F)$ given by $x \mapsto ((\varphi_V|p^{-1}(x))^{-1}) \circ (\varphi_U|p^{-1}(x))$ lands, in fact, in some subgroup $G \subset \mathrm{Homeo}(F, F)$ that with the relative topology is a topological group (this group G is the so-called *structure group* of p; see [69]). Then the assertions of the exercises also hold.

Given a right action of a (discrete) group G on a space X, we say that the action is *free* if given $g \in G$, then $g \neq 1$ implies $xg \neq x$ for all $x \in X$. We say that the action is *properly discontinuous* if every point $x \in X$ has a neighborhood V such that $V \cap Vg = \emptyset$ for every nontrivial permutation $g \in G$, where $Vg = \{xg \mid x \in V\}$. In particular, this implies that the action is free.

4.5.23 Definition. Let $p : E \longrightarrow B$ be a covering map. A *covering transformation* is a homemomorphism $F : E \longrightarrow E$ such that $p \circ F = F$. Clearly, the set of all covering transformations is a group under composition.

4.5.24 DEFINITION. A covering map $p : E \longrightarrow B$ is said to be *regular* if given any loop ω in B, then either every lifting of ω is a loop or none is a loop.

The following exercises will be needed to prove the important Theorem 4.5.29, below.

4.5.25 EXERCISE. Let $p : E \longrightarrow B$ be a covering map. Prove that p is regular if and only if $p_*\pi_1(E, e_0) = p_*\pi_1(E, e_1)$ whenever $p(e_0) = p(e_1)$.

4.5.26 EXERCISE. Let $p : E \longrightarrow B$ be a covering map and assume that E is path connected. Take $e_0, e_1 \in E$. Prove that there is a path $\omega : p(e_0) \simeq p(e_1)$ such that $p_*\pi_1(E, e_0) = \varphi_\omega p_*\pi_1(E, e_1)$. Conversely, given a path $\omega : p(e_0) \simeq x_1$ in B, prove that there is a point $e_1 \in p^{-1}(x_1)$ such that $\varphi_\omega p_*\pi_1(E, e_1) = p_*\pi_1(E, e_0)$. Here φ_ω is as defined in 2.5.18.

4.5.27 EXERCISE. Let $p : E \longrightarrow B$ be a covering map and assume that E is path connected. Take $x_0 \in B$. Prove that the family $\{p_*\pi_1(E, e) \mid e \in p^{-1}(x_0)\}$ is a conjugacy class in $\pi_1(B, x_0)$. (Hint: Use the exercise above, cf. 4.5.15(c).)

4.5.28 EXERCISE. Prove that if there is a properly discontinuous (right) action of a group G on a space E, then the quotient map $q : E \longrightarrow E/G$ mapping each element to its orbit is a covering map.

4.5.29 **Theorem**. *Let E be a path-connected space. If $q : E \longrightarrow E/G$ is the quotient map and $e_0 \in E$, then*

$$q_*\pi_1(E, e_0) \subset \pi_1(E/G, q(e_0))$$

is a normal subgroup, and there is a group isomorphism

$$\pi_1(E/G, q(e_0))/q_*\pi_1(E, e_0) \cong G\,.$$

Furthermore, the group of covering transformations of q is isomorphic to G.

Proof: By Exercise 4.5.28, $q : E \longrightarrow E/G$ is a covering map. Set $x_0 = q(e_0)$. Then there is an action $q^{-1}(x_0) \times \pi_1(E/G, x_0) \longrightarrow q^{-1}(x_0)$ given by $e \cdot [\omega] = \widetilde{\omega}(1)$, where $\widetilde{\omega}$ is the lifting of ω such that $\widetilde{\omega}(0) = e$. Since E is path connected, this action is transitive. The *isotropy subgroup* of e_0, that is, the subgroup of $\pi_1(E/G, x_0)$ leaving e_0 fixed, is clearly equal to $q_*\pi_1(E, e_0)$.

Let $\alpha_{e_0} : \pi_1(E/G, x_0) \longrightarrow q^{-1}(x_0)$ be given by $\alpha_{e_0}([\omega]) = e_0 \cdot [\omega]$. Then α_{e_0} induces a bijection $\overline{\alpha}_{e_0} : \pi_1(E/G, q(e_0))/q_*\pi_1(E, e_0) \longrightarrow q^{-1}(x_0)$ such that $\overline{\alpha}_{e_0}(\overline{[\omega]}) = \alpha_{e_0}([\omega])$. Since the action of G is free and the orbits of the action are precisely the fibers of q, we have another bijection $\beta_{e_0} : G \longrightarrow q^{-1}(x_0)$ given by $\beta_{e_0}(g) = e_0 \cdot g$. Therefore, we get a bijection

$$\varphi = \beta_{e_0}^{-1} \circ \overline{\alpha}_{e_0} : \pi_1(E/G, q(e_0))/q_*\pi_1(E, e_0) \longrightarrow G\,.$$

Now take another point $e_1 \in q^{-1}(x_0)$. Since $q^{-1}(x_0)$ is an orbit of the action of G, one has that $e_1 = e_0 \cdot g$ for some $g \in G$. But g induces a homeomorphism $R_g : E \longrightarrow E$ such that $R_g(e) = e \cdot g$, which is obviously a covering transformation. Using the functor π_1 we get the following commutative diagram:

$$\begin{tikzcd}
\pi_1(E, e_0) \arrow[rr, "R_{g*}"] \arrow[dr, "q_*"'] & & \pi_1(E, e_0) \arrow[dl, "q_*"] \\
& \pi_1(E/G, x_0)\,. &
\end{tikzcd}$$

Hence $q_*\pi_1(E, e_0) = q_*\pi_1(E, e_1)$, and by Exercise 4.5.25, q is regular.

Since E is path connected, by Exercise 4.5.27, the family of subgroups $\{q_*\pi_1(E, e) \mid e \in q^{-1}(x_0)\}$ is a conjugacy class in $\pi_1(E/G, x_0)$. But all of these subgroups coincide, so that $q_*\pi_1(E, e_0)$ is normal in $\pi_1(E/G, x_0)$, and then $\pi_1(E/G, x_0)/q_*\pi_1(E, e_0)$ is a group. By the definition of φ, we have that $\varphi(\overline{[\omega_1][\omega_2]}) = \beta_{e_0}^{-1}(e_0 \cdot ([\omega_1][\omega_2]))$. Let $\widetilde{\omega}_\nu$ be the unique lifting of ω_ν such that $\omega_\nu(0) = e_0$ and let $g_\nu \in G$ be the unique element such that $e_0 \cdot g_\nu = \widetilde{\omega}_\nu(1)$ $(\nu = 1, 2)$. To evaluate $e_0 \cdot ([\omega_1][\omega_2])$, let $\widehat{\omega}_2$ be the unique lifting of ω_2 such that $\widehat{\omega}_2(0) = \widetilde{\omega}_1(1)$. Then the product of paths $\widetilde{\omega}_1\widehat{\omega}_2$ is a lifting of $\omega_1\omega_2$ starting at e_0; hence $e_0 \cdot ([\omega_1][\omega_2]) = \widehat{\omega}_2(1)$.

Consider the homeomorphism $R_{g_1} : E \longrightarrow E$ and the path $R_{g_1} \circ \widehat{\omega}_2$. Since R_{g_1} is a covering transformation, $R_{g_1} \circ \widehat{\omega}_2$ is a lifting of ω_2 starting at $\widetilde{\omega}_1(1)$. Hence, $R_{g_1} \circ \widetilde{\omega}_2 = \widehat{\omega}_2$, and then $e_0 \cdot ([\omega_1][\omega_2]) = \widehat{\omega}_2(1) = R_{g_1} \circ \widetilde{\omega}_2(1) = \widetilde{\omega}_2(1) \cdot g_1 = (e_0 \cdot g_2) \cdot g_1 = e_0 \cdot (g_2 g_1)$. Therefore,

$$\varphi\left(\overline{[\omega_1][\omega_2]}\right) = g_2 g_1 = \varphi\left(\overline{[\omega_1]}\right)\varphi\left(\overline{[\omega_2]}\right)\,.$$

We define

$$\psi : \pi_1\left(E/G, q(e_0)\right)/q_*\pi_1\left(E, e_0\right) \longrightarrow G$$

by $\psi(\overline{[\omega]}) = \varphi(\overline{[\omega]}^{-1})$. Then ψ is an isomorphism.

Finally, let $\mathcal{G}$ be the group of covering transformations of q. There is a homomorphism $\gamma : G \longrightarrow \mathcal{G}$ given by $\gamma(g) = R_{g^{-1}}$. Since the action is free, it is also *effective*, so that γ is injective. Now let $F : E \longrightarrow E$ be any covering transformation and take $e_0 \in E$. Since e_0 and $F(e_0)$ are on the same fiber, there exists $g \in G$ such that $F(e_0) = e_0 \cdot g^{-1}$. Consider $R_{g^{-1}} \in \mathcal{G}$. Then $R_{g^{-1}}(e_0) = F(e_0)$. Since both $R_{g^{-1}}$ and F are liftings of q and E is path connected, thus connected, then by the uniqueness of the liftings, $R_{g^{-1}} = F$, hence γ is an isomorphism. □

4.5.30 Exercise. Let E be a Hausdorff space. Prove that if there is a free action of a finite group G on E, then the action is properly discontinuous. Conclude that the quotient map $q : E \longrightarrow E/G$ is a covering map.

4.6 Classification of Covering Maps over Paracompact Spaces

The purpose of this section is to classify covering maps, using similar methods and results to those that will be used in Section 8.5 to classify vector bundles over paracompact spaces. There the classifying spaces will be the Grassmann manifolds. Here they will be configuration spaces, which are certain subspaces of the symmetric products, which will be systematically analyzed in the next chapter.

Before starting with the classification, we need some general results on locally trivial bundles. These will also be of interest in Chapter 8.

4.6.1 **Lemma**. *Suppose that $p : E \longrightarrow B \times I$ is a locally trivial bundle whose restrictions to $B \times [0, a]$ and to $B \times [a, 1]$ are trivial for some $a \in I$. Then $p : E \longrightarrow B \times I$ itself is a trivial bundle.*

Proof: By assumption we have homeomorphisms $\varphi_1 : (B \times [0, a]) \times F \longrightarrow p^{-1}(B \times [0, a])$ and $\varphi_2 : (B \times [a, 1]) \times F \longrightarrow p^{-1}(B \times [a, 1])$. These in turn induce a map

$$(B \times \{a\}) \times F \xrightarrow{\varphi_1|} p^{-1}(B \times \{a\}) \xrightarrow{\varphi_2|^{-1}} (B \times \{a\}) \times F$$

of the form $(b, a, v) \mapsto (b, a, g(b)v)$, where $g : B \longrightarrow \operatorname{Homeo}(F)$ is continuous and $\operatorname{Homeo}(F)$ is the space of homeomorphisms of F onto itself with the compact-open topology and $\varphi_1|, \varphi_2|$ denote the appropriate restrictions.

Next we define $\varphi : (B \times I) \times F \longrightarrow E$ by

$$\varphi(b, t, v) = \begin{cases} \varphi_1(b, t, v) & \text{if } t \leq a, \\ \varphi_2(b, t, g(b)v) & \text{if } t \geq a. \end{cases}$$

Then φ is a trivialization, as desired. □

4.6.2 **Lemma**. *Let $p : E \longrightarrow B \times I$ be a locally trivial bundle. Then there exists an open cover $\{U\}$ of B such that $p^{-1}(U \times I) \longrightarrow U \times I$ is trivial for every U in the cover.*

Proof: Take $b \in B$. Then for each $t \in I$ there exists a neighborhood U_t of b in B and there exists a neighborhood V_t of t in I such that $p^{-1}(U_t \times V_t)$ is trivial. Since I is compact, there exists a finite subcover $\{V_{t_r} \mid r = 1, \ldots, m\}$ of the cover $\{V_t \mid t \in I\}$. Put $U_b = \bigcap_{r=1}^m U_{t_r}$ and choose $0 = s_0 < s_1 < \cdots < s_n = 1$ such that the differences $s_i - s_{i-1}$ for $i = 1, \ldots, n$ are all less than the Lebesgue number of the cover $\{V_{t_r}\}$. Then $p^{-1}(U_b \times [s_{i-1}, s_i]) \longrightarrow U_b \times [s_{i-1}, s_i])$ is trivial. And so by iterating and using Lemma 4.6.1 we have that $p^{-1}(U_b \times I)$ is trivial as well. Repeating this construction for every $b \in B$ we get an open cover $\{U_b\}$ of B such that each $p^{-1}(U_b \times I) \longrightarrow U_b \times I$ is trivial. □

4.6.3 **Proposition**. *Let $p : E \longrightarrow B \times I$ be a locally trivial bundle, where B is a paracompact space. Let $r : B \times I \longrightarrow B \times I$ be the retraction defined by $r(b,t) = (b,1)$ for $(b,t) \in B \times I$. Then there exists a bundle morphism*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "r"'] & B \times I. \end{tikzcd}$$

Therefore, $E \cong r^*E$.

Proof: Using 4.6.2 and the paracompactness of B there is a locally finite open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of B together with a subordinate partition of unity $\{\eta_\alpha\}_{\alpha \in \Lambda}$ such that $p^{-1}(U_\alpha \times I) \longrightarrow U_\alpha \times I$ is trivial. For each $\alpha \in \Lambda$, define $\mu_\alpha : B \longrightarrow I$ by

$$\mu_\alpha(x) = \frac{\eta_\alpha(x)}{\max\{\eta_\beta(x) \mid \beta \in \Lambda\}} \,.$$

Due to the fact that only a finite number of the $\eta_\beta(x)$ are nonzero, the function $\max\{\eta_\beta(x) \mid \beta \in \Lambda\}$ is continuous and nonzero. Therefore, μ_α is continuous, has support in U_α, and for each $x \in B$ satisfies $\max\{\mu_\alpha(x)\} = 1$.

Let $\varphi_\alpha : U_\alpha \times I \times F \longrightarrow p^{-1}(U_\alpha \times I)$ for each $\alpha \in \Lambda$ denote a local trivialization. For each $\alpha \in \Lambda$ we then define a bundle morphism

$$\begin{tikzcd} E \arrow[r, "f_\alpha"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "r_\alpha"'] & B \times I \end{tikzcd}$$

by setting, in the base space, $r_\alpha(b,t) = (b, \max(\mu_\alpha(b), t))$ for $(b,t) \in B \times I$ and by setting, in the total space, f_α to be the identity outside of $p^{-1}(U_\alpha \times I)$ and by setting $f_\alpha(\varphi_\alpha(b,t,v)) = \varphi_\alpha(b, \max(\mu_\alpha(b), t), v)$ inside of $p^{-1}(U_\alpha \times I)$. Let us choose a well-ordering $\prec$ on Λ. By local finiteness we have that for each $b \in B$ there exists a neighborhood W_b of b such that $W_b \cap U_\alpha$ is nonempty only for finitely many α in Λ, say for α in the finite subset $\Lambda_b = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ with $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_m$. We now define $r : B \times I \longrightarrow B \times I$ by $r|(W_b \times I) = r_{\alpha_m} \circ r_{\alpha_{m-1}} \circ \cdots \circ r_{\alpha_1}$, and we define $f : E \longrightarrow E$ by $f|p^{-1}(W_b \times I) = f_{\alpha_m} \circ f_{\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1}$. Since r_α on $W_b \times I$ and f_α on $p^{-1}(W_b \times I)$ are the identity if $\alpha \notin \Lambda_b$, we can view r and f as infinite composites of maps almost all of which are the identity in a neighborhood of any point. (Here "almost all" means "all except a finite number.") Since each f_α is an isomorphism on every fiber, the composite f also is an isomorphism on every fiber. $\square$

4.6.4 **Theorem**. *Let $p' : E' \longrightarrow B'$ be a locally trivial bundle and B a paracompact space, and suppose that we have two homotopic maps $f, g : B \longrightarrow B'$. Then we have a bundle isomorphism $f^*E' \cong g^*E'$.*

Proof: Let $F : B \times I \longrightarrow B'$ be a homotopy from f to g. Also let $i_\nu : B \longrightarrow B \times I$ be the inclusions $i_\nu(b) = (b, \nu)$ for $b \in B$ and $\nu = 0, 1$. It then follows that $f = F \circ i_0$ and $g = F \circ i_1$.

Let $r : B \times I \longrightarrow B \times I$ be the retraction defined by $r(b,t) = (b,1)$ for $(b,t) \in B \times I$. Then by applying 4.3.10, 4.6.3, and 4.5.21 we have that $f^*E' = (F \circ i_0)^*E' \cong i_0^*F^*E' \cong i_0^*r^*F^*E' \cong (r \circ i_0)^*F^*E' \cong i_1^*F^*E' \cong g^*E'$, where we have also used $r \circ i_0 = i_1$. $\square$

We move on to the solution of the classification problem.

4.6.5 DEFINITION. Let X be a topological space. We define its *nth configuration space* $F_n(X)$ by

$$F_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

If Σ_n denotes the symmetric (or permutation) group of the set $\{1, \ldots, n\}$, then there is a right free action of this group on $F_n(X)$ given by

$$(x_1, \ldots, x_n)\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \qquad x_i \in X.$$

The quotient space of this action can be considered as the space of subsets of cardinality n of X. This quotient space can be also viewed as a subspace of the nth symmetric product $\mathrm{SP}^n X$, which will be defined below (see 5.2.1). If X is a Hausdorff space, then by 4.5.30 the action is properly discontinuous. Hence the action is free, and by 4.5.29 the quotient map $p_n : F_n(X) \longrightarrow F_n(X)/\Sigma_n$ is a covering map. Since the fiber is Σ_n, the *multiplicity* of the covering map (that is, the cardinality of the fiber) is $n!$. There is also an *n-fold covering map*, that is, a covering map of multiplicity n, $\pi_n : E_n(X) \longrightarrow F_n(X)/\Sigma_n$ associated to $F_n(X)$ and defined as follows. The total space is given by $E_n(X) = \{(C, x) \in F_n(X)/\Sigma_n \times X \mid x \in C\}$ and the projection by $\pi_n(C, x) = C$.

We shall consider only the case $X = \mathbb{R}^k$, where $1 \leq k \leq \infty$. It can be shown that the space $F_n(\mathbb{R}^\infty)$ is contractible.

4.6.6 Definition. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be covering maps. We say that they are *equivalent* if there is a bundle isomorphism $\varphi : E \longrightarrow E'$, that is, a homeomorphism such that $p' \circ \varphi = p$. The map φ is called an *equivalence* of covering maps. In particular, fiberwise, φ is an equivalence of sets.

Corresponding to 4.5.20, one can directly prove the following special case.

4.6.7 Exercise. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be covering maps. Assume that $\varphi : E \longrightarrow E'$ is such that $p' \circ \varphi = p$ and $\varphi|p^{-1}(x) : p^{-1}(x) \longrightarrow p'^{-1}(x)$ for each $x \in X$ is an equivalence of sets, i.e., is bijective. Prove that φ is an equivalence.

4.6.8 **Lemma**. *Let $p : E \longrightarrow B$ and $q : E' \longrightarrow B'$ be covering maps. Assume that there are maps $F : E \longrightarrow E'$ and $f : B \longrightarrow B'$ such that*

(i) *$q \circ F = f \circ p$,*

(ii) *F restricted to each fiber of p is a bijection onto the corresponding fiber of q.*

Then $p : E \longrightarrow B$ is equivalent to the covering map $\overline{q} : f^*E' \longrightarrow B$ induced from q by f.

Proof: Consider the pullback diagram

$$\begin{tikzcd} f^*E' \arrow[r, "\overline{f}"] \arrow[d, "\overline{q}"'] & E' \arrow[d, "q"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

and the maps $F : E \longrightarrow E'$ and $p : E \longrightarrow B$. The map $\varphi : E \longrightarrow f^*E'$ given by $\varphi(e) = (p(e), F(e))$ coincides fiberwise with F. Therefore, it is a bijection of the fibers, and thus, by Exercise 4.6.7, φ is an equivalence. □

The following concept also has a version for vector bundles (see 8.5.2).

4.6.9 Definition. Let $p : E \longrightarrow B$ be an n-fold covering map. A *Gauss map* is a map $g : E \longrightarrow \mathbb{R}^k$, $1 \leq k \leq \infty$, such that $g|p^{-1}(x) : p^{-1}(x) \longrightarrow \mathbb{R}^k$ is injective for each $x \in B$.

4.6.10 **Proposition**. *Let $p : E \longrightarrow B$ be an n-fold covering map. Then there exists a Gauss map $g : E \longrightarrow \mathbb{R}^k$ for p if and only if there is a map $f : B \longrightarrow F_n(\mathbb{R}^k)/\Sigma_n$ such that E is equivalent to $f^*E_n(\mathbb{R}^k)$. The map f is called a* classifying map.

Proof: Let $g : E \longrightarrow \mathbb{R}^k$ be a Gauss map for p. Define $f : B \longrightarrow F_n(\mathbb{R}^k)/\Sigma_n$ as follows. For each $x \in B$, choose a bijection $h : \overline{n} \longrightarrow p^{-1}(x)$, where $\overline{n} = \{1, 2, \ldots, n\}$. Since $g \circ h : \overline{n} \longrightarrow \mathbb{R}^k$ is injective, set

$$f(x) = \pi_n(gh(1), \ldots, gh(n))\,.$$

This is well defined, since given any other bijection $h' : \overline{n} \longrightarrow p^{-1}(x)$, the composite $\sigma = h'^{-1} \circ h$ belongs to Σ_n and

$$(gh'(1), \ldots, gh'(n))\sigma = (gh(1), \ldots, gh(n))\,.$$

To see that f is continuous, take a trivializing cover $\{U\}$ with trivializing maps φ_U. Then, for each $x \in U$, the composite

$$p^{-1}(x) \xrightarrow{\varphi_U|} U \times \overline{n} \xrightarrow{\text{proj}} \overline{n}$$

is a bijection and $f(x) = \pi_n(g((\text{proj} \circ \varphi_U)^{-1}(x)), \ldots, g((\text{proj} \circ \varphi_U)^{-1}(x)))$.

Now we define $F : E \longrightarrow E_n(\mathbb{R}^k)$ by $F(e) = (fp(e), g(e))$ and get the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "F"] \arrow[d, "f"'] & E_n(\mathbb{R}^k) \arrow[d, "\pi_n"] \\
B \arrow[r, "f"'] & F_n(\mathbb{R}^k)/\Sigma_n\,.
\end{tikzcd}$$

Since F is a bijection on fibers, by Lemma 4.6.8, $f^*E_n(\mathbb{R}^k) \cong E$.

Conversely, let $h : E \longrightarrow f^*E_n(\mathbb{R}^k)$ be an equivalence of covering maps. Then $g : E \longrightarrow \mathbb{R}^k$ defined by

$$\begin{tikzcd} f^*E_n(\mathbb{R}^k) \arrow[r, "\overline{f}"] & E_n(\mathbb{R}^k) \arrow[r, hook] & F_n(\mathbb{R}^k)/\Sigma_n \times \mathbb{R}^k \arrow[d, "\mathrm{proj}"] \\ E \arrow[u, "h"] \arrow[rr, dashed, "g"'] & & \mathbb{R}^k \end{tikzcd}$$

is clearly a Gauss map. $\square$

4.6.11 Exercise. Let $p : E \longrightarrow B$ be an n-fold covering map.

(a) Prove that the above construction establishes a bijection between the set of bundle morphisms

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d, "p"'] & E_n(\mathbb{R}^k) \arrow[d] \\ B \arrow[r, "f"'] & F_n(\mathbb{R}^k)/\Sigma_n \end{tikzcd}$$

and the set of Gauss maps $g : E \longrightarrow \mathbb{R}^k$.

(b) Prove that if $G : E \times I \longrightarrow \mathbb{R}^k$ is a homotopy such that $G_t : E \longrightarrow \mathbb{R}^k$ is a Gauss map for every $t \in I$, where we define $G_t(e) = G(t, e)$ for $e \in E$, then we can use the above construction in order to obtain a bundle morphism

$$\begin{tikzcd} E \times I \arrow[r, "\overline{F}"] \arrow[d, "p\times\mathrm{id}"'] & E_n(\mathbb{R}^k) \arrow[d] \\ B \times I \arrow[r, "F"'] & F_n(\mathbb{R}^k)/\Sigma_n\,, \end{tikzcd}$$

with the following property. If $f_\nu : B \longrightarrow F_n(\mathbb{R}^k)/\Sigma_n$ for $\nu = 0, 1$ are the functions associated to G_ν for $\nu = 0, 1$, then F is a homotopy between f_0 and f_1.

In order to prove that every finite covering map over a paracompact space has a Gauss map we shall need the next important lemma, whose special case for covering maps we shall use below and whose special case for vector bundles will be used in Chapter 8.

4.6.12 **Lemma**. *Let $p : E \longrightarrow B$ be a locally trivial bundle over a paracompact space B. Then there exists a countable open cover of B, say $\{W_n\}$ with $n \geq 1$, such that $p^{-1}W_n$ is trivial for all $n \geq 1$.*

Proof: Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of B such that $p^{-1}(U_\alpha) \longrightarrow U_\alpha$ is trivial for all $\alpha \in \Lambda$. Since B is paracompact, there exists a partition of unity $\{\eta_\alpha\}_{\alpha\in\Lambda}$ subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$. For each $b \in B$ let us define $S(b)$ to be the finite set of those $\alpha \in \Lambda$ that satisfy $\eta_\alpha(b) > 0$. Also, for each finite subset $S \subset \Lambda$, let us define $W(S) = \{b \in B \mid \eta_\alpha(b) > \eta_\beta(b)$ whenever $\alpha \in S$ and $\beta \notin S\}$.

We claim that $W(S)$ is open in B. In fact, $B_{\alpha,\beta} = \{b \in B \mid \eta_\alpha(b) > \eta_\beta(b)\}$ is open, since $B_{\alpha,\beta} = (\eta_\alpha - \eta_\beta)^{-1}(0,1]$. Now for any given $b_0 \in W(S)$ there exists a neighborhood $V(b_0)$ of b_0 such that only $\eta_{\beta_1}, \eta_{\beta_2}, \dots, \eta_{\beta_r}$ are different from zero in $V(b_0)$ for some finite integer r. We put $N = \bigcap_{\alpha\in S}(B_{\alpha,\beta_1} \cap B_{\alpha,\beta_2} \cap \cdots \cap B_{\alpha,\beta_r})$, which is open, being a finite intersection of open sets. We then have $b_0 \in N \cap V(b_0) \subset W(S)$, and therefore $W(S)$ is open.

If S and S' are two distinct subsets of Λ each having m elements, then $W(S) \cap W(S') = \emptyset$. This is so, since there exists $\alpha \in S$ such that $\alpha \notin S'$ and there exists $\beta \in S'$ such that $\beta \notin S$ and therefore $b \in W(S) \cap W(S')$ would imply that $\eta_\alpha(b) > \eta_\beta(b)$ and that $\eta_\beta(b) > \eta_\alpha(b)$, a patent contradiction.

Now we define $W_n = \bigcup\{W(S(b)) \mid |S(b)| = n\}$ for every integer n, where here $|\cdot|$ denotes the cardinality of a set.

If $\alpha \in S(b)$, then $W(S(b)) \subset \eta_\alpha^{-1}(0,1] \subset U_\alpha$, and therefore we have that $p^{-1}W(S(b)) \longrightarrow W(S(b))$ is trivial. Since for each n the open set W_n is a disjoint union of sets of the form $W(S(b))$, it follows that $p^{-1}W_n \longrightarrow W_n$ is also trivial. □

4.6.13 Note. From the proof it is clear that any locally trivial bundle $p : E \longrightarrow B$ is a locally trivial bundle of *finite type* wherever B is paracompact; that is, it has a finite trivializing cover. This is because each $b \in B$ belongs to at most m subsets U_α, and so we have that $W_i = \emptyset$ for $i > m$. Therefore, there exists a finite open cover $\{W_i\}$ for $i = 1, \dots, m$ such that $p^{-1}W_i \longrightarrow W_i$ is trivial. And this establishes the claim.

4.6.14 **Proposition**. *Every n-fold covering map over a paracompact space has a Gauss map.*

Proof: Let B be paracompact and $p : E \longrightarrow B$ be an n-fold covering map. Since B is paracompact, by 4.6.12 there is a countable trivializing cover $\{W_i\}_{i=1}^\infty$ of B. Let $\varphi_i : p^{-1}(W_i) \longrightarrow W_i \times \overline{n}$ be a trivialization and let $\{\eta_i\}_{i=1}^\infty$ be a partition of unity subordinate to $\{W_i\}$. For each i, define $g_i : E \longrightarrow \mathbb{R}$ by

$$g_i(e) = \begin{cases} \eta_i(p(e)) \cdot \mathrm{proj}\varphi_i(e) & \text{if } e \in p^{-1}(W_i), \\ 0 & \text{if } e \notin p^{-1}(W_i), \end{cases}$$

where $\mathrm{proj} : W_i \times \overline{n} \longrightarrow \overline{n} \subset \mathbb{R}$ is the projection.

Now we define $g : E \longrightarrow \mathbb{R}^\infty$ by $g(e) = (g_1(e), \dots, g_i(e), \dots)$. $\square$

4.6.15 DEFINITION. Let X be a paracompact space. We denote by $\mathcal{C}_n(X)$ the set of equivalence classes of n-fold covering maps over X.

By Propositions 4.6.10 and 4.6.14, we have the following.

4.6.16 **Theorem**. *Let X be a paracompact space. Then there is a bijection*

$$[X, F_n(\mathbb{R}^\infty)/\Sigma_n] \longrightarrow \mathcal{C}_n(X)$$

given by $[f] \mapsto [f^* E_n(\mathbb{R}^\infty)]$.

Proof: By 4.6.4, the function is well defined. Propositions 4.6.10 and 4.6.14 show that the function is surjective.

To see that the function is injective, we consider $\mathbb{R}^\infty_1 = \{(t_i) \in \mathbb{R}^\infty \mid t_{2i} = 0, i = 1, 2, 3, \dots\}$ and $\mathbb{R}^\infty_2 = \{(t_i) \in \mathbb{R}^\infty \mid t_{2i+1} = 0,\ i = 0, 1, 2, \dots\}$, so that $\mathbb{R}^\infty = \mathbb{R}^\infty_1 \oplus \mathbb{R}^\infty_2$. Next we define two homotopies $h^1, h^2 : \mathbb{R}^\infty \times I \longrightarrow \mathbb{R}^\infty$ by

$$\begin{aligned} h^1((t_1, t_2, t_3, \dots), t) &= (1-t)(t_1, t_2, t_3, \dots) + t(t_1, 0, t_2, 0, t_3, \dots)\,, \\ h^2((t_1, t_2, t_3, \dots), t) &= (1-t)(t_1, t_2, t_3, \dots) + t(0, t_1, 0, t_2, 0, t_3, \dots)\,, \end{aligned}$$

where $(t_1, t_2, t_3, \dots) \in \mathbb{R}^\infty$ and $t \in I$. These homotopies start with the identity and end with maps that we denote by

$$h^1_1 : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_1 \subset \mathbb{R}^\infty \quad \text{and} \quad h^2_1 : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_2 \subset \mathbb{R}^\infty.$$

The composites $h^\nu_1 \circ p_2 : E_n(\mathbb{R}^\infty) \longrightarrow \mathbb{R}^\infty$ for $\nu = 1, 2$ are Gauss maps, where $p_2 : E_n(\mathbb{R}^\infty) \longrightarrow \mathbb{R}^\infty$ is the projection on the second coordinate. According to 4.6.11(a), these maps induce two morphisms of covering maps, namely,

$$\begin{tikzcd} E_n(\mathbb{R}^\infty) \arrow[r, "\overline{\varphi}_\nu"] \arrow[d] & E_n(\mathbb{R}^\infty) \arrow[d] \\ F_n(\mathbb{R}^\infty)/\Sigma_n \arrow[r, "\varphi_\nu"'] & F_n(\mathbb{R}^\infty)/\Sigma_n, \end{tikzcd} \qquad \nu = 1, 2.$$

The composites $h^\nu \circ (p_2 \times \mathrm{id}) : E_n(\mathbb{R}^\infty) \times I \longrightarrow \mathbb{R}^\infty$ for $\nu = 1, 2$ are homotopies that start with p_2, since $h^\nu(q \times \mathrm{id})(e, 0) = h^\nu(p_2(e), 0) = p_2(e)$ for $e \in E_n(\mathbb{R}^\infty)$, and that end with $h^\nu_1 \circ p_2$. Moreover, the restrictions of these homotopies to the slices at each fixed $t \in I$ are Gauss maps. Using 4.6.11(b) we then have that φ_ν for $\nu = 1, 2$ is homotopic to the map induced by p_2, which is obviously the identity. So we have shown that $\varphi_\nu \simeq \mathrm{id}$ for $\nu = 1, 2$.

We are now ready to show that the function is injective. Suppose that we are given $f_\nu : B \longrightarrow F_n(\mathbb{R}^\infty)/\Sigma_n$ for $\nu = 1, 2$ satisfying $f_1^* E_n(\mathbb{R}^\infty) \cong f_2^* E_n(\mathbb{R}^\infty)$. So to prove injectivity we must show that f_1 and f_2 are homotopic.

Denoting $f_1^* E_n(\mathbb{R}^\infty)$ by E and using the above isomorphism, we get two morphisms of covering maps

$$\begin{tikzcd} E \arrow[r, "\overline{f_\nu}"] \arrow[d] & E_n(\mathbb{R}^\infty) \arrow[d] \\ B \arrow[r, "f_\nu"'] & F_n(\mathbb{R}^\infty)/\Sigma_n\,, \end{tikzcd} \qquad \nu = 1, 2.$$

Let $g_\nu : E \longrightarrow \mathbb{R}^\infty$ for $\nu = 1, 2$ be the associated Gauss maps; that is, $g_\nu = p_2 \circ \overline{f}_\nu$.

Consider the composites $h_1^\nu \circ g_\nu : E \longrightarrow \mathbb{R}^\infty$ for $\nu = 1, 2$. These are Gauss maps, and according to 4.6.11(a) they induce two morphisms of covering maps of the form

$$\begin{tikzcd} E \arrow[r, "\overline{f}_\nu"] \arrow[d] & \mathrm{E}_k(\mathbb{R}^\infty) \arrow[r, "\overline{\varphi}_\nu"] \arrow[d] & \mathrm{E}_k(\mathbb{R}^\infty) \arrow[d] \\ B \arrow[r, "f_\nu"'] & F_n(\mathbb{R}^\infty)/\Sigma_n \arrow[r, "\varphi_\nu"'] & F_n(\mathbb{R}^\infty)/\Sigma_n\,, \end{tikzcd} \qquad \nu = 1, 2.$$

We then define $G : E \times I \longrightarrow \mathbb{R}^\infty$ by $G(e, t) = (1 - t)h_1^1 g_1(e) + t h_1^2 g_2(e)$ for $(e, t) \in E \times I$. This is a homotopy between $h_1^1 \circ g_1$ and $h_1^2 \circ g_2$. Since $h_1^1(\mathbb{R}^\infty) \cap h_1^2(\mathbb{R}^\infty) = 0$, it follows that G_t is a Gauss map for each $t \in I$. Therefore, using 4.6.11(b) we have that $\varphi_1 \circ f_1 \simeq \varphi_2 \circ f_2$. But we have already seen that $\varphi_\nu \simeq \mathrm{id}$ for $\nu = 1, 2$, and so $f_1 \simeq f_2$ follows. □

4.6.17 Remark. Consider the covering map

$$p_n : F_n(\mathbb{R}^k) \longrightarrow F_n(\mathbb{R}^k)/\Sigma_n\,.$$

Using the homotopy exact sequence of p_n, we have that

$$\pi_i(F_n(\mathbb{R}^k)/\Sigma_n) = \begin{cases} \Sigma_n & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Therefore, the space $F_n(\mathbb{R}^\infty)/\Sigma_n$ is an *Eilenberg–Mac Lane space* of type $(\Sigma_n, 1)$ (see 6.1.1). Since the space $F_n(\mathbb{R}^\infty) = \mathrm{colim}_k F_n(\mathbb{R}^k)$ is a CW-complex (see 5.1.1), it is paracompact. Moreover, because p_n is a closed map, $F_n(\mathbb{R}^\infty)/\Sigma_n$ is a Hausdorff space and hence paracompact. Therefore, p_n is a *numerable* principal Σ_n-bundle with contractible total space. By [24], this means that p_n is a *universal* Σ_n-bundle, and the space $F_n(\mathbb{R}^\infty)/\Sigma_n$ is then a *classifying space* for the group Σ_n; this space is usually denoted by $B\Sigma_n$. This argument shows that if X is paracompact, then there is a bijection $\mathcal{C}_n(X) \cong [X, B\Sigma_n]$.

Let X be a connected CW-complex with a 0-cell x_0 as base point. Let $\psi : [X, x_0; B\Sigma_n, *] \longrightarrow \operatorname{Hom}(\pi_1(X, x_0), \Sigma_n)$ be the function given by $\psi[f] = f_*$. Using obstruction theory (see [67]) one can show that ψ is a bijection. The action of the symmetric group Σ_n on $[X, x_0; B\Sigma_n, *]$ (see 4.4.1) corresponds under ψ to the action of Σ_n on $\operatorname{Hom}(\pi_1(X, x_0), \Sigma_n)$ given by conjugation. Therefore, there is a bijection

$$[X, B\Sigma_n] \cong \operatorname{Hom}^{\mathrm{conj}}(\pi_1(X, x_0), \Sigma_n)\,.$$

Hence, by Theorem 4.6.16, we get a bijection

$$\mathcal{C}_n(X) \cong \operatorname{Hom}^{\mathrm{conj}}(\pi_1(X, x_0), \Sigma_n)$$

for every connected CW-complex X.

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Chapter 5

CW-Complexes and Homology

We start this chapter by defining and studying a very important class of spaces, known as the CW-complexes; in the next chapters these will be the spaces with which we shall mainly work.

Some of their properties will be derived applying a very useful homotopy extension and lifting property. Therefrom, well-known results on the topic due to J.H.C. Whitehead will be obtained.

We shall introduce the notion of an "infinite symmetric product," which in the next chapter will be crucial for defining the Eilenberg–Mac Lane spaces, as was done by Dold and Thom in the beautiful article [26]. A key result for doing that is the Dold–Thom theorem, which will be discussed here. Its proof, however, will be postponed to Appendix A.

Using the results on infinite symmetric products, we shall define the homology groups and derive many of their properties.

5.1 CW-Complexes

As already announced, in this section we are going to introduce an important class of topological spaces, which is obtained by successively adjoining cells of dimension n, for each $n \geq 0$. Many of the interesting spaces that we study in algebraic topology are found in this class. Furthermore, many of the constructions discussed in this and the next chapter generate CW-complexes.

5.1.1 Definition. Let $\{I_n\}_{n=0}^{\infty}$ be a sequence of disjoint sets such that $I_0 \neq \emptyset$. Starting with this sequence we inductively construct a sequence of topological spaces $\{X^n\}$ as follows:

(i) For $n = 0$ we put $X^0 = I_0$ with the discrete topology on I_0.

(ii) If X^{n-1} has already been constructed, then put $X^n = X^{n-1}$ if $I_n = \emptyset$. However, if $I_n \neq \emptyset$, we assume that we have a family of maps $\{\varphi^i : \mathbb{S}^{n-1} \longrightarrow X^{n-1} \mid i \in I_n\}$, called *characteristic maps*, and we put $D_n = \coprod_{i\in I_n} \mathbb{D}^n_i$ and $S_n = \coprod_{i\in I_n} \mathbb{S}^{n-1}_i \subset D_n$, where $\mathbb{D}^n_i = \mathbb{D}^n$ and $\mathbb{S}^{n-1}_i = \mathbb{S}^{n-1}$. The family $\{\varphi^i\}$ determines a map $\varphi_n : S_n \longrightarrow X^{n-1}$ defined by $\varphi_n|\mathbb{S}^{n-1}_i = \varphi^i$. We then define $X^n = X^{n-1} \cup_{\varphi_n} D_n$.

(iii) Clearly, we have closed embeddings $X^{n-1} \subset X^n$. We define $X = \bigcup_{n=0}^{\infty} X^n$ with the union topology; namely, $K \subset X$ is closed $\iff$ $K \cap X^n$ is closed for all n.

A topological space homeomorphic to a space X obtained in this way is called a CW-*complex*. The subspace X^n is called the *n-skeleton* of X.

It is easy to prove that every CW-complex is Hausdorff and normal. Moreover, every CW-complex is even paracompact (see [59] and [45]) and locally path connected.

Let $q_n : D_n \sqcup X^{n-1} \longrightarrow X^n$ be the identification map of 5.1.1(ii), and put $\widetilde{\varphi}^i = q_n|\mathbb{D}^n_i$. We call $e^n_i = \widetilde{\varphi}^i(\overset{\circ}{\mathbb{D}}{}^n_i)$ an *open n-cell* of X, which is open in X^n though it is not open in general in X. It also is homeomorphic to $\overset{\circ}{\mathbb{D}}{}^n$. We call $\bar{e}^n_i = \widetilde{\varphi}^i(\mathbb{D}^n_i)$ a *closed n-cell* of X, which is closed both in X^n and in X. However, in general it is not homeomorphic to $\mathbb{D}^n$.

5.1.2 Exercise. Prove that a CW-complex X is the disjoint union (not the topological sum) of all its open cells e^n_i, $n \in \mathbb{N}$, $i \in I_n$.

5.1.3 Examples. The following are examples of CW-complexes.

(a) The projective space $\mathbb{CP}^n$. This can be constructed, as we shall see later on in 5.2.26, so that it has one 0-cell, one 2-cell,..., and one $2n$-cell.

(b) The sphere $\mathbb{S}^n$. This has two 0-cells (the poles), two 1-cells,..., and two n-cells (the two hemispheres).

(c) Simplicial complexes (polyhedra). See [67].

(d) Surfaces, as they were constructed in 3.2.12(c) and (d) or, more generally, differentiable manifolds.

5.1.4 Exercise. Prove that another possible decomposition of the sphere $\mathbb{S}^n$ as a CW-complex has one 0-cell and one n-cell. In fact, this particular decomposition is unique up to homeomorphism.

CW-complexes have important properties, which we state in what follows.

5.1.5 **Proposition**. *If X is a* CW-*complex, then the n-skeleton $X^n \subset X$ is closed for every n.* □

5.1.6 **Proposition**. *Let X be a* CW-*complex. Then the following hold:*

(a) *X is locally path connected.*

(b) *If X is connected, then it is path connected.*

Proof: (a) Attaching spaces obviously preserve the property of being locally path connected. Therefore, we have, inductively, that every skeleton X^n is locally path connected. Moreover, unions of closed locally path-connected spaces with the topology of the union are again locally path connected. Thus $X = \bigcup X^n$ is locally path connected.

(b) Any connected, locally path-connected space is path connected. Thus, if X is connected, then by (a) it is path connected. □

5.1.7 **Proposition**. *Let X be a* CW-*complex. Then the following hold.*

(a) *X is a T_1 space.*

(b) *X is a normal space, thus also Hausdorff.*

Proof: (a) By induction we have that X^n is a T_1 space. So, if $x \in X$, then $\{x\} \cap X^n$ is either empty or consists of one point; therefore, it is closed. Thus $\{x\}$ is closed in X, and so X is also T_1.

(b) Again using properties of attaching spaces and induction we have that X^n is normal for all n. Let $A, B \subset X$ be disjoint closed sets. Then there is a map $f_0 : X^0 \longrightarrow I$ with

$$f_0(x) = \begin{cases} 0 & \text{if } x \in A \cap X^0, \\ 1 & \text{if } x \in B \cap X^0. \end{cases}$$

Assume that we have already constructed a map $f_{n-1} : X^{n-1} \longrightarrow I$ with

$$f_{n-1}(x) = \begin{cases} 0 & \text{if } x \in A \cap X^{n-1}, \\ 1 & \text{if } x \in B \cap X^{n-1}, \end{cases}$$

such that $f_{n-1}|X^{n-2} = f_{n-2}$, $n > 1$.

Take $F = (A \cap X^n) \cup X^{n-1} \cup (B \cap X^n) \subset X$ and define $g_n : F \longrightarrow I$ by

$$g_n(x) = \begin{cases} 0 & \text{if } x \in A \cap X^n, \\ f_{n-1}(x) & \text{if } x \in X^{n-1}, \\ 1 & \text{if } x \in B \cap X^n. \end{cases}$$

Since X^n is normal and $F \subset X$ is closed, one can extend g to a map $f_n : X^n \longrightarrow I$ with the desired properties.

Define $f : X \longrightarrow I$ in such a way that $f|X^n = f_n$. This map is well defined, and since X has the topology of the union, it is continuous. Moreover, $f|A = 0$ and $f|B = 1$. Thus X is normal, and being also T_1, it is a Hausdorff space. □

5.1.8 EXERCISE. Prove that the given definition of the concept of a CW-complex is equivalent to the following one.

A CW-*complex* X is a Hausdorff space, together with index sets I_n, $n \geq 0$, and maps $\psi_n^i : \mathbb{D}^n \longrightarrow X$, $i \in I_n$, $I_0 \neq \emptyset$, such that the following conditions are fulfilled:

(i) $X = \bigcup_{n,i} \psi_n^i(\mathbb{D}^n)$.

(ii) $\psi_i^n(\overset{\circ}{\mathbb{D}}{}^n) \cap \psi_j^m(\overset{\circ}{\mathbb{D}}{}^n) = \emptyset$ unless $n = m$ and $i = j$.

(iii) $\psi_i^n|\overset{\circ}{\mathbb{D}}{}^n$ is bijective for all $n \geq 0$ and $i \in I_n$.

(iv) If $X^n = \bigcup_{k \leq n, i \in I_k} \psi_i^k(\overset{\circ}{\mathbb{D}}{}^k)$, $n \geq 0$, then $\psi_i^m(\mathbb{S}^{m-1}) \subset X^{m-1}$, for each $m \geq 1$ and $i \in I_m$.

(v) A subset $K \subset X$ is closed if and only if $(\psi_i^n)^{-1}(K)$ is closed in $\mathbb{D}^n$ for each $n \geq 0$ and $i \in I_n$.

(vi) For each $n \geq 0$ and $i \in I_n$, $\psi_i^n(\mathbb{D}^n)$ is contained in the union of finitely many sets of the form $\psi_j^m(\overset{\circ}{\mathbb{D}}{}^m)$.

An immediate consequence of (v) is the following.

5.1.9 **Proposition**. *A* CW-*complex X has the topology of the union of all its closed cells.* □

The following is also an important property of CW-complexes. However, we formulate it more generally for any Hausdorff space $X = \bigcup X_n$, where $X_1 \subset X_2 \subset X_3 \subset \cdots$ and where X has the topology of the union topology.

5.1.10 **Lemma**. *Let* $X = \bigcup X_n$*,* $X_1 \subset X_2 \subset X_3 \subset \cdots$*, be a Hausdorff space with the union topology. Then every compact subset* $K \subset X$ *lies inside* X_n *for some* n*.*

Proof: If the conclusion were not so, then there would exist a sequence $\{x_n\}$ in K satisfying $x_n \notin X_n$. Now, any such sequence forms a closed subset of X, since its intersection with each X_n is finite and hence closed in X_n. Here we are using the fact that X is Hausdorff, implying that X_n is also Hausdorff, so that points are closed in X_n. Therefore, the subsequences $\{x_m, x_{m+1}, x_{m+2}, \ldots\}$, $m = 1, 2, 3, \ldots$, form a nested system of closed subsets of K whose intersection is empty, although the intersection of every finite subsystem is nonempty. And this would give us a contradiction to the compactness of K. □

Note that in order to get the conclusion of 5.1.10, it is enough to assume that X is a T_1 space, that is, that every point $x \in X$ forms a closed subset of X.

5.1.11 Definition. If X is a CW-complex and $A \subset X$, then we say that A is a *subcomplex* of X if for every open cell e_i^n of X we have that $A \cap e_i^n \neq \emptyset \Rightarrow \bar{e}_i^n \subset A$. We call the pair of spaces (X, A) a CW-*pair*.

5.1.12 Example. Every n-skeleton X^n of a CW-complex X is a subcomplex.

We have the following consequence of Lemma 5.1.10.

5.1.13 **Corollary**. *Suppose that* X *is a* CW*-complex and* $K \subset X$ *is compact. Then we have* $K \subset X^n$ *for some* n*. More specifically,* $K \subset Y$ *for a subcomplex* $Y \subset X$*, where* Y *has only a finite number of cells.*

Proof: The first part follows immediately from Lemma 5.1.10. For the second part, in a similar way to the proof of 5.1.10, if K intersects an infinite number of open cells, then we would have an infinite number of points in K, each in an open cell. This set would contain a sequence $\{x_n\}$ that is similar to the one in the proof of 5.1.10, thus contradicting the compactness of K. □

5.1.14 **Proposition**. *Let* X *be a* CW*-complex and* $A \subset X$ *a subcomplex. Then* $A = \bigcup \{e_i^n \mid e_i^n \cap A \neq \emptyset\}$*.*

Proof: If $e_1^n \cap A \neq \emptyset$, then by definition, $\bar{e}_i^n \subset A$. Thus, if $J_n = \{i \in I_n \mid e_i^n \cap A \neq \emptyset\}$, then

$$\bigcup_{n\in\mathbb{N};i\in J_n} \bar{e}_i^n \subset A \subset \bigcup_{n\in\mathbb{N};i\in J_n} e_i^n \subset \bigcup_{n\in\mathbb{N};i\in J_n} \bar{e}_i^n .$$

Hence $A = \bigcup_{n\in\mathbb{N};i\in J_n} \bar{e}_i^n$. □

5.1.15 **Corollary**. *Let X be a* CW*-complex and $A \subset X$ a subcomplex. Then A is closed in X.*

Proof: Let e_i^n be some cell in X. Since $\bar{e}_i^n$ is compact, it meets only a finite number of open cells $e_1^{n_1}, \dots, e_k^{n_k}$ in A. Hence $\bar{e}_i^n \cap A = \bigcup_{j=1}^k \bar{e}_i^n \cap \bar{e}_j^{n_j}$, which is a finite union of closed sets and thus closed in $\bar{e}_i^n$. This holds for any n, i, and since X has the topology of the union of all its closed cells, A is closed. □

5.1.16 Exercise. Let X be a CW-complex. Prove that the following are equivalent:

(a) X is path connected.

(b) X is connected.

(c) X^1 is connected.

(d) X^1 is path connected.

(Hint: Since CW-complexes are locally path connected, (a)⇔(b), (c)⇔(d) follow immediately, as does (c)⇒(b). To prove (b)⇒(c), assume to the contrary the existence of a continuous surjective map $f_1 : X^1 \longrightarrow \{0,1\}$ and inductively extend it to $f : X \longrightarrow \{0,1\}$.)

The CW-complexes form the most convenient class of topological spaces for doing homotopy theory. In the following discussion we shall mention some very important results concerning these spaces.

5.1.17 Definition. Let $n \geq 1$ be an integer. A map $f : X \longrightarrow Y$ between arbitrary topological spaces is called an *n-equivalence* if for each $x \in X$ the homomorphism

$$f_* \ : \pi_q(X,x) \longrightarrow \pi_q(Y, f(x))$$

is an isomorphism for $q \leq n-1$ and is an epimorphism for $q = n$. We say that f is a *weak homotopy equivalence* if it is an n-equivalence for all $n \geq 1$. We also say that $f : (X,A) \longrightarrow (Y,B)$ is a *weak homotopy equivalence of pairs* if both $f : X \longrightarrow Y$ and $f|A : A \longrightarrow B$ are weak homotopy equivalences.

5.1.18 EXERCISE. Prove that if $f : X \longrightarrow Y$ is a homotopy equivalence, then f is a weak homotopy equivalence.

We shall show below in 5.1.37 that if X and Y are CW-complexes, then the converse of this statement is also true.

5.1.19 DEFINITION. Suppose that X is a pointed space and that $n \geq 0$. We say that X is *n-connected* if $\pi_r(X) = 0$ for $r \leq n$. In particular, X is 0-connected if and only if X is path connected. More generally, we say that a pair of spaces (X, A) is *n-connected* if $A \cap X_\nu \neq \emptyset$ for all path components X_ν of X and $\pi_r(X, A) = 0$ for $1 \leq r \leq n$; in particular, (X, A) is *0-connected* if the first condition holds.

These concepts of n-connectedness of a pair and n-equivalence are closely related as seen in the following exercises.

5.1.20 EXERCISE. Prove that the pair (X, A) is n-connected if and only if the inclusion map $i : A \hookrightarrow X$ is an n-equivalence. (Hint: Analyze the homotopy exact sequence of the pair.)

5.1.21 EXERCISE. More generally than in the previous exercise, prove that a map $f : X \longrightarrow Y$ is an n-equivalence if and only if the pair (M_f, X) is n-connected, where M_f is the mapping cylinder of f and X is considered as a subspace by identifying it with the top face.

5.1.22 EXAMPLE. *The sphere $\mathbb{S}^n$ is $(n-1)$-connected.* Indeed, take $q < n$; if $\xi \in \pi_q(\mathbb{S}^n)$ is represented by a map $\eta : \mathbb{S}^q \longrightarrow \mathbb{S}^n$, then take the composed map of pairs

$$\varphi : (\mathbb{D}^q, \mathbb{S}^{q-1}) \xrightarrow{p} (\mathbb{S}^q, *) \xrightarrow{\eta} (\mathbb{S}^n, *)\,,$$

where $*$ denotes the corresponding base points and p is the canonical quotient map. Then $\varphi^{-1}(\mathbb{S}^n - *) \subset \overset{\circ}{\mathbb{D}}{}^q$ is open, and using the smooth deformation theorem, one can find a map

$$\widetilde{\psi} : \overline{\varphi^{-1}(\mathbb{S}^n - *)} \longrightarrow \mathbb{S}^n$$

such that:

(1) $\widetilde{\psi}|\varphi^{-1}(\mathbb{S}^n - B) : \varphi^{-1}(\mathbb{S}^n - B) \longrightarrow \mathbb{S}^n - *$ is smooth (where B is a small ball containing $*$ and $\mathbb{S}^n - *$ is identified with $\mathbb{R}^n$ by the stereographic projection). Because $q < n$, this map misses a point.

(2) $\widetilde{\psi}|\partial\varphi^{-1}(\mathbb{S}^n - *) = \varphi|\partial\varphi^{-1}(\mathbb{S}^n - *)$, where the boundary is taken in the disk.

(See Theorem 2 in Basic Concepts and Notation.) Therefore, the map of pairs

$$\widetilde{\varphi} : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (\mathbb{S}^n, *)$$

such that

$$\widetilde{\varphi}|\mathbb{D}^q - \varphi^{-1}(\mathbb{S}^n - *) = \varphi|\mathbb{D}^q - \varphi^{-1}(\mathbb{S}^n - *) \quad \text{and} \quad \widetilde{\varphi}|\overline{\varphi^{-1}(\mathbb{S}^n - *)} = \widetilde{\psi}$$

is continuous and homotopic to φ relative to $\mathbb{S}^{q-1}$. Thus it induces a map $\widetilde{\eta} : (\mathbb{S}^q, *) \longrightarrow (\mathbb{S}^n, *)$ homotopic to η, and $\widetilde{\eta}$ is nullhomotopic, since it is not surjective. Hence $\xi = [\eta] = [\widetilde{\eta}] = 0$, and so $\pi_q(\mathbb{S}^n) = 0$ if $q \leq n-1$.

5.1.23 Example. *The pair $(\mathbb{D}^{n+1}, \mathbb{S}^n)$ is n-connected.* Indeed, since $\mathbb{S}^n$ is $(n-1)$-connected by 5.1.22 and $\mathbb{D}^{n+1}$ is contractible, the inclusion $\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ is an n-equivalence. Hence by 5.1.20, $(\mathbb{D}^{n+1}, \mathbb{S}^n)$ is n-connected.

5.1.24 **Proposition**. *Suppose $X \cup e^{n+1}$ is the result of attaching to the topological space X an $(n+1)$-cell. Then $X \subset X \cup e^{n+1}$, and the pair $(X \cup e^{n+1}, X)$ is n-connected.*

Proof: The proof is very similar to what we did in Example 5.1.22. Namely, if $\xi \in \pi_q(X \cup e^{n+1}, X)$ is represented by a map

$$\varphi : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X \cup e^{n+1}, X)\,,$$

and if $e^{n+1} = X \cup e^{n+1} - X$ is the open cell, then $\varphi^{-1}(e^{n+1}) \subset \mathbb{D}^q$ is open. As in 5.1.22, there exists

$$\widetilde{\psi} : \overline{\varphi^{-1}(e^{n+1})} \longrightarrow e^{n+1} \subset X \cup e^{n+1}$$

such that:

(1) $\widetilde{\psi}|\varphi^{-1}(e'^{n+1}) : \varphi^{-1}(e'^{n+1}) \longrightarrow e^{n+1}$ is smooth (where $e'^{n+1} \subset e^{n+1}$ is a slightly smaller subcell). Because $q < n$, the map misses a point.

(2) $\widetilde{\psi}|\partial\varphi^{-1}(\mathbb{S}^n - *) = \varphi|\partial\varphi^{-1}(e^{n+1})$, where the boundary is taken in the disk.

(See Theorem 2 in Basic Concepts and Notation.) Therefore, the map of pairs

$$\widetilde{\varphi} : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X \cup e^{n+1}, X)$$

such that

$$\widetilde{\varphi}|\mathbb{D}^q - \varphi^{-1}(e^{n+1}) = \varphi|\mathbb{D}^q - \varphi^{-1}(e^{n+1}) \text{ and } \widetilde{\varphi}|\overline{\varphi^{-1}(e^{n+1})} = \widetilde{\psi}$$

is continuous and homotopic to φ relative to $\mathbb{S}^{q-1}$. Since $\widetilde{\varphi}$ misses a point in the cell e^{n+1}, it can be deformed into a map with image in X, relative to X; that is, $\widetilde{\varphi}$ is nullhomotopic, and so too is φ. Hence $\pi_q(X \cup e^{n+1}, X) = 0$ if $q \leq n$. □

5.1.25 **Corollary**. *Let X be a* CW*-complex and let $i : X^n \hookrightarrow X$ be the inclusion map of the n-skeleton into X. Then the pair (X, X^n) is n-connected, and consequently i is an n-equivalence.*

Proof: Let $\varphi : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X, X^n)$ represent an element in $\pi_q(X, X^n)$. Since $\varphi(\mathbb{D}^q) \subset X$ is compact, by 5.1.13 it meets only a finite number of cells in X, say $\varphi(\mathbb{D}^q) \subset X^n \cup e_1^{m_1} \cup \cdots \cup e_k^{m_k}$, $n < m_1 \leq m_2 \leq \cdots \leq m_k$. Thus, if $q \leq n$, an iterated application of 5.1.24 a finite number of times shows that $\varphi : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X^n \cup e_1^{m_1} \cup \cdots \cup e_k^{m_k}, X^n) \hookrightarrow (X, X^n)$ is nullhomotopic. This shows that $\pi_q(X, X^n) = 0$ if $q \leq n$. □

The following homotopy extension and lifting property (HELP) will be a very useful tool in proving some properties of CW-complexes.

5.1.26 **Theorem**. (HELP) *Let A be a topological space and let X be the result of attaching to A successively cells of dimensions $0, 1, 2, \ldots, k \leq n$. Moreover, let $e : Y \longrightarrow Z$ be an n-equivalence. Then, given maps $f : X \longrightarrow Z$ and $g : A \longrightarrow Y$, and a homotopy $H : A \times I \longrightarrow Z$, $H : f|A \simeq e \circ g$, there are maps $\widetilde{g} : X \longrightarrow Y$ and $\widetilde{H} : X \times I \longrightarrow Z$ such that $\widetilde{g}|A = g$, $\widetilde{H}|A \times I = H$, and $\widetilde{H} : f \simeq e \circ \widetilde{g}$. Put in a diagram, if the following square commutes up to a homotopy H,*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, dashed, "\widetilde{g}"'] \\ Y \arrow[r, "e"'] & Z \end{tikzcd},$$

then there exists $\widetilde{g}$ such that the upper triangle is commutative and the lower triangle is commutative up to a homotopy $\widetilde{H}$ that extends H.

Proof: For convenience, we divide the proof into four steps.

First step. Assume $A = \mathbb{S}^{q-1}$, $X = \mathbb{D}^q$. We may replace $e : Y \longrightarrow Z$ by the inclusion of Y in its mapping cylinder M_e; that is, we may assume that $Y \subset Z$ and the pair (Z, Y) is n-connected (see 5.1.21). Since the inclusion

$\mathbb{S}^{q-1} \hookrightarrow \mathbb{D}^q$ is a cofibration (see 4.1.18), one can change f up to homotopy to f' such that the diagram commutes strictly. Renaming, we thus have a commutative diagram

$$\begin{tikzcd} \mathbb{S}^{q-1} \arrow[r, hook] \arrow[d, "g"'] & \mathbb{D}^q \arrow[d, "f"] \arrow[dl, dashed, "\widetilde{g}"'] \\ Y \arrow[r, hook] & Z\,; \end{tikzcd}$$

that is, $g = f|\mathbb{S}^{q-1}$, and we are looking for $\widetilde{g}$ extending g and homotopic to f when viewed as a map into Z. Then f is a map of pairs $(\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (Z, Y)$. But since $q \leq n$, this map is nullhomotopic; that is, there is a homotopy $\widetilde{H} : \mathbb{D}^q \times I \longrightarrow Z$, $\widetilde{H} : f \simeq \widetilde{g}$, where $\widetilde{g}(\mathbb{D}^q) \subset Y$. This proves this special case.

Second step. Assume $X = A \cup e^q$, where the q-cell is attached to A by a map $\varphi : \mathbb{S}^{q-1} \longrightarrow A$. Consider the diagram

$$\begin{tikzcd} \mathbb{S}^{q-1} \arrow[r, hook] \arrow[d, "\varphi"'] & \mathbb{D}^q \arrow[d, "\widetilde{\varphi}"] \\ A \arrow[r, hook] \arrow[d, "g"'] & A \cup e^q \arrow[d, "f"] \arrow[dl, dashed, "\widetilde{g}"'] \\ Y \arrow[r, "e"'] & Z\,. \end{tikzcd}$$

By the first step, there exist $\widetilde{g}' : \mathbb{D}^q \longrightarrow Y$ and $H' : \mathbb{D}^q \times I \longrightarrow Z$ such that $\widetilde{g}'|\mathbb{S}^{q-1} = g \circ \varphi$, $H'|\mathbb{S}^{q-1} \times I = H \circ (\varphi \times \mathrm{id}_I)$, and $H' : f \circ \widetilde{\varphi} \simeq e \circ \widetilde{g}'$. Thus $\widetilde{g}'$ and g determine $\widetilde{g} : A \cup e^q \longrightarrow Y$, while H' and H determine $\widetilde{H} : A \cup e^q \times I \longrightarrow Z$ with the desired properties.

Third step. Assume that A is any topological space and X is the result of attaching to A some number of q-cells. Specifically, suppose that there exists a map $\varphi : S_q = \coprod \mathbb{S}_i^{q-1} \longrightarrow A$ such that $X = A \cup_\varphi D_q$, where $D_q = \coprod \mathbb{D}_i^q$. Next consider the diagram

$$\begin{tikzcd} S_q \arrow[r, hook] \arrow[d, "\varphi"'] & D_q \arrow[d, "\widetilde{\varphi}"] \\ A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, dashed, "\widetilde{g}"'] \\ Y \arrow[r, "e"'] & Z\,. \end{tikzcd}$$

For each i, the restricted previous diagram is the one considered in the second step, so we have $\widetilde{g}_i' : \mathbb{D}_i^q \longrightarrow Y$ and $H_i' : \mathbb{D}_i^q \times I \longrightarrow Z$, which together are compatible with the attaching maps. Hence they determine $\widetilde{g}$ and $\widetilde{H}$ with the desired properties.

Fourth step. We prove now the general case. Let X_0 be the union of A with isolated points. Then the result is immediate. Thus we have $\widetilde{g}_0 : X_0 \longrightarrow Y$ and $\widetilde{H}_0 : X_0 \times I \longrightarrow Z$ such that $\widetilde{g}_0|A = g$, $\widetilde{H}_0|A \times I = H$, and $\widetilde{H}_0 : f|X_0 \simeq e \circ \widetilde{g}_0$. Assume that the result is already true for X_{q-1}, where we have attached cells to A up to dimension $q-1$; that is, we have $\widetilde{g}_{q-1} : X_{q-1} \longrightarrow Y$ and $\widetilde{H}_{q-1} : X_{q-1} \times I \longrightarrow Z$ such that $\widetilde{g}_{q-1}|A = g$, $\widetilde{H}_{q-1}|A \times I = H$, and $\widetilde{H}_{q-1} : f|X_{q-1} \simeq e \circ \widetilde{g}_{q-1}$. Now apply the third step to

$$\begin{tikzcd} X_{q-1} \arrow[r, hook] \arrow[d, "\widetilde{g}_{q-1}"'] & X_q \arrow[d, "f|X_q"] \arrow[dl, dashed, "\widetilde{g}_q"'] \\ Y \arrow[r, "e"'] & Z \end{tikzcd}$$

to obtain $\widetilde{g}_q : X_q \longrightarrow Y$ and $\widetilde{H}_q : X_0 \times I \longrightarrow Z$ such that $\widetilde{g}_q|X_{q-1} = \widetilde{g}_{q-1}$, $\widetilde{H}_q|X_{q-1} \times I = \widetilde{H}_{q-1}$, and $\widetilde{H}_q : f|X_q \simeq e \circ \widetilde{g}_q$.

By their compatibility, all the constructed maps $\widetilde{g}_q$ and $\widetilde{H}_q$ determine $\widetilde{g} : X \longrightarrow Y$ and $\widetilde{H} : X \times I \longrightarrow Z$ such that $\widetilde{g}|X_q = \widetilde{g}_q$ and $\widetilde{H}|X_q \times I = \widetilde{H}_q$. So $\widetilde{g}$ and $\widetilde{H}$ have the desired properties. □

5.1.27 EXERCISE. Assume in HELP that $Y = Z$ and $e = \mathrm{id}_Z$. Prove that in this case the statement of HELP is equivalent to the fact that the pair (X, E) has the HEP (homotopy extension property), i.e., $A \hookrightarrow X$ is a cofibration.

Assume in HELP, as in the previous exercise, that $Y = Z$ and $e = \mathrm{id}_Z$. Then e is an n-equivalence for all n and HELP implies that $A \hookrightarrow X$ is a cofibration. We thus have the following result.

5.1.28 **Lemma**. *Let A be a topological space and let X be the result of attaching to A successively cells of any dimensions. Then (X, A) has the homotopy extension property.* □

The following is a very important special case of the previous lemma.

5.1.29 **Theorem**. *Suppose that X is a* CW*-complex and that A is a subcomplex. Then (X, A) has the homotopy extension property.* □

From these last two results we obtain an interesting application.

5.1.30 **Corollary**. *Let X be a path-connected* CW*-complex of dimension n. Then we can cover X with $n+1$ open subsets that are contractible in X.*

Proof: We shall construct the open subsets by induction on the dimension of the skeletons. In the first place let us note that any given discrete subset $Y \subset X$ can be contracted in X to a point x_0. Specifically, for each point $x \in Y$ let $\omega_x : I \longrightarrow X$ be a path that starts at x and ends at x_0. Then the deformation $D_Y : Y \times I \longrightarrow X$ defined by $D_Y(x,t) = \omega_x(t)$ deforms Y to x_0 in X.

If X^0 is the 0-skeleton of X, then the pair (X, X^0) has the HEP by 5.1.29, and so there exists an open subset V^0 containing X^0 and a deformation $D^0 : V^0 \times I \longrightarrow X$ such that $D^0(x,0) = x$ and $D^0(x,1) \in X^0$ for $x \in V^0$. Then the homotopy defined by

$$H^0(x,t) = \begin{cases} D^0(x,2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ D_{X^0}(D^0(x,1), 2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

deforms the open subset V^0 to x_0 in X.

Let us assume now that we have already covered the $(k-1)$-skeleton X^{k-1} with open subsets $V^0, V^1, \dots, V^{k-1}$ in X each of which can be deformed to x_0 in X.

Then we have that the difference $X^k - X^{k-1} = \coprod e_i^k = W^k$ is an open set in X^k that can be deformed to the discrete set X_k consisting of the centers of each open cell e_i^k, since each one of these cells can be deformed to its center. Let $F^k : W^k \times I \longrightarrow X$ be such a deformation that starts with the inclusion and ends with a retraction $r' : W^k \longrightarrow X_k$. On the other hand, again using 5.1.29, the pair (X, X^k) has the HEP, so that there exists an open neighborhood V of X^k in X and a deformation $D : V \times I \longrightarrow X$ that starts with the inclusion and ends with a retraction $r : V \longrightarrow X^k$. We then define $V^k = r^{-1}(W^k) \subset V$. Then we have $X^k - X^{k-1} \subset V^k$, so that $\{V^0, V^1, \dots, V^{k-1}, V^k\}$ is a cover of X^k by open subsets of X. We next define $D^k = D|V^k \times I$, which then is a deformation that ends with the retraction $r|V^k : V^k \longrightarrow W^k$. Then $H^k : V^k \times I \longrightarrow X$ defined by

$$H^k(x,t) = \begin{cases} D^k(x,3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ F^k(r(x), 3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ D_{X_k}(r'r(x), 3t-2) & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

deforms the open subset V^k to x_0 in X.

In this way, X can be covered by $n+1$ open sets, namely,

$$V^0, V^1, \dots, V^{n-1}, V^n,$$

each of which is contractible in X. $\square$

5.1.31 Note. We define the *Lusternik–Schnirelmann category* of a topological space X as the smallest number k such that there are $k+1$ open subsets, say $V^0, \dots, V^k$, that are contractible in X and that cover X. So 5.1.30 states that *the Lusternik–Schnirelmann category of a connected* CW*-complex X of dimension n is less than or equal to n.*

The following result, due to J.H.C. Whitehead, is an immediate application of HELP 5.1.26.

5.1.32 **Theorem**. *If X is a* CW*-complex and $e : Y \longrightarrow Z$ is an n-equivalence, then $e_* : [X,Y] \longrightarrow [X,Z]$ is a bijection if* $\dim X < n$ *and a surjection if* $\dim X = n$*. Furthermore, this is also valid for pointed homotopy classes of pointed spaces.*

Proof: If $[f] \in [X,Z]$, take the pair $(X,\emptyset)$ and apply HELP if $\dim X \leq n$ to obtain $\widetilde{g} : X \longrightarrow Y$ such that $e \circ \widetilde{g} \simeq f$, i.e., $e_*[\widetilde{g}] = [f]$. This shows the surjectivity. In the pointed case, one takes instead the pair (X, x_0), where $x_0 \in X$ is the base point, and the constant map $\{x_0\} \longrightarrow Y$.

Now assume that $[g_0],[g_1] \in [X,Y]$ are such that $e_*[g_0] = e_*[g_1]$ and let $f : e \circ g_0 \simeq e \circ g_1$. Now take the pair $(X \times I, X \times \partial I)$ and the map $g : X \times \partial I \longrightarrow Y$ given by $g(x,\nu) = g_\nu(x)$, $\nu = 0,1$. If $\dim X < n$, apply HELP, taking H to be a constant homotopy, to obtain $\widetilde{g} : X \times I \longrightarrow Y$, which is a homotopy from g_0 to g_1. This proves the injectivity of e_*. In the pointed case, one takes instead the pair $(X, X \times \partial I \cup \{x_0\} \times I)$ and the map $g : X \times \partial I \cup \{x_0\} \times I \longrightarrow Y$ given by $g(x,\nu) = g_\nu(x)$, $\nu = 0,1$, and by $g(x_0,t) = y_0$, where $y_0 \in Y$ is the base point. □

5.1.33 **Corollary**. *If X is a* CW*-complex and $e : Y \longrightarrow Z$ is a weak homotopy equivalence, then $e_* : [X,Y] \longrightarrow [X,Z]$ is a bijection.* □

5.1.34 Definition. Given an arbitrary pair (X,A) of topological spaces, a CW-pair $(\widetilde{X},\widetilde{A})$ together with a weak homotopy equivalence of pairs $\varphi : (\widetilde{X},\widetilde{A}) \longrightarrow (X,A)$ is called a CW-*approximation* of (X,A).

5.1.35 **Theorem**. *If $\varphi : (\widetilde{X},\widetilde{A}) \longrightarrow (X,A)$ and $\psi : (\widetilde{Y},\widetilde{B}) \longrightarrow (Y,B)$ are* CW*-approximations and $f : (X,A) \longrightarrow (Y,B)$ is continuous, then there exists a map that is unique up to homotopy, say $\widetilde{f} : (\widetilde{X},\widetilde{A}) \longrightarrow (\widetilde{Y},\widetilde{B})$, such that the diagram*

$$\begin{tikzcd}
(\widetilde{X},\widetilde{A}) \arrow[r, "\varphi"] \arrow[d, "\widetilde{f}"'] & (X,A) \arrow[d, "f"] \\
(\widetilde{Y},\widetilde{B}) \arrow[r, "\psi"'] & (Y,B)
\end{tikzcd}$$

commutes up to homotopy, namely, $f \circ \varphi \simeq \psi \circ \widetilde{f}$ *(by means of a homotopy of pairs).*

Before passing to the proof, we state and prove the absolute case and then we give the proof in the relative case.

5.1.36 **Theorem**. *If* $\varphi : \widetilde{X} \longrightarrow X$ *and* $\psi : \widetilde{Y} \longrightarrow Y$ *are* CW*-approximations and* $f : X \longrightarrow Y$ *is continuous, then there exists a map that is unique up to homotopy, say* $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$*, such that the diagram*

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\varphi"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\
\widetilde{Y} \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

commutes up to homotopy, namely, $f \circ \varphi \simeq \psi \circ \widetilde{f}$.

Proof: Corollary 5.1.33 states that there is a bijection

$$\psi_* : [\widetilde{X}, \widetilde{Y}] \cong [\widetilde{X}, Y]\,.$$

Then there exists a map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$, unique up to homotopy, such that $\psi_*[\widetilde{f}] = [f \circ \varphi]$. That is, $\psi \circ \widetilde{f} \simeq f \circ \varphi$, as desired. □

Proof of 5.1.35. First apply 5.1.36 to see that there exists $\widetilde{f}_A : \widetilde{A} \longrightarrow \widetilde{B}$, unique up to homotopy, such that $H : \psi_B \circ \widetilde{f}_A \simeq f \circ \varphi_A$, where $\varphi_A = \varphi|\widetilde{A} : \widetilde{A} \longrightarrow A$ and $\psi_B = \psi|\widetilde{B} : \widetilde{B} \longrightarrow B$.

We now use HELP to extend $\widetilde{f}_A$ to $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$. Namely, we consider the diagram

$$\begin{tikzcd}
\widetilde{A} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\widetilde{f}_A"'] & \widetilde{X} \arrow[d, "\varphi"] \arrow[ddl, dashed, "\widetilde{f}"'] \\
\widetilde{B} \arrow[d, hook, "\widetilde{j}"'] & X \arrow[d, "f"] \\
\widetilde{Y} \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

together with the homotopy $H : f \circ \varphi \circ \widetilde{i} \simeq \psi \circ \widetilde{j} \circ \widetilde{f}_A$ given above. Then HELP implies the existence of $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $\widetilde{f} \circ \widetilde{i} = \widetilde{j} \circ \widetilde{f}_A$, i.e., $\widetilde{f}$ is a map of pairs $(\widetilde{X}, \widetilde{A}) \longrightarrow (\widetilde{Y}, \widetilde{B})$, and the existence of a homotopy of pairs $\widetilde{H} : f \circ \varphi \simeq \psi \circ \widetilde{f}$, as desired.

The uniqueness up to homotopy is another straightforward application of HELP and is left to the reader as an *exercise.* □

Later on, we shall prove the existence of a CW-approximation. See 6.3.20 and 6.3.21.

It is a consequence of this property that if the pairs $(\widetilde{X}, \widetilde{A})$, φ and $(\widetilde{X'}, \widetilde{A'})$, φ' are CW-approximations of (X, A), then there exists a (weak) homotopy equivalence $h : (\widetilde{X}, \widetilde{A}) \longrightarrow (\widetilde{X'}, \widetilde{A'})$, which is unique up to homotopy and satisfies $\varphi' \circ h \simeq \varphi$. (See 5.1.37 below.)

A well-known theorem of J.H.C. Whitehead is the following.

5.1.37 **Theorem**. *Every n-equivalence $e : Y \longrightarrow Z$ between* CW*-complexes of dimension less than n is a homotopy equivalence. Moreover, a weak homotopy equivalence between* CW*-complexes is a homotopy equivalence.*

Proof: Let $e : Y \longrightarrow Z$ fulfill one of the assumptions. Since in either case, by 5.1.32 or 5.1.33, $e_* : [Z, Y] \longrightarrow [Z, Z]$ is a bijection, there is a map $f : Z \longrightarrow Y$ such that $e \circ f \simeq \mathrm{id}_Z$. Then it follows that $e \circ f \circ e \simeq e$ and, since also $e_* : [Y, Y] \longrightarrow [Y, Z]$ is a bijection, $f \circ e \simeq \mathrm{id}_Y$. Thus e is a homotopy equivalence. □

A corresponding result holds also for CW-pairs; we have the following.

5.1.38 **Theorem**. *A weak homotopy equivalence between pairs of* CW *complexes is a homotopy equivalence. Therefore,* CW*-approximations are unique up to homotopy.*

Proof: If $e : (Y, B) \longrightarrow (Z, C)$ is a weak homotopy equivalence, then the restrictions $e_B : B \longrightarrow C$ and $e_Y : Y \longrightarrow Z$ are weak homotopy equivalences, and by the previous theorem, they are homotopy equivalences with homotopy inverses $f_B : C \longrightarrow B$ and $g_Y : Z \longrightarrow Y$. In principle, $g_Y|C \neq f_B$, but since these maps are homotopic and since the inclusion $C \hookrightarrow Z$ is a cofibration by 5.1.29, one can replace g_Y with a homotopic map $f_Y : Z \longrightarrow Y$ whose restriction to C satisfies $f_Y|C = f_B$. Then $f : (Z, C) \longrightarrow (Y, B)$, where $f|Z = f_Y$ and $f|C = f_B$, is a homotopy inverse of e. □

5.1.39 Exercise. Let X be an n-connected CW-complex for all $n \geq 0$. Prove that X is contractible.

5.1.40 Example. If X is not a CW-complex, then a weak homotopy equivalence need not be a homotopy equivalence. An example is the space defined as follows. Let $A = \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ y = \sin(\pi/x)\}$, $B = \{(0,y) \in \mathbb{R}^2 \mid -\frac{3}{2} \leq y \leq 1\}$ and $C = \{(0,y) \mid y \in [-\frac{3}{2},-1]\} \cup \{(x,-\frac{3}{2}) \in \mathbb{R}^2 \mid x \in [0,1]\} \cup \{(1,y) \mid y \in [-\frac{3}{2},0]\}$. Then the space $X = A \cup B \cup C$ is called the *Polish circle* (see Figure 5.1). So $\pi_n(X) = 0$ for all $n \geq 0$, since a map $\alpha : \mathbb{S}^n \longrightarrow X$ cannot be surjective. Therefore, X is n-connected for all n; that is, the projection $X \longrightarrow *$ is a weak homotopy equivalence. However, X is not contractible (cf. Exercise 5.1.39).

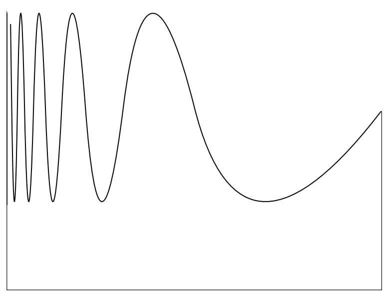

Figure 5.1

5.1.41 Exercise. Prove that the subspace $X = \{0\} \cup \{\frac{1}{n} \mid n{=}1,2,3,\ldots\} \subset \mathbb{R}$ is not a CW-complex. (Hint: If it were one, then the map $\mathbb{N} \cup \{0\} \longrightarrow X$, $n \mapsto \frac{1}{n}$, $0 \mapsto 0$, would be a homotopy equivalence.)

5.1.42 Exercise. Provide the details left out of the previous proof. Namely, prove that there exists $f_Y : Z \longrightarrow Y$ such that $f_Y \simeq g_Y$ and $f_Y|C = f_B$. Moreover, prove that f and e are homotopy inverses as maps of pairs.

5.1.43 Definition. Let (X,A) and (Y,B) be CW-pairs. A map of pairs $g : (X,A) \longrightarrow (Y,B)$ is called *cellular* if $g(X^n \cup A) \subset Y^n \cup B$ for every $n \geq 0$.

The next theorem on cellular approximation plays a very important role in the homotopy theory of CW-complexes.

5.1.44 **Theorem**. *Let* (X,A) *and* (Y,B) *be* CW*-pairs, and let* $f:(X,A)\longrightarrow(Y,B)$ *be a map of pairs. Then there exists a cellular map* $g:(X,A)\longrightarrow(Y,B)$ *such that* $g\simeq f\,\mathrm{rel}\,A$.

Proof: We proceed inductively over the skeletons. We need g homotopic to f such that for every n the following is a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "g"] & Y \\ X^n\cup A \arrow[u, hook] \arrow[r, "g_n"'] & Y^n\cup B, \arrow[u, hook, "i_n"'] \end{tikzcd}$$

where $g_n = g|X^n\cup A$. For $n=0$, just take a path $\gamma_i : f(x_i)\simeq y_i$ for every point $x_i\in X^0-A$, where y_i is any point in Y^0. Then define $H_0:(X^0\cup A)\times I\longrightarrow Y$ by $H_0(a,t)=f(a)$ for $a\in A$ and $H_0(x_i,t)=\gamma_i(t)$ for all $x_i\in X^0-A$. This is a homotopy from $f|A\cup X^0$ to $g_0:X^0\cup A\longrightarrow Y^0\cup B$ relative to A.

Assume inductively that we have g_n as in the diagram above, and that $H_n:(X^n\cup A)\times I\longrightarrow Y$ is such that $H_n: f|X^n\cup A\simeq i_n\circ g_n$, where $i_n:Y^n\cup B\hookrightarrow Y$ is the inclusion. For each attaching map $\varphi:\mathbb{S}^n\longrightarrow X^n$ of a cell $\widetilde{\varphi}:\mathbb{D}^{n+1}\longrightarrow X$, one applies HELP to

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, hook] \arrow[d, "g_n\circ\varphi"'] & \mathbb{D}^{n+1} \arrow[d, "f\circ\widetilde{\varphi}"] \arrow[dl, dashed, "g'_{n+1}"'] \\ Y^{n+1}\cup B \arrow[r, hook, "i_{n+1}"'] & Y \end{tikzcd}$$

and the homotopy $H_n\circ(\varphi\times\mathrm{id}_I)$ to obtain $g'_{n+1}:\mathbb{D}^{n+1}\longrightarrow Y^{n+1}\cup B$ and a homotopy $H'_{n+1}: f\circ\widetilde{\varphi}\simeq i_{n+1}\circ g'_{n+1}$.

All g'_{n+1} for the $(n+1)$-cells and $f|X^n\cup A$ glue together to produce $g_{n+1}:X^{n+1}\cup A\longrightarrow Y^{n+1}\cup B$ extending g_n, and the homotopies H'_n glue together to produce a homotopy $H_{n+1}: i_{n+1}\circ f|X^{n+1}\cup A\simeq g_{n+1}\,\mathrm{rel}\,A$.

Since X has the weak topology determined by its skeletons, we have that the maps g_n determine a cellular map $g:X\longrightarrow Y$ and that the homotopies H_n determine a homotopy $H:X\times I\longrightarrow Y$ such that $H:f\simeq g\,\mathrm{rel}\,A$. □

We obtain the next result as a consequence of 5.1.25.

5.1.45 **Corollary**. *Suppose that* X *is a* CW*-complex with exactly one* 0*-cell and with the rest of the cells all having dimension bigger than* n*. Then* X *is* n*-connected.*

Proof: By hypothesis we have $X^n = *$. Applying 5.1.25, we obtain that $i_* : \pi_r(X^n) \longrightarrow \pi_r(X)$ is an epimorphism for $r \leq n$, and consequently $\pi_r(X) = 0$ for $r \leq n$. $\square$

Suppose that X and Y are CW-complexes whose characteristic maps are $\{\varphi_i^n : \mathbb{D}^n \longrightarrow X \mid i \in I_n,\ n \geq 0\}$ and $\{\psi_j^m : \mathbb{D}^m \longrightarrow Y \mid j \in J_m,\ m \geq 0\}$, respectively. Next let us consider the product $X \times Y$ together with its characteristic maps $\{\varphi_i^n \times \psi_j^m : \mathbb{D}^n \times \mathbb{D}^m \approx \mathbb{D}^{m+n} \longrightarrow X \times Y \mid (i,j) \in I_n \times J_m,\ n \geq 0,\ m \geq 0\}$. In order for this to define a CW-complex structure on $X \times Y$ we have to impose some sort of restriction on $X \times Y$. One possibility is given in the next result, due to Milnor [45, II.5].

5.1.46 **Proposition**. *Let X and Y be* CW-*complexes. If*

(a) *either X or Y is locally compact, or if*

(b) *both X and Y have countably many cells,*

then $X \times Y$ is a CW-*complex.* $\square$

5.1.47 Note. Another way to realize $X \times Y$ as a CW-complex is to change its topology to the compactly generated topology of $k(X \times Y)$. See 4.3.20.

Suppose that X is a CW-complex whose characteristic maps are $\{\varphi_i^n : \mathbb{S}^{n-1} \longrightarrow X \mid i \in I_n,\ n \geq 0\}$ and that $A \subset X$ is a subcomplex whose cells are labeled by a subfamily $H_n \subset I_n$ for each $n \geq 0$. We define a family by $K_n = I_n - H_n$ for $n > 0$ and by $K_0 = (I_0 - H_0) \cup \{i_0\}$, where $i_0 \in H_0$. Let $p : X \longrightarrow X/A$ denote the quotient map, and consider the family of maps $\{p \circ \varphi_i^n : \mathbb{S}^{n-1} \longrightarrow X/A \mid i \in K_n,\ n \geq 0\}$.

5.1.48 Exercise. Prove that the family $\{p \circ \varphi_i^n : \mathbb{S}^{n-1} \longrightarrow X/A \mid i \in K_n,\ n \geq 0\}$, as just defined, determines a CW-complex structure on the quotient space X/A.

Let us now consider the following definition, which in some sense is dual to 2.9.1.

5.1.49 Definition. Let X and Y be pointed spaces with base points x_0 and y_0, respectively. We define their *smash product* $X \wedge Y$ to be the quotient

$$X \times Y / X \times \{y_0\} \cup \{x_0\} \times Y.$$

5.1.50 Exercise. Prove that the reduced suspension ΣX of a pointed space X (as defined in 2.10.1) is exactly the smash product $\mathbb{S}^1 \wedge X$ (at least when X is a CW-complex). Using this and the fact that the latter product is associative, show that $\mathbb{S}^n = \mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1$, where we take n copies of $\mathbb{S}^1$. Then conclude that the reduced n-suspension of X satisfies $\Sigma^n X = \mathbb{S}^n \wedge X$. (Just how general can we make this statement?)

5.1.51 **Proposition**. *Let X be a* CW*-complex with skeleton $X^{r-1} = \{*\}$, and let Y be a* CW*-complex with skeleton $Y^{s-1} = \{*\}$. Moreover, suppose that both of them have countably many cells and that their common base point is $*$. Then their smash product $X \wedge Y$ is an $(r+s-1)$-connected* CW*-complex.*

Proof: Using Proposition 5.1.46 we have that the product $X \times Y$ is a CW-complex with cells of the form $\{*\} \times e_j^n$, $e_i^m \times \{*\}$ or $e_i^m \times e_j^n$ for $m \geq r$ and $n \geq s$. The cells of the first two types form the subcomplex $X \vee Y$ of $X \times Y$. Then using Exercise 5.1.48 we get that $X \wedge Y = X \times Y / X \vee Y$ is a CW-complex with exactly one 0-cell and with the rest of its cells having dimension larger than $r + s - 1$. Then Corollary 5.1.45 implies that $\pi_q(X \wedge Y) = 0$ for $q \leq r + s - 1$. □

5.1.52 **Corollary**. *Let X be a pointed* CW*-complex. Then its n-suspension $\Sigma^n X$ is a* CW*-complex that is at least $(n-1)$-connected.*

Proof: This is an immediate consequence of 5.1.51 and Exercise 5.1.50. □

5.2 Infinite Symmetric Products

Up to now, we have met two instances of Eilenberg–Mac Lane spaces, both of type $(G, 1)$. Infinite symmetric products, which we are about to define, allow us to generalize the definition of the Eilenberg–Mac Lane spaces of type (G, n) for any abelian group G and any n, starting from certain spaces that are called Moore spaces.

Given a topological space X, however complicated from the homotopical point of view, its infinite symmetric product $\mathrm{SP}\, X$ is a homotopically simpler space still reflecting many topological properties of X. More precisely, these infinite symmetric products have the property of being topological abelian monoids. Since topological abelian monoids are characterized by their homotopy groups, as we shall see, then it is natural to consider these homotopy groups $\pi_n(\mathrm{SP}\, X)$.

We shall assume throughout this chapter that all the spaces considered are pointed spaces and that all the maps between them preserve the base points.

5.2.1 Definition. Let X be a pointed topological space, and let $\overline{X}^n = X \times \cdots \times X$ be its nth Cartesian product for $n \geq 1$. If Σ_n denotes the symmetric (or permutation) group of the set $\{1, \ldots, n\}$, then there is a right action of this group on $\overline{X}^n$, which permutes the coordinates, that is, for $\sigma \in \Sigma_n$ we define

$$(x_1, \ldots, x_n)\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \qquad x_i \in X\,.$$

The orbit space of this action

$$\mathrm{SP}^n X = \overline{X}^n / \Sigma_n$$

(i.e., we are identifying $\overline{x} \in \overline{X}^n$ with $\overline{x}\sigma \in \overline{X}^n$ for every $\sigma \in \Sigma_n$) provided with the quotient topology is called the *nth symmetric product* of X. The equivalence class of $(x_1, \ldots, x_n)$ will be denoted by $[x_1, \ldots, x_n]$. Using the base point $x_0 \in X$ we define inclusions

$$\mathrm{SP}^n X \longrightarrow \mathrm{SP}^{n+1} X$$

by

$$[x_1, \ldots, x_n] \mapsto [x_0, x_1, \ldots, x_n]$$

for $n \geq 1$. Then we can form the union

$$\mathrm{SP}\, X = \bigcup_n \mathrm{SP}^n X$$

equipped with the union topology; namely, $B \subset \mathrm{SP}\, X$ is closed if and only if $B \cap \mathrm{SP}^n X$ is closed for each $n \geq 1$. We call $\mathrm{SP}\, X$ the *infinite symmetric product* of X.

In this way the elements of $\mathrm{SP}\, X$ can also be considered as unordered n-tuples $[x_1, \ldots, x_n]$, where n is any positive integer. Then $\mathrm{SP}\, X$ turns out to be a pointed space with base point $0 = [x_0]$. Moreover, we have a natural inclusion $i : X \hookrightarrow \mathrm{SP}\, X$ since $X = \mathrm{SP}^1 X$.

5.2.2 Note. Let X be a CW-complex with countably many cells. One can give a natural cell structure to X^n such that each $\sigma \in \Sigma_n$ is either the identity on a cell or a homeomorphism of the cell onto some other (different) cell. In this way the quotient space $\mathrm{SP}^n X = X \times \cdots \times X/\Sigma_n$ has also a CW-complex structure such that $\mathrm{SP}^{n-1} X$ is a subcomplex, and since

$$\mathrm{SP}\, X = \operatorname*{colim}_n \mathrm{SP}^n X$$

has the colimit topology with respect to $\mathrm{SP}^n X$, for $n = 1, 2, \ldots$, then $\mathrm{SP}\, X$ is a CW-complex. If, more generally, X is an arbitrary CW-complex, then one should take the compactly generated topology in each product instead (see [80]).

5.2.3 EXERCISE. Let Λ be a partially ordered set of indices and let X_λ, $\lambda \in \Lambda$, be pointed spaces such that if $\lambda \leq \mu$, then $X_\lambda \subset X_\mu$ is a closed subset. Prove that if $X = \bigcup_\lambda X_\lambda$ has the union topology, then, for each n, $\bigcup_\lambda \mathrm{SP}^{\,n} X_\lambda = \mathrm{SP}^{\,n} X$.

5.2.4 EXAMPLE. Let us consider the 2-dimensional sphere $\mathbb{S}^2$ as the Riemann sphere consisting of the complex numbers together with the point at infinity, denoted by ∞. A point in $\mathrm{SP}^n(\mathbb{S}^2)$ is an unordered n-tuple $\alpha_1, \alpha_2, \ldots, \alpha_n$ of complex numbers or ∞. There exists a nonzero polynomial, unique up to a nonzero complex factor, of degree less than or equal to n whose roots are precisely $\alpha_1, \alpha_2, \ldots, \alpha_n$, where we consider ∞ to be a root of the polynomial if its degree is less than n. Considering the coefficients of this polynomial as homogeneous coordinates on the complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1} - 0/\!\sim$ (using the identification $x \sim \lambda x$ for nonzero $\lambda \in \mathbb{C}$) we get a homeomorphism $\mathrm{SP}^n\,(\mathbb{S}^2) \approx \mathbb{CP}^n$.

5.2.5 NOTE. We now give another way of understanding the infinite symmetric product $\mathrm{SP}\, X$ of a pointed space X. First define $\bigoplus X = \{(x_1, x_2, \ldots) \mid x_i \in X,\ x_i = * \text{ for all but finitely many indices } i \in \mathbb{N}\}$, considered as a set. We give $\bigoplus X$ the colimit topology induced by the subspaces $\overline{X}^n = \{(x_1, \ldots, x_n, *, *, \ldots)\}$, which themselves have the product topology. Now let Σ_∞ be the group of those permutations of the natural numbers $\mathbb{N}$ that leave pointwise fixed all but a finite number of the natural numbers. Then Σ_∞ acts on $\bigoplus X$ by defining $(x_1, x_2, x_3, \ldots)\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots)$ for $\sigma \in \Sigma_\infty$. Finally, we form the orbit space of this action, and we get $\bigoplus X/\Sigma_\infty = \mathrm{SP}\, X$.

5.2.6 EXERCISE. Prove that the alternative definition of $\mathrm{SP}\, X$, given in the previous note, in fact agrees with Definition 5.2.1.

If $f : X \longrightarrow Y$ is a (pointed) map, then it induces maps $f^n : \overline{X}^n \longrightarrow \overline{Y}^n$, which are compatible with the action of Σ_n. Actually, these maps in turn induce maps $f^{(n)} : \mathrm{SP}^n X \longrightarrow \mathrm{SP}^n Y$, which give us a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{SP}^n X \arrow[r] \arrow[d, "f^{(n)}"'] & \mathrm{SP}^{n+1} X \arrow[r] \arrow[d, "f^{(n+1)}"] & \cdots \\
\cdots \arrow[r] & \mathrm{SP}^n Y \arrow[r] & \mathrm{SP}^{n+1} Y \arrow[r] & \cdots
\end{tikzcd}$$

and therefore induce a map

$$\widehat{f} : \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, Y.$$

5.2.7 **Proposition**. *The construction* SP *has the following functorial properties:*

(a) $f = \mathrm{id}_X \Rightarrow \widehat{f} = \mathrm{id}_{\mathrm{SP}\, X}$.

(b) $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z \Rightarrow \widehat{(g \circ f)} = \widehat{g} \circ \widehat{f} : \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, Z$. □

5.2.8 **Proposition**. *Let A be a closed (respectively, open) subset of X that contains the base point, and let $i : A \hookrightarrow X$ denote the inclusion map. Then $i^{(n)} : \mathrm{SP}^n A \longrightarrow \mathrm{SP}^n X$ and $\widehat{i} : \mathrm{SP}\, A \longrightarrow \mathrm{SP}\, X$ also are inclusions.*

Proof: Let us consider the diagram

$$\begin{tikzcd}
\overline{A}^n \arrow[r, "i^n"] \arrow[d, "\rho"'] & \overline{X}^n \arrow[d, "\rho'"] \\
\mathrm{SP}^n A \arrow[r, "i^{(n)}"'] & \mathrm{SP}^n X,
\end{tikzcd}$$

where ρ and ρ' are the relevant identification maps. The maps i^n, ρ, and ρ' are closed (respectively, open), which means that they send closed subsets to closed subsets (respectively, open subsets to open subsets), and therefore, $i^{(n)}$ is also a closed (respectively, open) map. Thus $i^{(n)}$ is an inclusion, and its image $i^{(n)}(\mathrm{SP}^n A) \subset \mathrm{SP}^n X$ is closed (respectively, open) in $\mathrm{SP}^n X$.

Because $\mathrm{SP}\, X = \bigcup \mathrm{SP}^n X$ has the union topology, it follows that

$$i^{(n)}(\mathrm{SP}^n A) = \widehat{i}(\mathrm{SP}\, A) \cap \mathrm{SP}^n X$$

is closed (respectively, open) in $\mathrm{SP}^n X$. Thus, $\widehat{i}(\mathrm{SP}\, A)$ is closed (respectively, open) in $\mathrm{SP}\, X$ so that $\widehat{i} : \mathrm{SP}\, A \longrightarrow \mathrm{SP}\, X$ is an inclusion. □

If we are now given a (pointed) homotopy $F : X \times I \longrightarrow Y$, then we obtain homotopies

$$F^{(n)} : (\mathrm{SP}^n X) \times I \longrightarrow \mathrm{SP}^n Y$$

that are compatible with the inclusions. Consequently, we get a homotopy

$$\widehat{F} : (\mathrm{SP}\, X) \times I \longrightarrow \mathrm{SP}\, Y.$$

So we have proved the following result.

5.2.9 **Proposition**. *Suppose that X and Y are pointed spaces and that $f, g : X \longrightarrow Y$ are pointed maps. If $f \simeq g$, then $f^{(n)} \simeq g^{(n)}$ and $\widehat{f} \simeq \widehat{g}$.* □

We deduce the next property from 5.2.7 and 5.2.9.

5.2.10 **Corollary**. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then $\widehat{f} : \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, Y$ also is a homotopy equivalence.* □

5.2.11 EXAMPLE. The Riemann sphere without its poles (namely, $\mathbb{S}^2 - \{0, \infty\}$, where $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$), that is, the punctured plane $\mathbb{C} - 0$) has the same homotopy type of the circle $\mathbb{S}^1$. Specifically, the inclusion $\mathbb{S}^1 \subset \mathbb{S}^2 - \{0, \infty\} = \mathbb{C} - 0$ is a homotopy equivalence with inverse $\mathbb{C} - 0 \longrightarrow \mathbb{S}^1$ given by $z \mapsto z/|z|$ (see Figure 5.2).

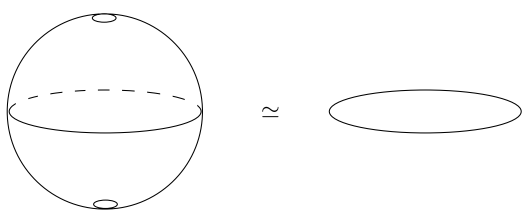

Figure 5.2

So from the point of view of homotopy theory, analyzing $\mathrm{SP}\, \mathbb{S}^1$ is equivalent to analyzing $\mathrm{SP}\, (\mathbb{S}^2 - \{0, \infty\})$. (See 5.2.23.)

Recall that a topological space X is *contractible* if there exists a homotopy equivalence between it and a one-point space or, equivalently, if there exists a homotopy $F : X \times I \longrightarrow X$ that starts with the identity and ends with the constant map $c(x) = x_0$, namely, if id_X is nullhomotopic. Such a homotopy F is called a *contraction*.

5.2.12 **Corollary**. *If X is contractible, then so also are $\mathrm{SP}^n X$ and $\mathrm{SP}\, X$.* □

5.2.13 Example. A typical example of a contractible space is the unit interval I. Specifically, we have a contraction given by

$$\begin{aligned} F : I \times I &\longrightarrow I\,, \\ F(s,t) &= 1-(1-s)(1-t)\,. \end{aligned}$$

More generally, the hypercube I^n is contractible with contraction given by

$$\begin{aligned} F : I^n \times I &\longrightarrow I^n\,, \\ F((s_1,\ldots,s_n),t) &= (1-(1-s_1)(1-t),\ldots,1-(1-s_n)(1-t)). \end{aligned}$$

Consequently, any space homeomorphic to I^n, such as the disk $\mathbb{D}^n$, for example, is contractible as well.

5.2.14 Example. Another typical example of a contractible space is the cone CX over any space X. In this case

$$\begin{aligned} F : CX \times I &\longrightarrow CX\,, \\ F(\overline{(x,s)},t) &= \overline{(x,1-(1-s)(1-t))}\,, \end{aligned}$$

defines a contraction.

5.2.15 Definition. We say that a neighborhood U of a subspace A of X can be *deformed to A in X*, or is *deformable to A in X*, if there exists a homotopy

$$D : X \times I \longrightarrow X$$

such that for all $x \in X$ we have

$$D(x,0) = x,$$

$$D(A \times I) \subset A,\ D(U \times I) \subset U,$$

$$D(U \times \{1\}) \subset A.$$

5.2.16 Example. Consider $A \subset X$. Let $X' = X \cup A \times I$ be the mapping cylinder of the inclusion map, and let $A' \subset X'$ be the image in X' of $A \times \{1\}$. Then A' has a neighborhood that is deformable to A' in X'. Specifically, we define U to be the image in X' of $A \times (\frac{1}{2},1]$, and we define $D : X' \times I \longrightarrow X'$ by $D(x,s) = x$ for $x \in X$ and by

$$D(a,t,s) = \begin{cases} (a,t(1+s)) & \text{if } t \leq \frac{1}{2}, \\ (a,t(1-s)+s) & \text{if } t \geq \frac{1}{2}, \end{cases}$$

for $(a,t) \in A \times I$. It is straightforward to verify that the homotopy D satisfies all the conditions of the previous definition.

The following is the key result of the paper by Dold and Thom. However, its proof is rather long, so we delay that until Appendix A.

5.2.17 **Theorem**. (Dold–Thom) *Suppose that X is a Hausdorff space and that A is a closed path-connected subspace that has a neighborhood deformable to A in X. Then the quotient map $p : X \longrightarrow X/A$ induces a quasifibration $\widehat{p} : \mathrm{SP}\,X \longrightarrow \mathrm{SP}\,(X/A)$ such that for every $\overline{x} \in \mathrm{SP}\,(X/A)$, we have $\widehat{p}^{-1}(\overline{x}) \simeq$* $\mathrm{SP}\,A$ (where $\simeq$ denotes homotopy equivalence). □

5.2.18 **Corollary**. *Suppose that X and Y are Hausdorff spaces with Y path connected and take $f : X \longrightarrow Y$. Consider the sequence of maps*

$$X \xrightarrow{f} Y \longrightarrow C_f \xrightarrow{\rho} \Sigma X.$$

Then

$$\widehat{\rho} : \mathrm{SP}\,(C_f) \longrightarrow \mathrm{SP}\,(\Sigma X)$$

is a quasifibration with fiber $\widehat{\rho}^{-1}(\bar{x}) \simeq \mathrm{SP}\,Y$.

Proof: The quotient map of C_f that identifies Y to a point, namely $\rho : C_f \longrightarrow \Sigma X$, satisfies the hypotheses of the Dold–Thom theorem; therefore, the result follows. □

So in particular, from the sequence

$$X \xrightarrow{\mathrm{id}} X \hookrightarrow CX \longrightarrow \Sigma X$$

we get the quasifibration

$$\mathrm{SP}\,(CX) \longrightarrow \mathrm{SP}\,(\Sigma X)$$

with fiber $\mathrm{SP}\,X$, and thereby the next result.

5.2.19 **Corollary**. *If X is Hausdorff and path connected, then for every $q \geq 0$ we have an isomorphism*

$$\pi_{q+1}(\mathrm{SP}\,(\Sigma X)) \cong \pi_q(\mathrm{SP}\,X).$$

Proof: First, we start with the quasifibration $\mathrm{SP}\,(CX) \longrightarrow \mathrm{SP}\,(\Sigma X)$ with fiber $\mathrm{SP}\,X$, and we apply the long exact sequence (see 4.3.41) to get

$$\cdots \longrightarrow \pi_{q+1}(\mathrm{SP}\,(CX)) \longrightarrow \pi_{q+1}(\mathrm{SP}\,(\Sigma X)) \longrightarrow$$
$$\longrightarrow \pi_q(\mathrm{SP}\,X) \longrightarrow \pi_q(\mathrm{SP}\,(CX)) \longrightarrow \cdots.$$

Then because CX is contractible, we know that $\mathrm{SP}\,(CX)$ is also contractible by applying 5.2.12. It follows that $\pi_q(\mathrm{SP}\,(CX)) = 0$ for $q \geq 0$. So we get the desired isomorphism from the previous exact sequence. □

5.2.20 Exercise. Prove that the inverse of the isomorphism given in the proof above is provided by

$$[f:\mathbb{S}^q \longrightarrow \mathrm{SP}\,X] \mapsto [\,\mathbb{S}^{q+1} \xrightarrow{\Sigma f} \Sigma\mathrm{SP}\,X \approx \mathrm{SP}\,\Sigma X\,]\,.$$

Let $X' = X \cup (A \times I)$ and $A' = A \times \{1\}$, as in 5.2.16. By the Dold–Thom theorem, the quotient map $p' : X' \longrightarrow X'/A'$ induces a quasifibration

$$\widehat{p'} : \mathrm{SP}\,X' \longrightarrow \mathrm{SP}\,(X'/A')$$

with fiber $\widehat{p'}^{-1}(\overline{x}) \simeq \mathrm{SP}\,A'$. Therefore, using Example 5.2.16 we have the next assertion.

5.2.21 **Proposition**. *Let X be a Hausdorff space and $A \subset X$ a path-connected subspace. Then the canonical map*

$$\mathrm{SP}\,(X \cup (A \times I)) \longrightarrow \mathrm{SP}\,(X \cup CA)$$

is a quasifibration with fiber $\mathrm{SP}\,A$. □

Suppose that X is a Hausdorff space with a subspace $A \subset X$ such that the inclusion is a cofibration. Then using 4.2.3 and the remarks that follow 4.2.7 we have that $X \cup A \times I$ has the same homotopy type of X and that $X \cup CA$ has the same homotopy type of X/A. Moreover, under these homotopy equivalences the quotient maps $X \longrightarrow X/A$ and $X \cup A \times I \longrightarrow X \cup CA$ correspond to each other, at least up to a homotopy equivalence. By applying 5.2.21, we obtain in this way the following version of the Dold–Thom theorem 5.2.17.

5.2.22 **Theorem**. *Suppose that X is a Hausdorff space with a path-connected subspace A such that the inclusion is a cofibration. Then the quotient map $p : X \longrightarrow X/A$ induces a quasifibration $\widehat{p} : \mathrm{SP}\,X \longrightarrow \mathrm{SP}\,(X/A)$ with fiber $\widehat{p}^{-1}(\overline{x}) \simeq \mathrm{SP}\,A$ for every $\overline{x} \in \mathrm{SP}\,(X/A)$.* □

This version of the Dold–Thom theorem is the most useful in applications, since usually either the hypothesis that $A \hookrightarrow X$ is a cofibration is easy to verify or it is well known that it holds in the given case.

We finish this section with two crucial results that will be useful for several applications.

5.2.23 **Proposition**. (Dold–Thom) *The natural inclusion $\mathbb{S}^1 \hookrightarrow \mathrm{SP}\,\mathbb{S}^1$ of the circle in its infinite symmetric product is a homotopy equivalence. Therefore,*

$$\pi_q(\mathrm{SP}\,\mathbb{S}^1) \cong \pi_q(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

Proof: As a representantive of the homotopy type of $\mathbb{S}^1$, let us take the Riemann sphere punctured in its poles, namely $\mathbb{S}^2 - \{0, \infty\}$ (see 5.2.11). According to 5.2.4, $\mathrm{SP}\,\mathbb{S}^2$ is nothing other than the space of nonzero polynomials $\sum_{i=0}^{n} a_i z^i$ of degree no greater than n. Then $\mathrm{SP}^n\,\mathbb{S}^1$ consists exactly of those polynomials that have neither 0 nor ∞ as a root, and this means those for which $a_0 \neq 0$ and $a_n \neq 0$. In other words, $\mathrm{SP}^n\,\mathbb{S}^1$ is obtained from the complex projective space $\mathbb{CP}^n \approx \mathrm{SP}^n\,\mathbb{S}^2$ by removing the hyperplanes $a_0 = 0$ and $a_n = 0$. When we restrict the quotient map $\mathbb{C}^n - 0 \longrightarrow \mathbb{CP}^{n-1}$ to the sphere $\mathbb{S}^{2n-1}$ we get another quotient map $\varphi : \mathbb{S}^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$. The reader can check that if we add the disk $\mathbb{D}^{2n}$ to $\mathbb{CP}^{n-1}$ using φ, then we get $\mathbb{CP}^n$ (see Exercise 5.2.25 below); namely, we have $\mathbb{D}^{2n} \cup_\varphi \mathbb{CP}^{n-1} = \mathbb{D}^{2n} \sqcup \mathbb{CP}^{n-1}/\!\sim\, = \mathbb{CP}^n$, where $\mathbb{D}^{2n} \supset \mathbb{S}^{2n-1} \ni x \sim \varphi(x) \in \mathbb{CP}^{n-1}$.

It follows from this that removing the hyperplanes $a_0 = 0$ and $a_n = 0$ from $\mathbb{CP}^n$ corresponds to removing two copies of $\mathbb{CP}^{n-1}$ that are embedded as $0 \times \mathbb{CP}^{n-1}$ and $\mathbb{CP}^{n-1} \times 0$. Removing the first of these leaves an open disk $\overset{\circ}{\mathbb{D}}{}^{2n}$, and then removing the second amounts to removing $(\mathbb{CP}^{n-1} \times 0) - (0 \times \mathbb{CP}^{n-2} \times 0) = (\mathbb{CP}^{n-1} - 0 \times \mathbb{CP}^{n-2}) \times 0 = \overset{\circ}{\mathbb{D}}{}^{2n-2} \times 0$. Then what remains is $\overset{\circ}{\mathbb{D}}{}^{2n-2} \times (\overset{\circ}{\mathbb{D}}{}^{2} - 0)$, which clearly has the same homotopy type of $\mathbb{S}^1$. Therefore, $\mathrm{SP}^n\,\mathbb{S}^1$ has the same homotopy type of the circle, and the injection $\mathbb{S}^1 \subset \mathrm{SP}^n\,\mathbb{S}^1$ is a homotopy equivalence.

Finally, by 4.5.13,

$$\pi_q(\mathrm{SP}\,\mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1, \end{cases}$$

and this proves the proposition. $\square$

Since as we have seen, the only nonvanishing homotopy group of $\mathrm{SP}\,\mathbb{S}^1$ is the fundamental group, which is isomorphic to $\mathbb{Z}$, and since $\mathbb{S}^2 = \Sigma\mathbb{S}^1$, we can use 5.2.19 and 5.2.23 to get

$$\pi_q(\mathrm{SP}\,\mathbb{S}^2) \cong \pi_{q-1}(\mathrm{SP}\,\mathbb{S}^1) \cong \pi_{q-1}(\mathbb{S}^1)\ ,$$

and so

$$\pi_q(\mathrm{SP}\,(\mathbb{S}^2)) = \begin{cases} \mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \neq 2. \end{cases}$$

By using $\Sigma\mathbb{S}^{n-1} = \mathbb{S}^n$ and by then applying 5.2.19 again, we inductively get the next assertion.

5.2.24 **Proposition**. *For each integer $n \geq 1$*

$$\pi_q(\mathrm{SP}\,\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

□

5.2.25 Exercise. Let X be a Hausdorff space and let $A \subset X$ be closed. Assume that there is a map $\varphi : \mathbb{D}^n \longrightarrow X$ such that $\varphi(\mathbb{S}^{n-1}) \subset A$ and such that $\varphi|\mathbb{D}^n - \mathbb{S}^{n-1} : \mathbb{D}^n - \mathbb{S}^{n-1} \approx X - A$. Prove that $X \approx A \cup_{\varphi|\mathbb{S}^{n-1}} \mathbb{D}^n$.

From the proof of 5.2.23 and the previous exercise we obtain the following result (cf. 5.1.3(a)).

5.2.26 **Corollary**. *For all $n \geq 1$ the complex projective space $\mathbb{CP}^n$ has the structure of a* CW*-complex with one $2k$-cell for each k, $0 \leq k \leq n$. Its $2k$-skeleton is $\mathbb{CP}^k$, and the attaching map of the $(2k+2)$-cell is the canonical quotient map $\varphi_{k+1} : \mathbb{S}^{2k+1} \longrightarrow \mathbb{CP}^k$.* □

5.3 Homology Groups

The infinite symmetric product $\mathrm{SP}\,X$ introduced in the previous section is determined by its homotopy groups, as we shall see later on in Section 6.1 (see 6.4.17). In this section we shall study these homotopy groups $\pi_n(\mathrm{SP}\,X)$, which will turn out to be the ordinary homology groups with integral coefficients of the given space X.

We shall first define the reduced groups, and from them shall define the relative groups.

5.3.1 Definition. Let X be a path-connected CW-complex with base point x_0. We define its *nth reduced homology group (with coefficients in $\mathbb{Z}$)* for $n \geq 0$ as

$$\widetilde{H}_n(X) = \pi_n(\mathrm{SP}\,X)\,,$$

where the homotopy group is defined with respect to the base point in $\mathrm{SP}\,X$ determined by x_0. For $n < 0$ we define $\widetilde{H}_n(X) = 0$.

In general, the functor π_0 does not give us a group. Nonetheless, according to 5.2.19, we immediately have the following statement.

5.3.2 **Proposition**. *If X is a pointed path-connected* CW*-complex, then*

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X),$$

for all n, where ΣX denotes the reduced suspension of X. □

On the one hand, this allows us to extend the definition of reduced homology groups to spaces that are not necessarily path connected. Specifically, since ΣX is always path connected, we define

$$\widetilde{H}_n(X) = \widetilde{H}_{n+1}(\Sigma X),$$

for every pointed CW-complex X and $n \geq 0$. On the other hand, it allows us to assert that $\widetilde{H}_0(X) \cong \widetilde{H}_1(\Sigma X) \cong \widetilde{H}_2(\Sigma^2 X) = \pi_2(\mathrm{SP}\,\Sigma^2 X)$ is not only a group but that it is abelian, as are all the other groups $\widetilde{H}_n(X)$.

If $f : X \longrightarrow Y$ is a pointed map of pointed CW-complexes, then the map $\widehat{f} : \mathrm{SP}\,X \longrightarrow \mathrm{SP}\,Y$ induces a homomorphism

$$f_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y).$$

These groups and homomorphisms have the following properties.

5.3.3 **Functoriality**. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are maps of pointed* CW*-complexes, then*

$$(g \circ f)_* = g_* \circ f_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Z).$$

Moreover, if $\mathrm{id}_X : X \longrightarrow X$ is the identity, then

$$\mathrm{id}_{X*} = 1_{\widetilde{H}_n(X)} : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X).$$

5.3.4 **Homotopy**. *If $f \simeq g : X \longrightarrow Y$ (a pointed homotopy), then*

$$f_* = g_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y).$$

5.3.5 **Exactness**. *For every pointed map $f : X \longrightarrow Y$ we have an exact sequence*

$$\widetilde{H}_q(X) \xrightarrow{f_*} \widetilde{H}_q(Y) \xrightarrow{i_*} \widetilde{H}_q(C_f)\,,$$

where C_f denotes the mapping cone of the map f and $i : Y \hookrightarrow C_f$ is the canonical inclusion.

5.3.6 **Dimension**. *For the* 0*-sphere* $\mathbb{S}^0$ *we have*

$$\widetilde{H}_n(\mathbb{S}^0) = \begin{cases} \mathbb{Z} & \textit{if } n = 0, \\ 0 & \textit{if } n \neq 0. \end{cases}$$

Proof: The functoriality property is an immediate consequence of the functoriality of the symmetric product construction (see Proposition 5.2.7) and the functoriality of homotopy groups (see Theorem 3.5.8). The homotopy property is an immediate consequence of Proposition 5.2.9. To prove the exactness property, we use the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, hook, "\simeq"'] \arrow[dr, "i"] & \\ & Z_f \arrow[r] & C_f, \end{tikzcd}$$

which is commutative up to homotopy. Here Z_f is the *reduced mapping cylinder* of f, which we define to be the result of identifying the line segment $\{x_0\} \times I$ in the mapping cylinder M_f (cf. (3.1.2)) of f to a single point. Moreover, the map $Z_f \longrightarrow C_f$ is the canonical identification of the cylinder to the cone, namely the map that identifies $X \times \{1\}$ to a single point. The canonical inclusion $Y \hookrightarrow Z_f$ is obviously a homotopy equivalence. (See Exercise 5.3.11 below.) By the Dold–Thom theorem 5.2.17, the induced map $\mathrm{SP}\,(Z_f) \longrightarrow \mathrm{SP}\,(C_f)$ is a quasifibration with fiber $\mathrm{SP}\,X$. By using Proposition 4.3.40, we therefore have an exact homotopy sequence

$$\pi_q(\mathrm{SP}\,X) \longrightarrow \pi_q(\mathrm{SP}\,(Z_f)) \longrightarrow \pi_q(\mathrm{SP}\,(C_f))\,,$$

which, up to the homotopy equivalence mentioned above, is equivalent to the exact sequence

$$\pi_q(\mathrm{SP}\,X) \xrightarrow{\widehat{f}_*} \pi_q(\mathrm{SP}\,Y) \xrightarrow{i_*} \pi_q(\mathrm{SP}\,(C_f))\,.$$

Then by using the definition of reduced homology group, we get the desired exact sequence.

Finally, the dimension property is an immediate consequence of Proposition 5.2.23, namely that the natural inclusion $\mathbb{S}^1 \hookrightarrow \mathrm{SP}\,\mathbb{S}^1$ is a homotopy equivalence. Therefore, we have that

$$\widetilde{H}_n(\mathbb{S}^0) = \widetilde{H}_{n+1}(\mathbb{S}^1) = \pi_{n+1}(\mathrm{SP}\,\mathbb{S}^1) \cong \pi_{n+1}(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

□

All given axioms of functoriality, homotopy, exactness and dimension are the so-called *Eilenberg–Steenrod axioms* for a reduced ordinary homology theory.

For the one-point space, or more generally for any contractible space, one sees immediately that it has trivial reduced homology. Specifically, we have the next assertion.

5.3.7 **Proposition**. *Let D be a contractible space. Then we have $\widetilde{H}_n(D) = 0$ for all n.* □

5.3.8 **Proposition**. *Suppose that $n > 0$. Then we have*

$$\widetilde{H}_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

Proof: Using Proposition 5.3.2 and the fact that $\mathbb{S}^n = \Sigma^n \mathbb{S}^0$ we obtain $\widetilde{H}_q(\mathbb{S}^n) \cong \widetilde{H}_{q-n}(\mathbb{S}^0)$, so that an application of the dimension property 5.3.6 gives us the desired result. Alternatively, the result follows immediately from 5.2.24. □

One very interesting and important consequence of Propositions 5.3.7 and 5.3.8 is the following famous theorem, known as the Brouwer fixed point theorem, whose special case for dimension $n = 2$ was proved in Chapter 2 (Theorem 2.4.23). The proof looks exactly the same, but instead of the degree, which (implicitly) uses the fundamental group, we use here the (reduced) nth homology groups.

5.3.9 **Theorem**. *Suppose that $n \geq 1$ and that $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ is continuous. Then there exists a point $x_0 \in \mathbb{D}^n$ satisfying $f(x_0) = x_0$.* We call x_0 a *fixed point* of f.

Proof: If there were no such x_0, then we would have $f(x) \neq x$ for every $x \in \mathbb{D}^n$. So the points x and $f(x)$ would determine a ray starting from $f(x)$. This ray would intersect $\mathbb{S}^{n-1}$ in exactly one point, say $r(x)$ (see Figure 5.3). The map $r : \mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$ is well defined and continuous and is actually a retraction. However, the existence of such a retraction contradicts the next proposition. □

5.3.10 **Proposition**. *Suppose that $n \geq 1$. Then there does not exist any retraction $r : \mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$.*

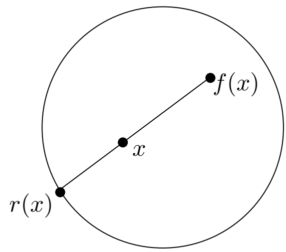

Figure 5.3

Proof: If such a retraction did exist, we would have the commutative triangle

$$\begin{tikzcd} & \mathbb{D}^n \arrow[dr, "r"] & \\ \mathbb{S}^{n-1} \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"'] & & \mathbb{S}^{n-1}, \end{tikzcd}$$

where $i : \mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n$ is the inclusion. Consequently, we would get the following commutative triangle of reduced homology groups with coefficients in $\mathbb{Z}$:

$$\begin{tikzcd} & \widetilde{H}_{n-1}(\mathbb{D}^n) \arrow[dr, "r_*"] & \\ \widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[ur, "i_*"] \arrow[rr, "1"'] & & \widetilde{H}_{n-1}(\mathbb{S}^{n-1}). \end{tikzcd}$$

But this is impossible, since according to Proposition 5.3.8, this would imply that $\mathbf{1}_{\mathbb{Z}}$ factors through the group $\widetilde{H}_{n-1}(\mathbb{D}^n)$, which is trivial by Proposition 5.3.7; however, $\widetilde{H}_{n-1}(\mathbb{S}^{n-1})$ is nontrivial. □

5.3.11 Exercise. Prove that the canonical inclusion $j : Y \hookrightarrow Z_f$ is a homotopy equivalence satisfying $j \circ f \simeq k$, where $k : X \hookrightarrow Z_f$ is the canonical inclusion induced by $x \mapsto (x, 1)$.

We can define homology groups of pairs as follows.

5.3.12 Definition. Let (X, A) be a CW-pair. We define the *nth homology group* of (X, A) to be

$$H_n(X, A) = \widetilde{H}_n(X \cup CA)\,,$$

where $X \cup CA$ is the mapping cone of the inclusion map of A into X. If $f : (X, A) \longrightarrow (Y, B)$ is a map of CW-pairs, then the induced map on the cones, namely $f' : X \cup CA \longrightarrow Y \cup CB$ defined by $f'(x) = f(x) \in Y$ for $x \in X$ and $f'(\overline{a,t}) = (\overline{f(a),t}) \in CB$ for $(\overline{a,t}) \in CA$, induces a homomorphism

$$f_* : H_n(X,A) \longrightarrow H_n(Y,B).$$

In particular, in the case $A = \emptyset$ we have that $H_n(X) = \widetilde{H}_n(X^+)$, since by definition $CA = *$ and $X \cup CA = X^+ = X \sqcup *$ in this case.

In the same way as in the cases of reduced homology, we have the following properties.

5.3.13 **Functoriality**. *If $f : (X,A) \longrightarrow (Y,B)$ and $g : (Y,B) \longrightarrow (Z,C)$ are maps of* CW-*pairs, then*

$$(g \circ f)_* = g_* \circ f_* : H_n(X,A) \longrightarrow H_n(Z,C).$$

Moreover, if $\mathrm{id}_{(X,A)} : (X,A) \longrightarrow (X,A)$ *is the identity, then*

$$\mathrm{id}_{(X,A)*} = 1_{H_n(X,A)} : H_n(X,A) \longrightarrow H_n(X,A).$$

5.3.14 **Homotopy**. *If $f \simeq g : (X,A) \longrightarrow (Y,B)$ (a homotopy of pairs), then*

$$f_* = g_* : H_n(X,A) \longrightarrow H_n(Y,B).$$

5.3.15 **Excision**. *Let $(X; X_1, X_2)$ be a* CW-triad*; that is, X_1 and X_2 are subcomplexes of X satisfying $X = X_1 \cup X_2$. Then the inclusion $j : (X_1, X_1 \cap X_2) \longrightarrow (X, X_2)$ induces an isomorphism*

$$j_* : H_n(X_1, X_1 \cap X_2) \longrightarrow H_n(X, X_2), \qquad n \geq 0.$$

5.3.16 **Exactness**. *For every* CW-*pair (X,A) there exists a long exact sequence*

$$\cdots \longrightarrow H_{q+1}(A) \longrightarrow H_{q+1}(X) \longrightarrow H_{q+1}(X,A) \xrightarrow{\partial} H_q(A) \longrightarrow \cdots,$$

where ∂ is called the connecting homomorphism *in homology, which is a natural homomorphism; namely, for every map of pairs $f : (X,A) \longrightarrow (Y,B)$, we have a commutative diagram:*

$$\begin{tikzcd}
H_{q+1}(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_q(A) \arrow[d, "(f|A)_*"] \\
H_{q+1}(Y,B) \arrow[r, "\partial"'] & H_q(B).
\end{tikzcd}$$

5.3.17 **Dimension**. *For the one-point space $*$ we have*

$$H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proof: The functoriality and homotopy properties follow immediately from the corresponding properties in the reduced case.

To prove the excision property, we first recall Corollary 4.2.3, namely that the identification $X \cup CA \longrightarrow X \cup CA/CA \approx X/A$ is a homotopy equivalence, which implies that

$$H_n(X, A) = \widetilde{H}_n(X/A) \tag{5.3.18}$$

for every CW-pair (X, A). So to prove 5.3.15 it is enough to note that the conditions imposed on X, X_1, and X_2 imply that

$$X/X_2 \quad \text{and} \quad X_1/X_1 \cap X_2$$

are homeomorphic.

In order to prove the exactness property we are going to define

$$\partial : H_{q+1}(X, A) \longrightarrow H_q(A)$$

by using the map

$$X/A \xrightarrow{p} X^+ \cup CA^+ \xrightarrow{p'} \Sigma A^+ ,$$

where p is the homotopy inverse of the quotient map $X^+ \cup CA^+ \longrightarrow X/A$ and p' is the quotient map that collapses X^+. Specifically, we define ∂ to be the composite

$$\begin{aligned} \partial : H_{q+1}(X, A) \longrightarrow \widetilde{H}_{q+1}(X/A) \xrightarrow{(p' \circ p)_*} \widetilde{H}_{q+1}(\Sigma A^+) \cong \\ \cong \widetilde{H}_q(A^+) = H_q(A) . \end{aligned}$$

We prove the exactness at $H_{q+1}(X)$ by taking the exact sequence for the reduced case for the inclusion $i : A^+ \hookrightarrow X^+$ and so get the exact sequence

$$\widetilde{H}_{q+1}(A^+) \longrightarrow \widetilde{H}_{q+1}(X^+) \longrightarrow \widetilde{H}_{q+1}(C_i) ,$$

which is the same as

$$H_{q+1}(A) \longrightarrow H_{q+1}(X) \longrightarrow H_{q+1}(X, A) ,$$

since $\widetilde{H}_{q+1}(C_i) = H_{q+1}(X, A)$.

The exactness at $H_{q+1}(X,A)$ is now shown by taking the exact sequence for the reduced case for the inclusion $j : X \hookrightarrow X^+ \cup CA^+$, namely

$$\widetilde{H}_{q+1}(X^+) \longrightarrow \widetilde{H}_{q+1}(X^+ \cup CA^+) \longrightarrow \widetilde{H}_{q+1}(C_j).$$

It is easy to prove that $C_j \approx \Sigma A^+$ (see Section 3.3, particularly equation (3.3.4)), which implies that the previous exact sequence becomes

$$H_{q+1}(X) \longrightarrow H_{q+1}(X,A) \longrightarrow H_q(A),$$

where the last homomorphism is precisely ∂.

Finally, the exactness at $H_q(A)$ is proved by considering the exact sequence for the reduced case for the identification $p : X^+ \cup CA^+ \longrightarrow \Sigma A^+$ and by noting that $C_p \simeq \Sigma X^+$. This means that the sequence

$$\widetilde{H}_{q+1}(X^+ \cup CA^+) \longrightarrow \widetilde{H}_{q+1}(\Sigma A^+) \longrightarrow \widetilde{H}_{q+1}(\Sigma X^+)$$

becomes the sequence

$$H_{q+1}(X,A) \longrightarrow H_q(A) \longrightarrow H_q(X),$$

where the first homomorphism is exactly ∂ and the second homomorphism is induced by the inclusion $A \hookrightarrow X$.

The dimension property follows immediately from the fact that $H_n(*) = \widetilde{H}_n(\mathbb{S}^0)$. □

All axioms of functoriality, homotopy, exactness and dimension given above are the so-called *Eilenberg–Steenrod axioms* for an ordinary (unreduced) homology theory.

5.3.19 Exercise. Verify the details of the proof of the exactness property. In particular, show that $C_p \simeq \Sigma X^+$ and that up to precisely this homotopy equivalence, the map $\Sigma A^+ \longrightarrow C_p$ corresponds to the inclusion.

5.3.20 Note. The proof that we have given of the exactness property for the case of pairs (starting from the corresponding property for the reduced case) is not the simplest. However, it is worthwhile to present this, since it gives a general way of proving that any functor satisfies a relative exactness axiom (such as 5.3.16), provided that it satisfies a reduced exactness axiom (such as 5.3.5). This is particularly important in the study of generalized theories of homology (or cohomology). A simpler proof of 5.3.16 is possible, as we request the reader to provide in the next exercise.

5.3.21 Exercise. Construct an alternative proof of the relative exactness axiom (5.3.16) using the long exact homotopy sequence (4.3.41) of the quasifibration $\mathrm{SP}\, Z_i \longrightarrow \mathrm{SP}\, C_i$ that is induced by the identification map $Z_i \longrightarrow C_i$, where $i : A \longrightarrow X$ is the inclusion.

5.3.22 Exercise. Assume that X is contractible. Prove that

$$H_q(X, A) \cong H_{q-1}(A)$$

if $q > 1$, and

$$H_1(X, A) \cong \widetilde{H}_0(A)\,.$$

5.3.23 Exercise. Take $A \subset B \subset X$ and assume that the inclusion $A \hookrightarrow B$ is a homotopy equivalence. Prove that the inclusion of pairs $(X, A) \hookrightarrow (X, B)$ induces an isomorphism

$$H_q(X, A) \longrightarrow H_q(X, B)$$

for all q.

5.3.24 Note. We can extend Definition 5.3.12 to arbitrary pairs (X, A) by defining $H_n(X, A) = H_n(\widetilde{X}, \widetilde{A})$, where $(\widetilde{X}, \widetilde{A})$ is a CW-approximation of (X, A). For any continuous $f : (X, A) \longrightarrow (Y, B)$ we define $f_* = \widetilde{f}_*$. These definitions are well defined due to Theorems 5.1.35 and 5.1.44.

5.3.25 Exercise. Prove that if $f : (X, A) \longrightarrow (Y, B)$ is a weak homotopy equivalence of pairs of topological spaces, then

$$f_* : H_q(X, A) \longrightarrow H_q(Y, B)$$

is an isomorphism for all q. This is the so-called *weak homotopy equivalence axiom*. (Hint: See 5.1.35.)

The definitions of the previous paragraph clearly satisfy the axioms of functoriality, homotopy, exactness, and dimension as formulated above. But they also satisfy the following excision axiom that corresponds to 5.3.15.

5.3.26 **Excision**. (For excisive triads) *Let* $(X; A, B)$ *be an* excisive triad*; that is,* X *is a topological space with subspaces* A *and* B *such that* $\overset{\circ}{A} \cup \overset{\circ}{B} = X$*, where* $\overset{\circ}{A}$ *and* $\overset{\circ}{B}$ *denote the interiors of* A *and* B*, respectively. Then the inclusion* $j : (A, A \cap B) \longrightarrow (X, B)$ *induces an isomorphism*

$$J_* : H_n(A, A \cap B) \longrightarrow H_n(X, B), \qquad n \geq 0.$$

Proof: In order to show that we have this property we take a CW-approximation of $A \cap B$, say $\varphi : \widetilde{A \cap B} \longrightarrow A \cap B$, and extend it to an approximation of A, say $\varphi_1 : \widetilde{A} \longrightarrow A$, and to an approximation of B, say $\varphi_2 : \widetilde{B} \longrightarrow B$, in such a way that $\widetilde{A \cap B} = \widetilde{A} \cap \widetilde{B}$. Thus we can define a map $\widetilde{\varphi} : \widetilde{X} = \widetilde{A} \cup \widetilde{B} \longrightarrow A \cup B = X$ such that $\widetilde{\varphi}|\widetilde{A} = \varphi_1$, $\widetilde{\varphi}|\widetilde{B} = \varphi_2$, and $\widetilde{\varphi}|\widetilde{A \cap B} = \varphi$. Using the hypothesis $\overset{\circ}{A} \cup \overset{\circ}{B} = X$ we can now prove that $\widetilde{\varphi}$ is a weak homotopy equivalence; that is, $\widetilde{\varphi}$ is a CW-approximation of X (see 5.1.34). Using this result it is clear that the excision axiom for excisive triads follows from the excision axiom (5.3.15) for CW-triads. □

5.3.27 EXERCISE. Prove that the excision axiom for excisive triads is equivalent to the following axiom. Suppose that (X, A) is a pair of spaces and that $U \subset A$ satisfies $\overline{U} \subset \overset{\circ}{A}$. Then the inclusion $i : (X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism $H_n(X - U, A - U) \cong H_n(X, A)$ for each $n \geq 0$.

5.3.28 **Lemma**. *For every pointed topological space X we have that*

$$H_q(X) = \begin{cases} \widetilde{H}_q(X) & \textit{if } q \neq 0, \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & \textit{if } q = 0. \end{cases}$$

Proof: The cone $C_j = CX \cup *$ of the natural inclusion $j : X \hookrightarrow X^+$ has the same homotopy type of the 0-sphere $\mathbb{S}^0$. So for each $q \geq 0$ there exists an exact sequence

$$\widetilde{H}_q(X) \xrightarrow{j_*} \widetilde{H}_q(X^+) \longrightarrow \widetilde{H}_q(\mathbb{S}^0).$$

Note that the natural projection $p : X^+ \longrightarrow X$, which sends $*$ to the base point of X, satisfies $p \circ j = \mathrm{id}_X$. So the previous exact sequence splits, implying that the group in the middle can be expressed as the sum of the other two groups; that is,

$$H_q(X) = \widetilde{H}_q(X^+) = \widetilde{H}_q(X) \oplus \widetilde{H}_q(\mathbb{S}^0).$$

The statement now follows immediately from the dimension property 5.3.6. □

As an immediate consequence of 5.3.8 we obtain the following result.

5.3.29 **Proposition**. *Suppose that $n > 0$. Then we have that*

$$H_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \textit{if } q = 0, n, \\ 0 & \textit{if } q \neq 0, n. \end{cases}$$

□

Reduced homology groups have an additional property with respect to infinite unions of CW-complexes. In particular, this allows us to compute the homology of any CW-complex from the homology of its finite subcomplexes.

5.3.30 **Proposition**. *Suppose that X is a pointed topological space and that $\{X_\lambda\}$ is a system of closed (or open) subspaces that contain the base point of X. The inclusions $i_\lambda^\mu : X_\lambda \longrightarrow X_\mu$ determine a directed system of groups when one applies to them the functor $\widetilde{H}_q$ for any q. We then have an isomorphism*

$$\operatorname{colim} \widetilde{H}_q(X_\lambda) \cong \widetilde{H}_q(X),$$

which is determined by the inclusions $i_\lambda : X_\lambda \longrightarrow X$.

Proof: The inclusions i_λ^μ and i_λ induce inclusions $\widehat{i}_\lambda^\mu : \mathrm{SP}\, X_\lambda \longrightarrow \mathrm{SP}\, X_\mu$ and $\widehat{i}_\lambda : \mathrm{SP}\, X_\lambda \longrightarrow \mathrm{SP}\, X$, which in turn induce a continuous and bijective map $\operatorname{colim} \mathrm{SP}\, X_\lambda \longrightarrow \mathrm{SP}\, X$. In general, the inverse function is not continuous, except on compact subsets. For example, it is continuous if X is compactly generated (see 4.3.22). However, what we know is enough to guarantee that the inverse function induces isomorphisms of homotopy groups. For just this very reason we have $\pi_q(\operatorname{colim} \mathrm{SP}\, X_\lambda) = \operatorname{colim} \pi_q(\mathrm{SP}\, X_\lambda)$. And so we have the desired result. □

The next result establishes the so-called *wedge axiom* for homology.

5.3.31 **Proposition**. *If $X = \bigvee_{\lambda\in\Lambda} X_\lambda$, then*

$$\widetilde{H}_q(X) \cong \bigoplus_{\lambda\in\Lambda} \widetilde{H}_q(X_\lambda)\,.$$

Proof: First case. Assume that X_1 and X_2 are CW-complexes whose base point is a 0-cell and take $X = X_1 \vee X_2$. If $i : X_1 \hookrightarrow X$ is the canonical inclusion, then by 4.2.3 the canonical quotient map $C_i = X \cup CX_1 \twoheadrightarrow X/X_1 \approx X_2$ is a homotopy equivalence. Therefore, by the homotopy property of $\widetilde{H}$, one has a short exact sequence

$$0 \longrightarrow \widetilde{H}_q(X_1) \longrightarrow \widetilde{H}_q(X) \longrightarrow \widetilde{H}_q(X_2) \longrightarrow 0\,,$$

which obviously splits. Therefore,

$$\widetilde{H}_q(X) \cong \widetilde{H}_q(X_1) \oplus \widetilde{H}_q(X_2)\,.$$

This implies that for any finite wedge $X = \bigvee_{\lambda=1}^k X_\lambda$,

$$\widetilde{H}_q(X) \cong \bigoplus_{\lambda=1}^k \widetilde{H}_q(X_\lambda)\,.$$

General case. If $X = \bigvee_{\lambda\in\Lambda} X_\lambda$ is an arbitrary wedge of CW-complexes, then we can take the system of all finite wedges $X_\Gamma = \bigvee_{\lambda\in\Gamma} X_\lambda$, $\Gamma \subset \Lambda$ finite. By the previous step,

$$\widetilde{H}^q(X_\Gamma) \cong \bigoplus_{\lambda\in\Gamma} \widetilde{H}^q(X_\lambda)\,.$$

Therefore, by 5.3.30,

$$\widetilde{H}^q(X) \cong \operatorname*{colim}_{\Gamma\subset\Lambda} \bigoplus_{\lambda\in\Gamma} \widetilde{H}^q(X_\lambda) = \bigoplus_{\lambda\in\Lambda} \widetilde{H}_q(X_\lambda)\,.$$

In the general case, if the given spaces are not CW-complexes, then one takes a CW-approximation $\widetilde{X}_\lambda \longrightarrow X_\lambda$ for each λ and takes as a CW-approximation of X precisely the wedge

$$\widetilde{X} = \bigvee \widetilde{X}_\lambda \longrightarrow \bigvee X_\lambda = X\,.$$

Then the result follows from the CW-case for any spaces. □

5.3.32 Exercise. Let $(X,A) = \coprod(X_\lambda, A_\lambda)$. Prove that for all q,

$$H_q(X,A) \cong \bigoplus_\lambda H_q(X_\lambda, A_\lambda)\,.$$

This is the so-called *additivity axiom* for homology.

5.3.33 Note. In homology there is a way of introducing coefficients in a general group G. This will be done at the end of the next chapter in Section 6.3. In what follows, among other things, we shall show another way of introducing coefficients in a cyclic group using a variation of the infinite symmetric product.

In the article [26], Dold and Thom introduce another construction related to the infinite symmetric product of a space. This is the *free topological abelian group* over a topological space X with base point x_0, which serves as the zero element of the group. This topological group has properties analogous to those of the infinite symmetric product. The construction enjoys the desired properties when the space X is a connected CW-complex with x_0 as one of its vertices.

To define this topological group we construct the wedge $X \vee X$ of two copies of X and then take the map $\tau : X \vee X \longrightarrow X \vee X$ that interchanges the two terms and next define an equivalence relation on $\mathrm{SP}\,(X \vee X)$ by

$$x \sim x + x' + \widehat{\tau}(x'),$$

where x and x' are elements in X (considered as a subset of $X \vee X$, which in turn is a subset of $\mathrm{SP}\,(X \vee X)$) and the sum $+$ is that of the symmetric product $\mathrm{SP}\,(X \vee X)$ given by juxtaposition of the elements. The resulting quotient space $\mathrm{AG}\,X$ of equivalence classes is an abelian topological group. Obviously, this construction is functorial. *If X is a countable simplicial complex, then $\mathrm{AG}\,X$ has the structure of a CW-complex.* If, instead, X is a countable CW-complex, then $\mathrm{AG}\,X$ has the homotopy type of a CW-complex (see [45]). (In case of a general CW-complex, one should take the compactly generated topology in the products.)

For any positive integer m we can consider the subgroup $m{\cdot}\mathrm{AG}\,X$ of $\mathrm{AG}\,X$ consisting of the elements divisible by m. And then we can form the quotient group $\mathrm{AG}\,X/m \cdot \mathrm{AG}\,X$ and so functorially get a new topological group $\mathrm{AG}\,(X;m)$, which is nothing other than the *free topological $\mathbb{Z}/m$-module* over the space X.

Corresponding to the Dold–Thom Theorem 5.2.17, which is the principal result about infinite symmetric products, we have a result for free abelian topological groups. Let A be a subcomplex of a countable simplicial complex X, which has x_0 as a vertex. *If $p : X \longrightarrow X/A$ is the quotient map, then the induced map $\widetilde{p} : \mathrm{AG}\,X \longrightarrow \mathrm{AG}\,(X/A)$ is a locally trivial bundle with fiber* $\mathrm{AG}\,A$. Actually, this is a principal fiber bundle with both fiber and structure group equal to $\mathrm{AG}\,A$.

It follows analogously to the construction SP that the construction AG is such that *the groups $\widetilde{H}'_q(X) = \pi_q(\mathrm{AG}\,X)$ and $\widetilde{H}'_q(X;m) = \pi_q(\mathrm{AG}(X;m))$ coincide with the reduced ordinary homology of X with coefficients in $\mathbb{Z}$, respectively in $\mathbb{Z}/m$, in the category of countable simplicial complexes.*

Chapter 6

Homotopy Properties of CW-Complexes

In order to define cohomology groups, as we shall do in the next chapter, we have to define some special spaces, called Eilenberg–Mac Lane spaces, which we have already mentioned. These spaces will be constructed starting from the concept of an infinite symmetric product introduced in the last chapter. This construction will be applied to the so-called Moore spaces, which by construction are CW-complexes.

A very useful tool for analyzing properties of CW-complexes and especially of the Moore spaces is the homotopy excision theorem of Blakers–Massey, which will be proved here.

6.1 Eilenberg–Mac Lane and Moore Spaces

As we have already noted, one way of defining cohomology groups is by means of Eilenberg–Mac Lane spaces. In this section we shall construct these spaces and study some of their properties. In order to define Eilenberg–Mac Lane spaces we shall need some knowledge of a family of spaces that are associated to abelian groups or, more precisely, to their primary decomposition. These are the so-called Moore spaces, they possess some interesting homotopy properties which we shall present in this section.

6.1.1 Definition. A space A is said to be an *Eilenberg–Mac Lane space of type* $K(G,n)$ or, more briefly, to be a $K(G,n)$, if

$$\pi_q(A) \cong \begin{cases} G & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

To prove the existence of these spaces we shall avail ourselves of the ideas about infinite symmetric products that are developed in [26] and were discussed in the last chapter.

Proposition 5.2.24 provides us with the first and very important examples of Eilenberg–Mac Lane spaces.

6.1.2 **Proposition**. *For each integer $n \geq 1$ the infinite symmetric product $\mathrm{SP}\,\mathbb{S}^n$ is a $K(\mathbb{Z}, n)$.* □

Since the space $\mathbb{S}^1$ is an H-cogroup (see 2.10.2 and 2.10.8), we can consider the composite map

$$\alpha_2 : \mathbb{S}^1 \xrightarrow{\nu} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\rho} \mathbb{S}^1\,,$$

where ν is the comultiplication and ρ maps each copy of $\mathbb{S}^1$ in the wedge by the identity. Clearly, we have that

$$\alpha_{2_*} : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1)$$

is multiplication by 2 in $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Analogously,

$$\alpha_3 : \mathbb{S}^1 \xrightarrow{\quad\nu\quad} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\alpha_2 \vee \mathrm{id}} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\quad\rho\quad} \mathbb{S}^1$$

induces

$$\alpha_{3_*} : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1),$$

which is multiplication by 3. Inductively we can define

$$\alpha_k : \mathbb{S}^1 \xrightarrow{\quad\nu\quad} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\alpha_{k-1} \vee \mathrm{id}} \mathbb{S}^1 \vee \mathbb{S}^1 \xrightarrow{\quad\rho\quad} \mathbb{S}^1\,,$$

so that

$$\alpha_{k_*} : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1)\,,$$

is multiplication by k. Let us consider the sequence of maps

$$\mathbb{S}^1 \xrightarrow{\quad\alpha_k\quad} \mathbb{S}^1 \xrightarrow{\qquad} C_{\alpha_k} \xrightarrow{\qquad} \mathbb{S}^2. \tag{6.1.3}$$

We usually write C_{α_k} as the attaching space $\mathbb{S}^1 \cup_{\alpha_k} e^2$, since it is the result of attaching to $\mathbb{S}^1$ the cell $C\mathbb{S}^1 \approx \bar{e}^2$ by means of the map α_k on its boundary. The portion

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^1 \cup_{\alpha_k} e^2 \longrightarrow \mathbb{S}^2$$

of the previous sequence induces a quasifibration

$$\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2) \longrightarrow \mathrm{SP}\,\mathbb{S}^2$$

with fiber $\mathrm{SP}\,\mathbb{S}^1$. So we get a long exact sequence

$$\cdots \longrightarrow \pi_q(\mathrm{SP}\,\mathbb{S}^1) \longrightarrow \pi_q(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) \longrightarrow \pi_q(\mathrm{SP}\,\mathbb{S}^2) \longrightarrow \\ \longrightarrow \pi_{q-1}(\mathrm{SP}\,\mathbb{S}^1) \longrightarrow \cdots$$

from which we obtain

$$\pi_q(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) = 0 \text{ if } q \neq 1, 2$$

and

$$0 \longrightarrow \pi_2(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) \longrightarrow \pi_2(\mathrm{SP}\,(\mathbb{S}^2)) \longrightarrow \pi_1(\mathrm{SP}\,(\mathbb{S}^1)) \longrightarrow \\ \longrightarrow \pi_1(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) \longrightarrow 0\,.$$

Using (6.1.3) we can deduce that the map $\pi_2(\mathrm{SP}\,\mathbb{S}^2) \longrightarrow \pi_1(\mathrm{SP}\,\mathbb{S}^1)$ in this last sequence is just multiplication by k in $\mathbb{Z}$, and so we have $\pi_2(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) = 0$ and $\pi_1(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) = \mathbb{Z}/k$. So we have proved that $\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)$ is a $K(\mathbb{Z}/k, 1)$; that is, we have the next result.

6.1.4 **Proposition**. *The infinite symmetric product* $\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)$ *is an Eilenberg–Mac Lane space of type* $(\mathbb{Z}/k, 1)$*; that is,*

$$\pi_q(\mathrm{SP}\,(\mathbb{S}^1 \cup_{\alpha_k} e^2)) = \begin{cases} \mathbb{Z}/k & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

□

If we generalize this construction, we obtain the next definition.

6.1.5 Definition. The attaching spaces $\mathbb{S}^n \cup_{\alpha_k} e^{n+1}$, $k \geq 2$, are called *Moore spaces of type* $(\mathbb{Z}/k, n)$, where now $\alpha_k : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ denotes the $(n-1)$-fold suspension of the map α_k defined above, $n \geq 1$..

Now applying 6.1.2 and the same reasoning that led us to 6.1.4, we get the next result.

6.1.6 **Theorem**. *The infinite symmetric product of a Moore space,*

$$\mathrm{SP}\,(\mathbb{S}^n \cup_{\alpha_k} e^{n+1})\,, \qquad \textit{where} \qquad n \geq 1\,,$$

is a $K(\mathbb{Z}/k, n)$*; that is,*

$$\pi_q(\mathrm{SP}\,(\mathbb{S}^n \cup_{\alpha_k} e^{n+1})) = \begin{cases} \mathbb{Z}/k & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

□

6.1.7 EXERCISE. Consider $\mathbb{S}^1$ as the unit circle in the complex plane $\mathbb{C}$, and define $\widetilde{\alpha}_k : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ by $\widetilde{\alpha}_k(e^{2\pi i t}) = e^{2\pi i k t}$. Here k can be any real number. Prove the following:

(a) For each integer $k \geq 1$, we have a homotopy $\widetilde{\alpha}_k \simeq \alpha_k : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

(b) For $k \geq -1$ we have that $\widetilde{\alpha}_{-1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is the reflection in the real axis of $\mathbb{C}$.

(c) If we now let $\widetilde{\alpha}_{-1} : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ denote the $(n-1)$-fold suspension of the map $\widetilde{\alpha}_{-1}$ of part (b), then this new $\widetilde{\alpha}_{-1}$ is a reflection in a hyperplane.

Suppose that X and Y are *well-pointed* spaces. Recall that this means that the inclusions of the base points x_0 and y_0 in X and Y, respectively, are closed cofibrations. Then the inclusions $Y \hookrightarrow X \vee Y$ and $X \hookrightarrow X \vee Y$ also are (closed) cofibrations. Consequently, if X and Y are 0-connected, then they satisfy the hypotheses of version 5.2.22 of the Dold–Thom theorem. And then since $X \vee Y/Y \approx X$ and $X \vee Y/X \approx Y$, we have that the canonical maps $X \vee Y \longrightarrow X$ and $X \vee Y \longrightarrow Y$ induce quasifibrations

$$\mathrm{SP}\,(X \vee Y) \longrightarrow \mathrm{SP}\,X\,, \qquad \mathrm{SP}\,(X \vee Y) \longrightarrow \mathrm{SP}\,Y\,,$$

with fibers $\mathrm{SP}\,Y$ and $\mathrm{SP}\,X$, respectively. Since the inclusions $X \hookrightarrow X \vee Y$ and $Y \hookrightarrow X \vee Y$ induce sections of these quasifibrations, we have that the canonical map $\mathrm{SP}\,(X \vee Y) \longrightarrow \mathrm{SP}\,X \times \mathrm{SP}\,Y$ induces isomorphisms

$$\pi_q(\mathrm{SP}\,(X \vee Y)) \cong \pi_q(\mathrm{SP}\,X) \oplus \pi_q(\mathrm{SP}\,Y) \tag{6.1.8}$$

for every q. Moreover, if $i : X \hookrightarrow \mathrm{SP}\,X$ and $j : Y \hookrightarrow \mathrm{SP}\,Y$ are the canonical inclusions, we have the commutative diagram

$$\begin{tikzcd}
\pi_q(X \vee Y) \arrow[r] \arrow[d, "(i\vee j)_*"'] & \pi_q(X) \oplus \pi_q(Y) \arrow[d, "i_*\oplus j_*"] \\
\pi_q(\mathrm{SP}\,(X \vee Y)) \arrow[r, "\cong"'] & \pi_q(\mathrm{SP}\,X) \oplus \pi_q(\mathrm{SP}\,Y)\,.
\end{tikzcd} \tag{6.1.9}$$

Because π_q and SP commute with colimit, (6.1.8) and (6.1.9) hold for infinite wedges. We should mention here that there is a direct proof of (6.1.8). Or more accurately put, without using the Dold–Thom theorem and without the hypothesis that X and Y are well pointed, one can show that the canonical map that induces the isomorphism (6.1.8) is a weak homotopy equivalence. (See [26, 3.14].)

Let us recall from algebra the well-known result called the primary decomposition theorem. This says that if G is a finitely generated abelian group, then there is a unique decomposition

$$G = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}\,, \tag{6.1.10}$$

where $d_1|d_2, d_2|d_3, \ldots, d_{k-1}|d_k$. Corresponding to such a decomposition of G we define a space X by

$$X = \underbrace{\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n}_{r} \vee (\mathbb{S}^n \cup_{\alpha_{d_1}} e^{n+1}) \vee \cdots \vee (\mathbb{S}^n \cup_{\alpha_{d_k}} e^{n+1}). \tag{6.1.11}$$

6.1.12 DEFINITION. The space X defined in (6.1.11) is called a *Moore space of type* (G, n).

Note that by construction, a Moore space of type (G, n) is a CW-complex with exactly one 0-cell and with all the other cells in dimensions n and $n+1$.

We deduce the next theorem from (6.1.8) and 6.1.6.

6.1.13 **Theorem**. *Let X be a Moore space of type (G, n). Then* $\mathrm{SP}\,X$ *is an Eilenberg–Mac Lane space of type $K(G, n)$. In other words, this means that for $n \geq 1$ and for all q we have*

$$\pi_q(\mathrm{SP}\,(\underbrace{\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n}_{r} \vee (\mathbb{S}^n \cup_{\alpha_{d_1}} e^{n+1}) \vee \cdots \vee (\mathbb{S}^n \cup_{\alpha_{d_k}} e^{n+1})))$$
$$= \begin{cases} G & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

□

6.2 HOMOTOPY EXCISION AND RELATED RESULTS

We start this section with the following very important result in homotopy theory, which can be interpreted as the homotopy version of the cohomology (or homology) excision theorem (see 7.1.16). We give here a homotopical proof following [51].

6.2.1 **Theorem**. (Blakers–Massey) *Suppose that X is a pointed space and that A and B are pointed subspaces of X such that*

(i) $X = A \cup B$ *and*

(ii) *the inclusions* $A \cap B \hookrightarrow A$ *and* $A \cap B \hookrightarrow B$ *are cofibrations.*

If the pair $(A, A\cap B)$ *is* $(m-1)$*-connected and the pair* $(B, A\cap B)$ *is* $(n-1)$*-connected,* $m \geq 2$*,* $n \geq 1$*, then the homomorphism induced by the inclusion, namely* $i_* : \pi_q(A, A \cap B) \longrightarrow \pi_q(X, B)$*, is an isomorphism for* $q < m+n-2$ *and is an epimorphism for* $q = m+n-2$*.*

Before passing to the proof of this theorem we can obtain as a consequence two very useful results, which we shall present in the following discussion.

6.2.2 **Proposition**. *Suppose that* $Y_0 \hookrightarrow Y$ *is a cofibration, that the pair* (Y, Y_0) *is* $(r-1)$*-connected, and that the subspace* Y_0 *is* $(s-1)$*-connected. Then the homomorphism induced by the quotient map, namely*

$$p_* : \pi_q(Y, Y_0) \longrightarrow \pi_q(Y/Y_0)\,,$$

is an isomorphism for $q < r+s-1$ *and is an epimorphism for* $q = r+s-1$ *(*$r > 0$*).*

Proof: By hypothesis $Y_0 \hookrightarrow Y$ is a cofibration, as also is the inclusion $Y_0 \hookrightarrow CY_0$ in the cone (see 3.1.6). Since Y_0 is $(s-1)$-connected, we can use the exact homotopy sequence of the pair (CY_0, Y_0) to show that the pair is s-connected. Then using Theorem 6.2.1, we get that $i : \pi_q(Y, Y_0) \longrightarrow \pi_q(Y \cup CY_0, CY_0)$ is an isomorphism for $q < r+s-1$ and is an epimorphism for $q = r+s-1$. But 4.2.3 says that the quotient map $(Y \cup CY_0, CY_0) \longrightarrow (Y/Y_0, *)$ is a homotopy equivalence. Therefore, we have the desired result. □

Let us recall that the suspension of a pointed map $f : X \longrightarrow Y$ between pointed spaces is denoted by $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ and is defined by $\Sigma f(x \wedge t) = f(x) \wedge t$ (see 2.10.1). For a pointed space X we define the *suspension homomorphism* $\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ by $\Sigma[\alpha] = [\Sigma\alpha]$, where $\alpha : \mathbb{S}^q \longrightarrow X$ represents a pointed homotopy class and $\Sigma\alpha : \mathbb{S}^{q+1} \longrightarrow \Sigma X$ is its suspension.

6.2.3 EXERCISE. Suppose that X is a pointed space,

$$p : (CX, X) \longrightarrow (CX/X, \{*\})$$

is the quotient map and that

$$\partial : \pi_{q+1}(CX, X) \longrightarrow \pi_q(X)$$

is the connecting homomorphism (see 3.4.5). Prove that the diagram

$$\begin{tikzcd}
\pi_{q+1}(CX,X) \arrow[r, "p_*"] \arrow[d, "\partial"'] & \pi_{q+1}(CX/X,\{*\}) \arrow[d, "\cong"] \\
\pi_q(X) \arrow[r, "\Sigma"'] & \pi_{q+1}(\Sigma X)
\end{tikzcd}$$

commutes up to sign. (Hint: see 3.3.6 and 2.10.7.) Moreover, we have a commutative diagram, up to sign,

$$\begin{tikzcd}
\pi_q(X) \arrow[r, "\Sigma"] \arrow[d, "j_*"'] & \pi_{q+1}(\Sigma X) \arrow[d, "(\Sigma j)_*"] \\
\pi_q(\mathrm{SP}\, X) \arrow[r] & \pi_{q+1}(\mathrm{SP}\,(\Sigma X)),
\end{tikzcd}$$

where the lower horizontal arrow is the isomorphism in Corollary 5.2.19.

From the first part of Exercise 6.2.3, from the exact homotopy sequence of the pair (CX,X) (see 3.5.8(e)) and from the fact that CX is contractible, we get that $\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ is an isomorphism if and only if $p_* : \pi_{q+1}(CX,X) \longrightarrow \pi_{q+1}(CX/X,\{*\})$ is an isomorphism. If X is $(n-1)$-connected, then the pair (CX,X) is n-connected. So by applying 6.2.2 we get the next result, which is known as the Freudenthal suspension theorem, where we shall call a pointed space *well pointed* if the inclusion map of the base point into the space is a cofibration.

6.2.4 **Theorem**. *Let X be an $(n-1)$-connected well-pointed space. Then $\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ is an isomorphism for $q < 2n-1$ and an epimorphism for $q = 2n-1$.* □

6.2.5 Exercise.

(a) Prove that $\pi_2(\mathbb{S}^2) \cong \mathbb{Z}$ by using the Hopf fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \xrightarrow{p} \mathbb{S}^2$$

defined in 4.5.10. (Hint: From the exact homotopy sequence of the fibration p we get the exact sequence

$$\pi_2(\mathbb{S}^3) \longrightarrow \pi_2(\mathbb{S}^2) \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^3),$$

where the groups on either end are zero by 5.1.25. Then apply 4.5.13.)

(b) Prove that $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ for $n > 2$. (Hint: Apply 6.2.4.)

(c) Prove that $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. (Hint: In the portion

$$\pi_3(\mathbb{S}^1) \longrightarrow \pi_3(\mathbb{S}^3) \longrightarrow \pi_3(\mathbb{S}^2) \longrightarrow \pi_2(\mathbb{S}^1)$$

of the exact homotopy sequence of the fibration p the groups on the ends are zero.)

(d) Prove that in parts (a) and (b) the class $[\mathrm{id}_{\mathbb{S}^n}]$ is a generator of $\pi_n(\mathbb{S}^n)$ for $n \geq 1$, and thereby conclude that in part (c) the class $[p]$ is a generator of $\pi_3(\mathbb{S}^2)$.

6.2.6 EXERCISE. Conclude from Exercise 6.2.5(b) that

$$\pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z} \quad \text{for} \quad n \geq 1\,,$$

and that a generator of this group is represented by $\mathrm{id}_{(\mathbb{D}^n,\mathbb{S}^{n-1})}$. (Hint: Use the exact homotopy sequence of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.)

We have a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathbb{S}^2) \arrow[r, "\cong"] \arrow[d] & \pi_1(\mathbb{S}^1) \arrow[d, "\cong"] \\
\pi_2(\mathrm{SP}\,\mathbb{S}^2) \arrow[r, "\cong"'] & \pi_1(\mathrm{SP}\,\mathbb{S}^1),
\end{tikzcd}$$

where the isomorphism on the top of the diagram comes from the exact sequence in 6.2.5(a), the isomorphism on the bottom of the diagram comes from 5.2.19, and the isomorphism on the right comes from 5.2.23. It follows that the homomorphism on the left of the diagram is an isomorphism.

Suppose that $n \geq 2$. From the second part of Exercise 6.2.3 we have, up to sign, a commutative diagram

$$\begin{tikzcd}
\pi_n(\mathbb{S}^n) \arrow[r, "\Sigma"] \arrow[d] & \pi_{n+1}(\mathbb{S}^{n+1}) \arrow[d] \\
\pi_n(\mathrm{SP}\,\mathbb{S}^n) \arrow[r, "\cong"'] & \pi_{n+1}(\mathrm{SP}\,\mathbb{S}^{n+1})\,,
\end{tikzcd}$$

where the isomorphism on the bottom of the diagram is from 5.2.19. Using 6.2.4 the homomorphism on the top is an isomorphism as well.

In the case $n = 2$, the homomorphism on the left is an isomorphism. Therefore, the homomorphism on the right is also an isomorphism in this case, but this is precisely the isomorphism on the left for the case $n = 3$. Continuing inductively we can prove the next result.

6.2.7 **Proposition**. *The natural inclusion $i : \mathbb{S}^n \hookrightarrow \mathrm{SP}\,\mathbb{S}^n$ is an n-equivalence.* □

To prepare for the proof of 6.2.1, we need some concepts.

6.2.8 Definition. Given a triad $(X; A, B)$ with base point $x_0 \in C = A \cap B$, we define the *triad homotopy group*

$$\pi_q(X; A, B) = \pi_{q-1}(P(X; x_0, B), P(A; x_0, C))\,,$$

where $P(X; x_0, B)$, respectively $P(A; x_0, C)$, is the homotopy fiber of the inclusion $B \hookrightarrow X$, respectively $A \hookrightarrow X$, namely the set of paths in X, respectively A, starting in x_0 and ending in B, respectively C, and $q \geq 2$. Specifically, $\pi_q(X; A, B)$ is the set of homotopy classes of maps of tetrads

$$\begin{array}{c}(I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \\ \big\downarrow \\ (X; A, B, x_0)\,.\end{array}$$

From the exact homotopy sequence of the pair

$$(P(X; x_0, B), P(A; x_0, C))\,,$$

we obtain

$$\cdots \to \pi_{q+1}(X; A, B) \longrightarrow \pi_q(A, C) \longrightarrow \pi_q(X, B) \longrightarrow \pi_q(X; A, B) \to \cdots$$

(see (3.4.6)).

Coming back to the Blakers–Massey theorem 6.2.1, we have that the conditions $m \geq 1$ and $n \geq 1$ imply only that $\pi_0(A \cap B) \longrightarrow \pi_0(A)$ and $\pi_0(A \cap B) \longrightarrow \pi_0(B)$ are surjective. The condition $m \geq 2$ guarantees that $\pi_1(X, B) = 0$. By the long exact sequence just given, 6.2.1 is equivalent to the following.

6.2.9 **Theorem**. *Under the same assumptions as the Blakers–Massey theorem,*

$$\pi_q(X; A, B) = 0 \quad \textit{for} \quad 2 \leq q \leq m + n - 2$$

and for any base point $x_0 \in A \cap B$.

Proof: We prove the theorem only in the case that $(X; A, B)$ is a CW-triad. We do it in several steps.

First step. Assume that X is a CW-complex and that A and B are subcomplexes, each obtained from $C = A \cap B$ by attaching a cell.

We have that $A = C \cup e^m$ and $B = C \cup e^n$, where $m \geq 2$ and $n \geq 1$. Also take $x_0 \in C$.

Given a map of tetrads

$$\begin{array}{c}(I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \\ \big\downarrow f \\ (X; A, B, x_0),\end{array}$$

where $2 \leq q \leq m + n - 2$, we must prove that f is nullhomotopic as a map of tetrads. Given interior points $x \in e^m$ and $y \in e^n$, there are inclusions of pointed triads

$$\begin{aligned}(A; A, A - \{x\}) \subset (X - \{y\}; A, X - \{x, y\}) \subset \\ \subset (X; A, X - \{x\}) \supset (X; A, B)\,.\end{aligned}$$

The first and third inclusions induce isomorphisms in triad homotopy groups, thanks to the radial deformations away from x of $X - \{x\}$ onto B and away from y of $X - \{y\}$ onto A. It is immediate to verify that $\pi_r(A; A, A') = 0$ for all r and any $A' \subset A$. We shall be done if we can show for adequate x, y that f regarded as a map of pointed triads into $(X; A, X - \{x\})$ is homotopic to a map f' whose image lies in $(X - \{y\}; A, X - \{x, y\})$, since this will imply that f is nullhomotopic.

Let $e^m_{1/2} \subset e^m$ and $e^n_{1/2} \subset e^n$ be the subcells of half of the radius. We may subdivide the cube I^q into subcubes I^q_α in such a way that for each α, $f(I^q_\alpha)$ lies in the interior of e^m if it intersects $e^m_{1/2}$ and lies in the interior of e^n if it intersects $e^n_{1/2}$. We may now deform f to be homotopic as a map of tetrads to a map g whose restriction to the $(n-1)$-skeleton of I^q with its cubically subdivided CW-structure does not cover $e^n_{1/2}$ and whose restriction to the $(m-1)$-skeleton of I^q does not cover $e^m_{1/2}$. Moreover, one may assume that g can be so selected that the dimension of $g^{-1}(y)$ is at most $q - n$ for some point $y \in e^n_{1/2}$ that is not in the image under g of the $(n-1)$-skeleton of I^q. (This very important step in the proof can be achieved if one uses Theorem 2 in Basic Concepts and Notation to deform f to g in such a way that g is smooth in a smaller subcell, and then choose y as a common regular value of the restriction of g to each cell.)

Now let $\pi : I^q \longrightarrow I^{q-1}$ be the projection on the first $q - 1$ coordinates and let $K = \pi^{-1}(\pi(g^{-1}(y)))$. Then the dimension of K can exceed by at most one the dimension of $g^{-1}(y)$, so that

$$\dim K \leq q - n + 1 \leq m - 1\,.$$

Therefore, $g(K)$ cannot cover $e^m_{1/2}$. Choose a point $x \in e^m_{1/2}$ such that $x \notin g(K)$. Since $g(\partial I^{q-1} \times I) \subset A$, we have that the sets $\pi(g^{-1}(x)) \cup \partial I^{q-1}$ and $g^{-1}(y)$ are disjoint closed subsets of I^{q-1}. Applying Urysohn's lemma (see [60]), one can find a map $v : I^{q-1} \longrightarrow I$ such that

$$v(\pi(g^{-1}(x)) \cup \partial I^{q-1}) = 0 \quad \text{and} \quad v(g^{-1}(y)) = 1\,.$$

Define now $h : I^{q+1} \longrightarrow I^q$ by

$$h(r,s,t) = (r, s - stv(r)) \quad \text{for} \quad r \in I^{q-1} \quad \text{and} \quad s,t \in I\,.$$

Then let $f' = g \circ h_1$, where $h_1(r,s) = h(r,s,1)$. We claim that f' is as desired. First observe that

$$h(r,s,0) = (r,s),\ h(r,0,t) = (r,0) \text{ and } h(r,s,t) = (r,s) \text{ if } r \in \partial I^q.$$

Moreover,

$$h(r,s,t) = (r,s) \ \text{ if } \ h(r,s,t) \in g^{-1}(x)\,,$$

since $r \in \pi(g^{-1}(x))$ implies $v(r) = 0$, and

$$h(r,s,t) = (r, s - st) \ \text{ if } \ h(r,s,t) \in g^{-1}(y)\,,$$

since $r \in \pi(g^{-1}(y))$ implies $v(r) = 1$. Then $g \circ h$ is a homotopy of maps of tetrads

$$\begin{array}{c} (I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \\ \big\downarrow \\ (X; A, X - \{x\}, x_0) \end{array}$$

from g to f', and f' has image in $(X - \{y\}; A, X - \{x,y\})$, as we wished.

Second step. Assume that X is a CW-complex and that A and B are subcomplexes, each obtained from $C = A \cap B$ by attaching a finite number of cells.

We suppose that $C \subset A' \subset A$, where A is obtained from A' by attaching a single cell. Let $X' = A' \cup_C B$. If the statement of the theorem holds for the triads $(X'; A', B)$ and $(X; A, A')$, then the result holds for $(X; A, B)$. To see this, we apply the five lemma to the following commutative diagram, which is obtained from the naturality of the exact sequence of a triple (see 3.5.10):

$$\begin{array}{ccccccccc} \pi_{q+1}(A,A') & \to & \pi_q(A',C) & \to & \pi_q(A,C) & \to & \pi_q(A,A') & \to & \pi_{q-1}(A',C) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_{q+1}(X,X') & \to & \pi_q(X',B) & \to & \pi_q(X,B) & \to & \pi_q(X,X') & \to & \pi_{q-1}(X',B). \end{array}$$

We suppose now that $C \subset B' \subset B$, where B is obtained from B' by attaching a single cell. Let $X'' = A \cup_C B'$. If the statement of the theorem holds for the triads $(X''; A, B')$ and $(X; X', B)$, then the result holds for $(X; A, B)$, since the inclusion $(A, C) \hookrightarrow (X, B)$ factors as the composite

$$(A, C) \hookrightarrow (X'', B') \hookrightarrow (X, B)\,.$$

Third step. Assume that X is a CW-complex and that A and B are subcomplexes such that $X = A \cup B$.

Let again $C = A \cap B$. Since (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, we may assume that there are only q-cells in $A - C$ with $q \geq m$ and in $B - C$ with $q \geq n$. We may also assume that (A, C) and (B, C) have at least one cell since otherwise the result would hold trivially. □

6.2.10 REMARK. The general case of 6.2.9 for any excisive triad $(X; A, B)$ follows from the cellular case just proved by approximating it with a CW-triad of the same weak homotopy type. One easily sees that this does not change the triad homotopy groups. This approximation can be achieved using the cellular approximation theorem 5.1.44. As a matter of fact, in the third step of the proof we assume that there are only q-cells in $A - C$ with $q \geq m$ and in $B - C$ with $q \geq n$. This follows also from 6.3.20. We should remark that the proof of 6.3.20 is straightforward and does not require the Blakers–Massey theorem 6.2.1.

The next proposition allows us to study some of the homotopy properties of the Moore spaces.

6.2.11 **Proposition**. *Let (X, A) be a* CW-*pair such that all of the cells of $X - A$ have dimension larger than n. Then the pair (X, A) is n-connected.*

Proof: We have to prove that $\pi_q(X, A) = 0$ for $q \leq n$. Suppose that $f : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X, A)$ represents an arbitrary element $[f] \in \pi_q(X, A)$. Now $[f] = 0$ if and only if $f \simeq g \operatorname{rel} \mathbb{S}^{q-1}$ for some $g : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X, A)$ that satisfies $g(\mathbb{D}^q) \subset A$. According to 5.1.44 there exists a cellular map $\varphi : (\mathbb{D}^q, \mathbb{S}^{q-1}) \longrightarrow (X, A)$ such that $f \simeq \varphi \operatorname{rel} \mathbb{S}^{q-1}$. But by hypothesis we have $X^n \subset A$, and then since φ is cellular, it follows that $\varphi(\mathbb{D}^q) \subset X^q \cup A \subset X^n \cup A = A$. □

6.2.12 DEFINITION. Suppose that X is a CW-complex and that $i : X \hookrightarrow \operatorname{SP} X$ is the canonical inclusion into its infinite symmetric product. Then i induces a homomorphism

$$h_X : \pi_q(X) \longrightarrow H_q(X)$$

for each q; this is called the *Hurewicz homomorphism.*

In the following section, in Theorem 6.3.24, which is the famous and important Hurewicz theorem, we shall analyze under what circumstances it is an isomorphism.

6.2.13 Remark. Equation (6.1.8) can be rewritten in terms of homology as

$$H_q(X \vee Y) \cong H_q(X) \oplus H_q(Y)$$

for any (pointed) spaces X, Y and $q > 0$ (cf. 5.3.31). Moreover, (6.1.9) implies the compatibility of the Hurewicz homomorphism with the sum decomposition; namely, one has a commutative diagram

$$\begin{tikzcd} \pi_q(X \vee Y) \arrow[r] \arrow[d, "h_{X\vee Y}"'] & \pi_q(X) \oplus \pi_q(Y) \arrow[d, "h_X \oplus h_Y"] \\ H_q(X \vee Y) \arrow[r, "\cong"'] & H_q(X) \oplus H_q(Y)\,. \end{tikzcd}$$

6.3 Homotopy Properties of the Moore Spaces

In this section we shall go deeper into the study of the properties of Moore spaces. This will be useful for us in later chapters.

In order to apply Proposition 6.2.11 in the previous section to Moore spaces we shall use the following result.

6.3.1 **Lemma**. *The nth homotopy group of the wedge $\bigvee_\alpha \mathbb{S}^n_\alpha$ of n-spheres is given in terms of the inclusion maps $i_\alpha : \mathbb{S}^n = \mathbb{S}^n_\alpha \hookrightarrow \bigvee_\alpha \mathbb{S}^n_\alpha$ as follows.*

(a) *For $n > 1$ we have that $\pi_n(\bigvee_\alpha \mathbb{S}^n_\alpha)$ is the free abelian group generated by the classes $[i_\alpha]$.*

(b) *For $n = 1$ we have that $\pi_1(\bigvee_\alpha \mathbb{S}^1_\alpha)$ is the free group generated by the classes $[i_\alpha]$.*

Proof: First we shall consider the case of a finite wedge $\mathbb{S}^n_1 \vee \mathbb{S}^n_2 \vee \cdots \vee \mathbb{S}^n_r$ for some finite $r > 1$. (The case $r = 1$ is already known.) Assuming that each sphere $\mathbb{S}^n_i$ has a CW-structure with one 0-cell and one n-cell, then by Proposition 5.1.46 the product $\mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r$ is a CW-complex that contains the wedge as the subcomplex consisting of those products of cells, say $e_1 \times \cdots \times e_r$, where all except for possibly one of these cells is the 0-cell of $\mathbb{S}^n$. Consequently, the cells of $\mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r - \mathbb{S}^n_1 \vee \mathbb{S}^n_2 \vee \cdots \vee \mathbb{S}^n_r$ have dimension greater than or equal to $2n$, and so, by Proposition 6.2.11, we have that

$$\pi_q(\mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r, \mathbb{S}^n_1 \vee \mathbb{S}^n_2 \vee \cdots \vee \mathbb{S}^n_r) = 0$$

for $q \leq 2n-1$. Using the exact homotopy sequence of a pair (see 3.5.8(e)), we get that the inclusion $j : \mathbb{S}^n_1 \vee \mathbb{S}^n_2 \vee \cdots \vee \mathbb{S}^n_r \hookrightarrow \mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r$ induces an isomorphism $\pi_q(\mathbb{S}^n_1 \vee \mathbb{S}^n_2 \vee \cdots \vee \mathbb{S}^n_r) \longrightarrow \pi_q(\mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r)$ for $q \leq 2n-2$.

On the other hand, for $q \geq 1$ we have an isomorphism $(p_{1_*}, p_{2_*}, \ldots, p_{r_*}) : \pi_q(\mathbb{S}^n_1 \times \mathbb{S}^n_2 \times \cdots \times \mathbb{S}^n_r) \longrightarrow \pi_q(\mathbb{S}^n_1) \times \pi_q(\mathbb{S}^n_2) \times \cdots \times \pi_q(\mathbb{S}^n_r)$ induced by the projections of the product onto its factors. Because $p_k \circ j \circ i_k = \mathrm{id}$ holds, it follows that $(p_{1_*}, p_{2_*}, \ldots, p_{r_*}) \circ j_* \circ \bigoplus_{k=1}^r i_{k*} = \mathbf{1}$, and thus $\bigoplus_{k=1}^r i_{k*}$ is an isomorphism for $q \leq 2n-2$.

Suppose now that the set of indices is infinite. But any (pointed) map $f : \mathbb{S}^n \longrightarrow \bigvee_\alpha \mathbb{S}^n_\alpha$ has a compact image, which is therefore contained in a subwedge $\bigvee_{k=1}^r \mathbb{S}^n_{\alpha_j}$ for some finite $r > 1$. Since the diagram

$$\begin{tikzcd}
\bigoplus_\alpha \pi_q\left(\mathbb{S}^n_\alpha\right) \arrow[r, "\bigoplus_\alpha i_{\alpha *}"] & \pi_q\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \\
\bigoplus_{j=1}^r \pi_q\left(\mathbb{S}^n_{\alpha_j}\right) \arrow[u, hook] \arrow[r, "\bigoplus_{j=1}^r i_{\alpha_j *}", "\cong"'] & \pi_q\left(\bigvee_{j=1}^r \mathbb{S}^n_{\alpha_j}\right) \arrow[u]
\end{tikzcd}$$

commutes, we conclude that $\bigoplus_\alpha i_{\alpha *}$ is surjective.

Similarly, any homotopy $H : \mathbb{S}^q \times I \longrightarrow \bigvee_\alpha \mathbb{S}^n_\alpha$ has a compact image, so that $\bigoplus_\alpha i_{\alpha *}$ is injective. Therefore, $\bigoplus_\alpha i_{\alpha *}$ is an isomorphism for $q \leq 2n-2$. Part (a) is then obtained from the fact that $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ (see 6.2.5(b)).

Finally, part (b) is obtained inductively from the Seifert–van Kampen theorem for the fundamental group; see 3.2.6. □

6.3.2 **Lemma**. *Let $n \geq 1$ be an integer. Suppose that $L(\mathcal{A})$ and $L(\mathcal{B})$ are the free groups (abelian if $n > 1$) generated by the elements of two given sets $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose that $f : L(\mathcal{A}) \longrightarrow L(\mathcal{B})$ is a homomorphism. Then there exists a map $\varphi : \bigvee_{\alpha \in \mathcal{A}} \mathbb{S}^n_\alpha \longrightarrow \bigvee_{\beta \in \mathcal{B}} \mathbb{S}^n_\beta$, unique up to homotopy, such that $f = \varphi_* : \pi_n\left(\bigvee_{\alpha \in \mathcal{A}} \mathbb{S}^n_\alpha\right) \longrightarrow \pi_n\left(\bigvee_{\beta \in \mathcal{B}} \mathbb{S}^n_\beta\right)$.*

Proof: According to Lemma 6.3.1, there are isomorphisms

$$L(\mathcal{A}) \cong \pi_n\left(\bigvee_{\alpha \in \mathcal{A}} \mathbb{S}^n_\alpha\right) \quad \text{and} \quad L(\mathcal{B}) \cong \pi_n\left(\bigvee_{\beta \in \mathcal{B}} \mathbb{S}^n_\beta\right),$$

given on generators by $\alpha \mapsto (i_\alpha : \mathbb{S}^n = \mathbb{S}^n_\alpha \hookrightarrow \bigvee_\alpha \mathbb{S}^n_\alpha)$ and by $\beta \mapsto (i_\beta : \mathbb{S}^n = \mathbb{S}^n_\beta \hookrightarrow \bigvee_\beta \mathbb{S}^n_\beta)$, respectively. Then $f(\alpha)$ corresponds to the homotopy class of some map, say $\varphi(\alpha) : \mathbb{S}^n \longrightarrow \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta$. We define $\varphi : \bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha \longrightarrow \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta$ by $\varphi|\mathbb{S}^n_\alpha = \varphi(\alpha)$ for each $\alpha \in \mathcal{A}$. Obviously, we have $\varphi_* = f$.

In order to prove uniqueness up to homotopy, consider any map $\psi : \bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha \longrightarrow \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta$ that satisfies $\psi_* = f$. Then for each $\alpha \in \mathcal{A}$ we have $\psi_*[i_\alpha] = \varphi_*[i_\alpha]$. This means that $\psi|\mathbb{S}^n_\alpha \simeq \varphi|\mathbb{S}^n_\alpha \operatorname{rel}\{*\}$, and therefore it also follows that $\psi \simeq \varphi \operatorname{rel}\{*\}$. □

We shall now examine in more detail the construction of Moore spaces.

For every integer $n \geq 1$ and every group G (which is assumed to be abelian if $n > 1$) there is a CW-complex, denoted by $M(G,n)$, that has exactly one 0-cell and, at the most, cells of dimension n and $n+1$ and that also satisfies $\pi_n(M(G,n)) \cong G$. If G is free, then according to 6.3.1 it follows that the space $M(G,n) = \bigvee_\alpha \mathbb{S}^n_\alpha$ fulfills these conditions, where $\{\alpha\}$ is a set of generators of G. If G is not free, then we consider a free resolution of G, that is, a short exact sequence

$$0 \longrightarrow L_n(\mathcal{A}) \xrightarrow{f} L_n(\mathcal{B}) \longrightarrow G \longrightarrow 1\,.$$

By Lemma 6.3.2 there exists a map $\varphi : \bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha \longrightarrow \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta$ satisfying $f = \varphi_* : \pi_n\left(\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha\right) \longrightarrow \pi_n\left(\bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta\right)$. Using this discussion, we arrive at the following alternative definition of a Moore space.

6.3.3 Definition. We define a *Moore space of type* (G,n), denoted by $M(G,n)$, to be precisely the mapping cone C_φ of (some) φ. If ψ is another map such that $\psi_* = f$, then the mapping cones C_φ and C_ψ have the same homotopy type.

6.3.4 Note. Suppose that the abelian group G is finitely generated. Then we consider its primary decomposition as given in (6.1.10), and we use the notation of (6.1.10) in the following. Now we can take a free resolution of G, as discussed above, such that $\mathcal{B}$ has $r+k$ elements, say $\beta_1,\ldots,\beta_r,\beta_{r+1},\ldots,\beta_{r+k}$, and $\mathcal{A}$ has k elements, say $\alpha_1,\ldots,\alpha_k$. Moreover, we define $f : L_n(\mathcal{A}) \longrightarrow L_n(\mathcal{B})$ by $f(\alpha_j) = d_j\beta_{r+j}$ for $j = 1,\ldots,k$. In this case, the map $\varphi : \bigvee_{j=1}^k \mathbb{S}^n_{\alpha_j} \longrightarrow \bigvee_{i=1}^{r+k} \mathbb{S}^n_{\beta_i}$ that corresponds to f has the property that $C_\varphi = (\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n) \vee (\mathbb{S}^n \cup_{\alpha_{d_1}} e^{n+1}) \vee \cdots \vee (\mathbb{S}^n \cup_{\alpha_{d_k}} e^{n+1}) = X$, where X is defined in (6.1.11). Therefore, the previous definition of $M(G,n)$ extends that of 6.1.12.

The space $M(G,n)$, that we have just defined has the property that $\pi_n(M(G,n)) \cong G$. In order to see this let us recall that in general, if $\varphi : X \longrightarrow Y$ is continuous, then its mapping cone C_φ satisfies $C_\varphi = M_\varphi/X$, where M_φ is the mapping cylinder of φ and X is included as the top face of M_φ. As we already have mentioned (see 4.2.8), the inclusion into the upper face $i : X \hookrightarrow M_\varphi$ is a cofibration, the canonical inclusion $j : Y \hookrightarrow M_\varphi$ is a homotopy equivalence, and $j \circ \varphi \simeq i$ holds.

Let us now consider the exact homotopy sequence of the pair of spaces $(M_\varphi, \bigvee_\alpha \mathbb{S}^n_\alpha)$ for the case $n > 1$, namely the top of the following diagram:

$$\begin{tikzcd}
\arrow[r] & \pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \arrow[r, "i_*"] \arrow[dr, "\varphi_*"'] & \pi_n(M_\varphi) \arrow[r] & \pi_n(M_\varphi, \bigvee_\alpha \mathbb{S}^n_\alpha) \arrow[r] \arrow[dd, "p_*"', "\cong"] & \pi_{n-1}\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \arrow[r] & {} \\
& & \pi_n\left(\bigvee_\beta \mathbb{S}^n_\beta\right) \arrow[u, "\cong", "j_*"'] & & & \\
& L_n(\mathcal{A}) \arrow[uu, "\cong"'] \arrow[r, "f"] & L_n(\mathcal{B}) \arrow[u, "\cong"] & \pi_n(M(G,n)), & &
\end{tikzcd}$$

where $p : (M_\varphi, \bigvee_\alpha \mathbb{S}^n_\alpha) \longrightarrow (M(G,n), *)$ is the identification map. Because $M_\varphi - \bigvee_\alpha \mathbb{S}^n_\alpha$ only has cells of dimension n and $n+1$, we then have by Proposition 6.2.11 that the pair $(M_\varphi, \bigvee_\alpha \mathbb{S}^n_\alpha)$ is $(n-1)$-connected. Analogously, the wedge $\bigvee_\alpha \mathbb{S}^n_\alpha$ is $(n-1)$-connected as well. Thus from Proposition 6.2.2 we get that p_* is an isomorphism. Since $\pi_{n-1}\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) = 0$, the exact sequence can be rewritten as follows:

$$\begin{tikzcd}
0 \arrow[r] & \pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \arrow[r, "i_*"] \arrow[d, leftrightarrow, "\cong"'] & \pi_n(M_\varphi) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \pi_n\left(M_\varphi, \bigvee_\alpha \mathbb{S}^n_\alpha\right) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & 0 \\
& L_n(\mathcal{A}) \arrow[r, "f"'] & L_n(\mathcal{B}) & \pi_n(M(G,n)). &
\end{tikzcd}$$

Consequently, this gives us the isomorphism $\pi_n(M(G,n)) \cong G$.

On the other hand, by applying the Seifert–van Kampen theorem, it is easy to prove that $\pi_1(M(G,1)) \cong G$.

The next proposition shows that not only groups can be realized topologically using Moore spaces, but that we can also realize group homomorphisms.

6.3.5 **Proposition**. *Let $f : A \longrightarrow A'$ be a homomorphism between the groups A and A'. Then there exists a map $\varphi : M(A,n) \longrightarrow M(A',n)$ such that $\varphi_* = f$.*

Proof: Let us consider the following free resolutions of A and B:

$$\begin{tikzcd} 0 \arrow[r] & L_n(\mathcal{A}) \arrow[r, "l"] \arrow[d, "h"'] & L_n(\mathcal{B}) \arrow[r, "p"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & L_n(\mathcal{A}') \arrow[r, "l'"'] & L_n(\mathcal{B}') \arrow[r, "p'"'] & A' \arrow[r] & 0. \end{tikzcd}$$

Because the rows are exact, we clearly can define g and h so that the diagram commutes. According to Lemma 6.3.2 there exist maps λ, λ', χ, and γ such that the left square in the previous diagram can be realized as the nth homotopy functor applied to the left square in the diagram

$$\begin{tikzcd} \bigvee_\alpha \mathbb{S}^n_\alpha \arrow[r, "\lambda"] \arrow[d, "\chi"'] & \bigvee_\beta \mathbb{S}^n_\beta \arrow[r, hook, "j"] \arrow[d, "\gamma"] & C_\lambda \arrow[d, dashed, "\varphi"] \\ \bigvee_{\alpha'} \mathbb{S}^n_{\alpha'} \arrow[r, "\lambda'"'] & \bigvee_{\beta'} \mathbb{S}^n_{\beta'} \arrow[r, hook, "j'"'] & C_{\lambda'}. \end{tikzcd}$$

Because the topological realization of a homomorphism is unique up to homotopy by Lemma 6.3.2, it follows that $\gamma \circ \lambda \simeq \lambda' \circ \chi$. Using Proposition 3.1.7, we have that $j' \circ \lambda' \simeq 0$, and therefore $j' \circ \gamma \circ \lambda \simeq j' \circ \lambda' \circ \chi \simeq 0$ holds. Using Proposition 3.1.7 again, there exists a map $\varphi : C_\lambda \longrightarrow C_{\lambda'}$ such that the above diagram of spaces commutes.

Now let us consider the exact homotopy sequence of the pair $(M_\lambda, \bigvee_\alpha \mathbb{S}^n_\alpha)$, namely

$$0 \longrightarrow \pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \xrightarrow{i_*} \pi_n(M_\lambda) \longrightarrow \pi_n\left(M_\lambda, \bigvee_\alpha \mathbb{S}^n_\alpha\right) \longrightarrow 0\,,$$

which we have already studied earlier, and let us also consider the diagram

$$\begin{tikzcd} \bigvee_\alpha \mathbb{S}^n_\alpha \arrow[r, hook] \arrow[d, leftrightarrow, "\mathrm{id}"'] & M_\lambda \arrow[r, hook] & (M_\lambda, \bigvee_\alpha \mathbb{S}^n_\alpha) \arrow[d, "p"] \\ \bigvee_\alpha \mathbb{S}^n_\alpha \arrow[r, "\lambda"'] & \bigvee_\beta \mathbb{S}^n_\beta \arrow[r] \arrow[u, hook] & (C_\lambda, *). \end{tikzcd}$$

The left square commutes up to homotopy by 4.2.8(c), and the right square obviously commutes. In this way, the exact sequence of the pair $(M_\lambda, \bigvee_\alpha \mathbb{S}^n_\alpha)$ can be rewritten as

$$0 \longrightarrow \pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \xrightarrow{\lambda_*} \pi_n\left(\bigvee_\beta \mathbb{S}^n_\beta\right) \xrightarrow{j_*} \pi_n(C_\lambda) \longrightarrow 0\,. \tag{6.3.6}$$

A similar result holds for λ'. So we have obtained the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \pi_n\left(\bigvee_\alpha \mathbb{S}^n_\alpha\right) \arrow[r, "\lambda_*"] \arrow[d, "\chi_*=h"'] & \pi_n\left(\bigvee_\beta \mathbb{S}^n_\beta\right) \arrow[r, "j_*"] \arrow[d, "\gamma_*=g"] & \pi_n(C_\lambda) \arrow[r] \arrow[d, "\varphi_*"] & 0 \\ 0 \arrow[r] & \pi_n\left(\bigvee_{\alpha'} \mathbb{S}^n_{\alpha'}\right) \arrow[r, "\lambda'_*"'] & \pi_n\left(\bigvee_{\beta'} \mathbb{S}^n_{\beta'}\right) \arrow[r, "j'_*"'] & \pi_n(C_{\lambda'}) \arrow[r] & 0. \end{tikzcd}$$

By the universal property of the cokernel, we then have that $\varphi_* = f$, as desired. $\square$

6.3.7 **Proposition**. *Suppose that $f : X \longrightarrow Y$ is continuous, that X is $(s-1)$-connected, and that f is an $(r-1)$-equivalence. Then there exists the following exact sequence truncated on the left:*

$$\pi_{r+s-2}(X) \xrightarrow{f_*} \pi_{r+s-2}(Y) \xrightarrow{i_*} \pi_{r+s-2}(C_f) \longrightarrow$$
$$\longrightarrow \pi_{r+s-3}(X) \xrightarrow{f_*} \pi_{r+s-3}(Y) \longrightarrow \cdots .$$

Proof: Let us consider the exact sequence of the pair (M_f, X),

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{i_*} \pi_k(M_f) \xrightarrow{j_*} \pi_k(M_f, X) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \cdots ,$$

and the diagram

$$\begin{tikzcd} & M_f \arrow[d, shift right=1.5ex, "p"'] \arrow[d, phantom, "\simeq"] \\ X \arrow[ur, hook, "i"] \arrow[r, "f"'] & Y, \arrow[u, shift right=1.5ex, "j"'] \end{tikzcd}$$

where $p \circ i = f$ and $j \circ f \simeq i$. Moreover, i is a cofibration. Since f is an $(r-1)$-equivalence, that is, $f_* : \pi_q(X) \longrightarrow \pi_q(Y)$ is an isomorphism for $q \leq r-2$ and an epimorphism for $q = r-1$, we have that the pair (M_f, X) is $(r-1)$-connected. So by Proposition 6.2.2 the quotient map induces an isomorphism $\pi_k(M_f, X) \longrightarrow \pi_k(C_f)$ for $k < r+s-1$. When we substitute $\pi_k(M_f)$ by $\pi_k(Y)$ and $\pi_k(M_f, X)$ by $\pi_k(C_f)$ in the portion of the homotopy sequence of the pair (M_f, X), where $k \leq r+s-2$, we obtain the desired exact sequence. $\square$

The following exercises will be used later in some applications.

6.3.8 Exercise. Let X be any space and let M be locally compact. Prove that $(CX) \wedge M \approx C(X \wedge M)$, where C represents the reduced cone construction. Conclude that for every pointed map $f : X \longrightarrow Y$, $C_{f \wedge \mathrm{id}_M} \approx C_f \wedge M$.

6.3.9 Exercise. Given a pointed pair (X, A) and a pointed space Z, prove that $(X/A) \wedge Z \approx (X \wedge Z)/(A \wedge Z)$.

6.3.10 Exercise. Given the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"] & C_f \arrow[d, dashed, "\gamma"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r] & C_{f'}, \end{tikzcd}$$

where the left square is homotopy commutative, prove that there exists a map $\gamma : C_f \longrightarrow C_{f'}$ that makes the right square commutative. (This amounts to saying that the mapping cone construction is a functor.)

The assertions of 6.3.8 still hold up to homotopy if M is not locally compact. One has the following results.

6.3.11 **Proposition**. *If $f : X \longrightarrow Y$ is a map between pointed spaces and Z is a pointed space, then $C_{f\wedge \mathrm{id}_Z} \simeq C_f \wedge Z$ holds.*

Proof: Recall that if $g : B \hookrightarrow Y$ is a cofibration, we have that $C_g \simeq Y/B$ (see 4.2.3) and that $g \wedge \mathrm{id}_Z : B \wedge Z \hookrightarrow Y \wedge Z$ is also a cofibration. Then we have $C_{g\wedge \mathrm{id}_Z} \simeq (Y \wedge Z)/(B \wedge Z)$, and the latter space is homeomorphic to $(Y/B) \wedge Z \simeq C_g \wedge Z$, according to 6.3.9. It follows that

$$C_{g\wedge \mathrm{id}_Z} \simeq C_g \wedge Z \tag{6.3.12}$$

whenever g is a cofibration.

Now let us transform an arbitrary map f into a cofibration i in the usual way. So we have the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\mathrm{id}"'] & M_f \arrow[r] \arrow[d, "\pi"] & C_i \arrow[d] \\ X \arrow[r] & Y \arrow[r] & C_f, \end{tikzcd} \tag{6.3.13}$$

and then, according to 6.3.10,

$$C_i \simeq C_f. \tag{6.3.14}$$

We can now apply (6.3.12) to $g = i : X \longrightarrow M_f$, thereby obtaining $C_{i\wedge \mathrm{id}_Z} \simeq C_i \wedge Z$, and so, by using (6.3.14), it follows that

$$C_{i\wedge \mathrm{id}_Z} \simeq C_f \wedge Z. \tag{6.3.15}$$

Next we apply 6.3.10 to the diagram

$$\begin{tikzcd} X \wedge Z \arrow[r, "i\wedge \mathrm{id}_Z"] \arrow[d, "\mathrm{id}"'] & M_f \wedge Z \arrow[r] \arrow[d, "\pi\wedge \mathrm{id}_Z"', "\simeq"] & C_{i\wedge \mathrm{id}_Z} \arrow[d, "\simeq"] \\ X \wedge Z \arrow[r, "f\wedge \mathrm{id}_Z"'] & Y \wedge Z \arrow[r] & C_{f\wedge \mathrm{id}_Z} \end{tikzcd}$$

and get that $C_{f\wedge \mathrm{id}_Z} \simeq C_{i\wedge \mathrm{id}_Z} \simeq C_f \wedge Z$, where the latter homotopy equivalence is just (6.3.15). $\square$

6.3.16 **Proposition**. *Suppose that X and Y are* CW*-complexes, each one having countably many cells. Moreover, suppose that for some r, $s > 1$ we have trivial skeletons $X^{r-1} = \{*\}$ and $Y^{s-1} = \{*\}$. Then the homomorphism*

$$h : \pi_r(X) \otimes \pi_s(Y) \longrightarrow \pi_{r+s}(X \wedge Y)$$

defined by

$$[\alpha] \otimes [\beta] \longmapsto [\alpha \wedge \beta]$$

is an isomorphism.

Proof: Since $\pi_r(X) = \pi_r(X^{r+1})$ by Proposition 5.1.25 and since $X^{r-1} = \{*\}$, we have that $X^r = \bigvee_\beta \mathbb{S}^r_\beta$ and that $X^{r+1} = C_\lambda$, for some map $\lambda : \bigvee_\alpha \mathbb{S}^r_\alpha \longrightarrow \bigvee_\beta \mathbb{S}^r_\beta$. Let us consider the diagram

$$\begin{tikzcd}
\pi_r\left(\bigvee_\alpha \mathbb{S}^r_\alpha\right) \otimes \pi_s(Y) \arrow[r, "\lambda_*\otimes 1"] \arrow[d, "f"'] & \pi_r\left(\bigvee_\beta \mathbb{S}^r_\beta\right) \otimes \pi_s(Y) \arrow[r, two heads, "j_*\otimes 1"] \arrow[d, "g"] & \pi_r(X^{r+1}) \otimes \pi_s(Y) \arrow[d, "h'"] \\
\pi_{r+s}\left(\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y\right) \arrow[r] & \pi_{r+s}\left(\bigvee_\beta \mathbb{S}^r_\beta \wedge Y\right) \arrow[r, two heads] & \pi_{r+s}(X^{r+1} \wedge Y),
\end{tikzcd}$$

where f, g, and h' are defined in the same way as h was.

The first row in this diagram is the tensor product of the exact sequence (6.3.6) with $\pi_s(Y)$, so that it is also exact, except that $\lambda_* \otimes 1$ is not necessarily a monomorphism.

Let us now take the map $\lambda \wedge \mathrm{id}_Y : \bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y \longrightarrow \bigvee_\beta \mathbb{S}^r_\beta \wedge Y$. As we saw in the proof of Proposition 5.1.51, we have trivial skeletons $\left(\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y\right)^{r+s-1} = \left(\bigvee_\beta \mathbb{S}^r_\beta \wedge Y\right)^{r+s-1} = \{*\}$. So each of these spaces is $(r+s-1)$-connected. Moreover, by using Proposition 6.3.11, we have $C_{\lambda \wedge \mathrm{id}_Y} \simeq C_\lambda \wedge Y = X^{r+1} \wedge Y$, and so the second row of the diagram is the exact sequence of Proposition 6.3.7.

Obviously, we have $\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y \approx \bigvee_\alpha (\mathbb{S}^r_\alpha \wedge Y)$, which implies $\pi_{r+s}\left(\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y\right) \cong \pi_{r+s}\left(\bigvee_\alpha (\mathbb{S}^r_\alpha \wedge Y)\right)$. Using the same method as in the proof of Lemma 6.3.1 we get that

$$\pi_{r+s}\left(\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y\right) \cong \bigoplus_\alpha \pi_{r+s}\left(\mathbb{S}^r_\alpha \wedge Y\right).$$

But by the Freudenthal suspension theorem 6.2.4, we have that $\pi_{r+s}(\mathbb{S}^r \wedge Y) \cong \pi_s(Y)$, and so $\pi_{r+s}(\bigvee_\alpha \mathbb{S}^r_\alpha \wedge Y) \cong \bigoplus_\alpha \pi_s(Y)$. By Lemma 6.3.1, we have $\pi_r\left(\bigvee_\alpha \mathbb{S}^r_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$, which then gives us

$$\pi_r\left(\bigvee_\alpha \mathbb{S}^r_\alpha\right) \otimes \pi_s(Y) \cong \bigoplus_\alpha \pi_s(Y).$$

From this we get that f is an isomorphism, and in exactly the same manner we obtain that g is an isomorphism. It then follows from the five lemma that h' is also an isomorphism. Finally, because $(X^{r+1} \wedge Y)^{r+s+1} = (X \wedge Y)^{r+s+1}$ holds, we have that the inclusion $X^{r+1} \wedge Y \hookrightarrow X \wedge Y$ is an $(r+s+1)$-equivalence and that the square

$$\begin{tikzcd}
\pi_r(X^{r+1}) \otimes \pi_s(Y) \arrow[r, "h'", "\cong"'] \arrow[d, "\cong"'] & \pi_{r+s}(X^{r+1} \wedge Y) \arrow[d, "\cong"] \\
\pi_r(X) \otimes \pi_s(Y) \arrow[r, "h"'] & \pi_{r+s}(X \wedge Y)
\end{tikzcd}$$

commutes, implying that h is an isomorphism as well. $\square$

We shall now show that the natural inclusion $i : M(G,n) \longrightarrow \mathrm{SP}\, M(G,n)$ induces isomorphisms in homotopy up to dimension n and an epimorphism in dimension $n+1$; that is, i is an $(n+1)$-equivalence. In order to do this we shall need the following fundamental lemma.

6.3.17 **Lemma**. *Let $\varphi : X \longrightarrow Y$ be a map, where X and Y are $(n-1)$-connected spaces. If the inclusion maps $i : X \hookrightarrow \mathrm{SP}\, X$ and $j : Y \hookrightarrow \mathrm{SP}\, Y$ are $(n+1)$-equivalences (see Definition 5.1.17), then the inclusion $k : C_\varphi \hookrightarrow \mathrm{SP}\, C_\varphi$ is also an $(n+1)$-equivalence.*

Proof: We shall assume that $n > 1$ and shall leave the case $n = 1$ to the reader. Let M_φ be the mapping cylinder of φ. We consider the exact homotopy sequence of the pair (M_φ, X),

$$\cdots \longrightarrow \pi_q(X) \longrightarrow \pi_q(M_\varphi) \longrightarrow \pi_q(M_\varphi, X) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots,$$

as given in 3.5.8(e). Since both X and $M_\varphi \simeq Y$ are $(n-1)$-connected (which means that $\pi_q(M_\varphi) = 0 = \pi_{q-1}(X)$ for $q \leq n-1$), it follows that the pair $(M\varphi, X)$ is $(n-1)$-connected. By Propostion 6.2.2, the quotient map $p : M_\varphi \longrightarrow M_\varphi/X = C_\varphi$ induces an isomorphism $p_* : \pi_q(M\varphi, X) \longrightarrow \pi_q(C_\varphi)$ for $q < 2n-1$. Then for every such q the diagram

$$\begin{tikzcd}
\pi_q(X) \arrow[r, "\varphi_*"] \arrow[d, "i_*"'] & \pi_q(Y) \arrow[r] \arrow[d, "j_*"'] & \pi_q(C_\varphi) \arrow[r] \arrow[d, "k_*"] & \pi_{q-1}(X) \arrow[r] \arrow[d, "i_*"] & \cdots \\
\pi_q(\mathrm{SP}\, X) \arrow[r] & \pi_q(\mathrm{SP}\, Y) \arrow[r] & \pi_q(\mathrm{SP}\, C_\varphi) \arrow[r] & \pi_{q-1}(\mathrm{SP}\, X) \arrow[r] & \cdots
\end{tikzcd}$$

commutes, where the horizontal sequences are exact. (The lower one is exact by the Dold–Thom theorem.) Using the five lemma, we immediately conclude that the inclusion k is an $(n+1)$-equivalence. $\square$

6.3.18 **Lemma**. *Let $\mathbb{S}^n_\alpha$ be a copy of the n-sphere $\mathbb{S}^n$ for every $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary set. Then $X = \bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha$ is $(n-1)$-connected, and the canonical inclusion $i : X \hookrightarrow \operatorname{SP} X$ is an $(n+1)$-equivalence.*

Proof: We assume that $n > 1$. Since the $(n-1)$-skeleton of X is a point, X is $(n-1)$-connected. First let us assume that the set $\mathcal{A}$ is finite. According to 6.3.1(a) the canonical inclusions $i_\alpha : \mathbb{S}^n_\alpha \hookrightarrow X$ induce an isomorphism $\bigoplus \pi_n(\mathbb{S}^n_\alpha) \longrightarrow \pi_n(X)$. Moreover, by induction, the commutativity of the diagram (6.1.9) implies that we have a commutative diagram

$$\begin{tikzcd}
\bigoplus \pi_n(\mathbb{S}^n_\alpha) \arrow[r] \arrow[d] & \pi_n(X) \arrow[d] \\
\bigoplus \pi_n(\operatorname{SP}\mathbb{S}^n_\alpha) \arrow[r] & \pi_n(\operatorname{SP} X)\,,
\end{tikzcd}$$

where the horizontal arrows are isomorphisms. By Proposition 6.2.7 the vertical arrow on the left is also an isomorphism, and so it follows that the vertical arrow on the right is an isomorphism as well.

Using Theorem 6.1.13, we have that $\pi_{n+1}(\operatorname{SP} X) = 0$, and so the inclusion i is an $(n+1)$-equivalence in this case, namely, in the case that $\mathcal{A}$ is finite.

In the case that the set $\mathcal{A}$ is infinite, we use the fact that X is the colimit of finite wedges and that $\operatorname{SP} X$ is the colimit of infinite symmetric products of finite wedges. Since the infinite direct sum is also the colimit of its finite subsums, by passing to the colimit we extend the result of the finite case to the present case.

The case $n = 1$, with due care, follows analogously using 6.3.1(b) instead. $\square$

6.3.19 **Theorem**. *Let X be a* CW*-complex whose $(n-1)$-skeleton is a point. Then the inclusion $i : X \longrightarrow \operatorname{SP} X$ is an $(n+1)$-equivalence.*

Proof: Because the $(n-1)$-skeleton of X is a point, its n-skeleton X^n is a wedge of n-spheres $\bigvee \mathbb{S}^n_\beta$, and its $(n+1)$-skeleton is obtained as a mapping cone; that is, there is a map $\varphi^n : \bigvee \mathbb{S}^n_\alpha \longrightarrow \bigvee \mathbb{S}^n_\beta$ such that $X^{n+1} = C_{\varphi^n}$. Therefore, by Lemma 6.3.18 the hypotheses of Lemma 6.3.17 are satisfied, and consequently, the canonical inclusion $i^{n+1} : X^{n+1} \hookrightarrow \operatorname{SP} X^{n+1}$ is an $(n+1)$-equivalence.

Let us assume inductively that the canonical inclusion $i^{n+k} : X^{n+k} \hookrightarrow \operatorname{SP} X^{n+k}$ is an $(n+1)$-equivalence. Once again, the $(n+k+1)$-skeleton is obtained as a mapping cone of some $\varphi^{n+k} : \bigvee \mathbb{S}^{n+k}_\beta \longrightarrow X^{n+k}$, so that

$X^{n+k+1} = C_{\varphi^{n+k}}$. Since $\bigvee \mathbb{S}^{n+k}_\beta$ and X^{n+k} are $(n-1)$-connected, $i^{n+k+1}: X^{n+k+1} \longrightarrow \mathrm{SP}\, X^{n+k+1}$ is an $(n+1)$-equivalence by Lemma 6.3.17.

Finally, since X and $\mathrm{SP}\, X$ are colimits of X^{n+k} and $\mathrm{SP}\, X^{n+k}$, respectively, we have the desired result. □

The next result that we prove gives us, in particular, the CW-approximation of any topological space (see 5.1.35).

6.3.20 **Theorem**. *Let X be an $(n-1)$-connected pointed topological space. Then there exists a* CW-*approximation $\widetilde{X}$; that is, there exist both a* CW-*complex whose $(n-1)$-skeleton $\widetilde{X}^{n-1}$ is a point and a weak homotopy equivalence $\varphi : \widetilde{X} \longrightarrow X$. If, in particular, X is a* CW-*complex, then φ is a homotopy equivalence.*

Proof: First we assume that X is connected, which means that $n \geq 1$. Then we have that $\pi_q(X) = 0$ for $q < n$. Put $* = Y^0 = \cdots = Y^{n-1}$ and define $\varphi_{n-1} : Y^{n-1} \longrightarrow X$ by $\varphi_{n-1}(*) = *$, where $*$ denotes also the base point of X. Then φ_{n-1} is an $(n-1)$-equivalence.

Let us assume inductively that we have already constructed an m-equivalence $\varphi_m : Y^m \longrightarrow X$ for $m \geq n-1$. Then $(\varphi_m)_* : \pi_q(Y^m) \longrightarrow \pi_q(X)$ is an isomorphism for $q \leq m-1$ and an epimorphism for $q = m$. In order to change this last map into an isomorphism, we shall do the following.

Let $\Phi : L(\mathcal{B}) \twoheadrightarrow \ker((\varphi_m)_*) \subset \pi_m(Y^m)$ be a free resolution, and define $h : \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^m_\beta \longrightarrow Y^m$, where $\mathbb{S}^m_\beta = \mathbb{S}^m$ for all β, so that each $h_\beta = h|\mathbb{S}^m_\beta : \mathbb{S}^m \longrightarrow Y^m$ represents a generator of $\ker((\varphi_m)_*)$. Therefore, $\varphi_m \circ h \simeq 0$, and so φ_m determines a map $\psi_{m+1} : C_h \longrightarrow X$ such that the diagram

$$\begin{tikzcd} \bigvee \mathbb{S}^m_\beta \arrow[r, "h"] & Y^m \arrow[r] \arrow[dr, "\varphi_m"'] & C_h \arrow[d, dashed, "\psi_{m+1}"] \\ & & X \end{tikzcd}$$

commutes. The map ψ_{m+1} induces isomorphisms in homotopy up to dimension m. Then $Z^{m+1} = C_h$ is a CW-complex of dimension $m+1$, whose m-skeleton is Y^m.

However, the homomorphism

$$(\psi_{m+1})_* : \pi_{m+1}(Z^{m+1}) \longrightarrow \pi_{m+1}(X)$$

is not necessarily an epimorphism.

Define the set $\mathcal{A} = \pi_{m+1}(X) - (\psi_{m+1})_*(\pi_{m+1}(Z^{m+1}))$. The map $\varphi_{m+1} = (\psi_{m+1}, (\gamma_\alpha)) : Y^{m+1} = Z^{m+1} \vee (\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^{m+1}_\alpha) \longrightarrow X$, where $\gamma_\alpha : \mathbb{S}^{m+1}_\alpha = \mathbb{S}^{m+1} \longrightarrow X$ represents the element $\alpha \in \mathcal{A}$, induces isomorphisms in homotopy up to dimension m and an epimorphism in dimension $m+1$; namely, φ_{m+1} is an $(m+1)$-equivalence that extends φ_m.

So we have constructed a chain of CW-complexes

$$* = \cdots = Y^{n-1} \subset Z^n \subset Y^n \subset \cdots \subset Y^m \subset Z^{m+1} \subset Y^{m+1} \subset \cdots$$

such that the maps $\varphi_m : Y^m \longrightarrow X$ are compatible in the union $\widetilde{X} = \bigcup_m Y^m$ and determine the desired weak homotopy equivalence $\varphi : \widetilde{X} \longrightarrow X$.

If X is not connected (which means that $n = 0$), then we construct a CW-approximation for each connected component of X as above. □

Here is the relative version of the previous theorem.

6.3.21 **Theorem**. *Let (X, A) be an $(n-1)$-connected pair of spaces. Then there exists a* CW-*approximation $(\widetilde{X}, \widetilde{A})$; that is, there exist both a* CW-*pair whose $(n-1)$-skeleton is such that $\widetilde{X}^{n-1} = \widetilde{A}$ and a weak homotopy equivalence $\varphi : (\widetilde{X}, \widetilde{A}) \longrightarrow (X, A)$. If, in particular, (X, A) is a* CW-*pair, then φ is a homotopy equivalence of pairs.*

Proof: The proof is very similar to the above. Namely, first construct a CW-approximation $\varphi_A : \widetilde{A} \longrightarrow A$ as pointed spaces, and then proceed as in the former proof, but instead of starting the construction with a singleton $*$ we do it starting with $\widetilde{A}$.

More specifically, we take $\widetilde{A} = Y^0 = Y^1 = \cdots = Y^{n-1}$ and take $\varphi_{n-1} = \varphi_A$. Then $\varphi_{n-1} : Y^{n-1} \longrightarrow X$ is obviously an $(n-1)$-equivalence, since the pair (X, A) is $(n-1)$-connected (see 5.1.20).

Inductively we may assume that we have already constructed an m-equivalence $\varphi_m : Y^m \longrightarrow X$, $m \geq n-1$ such that $\varphi_m|\widetilde{A} = \varphi_A$. Then $\varphi_{m*} : \pi_q(Y^m) \longrightarrow \pi_q(X)$ is an isomorphism for $q \leq m-1$ and an epimorphism for $q = m$. The rest of the proof follows exactly as before.

At the end, we obtain a weak homotopy equivalence $\varphi : \widetilde{X} \longrightarrow X$ such that $\varphi|\widetilde{A} = \varphi_A : \widetilde{A} \longrightarrow A$ is also a weak homotopy equivalence. Thus, $\varphi : (\widetilde{X}, \widetilde{A}) \longrightarrow (X, A)$ is a weak homotopy of pairs, as desired. □

6.3.22 EXERCISE. Given a CW-complex Y and maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}_X$, prove that X has the homotopy type of a CW-complex. In this case one says that Y *dominates* X. (Hint: Prove that every CW-approximation $\varphi : \widetilde{X} \longrightarrow X$ is a homotopy equivalence.)

The next theorem, which follows from 6.3.16 and 6.3.20, will be handy in the next section.

6.3.23 **Theorem**. *Suppose that X and Y are* CW*-complexes, each with countably many cells that are $(r-1)$-connected and $(s-1)$-connected, respectively. Then the homomorphism*

$$h : \pi_r(X) \otimes \pi_s(Y) \longrightarrow \pi_{r+s}(X \wedge Y)$$

defined by

$$[\alpha] \otimes [\beta] \mapsto [\alpha \wedge \beta]$$

is an isomorphism, provided that $r, s > 1$.

Proof: According to Theorem 6.3.20, X and Y have the same homotopy type as some CW-complexes $\widetilde{X}$ and $\widetilde{Y}$ that satisfy $\widetilde{X}^{r-1} = \{*\}$ and $\widetilde{Y}^{s-1} = \{*\}$. Since $\widetilde{h} : \pi_r(\widetilde{X}) \otimes \pi_s(\widetilde{Y}) \longrightarrow \pi_{r+s}(\widetilde{X} \wedge \widetilde{Y})$ is an isomorphism by Proposition 6.3.16, we can substitute $\widetilde{X}$ with X and $\widetilde{Y}$ with Y and thereby get that $h : \pi_r(X) \otimes \pi_s(Y) \longrightarrow \pi_{r+s}(X \wedge Y)$ also is an isomorphism. □

As one consequence of Theorems 6.3.20 and 6.3.19 we have the following fundamental result. This will be reformulated below as the Hurewicz theorem (6.3.25).

6.3.24 **Theorem**. *Let X be an $(n-1)$-connected* CW*-complex. Then the canonical inclusion $i : X \longrightarrow \mathrm{SP}\, X$ into the infinite symmetric product is an $(n+1)$-equivalence.*

Proof: We have to show that $i_* : \pi_q(X) \longrightarrow \pi_q(\mathrm{SP}\, X)$ is an isomorphism for $q \leq n$ and an epimorphism for $q = n+1$. By Theorem 6.3.20, we have a weak homotopy equivalence $h : \widetilde{X} \longrightarrow X$, where $\widetilde{X}$ is a CW-complex whose $(n-1)$-skeleton is a point. Actually, because X is a CW-complex, it follows that h is a homotopy equivalence. By applying 5.2.10, we then have that $\widehat{h} : \mathrm{SP}\, \widetilde{X} \longrightarrow \mathrm{SP}\, X$ is a homotopy equivalence. On the other hand 6.3.19 implies that the natural inclusion $\widetilde{i} : \widetilde{X} \longrightarrow \mathrm{SP}\, \widetilde{X}$ is an $(n+1)$-equivalence. Consequently, since we have a commutative diagram

$$\begin{tikzcd}
\pi_q(\widetilde{X}) \arrow[r, "\widetilde{i}_*"] \arrow[d, "h_*"', "\cong"] & \pi_q(\mathrm{SP}\, \widetilde{X}) \arrow[d, "\widehat{h}_*", "\cong"'] \\
\pi_q(X) \arrow[r, "i_*"'] & \pi_q(\mathrm{SP}\, X) ,
\end{tikzcd}$$

whenever $\widetilde{i}_*$ is an isomorphism (respectively, epimorphism), then i_* is an isomorphism (respectively, epimorphism). □

As an immediate consequence of the previous theorem, we obtain the famous and important Hurewicz theorem.

6.3.25 **Theorem**. (Hurewicz isomorphism theorem) *Let X be an $(n-1)$-connected* CW*-complex. Then the Hurewicz homomorphism $h_X : \pi_q(X) \longrightarrow H_q(X)$ is an isomorphism for $q \leq n$ and an epimorphism for $q = n+1$.* □

The following proposition relates our definition of the Hurewicz homomorphism with the most usual one as given by other authors. Recall that $H^q(\mathbb{S}^q) \cong Z$ and that the canonical generator $g_q \in H^q(\mathbb{S}^q)$ is the image of $[\mathrm{id}_{\mathbb{S}^q}]$ under the Hurewicz homomorphism $h_{\mathbb{S}^q} : \pi_q(\mathbb{S}^q) \longrightarrow \pi_q(\mathrm{SP}\,\mathbb{S}^q) = H_q(\mathbb{S}^q)$.

6.3.26 **Proposition**. *If $\xi \in \pi_q(X)$ is represented by a map $\alpha : \mathbb{S}^q \longrightarrow X$, then $h_X(\xi) = \alpha_*(g_q)$.*

Proof: Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_q(\mathbb{S}^q) \arrow[r, "h_{\mathbb{S}^q}"] \arrow[d, "\alpha_*"'] & \pi_q(\mathrm{SP}\,\mathbb{S}^q) \arrow[d, "\widehat{\alpha}_*"] \\
\pi_q(X) \arrow[r, "h_X"] & \pi_q(\mathrm{SP}\,X)\,.
\end{tikzcd}$$

Chasing $[\mathrm{id}_{\mathbb{S}^q}] \in \pi_q(\mathbb{S}^q)$ along the diagram shows the desired result. □

There is a relative version of the Hurewicz isomorphism theorem. First we have a relative version of the Hurewicz homomorphism. To that end recall that by 6.2.6, $\pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$ for $n \geq 1$, generated by $g'_N = [\mathrm{id}_(\mathbb{D}^n, \mathbb{S}^{n-1})]$.

6.3.27 Definition. Suppose that (X, A) is a CW-pair. Then the homomorphism

$$h_{(X,A)} : \pi_q(X, A) \longrightarrow H_q(X, A)$$

for $q \geq 1$ such that for an element $\eta \in \pi_q(X, A)$, represented by a map $\beta : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$, we have $h_{(X,A)}(\eta) = \beta_*(g'_n)$, where $g'_n \in \pi_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ is the generator, is called the *relative Hurewicz homomorphism*.

The next result follows immediately from 6.3.26.

6.3.28 **Proposition**. *Let (X, A) be a pair of spaces. Then for $q \geq 1$, the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_q(A) \arrow[r] \arrow[d, "h_A"'] & \pi_q(X) \arrow[r] \arrow[d, "h_X"'] & \pi_q(X,A) \arrow[r] \arrow[d, "h_{(X,A)}"'] & \pi_{q-1}(A) \arrow[r] \arrow[d, "h_A"'] & \cdots \\
\cdots \arrow[r] & \widetilde{H}_q(A) \arrow[r] & \widetilde{H}_q(X) \arrow[r] & H_q(X,A) \arrow[r] & \widetilde{H}_{q-1}(A) \arrow[r] & \cdots ,
\end{tikzcd}$$

where on the top it is the homotopy exact sequence of the pair and on the bottom its homology exact sequence. □

6.3.29 **Theorem**. (Relative Hurewicz isomorphism theorem) *Let (X, A) be an $(n-1)$-connected* CW*-pair such that $A \neq \emptyset$ and $n \geq 2$. If A is 1-connected, then the Hurewicz homomorphism $h_{(X,A)} : \pi_q(X, A) \longrightarrow H_q(X, A)$ is an isomorphism for $1 \leq q \leq n$ and an epimorphism for $q = n + 1$. In particular, $H_q(X, A) = 0$ for $1 \leq q \leq n - 1$. Furthermore, $H_0(X, A) = 0$.*

Proof: Let $p : (X, A) \longrightarrow (X/A, *)$ be the quotient map. By Proposition 6.2.2,

$$p_* : \pi_q(X, A) \longrightarrow \pi_q(X/A, *)$$

is an isomorphism for $1 \leq q \leq n$ and an epimorphism for $q = n + 1$. By (5.3.18), $H_q(X, A) = \widetilde{H}_q(X/A)$ for all q. Moreover, by the Hurewicz isomorphism theorem 6.3.25, we have that

$$h_{X/Y} : \pi_q(X/Y) \longrightarrow \widetilde{H}_q(X/Y)$$

is an isomorphism for $q \leq n$ and an epimorphism for $q = n + 1$. From the naturality of the Hurewicz homomorphism, it follows that the following diagram is commutative

$$\begin{tikzcd}
\pi_q(X,A) \arrow[r, "p_*"] \arrow[d, "h_{(X,A)}"'] & \pi_q(X/A) \arrow[d, "p_*"] \\
H_q(X,A) \arrow[r, "p_*"] & H_q(X/A) .
\end{tikzcd}$$

Hence $h_{(X,A)} : \pi_q(X, A) \longrightarrow H_q(X, A)$ is an isomorphism for $1 \leq q \leq n$ and an epimorphism for $q = n + 1$.

Since (X, A) is 0-connected, that is, A is 0-connected and intersects each path component of X, it follows that $H_0(X, A) = 0$ (in fact, since A is 0-connected, so also is X). □

6.3.30 REMARK. For general $(n-1)$-connected spaces X, respectively pairs of spaces (X, A), recall that their homology is defined by taking a CW-approximation $\widetilde{X} \longrightarrow X$, respectively $(\widetilde{X}, \widetilde{A}) \longrightarrow (X, A)$, and then defining

$$\widetilde{H}_q(X) = \widetilde{H}_q(\widetilde{X}), \quad \text{respectively} \quad H_q(X, A) = H_q(\widetilde{X}, \widetilde{A}).$$

Since by the very definition of a CW-approximation

$$\pi_q(X) \cong \pi_q(\widetilde{X}), \quad \text{respectively} \quad \pi_q(X, A) \cong \pi_q(\widetilde{X}, \widetilde{A}),$$

then both the Hurewicz isomorphism theorem and the relative Hurewicz isomorphism theorem hold immediately in the general case.

A nice and important consequence of Proposition 6.3.28 and both Hurewicz isomorphism theorems is the following result, known as the Whitehead theorem.

6.3.31 **Theorem**. *Let X and Y be simply connected pointed spaces. Let $f : X \longrightarrow Y$ be a map such that $f_* : H_q(X) \longrightarrow H_q(Y)$ is an isomorphism for all q. Then f is a weak homotopy equivalence. In particular, if X and Y are* CW-*complexes, then f is a homotopy equivalence.*

Proof: By Theorem 4.2.8, one can replace f, up to homotopy equivalence, by the inclusion $j : X \hookrightarrow M_f$ of X in the top face of its mapping cylinder. Therefore, without losing generality, we can assume that $f : X \hookrightarrow Y$ is an inclusion.

By 6.3.28, for all $q \geq 1$, we have a commutative diagram

$$\begin{array}{ccccccccc}
\pi_q(X) & \xrightarrow{f_*} & \pi_q(Y) & \longrightarrow & \pi_q(Y, X) & \longrightarrow & \pi_{q-1}(X) & \xrightarrow{f_*} & \pi_{q-1}(Y) \\
{\scriptstyle h_X}\downarrow & & {\scriptstyle h_Y}\downarrow & & {\scriptstyle h_{(Y,X)}}\downarrow & & {\scriptstyle h_X}\downarrow & & {\scriptstyle h_Y}\downarrow \\
\widetilde{H}_q(X) & \xrightarrow{f_*} & \widetilde{H}_q(Y) & \longrightarrow & H_q(Y, X) & \longrightarrow & \widetilde{H}_{q-1}(X) & \xrightarrow{f_*} & \widetilde{H}_{q-1}(Y),
\end{array}$$

where the vertical arrows are the corresponding Hurewicz homomorphisms. By assumption, $\pi_1(Y) = 0$ and $\pi_0(X) = 0$, and hence from the exactness of the top row in the diagram, also $\pi_1(Y, X) = 0$. Furthermore, $\pi_1(X) = 0$, so by the relative Hurewicz isomorphism theorem, $H_1(Y, X) = 0$ and $\pi_2(Y, X) \cong H_2(Y, X)$. Since f induces isomorphisms in homology, now from the exactness of the bottom row $H_2(Y, X) = 0$, and so $\pi_2(Y, X) = 0$. By induction, $\pi_q(Y, X) = 0$ for all $q \geq 1$. Again the exactness of the top row shows that $f_* : \pi_q(X) \longrightarrow \pi_q(Y)$ is an isomorphism for all q, and hence f is a weak homotopy equivalence. □

We finish this section by stating a very interesting result of J.P. Serre, whose proof can be consulted in [66].

6.3.32 **Theorem**. *Let X be a finite, simply connected, noncontractible* CW-*complex with dimension at least* 2, *e.g.,* $X = \mathbb{S}^2$. *Then X has infinitely many nonzero homotopy groups.* □

The Whitehead theorem 6.3.31 is thus surprisingly strong. If the (finitely many) homology groups of two such CW-complexes are mapped isomorphically, then so are all homotopy groups of those spaces.

6.4 Homotopy Properties of the Eilenberg–Mac Lane Spaces

In Section 6.1 we constructed the Eilenberg–Mac Lane spaces $K(A,n)$ for A a finitely generated (abelian) group. For the general case, recall that if A is an abelian group, then there exists a short exact sequence

$$0 \longrightarrow L(\mathcal{A}) \xrightarrow{\quad f \quad} L(\mathcal{B}) \xrightarrow{\qquad} A \longrightarrow 0$$

such that $L(\mathcal{A})$ and $L(\mathcal{B})$ are free groups generated by the sets $\mathcal{A}$ and $\mathcal{B}$, respectively. We have also already shown that this sequence can be realized by a sequence of topological spaces and maps

$$\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha \xrightarrow{\ \varphi\ } \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta \longrightarrow C_\varphi$$

in such a way that $C_\varphi = M(A,n)$ is a Moore space of type (A,n). This sequence can be replaced by the sequence

$$\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha \xhookrightarrow{\ i\ } M_\varphi \longrightarrow C_\varphi\,,$$

where M_φ is the mapping cylinder of the map φ, the inclusion i is a cofibration, and the mapping cone of φ satisfies $C_\varphi = M_\varphi / \bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha$.

The Dold–Thom theorem 5.2.22 implies that we have a quasifibration

$$\mathrm{SP}\, M_\varphi \longrightarrow \mathrm{SP}\, C_\varphi$$

with fiber $\mathrm{SP}\,(\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha)$. Since $M_\varphi \simeq \bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta$, we have a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \pi_q(\mathrm{SP}\,(\textstyle\bigvee_{\beta\in\mathcal{B}} \mathbb{S}^n_\beta)) \longrightarrow \pi_q(\mathrm{SP}\, C_\varphi) \longrightarrow \\ \longrightarrow \pi_{q-1}(\mathrm{SP}\,(\textstyle\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^n_\alpha)) \xrightarrow{\quad \lambda \quad} \cdots\,, \end{aligned} \tag{6.4.1}$$

where $\lambda = \widehat{\varphi}_*$. By the infinite version of (6.1.8), we have isomorphisms

$$\pi_q\left(\mathrm{SP}\left(\bigvee_{\beta\in\mathcal{B}}\mathbb{S}^n_\beta\right)\right) \cong \bigoplus_{\beta\in\mathcal{B}}\pi_q(\mathrm{SP}\,\mathbb{S}^n_\beta)$$

and

$$\pi_q\left(\mathrm{SP}\left(\bigvee_{\alpha\in\mathcal{A}}\mathbb{S}^n_\alpha\right)\right) \cong \bigoplus_{\alpha\in\mathcal{A}}\pi_q(\mathrm{SP}\,\mathbb{S}^n_\alpha)\,.$$

Moreover, if $q \neq n$, then $\pi_q(\mathrm{SP}\,\mathbb{S}^n) = 0$ by Proposition 6.1.2, and this in turn implies that $\pi_q(\mathrm{SP}\,C_\varphi) = 0$ if $q \neq n, n+1$. Furthermore, if $q = n+1$, then we have that the homomorphism λ can be factored as the composite

(6.4.2)
$$\begin{aligned}\lambda : \pi_n(\mathrm{SP}\,(\textstyle\bigvee_{\alpha\in\mathcal{A}}\mathbb{S}^n_\alpha)) \cong \pi_n(\textstyle\bigvee_{\alpha\in\mathcal{A}}\mathbb{S}^n_\alpha) \cong L(\mathcal{A})\xrightarrow{\quad f\quad}\\ \longrightarrow L(\mathcal{B}) \cong \pi_n(\textstyle\bigvee_{\beta\in\mathcal{B}}\mathbb{S}^n_\beta) \cong \pi_n(\mathrm{SP}\,(\textstyle\bigvee_{\beta\in\mathcal{B}}\mathbb{S}^n_\beta))\,.\end{aligned}$$

It follows that λ is a monomorphism and also that we have

(6.4.3)
$$\pi_{n+1}(\mathrm{SP}\,C_\varphi) = 0\,.$$

This means that $\mathrm{SP}\,C_\varphi$ is an Eilenberg–Mac Lane space. We therefore get that the sequence (6.4.1) can be reduced to a short exact sequence

$$0 \longrightarrow \pi_n\left(\mathrm{SP}\left(\bigvee_{\alpha\in\mathcal{A}}\mathbb{S}^n_\alpha\right)\right) \longrightarrow \pi_n\left(\mathrm{SP}\left(\bigvee_{\beta\in\mathcal{B}}\mathbb{S}^n_\beta\right)\right) \longrightarrow \pi_n\,(\mathrm{SP}\,C_\varphi) \longrightarrow 0\,,$$

which, by using (6.4.2), is isomorphic to

$$0 \longrightarrow L(\mathcal{A})\xrightarrow{\quad f\quad}L(\mathcal{B}) \longrightarrow \pi_n(\mathrm{SP}\,C_\varphi) \longrightarrow 0\,.$$

So we have arrived at the next result.

6.4.4 **Theorem**. *Suppose that A is an abelian group and that $n \geq 1$. Then $\mathrm{SP}\,M(A,n) = \mathrm{SP}\,C_\varphi$ is an Eilenberg–Mac Lane space of type (A,n), namely,*

$$\mathrm{SP}\,M(A,n) = K(A,n)\,.$$

□

For an alternative construction of $K(A,n)$ see 6.4.20.

The properties of Eilenberg–Mac Lane spaces that we shall study in this section will be used to establish the multiplicative structure of cohomology groups in the next chapter.

Given that A is an abelian group with countably many generators, it follows that the Moore space $M(A,n)$ is a CW-complex with countably many cells, one in dimension 0 and the rest in dimensions n and $n+1$. According to 5.2.2 the corresponding Eilenberg–Mac Lane space $K(A,n) = \mathrm{SP}\, M(A,n)$ is a CW-complex, which, in particular, is $(n-1)$-connected.

Suppose that $r, s > 1$. Since the Eilenberg–Mac Lane spaces $X = K(A,r)$ and $Y = K(B,s)$ satisfy the hypotheses of Theorem 6.3.23, we obtain the next result.

6.4.5 **Proposition**. *Suppose that $r, s > 1$. Then h induces an isomorphism*

$$h_{r,s} : \pi_r(K(A,r)) \otimes \pi_s(K(B,s)) \longrightarrow \pi_{r+s}(K(A,r) \wedge K(B,s)). \qquad \square$$

The next proposition gives a sufficient condition for realizing a given homomorphism of homotopy groups as the homomorphism induced by a continuous map.

6.4.6 **Proposition**. *Let X be a* CW*-complex whose $(n-1)$-skeleton X^{n-1} is equal to $\{*\}$ for some $n \geq 1$ and let Y be a pointed space satisfying $\pi_j(Y) = 0$ for $j > n$. Let $f : \pi_n(X) \longrightarrow \pi_n(Y)$ be a homomorphism. Then there exists a pointed map $\varphi : X \longrightarrow Y$, unique up to homotopy, such that $\varphi_* = f$.*

Proof: Because $X^{n-1} = \{*\}$, we have that $X^n = \bigvee_\alpha \mathbb{S}^n_\alpha$. Let $i : X^n \hookrightarrow X$ be the inclusion. By Proposition 5.1.25, $i_* : \pi_n(X^n) \longrightarrow \pi_n(X)$ is surjective, and by Lemma 6.3.1, $\pi_n(X^n) = \pi_n(\bigvee_\alpha \mathbb{S}^n_\alpha)$ is a free abelian group generated by the inclusions $i_\alpha : \mathbb{S}^n = \mathbb{S}^n_\alpha \hookrightarrow \bigvee_\alpha \mathbb{S}^n_\alpha$. If we define $\varphi_n : \bigvee_\alpha \mathbb{S}^n_\alpha \longrightarrow Y$ so that $\varphi_n|\mathbb{S}^n_\alpha$ is a representative of the class $fi_*([i_\alpha]) \in \pi_n(Y)$ for each α, then we have the following commutative diagram:

$$\begin{tikzcd}
\pi_n(\bigvee_\alpha \mathbb{S}^n_\alpha) \arrow[r, two heads, "i_*"] \arrow[dr, "\varphi_{n*}"'] & \pi_n(X) \arrow[d, "f"] & \cong \pi_n(\bigvee_\alpha \mathbb{S}^n_\alpha)/\ker(i_*) \\
 & \pi_n(Y), &
\end{tikzcd} \tag{6.4.7}$$

where the horizontal arrow i_* is an epimorphism. We can now extend φ_n to the $(n+1)$-skeleton, which is obtained by adding $(n+1)$-cells e^{n+1}_j by using attaching maps $g_j : \mathbb{S}^n \longrightarrow X^n$. In order to extend φ_n to $X^n \cup_{g_j} e^{n+1}_j$, we consider the following diagram:

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "g_j"] & X^n \arrow[r, hook] \arrow[d, "\varphi_n"'] & C_{g_j} = X^n \cup_{g_j} e^{n+1}_j \arrow[dl, dashed, "\widetilde{\varphi}_n"] \\
 & Y. &
\end{tikzcd}$$

According to Propositiom 3.1.7, $\widetilde{\varphi}_n$ exists if and only if $\varphi_n \circ g_j$ is nullhomotopic, that is, if and only if $\varphi_{n*}[g_j] = 0$. But again by Proposition 3.1.7 we have $i_*[g_j] = 0$, so that using (6.4.7), it follows that $\varphi_{n*}[g_j] = fi_*[g_j] = 0$ holds. Doing the same for every cell, we get the desired extension $\varphi_{n+1} : X^{n+1} \longrightarrow Y$.

In order then to extend φ_{n+1} to the rest of the skeletons, we use Proposition 3.1.8, since $\pi_k(Y) = 0$ for $k > n$, thereby obtaining a map $\varphi : X \longrightarrow Y$. Because φ is an extension of φ_n, we have that $\varphi_* \circ i_* = \varphi_{n*}$, and so $f \circ i_* = \varphi_{n*}$ by using 6.4.7. Thus we get $\varphi_* i_*([i_\alpha]) = fi_*([i_\alpha])$, which in turn implies $\varphi_* = f$.

Uniqueness up to homotopy is proved similarly. □

6.4.8 EXERCISE. Prove the uniqueness up to homotopy of the map φ whose existence was just shown above.

6.4.9 EXERCISE. Prove that the previous result is true if instead of requiring $X^{n-1} = \{*\}$, we require only that X be $(n-1)$-connected. (Hint: Using Theorem 6.3.20, substitute X with a CW-complex whose $(n-1)$-skeleton is one point.)

We now present the next definition, which we shall use in Section 7.2 of the next chapter and which will play a critical role in defining the multiplicative structure of cohomology groups.

6.4.10 DEFINITION. Let A and B be groups with countably many generators. We define maps

$$\gamma_{r,s} : K(A,r) \wedge K(B,s) \longrightarrow K(A \otimes B, r+s)$$

as follows.

We first note that $K(A,r) \wedge K(B,s) = \mathrm{SP}\, M(A,r) \wedge \mathrm{SP}\, M(B,s)$ is an $(r+s-1)$-connected CW-complex. Next, by Proposition 6.4.5, we have that

$$\pi_{r+s}(K(A,r) \wedge K(B,s)) \cong A \otimes B.$$

Then, if we consider the composition of this isomorphism with

$$A \otimes B \xrightarrow{\quad 1 \quad} A \otimes B \cong \pi_{r+s}(K(A \otimes B, r+s)),$$

then by 6.4.9 we get the map $\gamma_{r,s}$, which induces this composition in homotopy.

Once one has Moore spaces, it is possible to introduce coefficients in homology, as follows. This could already have been done in Section 6.1 for finitely generated coefficient groups.

6.4.11 DEFINITION. Let G be an abelian group and let X be a pointed CW-complex. We define its *nth reduced homology group with coefficients in* G for $n \geq 0$ as
$$\widetilde{H}_n(X;G) = \pi_{n+1}(\mathrm{SP}\,(X \wedge M(G,1)))\,.$$
For $n < 0$ we define $\widetilde{H}_n(X;G) = 0$.

Observe that $\widetilde{H}_n(X;G) = \widetilde{H}_{n+1}(X \wedge M(G,1))$. Then, it is easy to verify that these groups satisfy the Eilenberg–Steenrod axioms for a reduced homology theory with coefficients in G. **Functoriality** follows simply because the smash product with the Moore space $M(G,1)$ is already a functor from $\mathcal{T}op_*$ to $\mathcal{T}op_*$; **Homotopy** follows from the fact that smashing pointed homotopic maps with any map (in this case $\mathrm{id}_{M(G,1)}$) yields homotopic maps. For **Exactness**, it is enough to observe that given any pointed map $f : X \longrightarrow Y$, 5.3.5 applied to $f \wedge \mathrm{id}_{M(G,1)} : X \wedge M(G,1) \longrightarrow Y \wedge M(G,1)$ implies the exactness of
$$\widetilde{H}_{q+1}(X \wedge M(G,1)) \xrightarrow{(f\wedge \mathrm{id}_{M(G,1)})_*} \widetilde{H}_{q+1}(Y \wedge M(G,1)) \longrightarrow$$
$$\xrightarrow{(i\wedge \mathrm{id}_{M(G,1)})_*} \widetilde{H}_{q+1}(C_{f\wedge M(G,1)})\,.$$
Since there is a homotopy equivalence $C_{f\wedge \mathrm{id}_{M(G,1)}} \simeq C_f \wedge M(G,1)$, the previous exact sequence becomes
$$\widetilde{H}_q(X;G) \xrightarrow{f_*} \widetilde{H}_q(Y;G) \xrightarrow{i_*} \widetilde{H}_q(C_f;G)\,.$$
Finally, since for the 0-sphere $\mathbb{S}^0$ one has $\mathrm{SP}\,(\mathbb{S}^0 \wedge M(G,1)) \approx \mathrm{SP}\,M(G,1) = K(G,1)$, we have
$$\widetilde{H}_n(\mathbb{S}^0;G) = \pi_{n+1}(K(G,1)) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$
so that **Dimension** for coefficients in G is proved.

6.4.12 EXERCISE. Prove that for any pointed CW-complex X, a group homomorphism $\varphi : G \longrightarrow G'$ induces another group homomorphism
$$\widetilde{H}_q(X;G) \xrightarrow{\varphi_\#} \widetilde{H}_q(X;G')$$
in such a way that the association $G \mapsto \widetilde{H}_q(X;G)$ becomes a functor. (Hint: By 6.3.5, φ determines a pointed map $\varphi_\bullet : M(G,1) \longrightarrow M(G',1)$.)

As in 5.3.12, if (X, A) is a CW-pair, we define the *nth homology group of* (X, A) *with coefficients in* G to be

$$H_n(X, A; G) = \widetilde{H}_n(X \cup CA; G)\,,$$

where $X \cup CA$ is the mapping cone of the inclusion map of A into X. In particular, $H_n(X; G) = H_n(X, \emptyset; G)$.

As in (5.3.18), we have

$$H_n(X, A; G) = \widetilde{H}_n(X/A; G)$$

for every CW-pair (X, A).

There is, of course, a version of the axioms 5.3.13, 5.3.14, 5.3.15, 5.3.16, and 5.3.17 for the unreduced homology with coefficients in G, whose formulation and proof are left to the reader as an *exercise.*

In particular, a version of Lemma 5.3.28 holds; namely, for any pointed topological space X we have that

$$H_n(X; G) = \begin{cases} \widetilde{H}_n(X; G) & \text{if } n \neq 0, \\ \widetilde{H}_0(X) \oplus G & \text{if } n = 0. \end{cases}$$

To finish this chapter we are going to consider the properties of the infinite symmetric product as a topological abelian monoid. First we need another concept.

The *weak product* $\overset{\circ}{\prod}_{i=1}^{\infty} Z_i$ of pointed spaces Z_i consists of all elements $x \in \prod_{i=1}^{\infty} Z_i$ such that all but a finite number of coordinates x_i of x are the base points. However, its topology is not the relative topology, but the topology of the union of the finite products $\prod_{i=1}^{k} Z_i \subset \overset{\circ}{\prod}_{i=1}^{\infty} Z_i$.

6.4.13 EXERCISE. Prove that $\pi_k\left(\overset{\circ}{\prod}_{i=1}^{\infty} Z_i\right) \cong \oplus_{i=1}^{\infty} \pi_k(Z_i)$.

6.4.14 EXAMPLE. Consider any pointed space X and the weak product $\overset{\circ}{\prod}_{i=1}^{\infty} K(\pi_i(X), i)$. Then by the previous exercise, both of these spaces have the same homotopy groups. However, in general, there is no weak homotopy equivalence between them. We shall state sufficient conditions for this to happen.

More precisely the next theorem, generalizing a result of J.C. Moore, shows that the infinite symmetric product of X is determined by its homotopy groups. First we define a *weak topological abelian monoid* to be a space Y provided with an associative and commutative multiplication $Y \times Y \longrightarrow Y$ with a neutral element and such that the multiplication is continuous on compact subsets of $Y \times Y$.

6.4.15 **Theorem**. *Let Y be a path-connected weak topological abelian monoid. Then there is a weak homotopy equivalence*

$$\overset{\circ}{\prod}_{i=1}^{\infty} K(\pi_i(Y), i) \longrightarrow Y\,.$$

For the *proof* we refer the reader to [26]. □

6.4.16 **Corollary**. *Let Y and Y' be path-connected topological abelian monoids that have the homotopy type of* CW*-complexes. If $\pi_i(Y) \cong \pi_i(Y')$ for all $i \geq 1$, then Y and Y' have the same homotopy type.*

Proof: By the previous theorem there are weak homotopy equivalences

$$\overset{\circ}{\prod}_{i=1}^{\infty} K(\pi_i(Y), i) \longrightarrow Y\,, \quad \overset{\circ}{\prod}_{i=1}^{\infty} K(\pi_i(Y'), i) \longrightarrow Y'\,.$$

Since Y and Y' have the homotopy type of CW-complexes, these are indeed homotopy equivalences. On the other hand, by 6.4.6 and 5.1.37, the isomorphisms $\pi_i(Y) \cong \pi_i(Y')$ induce homotopy equivalences $K(\pi_i(Y), i) \simeq K(\pi_i(Y'), i)$ for all $i \geq 1$, and these in turn induce a homotopy equivalence between the corresponding weak products. This proves the result. □

Given any pointed topological space X, we have a multiplication $\mathrm{SP}\,X \times \mathrm{SP}\,X \longrightarrow \mathrm{SP}\,X$ given by juxtaposition of the elements. It is easy to prove that this provides $\mathrm{SP}\,X$ with the structure of a weak topological abelian monoid (see [26, 3.8]). In fact, it is the *free* topological abelian monoid generated by X, where the base point of X plays the role of the neutral element (see Exercise 6.4.19 below).

Since $\pi_i(\mathrm{SP}\,X) = H_i(X)$, we have the following consequence of 6.4.15.

6.4.17 **Corollary**. *Let X be a path-connected space. Then there is a weak homotopy equivalence*

$$\overset{\circ}{\prod}_{i=1}^{\infty} K(H_i(X), i) \longrightarrow \mathrm{SP}\,X. \qquad \square$$

Moreover, from Corollary 6.4.16 and 5.2.2, we have the following result.

6.4.18 **Corollary**. *Let X, X' be path-connected spaces that have the homotopy type of a* CW*-complex. If $H_i(X) \cong H_i(X')$ for all $i \geq 1$, then* $\mathrm{SP}\,X$ *and* $\mathrm{SP}\,X'$ *have the same homotopy type.* □

6.4.19 Exercise. Prove that there is a bijection which is an isomorphism of monoids

$$\begin{aligned} \mathrm{SP}\,X &\longrightarrow F(X, \mathbb{N} \cup \{0\}) \\ &= \{\alpha : X \longrightarrow \mathbb{N} \cup \{0\} \mid \alpha(x_0) = 0, \text{ and } \alpha(x) = 0 \text{ for almost all } x\} \end{aligned}$$

such that $\widehat{x} = [x_1, \ldots, x_n, x_0, x_0, \ldots] \mapsto \alpha_{\widehat{x}}$, where $\alpha_{\widehat{x}} = \sum \overline{x_i}$, and $\overline{x} : X \longrightarrow \mathbb{N} \cup \{0\}$ is defined by $\overline{x_0} = 0$ and

$$\overline{x}(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

if $x \neq x_0$. Moreover, prove that there is a similar bijection

$$\begin{aligned} &\mathrm{SP}^r\,X \longrightarrow F_r(X, \mathbb{N} \cup \{0\}) \\ &= \{\alpha : X \longrightarrow \mathbb{N} \cup \{0\} \mid \alpha(x_0) = 0, \text{ and } \alpha(x) \neq 0 \text{ for at most } r \text{ points } x\}\,. \end{aligned}$$

According to the previous exercise, one can alternatively define $\mathrm{SP}\,X$ as a certain set of functions $F(X, \mathbb{N} \cup \{0\})$. By 6.4.4, $\mathrm{SP}\,\mathbb{S}^n$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}, n)$. Therefore, $\mathrm{SP}\,\mathbb{S}^n \cong F(X, \mathbb{N} \cup \{0\})$ is a $K(\mathbb{Z}, n)$ with the structure of a topological abelian monoid (in this case the operation is globally continuous and not only on compact subsets of $\mathrm{SP}\,\mathbb{S}^n \times \mathrm{SP}\,\mathbb{S}^n$, as we shall see below). With this interpretation of $\mathrm{SP}\,\mathbb{S}^n$, it is clear how to get a topological abelian group of type $(\mathbb{Z}, n)$; namely, one simply takes $F(\mathbb{S}^n, \mathbb{Z})$. More generally, following [52] and assuming that G is a countable abelian group, we shall similarly construct an Eilenberg–Mac Lane space of type (G, n).

6.4.20 Definition. Let G be an abelian (additive) group. We denote by $F(\mathbb{S}^n, G)$ the set of pointed functions $\alpha : (\mathbb{S}^n, x_0) \longrightarrow (G, 0)$ such that $\alpha(x) = 0$ for almost all $x \in \mathbb{S}^n$, where x_0 is some base point in $\mathbb{S}^n$. $F(\mathbb{S}^n, G)$ is then an abelian group under pointwise addition of functions.

In order to endow $F(\mathbb{S}^n, G)$ with a topology, we consider a filtration of $F(\mathbb{S}^n, G)$ as follows. Let $F_r(\mathbb{S}^n, G) = \{\alpha \in F(\mathbb{S}^n, G) \mid \alpha(x) \neq 0 \text{ for at most } r$ points $x\}$. Then

$$F_0(\mathbb{S}^n, G) \subset F_1(\mathbb{S}^n, G) \subset \cdots \subset F_r(\mathbb{S}^n, G) \subset F_{r+1}(\mathbb{S}^n, G) \subset \cdots \subset F(\mathbb{S}^n, G)\,.$$

Now, for every $x \in \mathbb{S}^n - \{x_0\}$ and every $g \in G$, we define a function $gx \in F(\mathbb{S}^n, G)$ by

$$gx(x') = \begin{cases} g & \text{if } x = x', \\ 0 & \text{if } x \neq x', \end{cases}$$

and $gx_0(x) = 0$ for all $x \in \mathbb{S}^n$.

Let now $p_r : (G \times \mathbb{S}^n)^r \longrightarrow F_r(\mathbb{S}^n, G)$ be given by

$$p_r\left((g_1, x_1), (g_2, x_2), \dots, (g_r, x_r)\right) = g_1x_1 + g_2x_2 + \cdots + g_rx_r\,.$$

We consider $(G \times \mathbb{S}^n)^r$ with the product topology and give $F_r(\mathbb{S}^n, G)$ the identification topology. One can easily show that $p_{r+1}^{-1}F_r(\mathbb{S}^n, G)$ is a finite union of closed subsets of $(G \times \mathbb{S}^n)^{r+1}$. Therefore, $F_r(\mathbb{S}^n, G)$ is closed in $F_{r+1}(\mathbb{S}^n, G)$, and we endow $F(\mathbb{S}^n, G) = \bigcup_r F_r(\mathbb{S}^n, G)$ with the union topology.

Since $\mathbb{S}^n$ is triangulable and G is discrete, there is a canonical simplicial structure on $(G\times\mathbb{S}^n)^r$. Hence $\coprod_r (G\times\mathbb{S}^n)^r$ has also a simplicial structure. Let $p : \coprod_r (G \times \mathbb{S}^n)^r \longrightarrow F(\mathbb{S}^n, G)$ be the identification defined by $p|(G \times \mathbb{S}^n)^r = i_r \circ p_r$, where $i_r : F_r(\mathbb{S}^n, G) \hookrightarrow F(\mathbb{S}^n, G)$ is the inclusion. Using the simplicial structure on $\coprod_r (G \times \mathbb{S}^n)^r$ and the map p one can provide $F(\mathbb{S}^n, G)$ with a CW-structure (see [52]). Since the group G is countable, $F(\mathbb{S}^n, G)$ is a countable CW-complex.

6.4.21 **Proposition**. *If G is a countable abelian group, then $F(\mathbb{S}^n, G)$ is a topological abelian group.*

Proof: Consider the following commutative diagram:

$$\begin{tikzcd} \coprod_r (G \times \mathbb{S}^n)^r \times \coprod_r (G \times \mathbb{S}^n)^r \arrow[r] \arrow[d, "p\times p"'] & \coprod_r (G \times \mathbb{S}^n)^r \arrow[d, "p"] \\ F(\mathbb{S}^n, G) \times F(\mathbb{S}^n, G) \arrow[r] & F(\mathbb{S}^n, G)\,. \end{tikzcd}$$

The map at the top is induced by the obvious homeomorphisms

$$(G \times \mathbb{S}^n)^r \times (G \times \mathbb{S}^n)^s \longrightarrow (G \times \mathbb{S}^n)^{r+s}\,,$$

and the one at the bottom is the sum in $F(\mathbb{S}^n, G)$.

Since $\coprod_r (G \times \mathbb{S}^n)^r$ is a countable simplicial complex and $F(\mathbb{S}^n, G)$ is a countable CW-complex, by [45] the usual topological product coincides with the compactly generated one. Therefore, by [70], $p \times p$ is an identification and hence the sum is continuous.

The continuity of the inverse follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
\coprod_r (G\times\mathbb{S}^n)^r & \longrightarrow & \coprod_r (G\times\mathbb{S}^n)^r \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
F(\mathbb{S}^n, G) & \longrightarrow & F(\mathbb{S}^n, G)\,,
\end{array}$$

where the top map is induced by the maps $(G\times\mathbb{S}^n)^r \longrightarrow (G\times\mathbb{S}^n)^r$ given by

$$((g_1,x_1),(g_2,x_2),\dots,(g_r,x_r)) \longmapsto ((-g_1,x_1),(-g_2,x_2),\dots,(-g_r,x_r))\ ,$$

and the bottom map is the inverse. □

We consider the circle $\mathbb{S}^1$ as the quotient space $I/\partial I$, and we denote a point in $\mathbb{S}^1$ by $\bar{t}$, where $t\in I$.

Let G be a countable abelian group. Since $F(\mathbb{S}^n, G)$ is a CW-complex, by [54] $\Omega F(\mathbb{S}^n, G)$ has the homotopy type of a CW-complex. Therefore, by 4.3.22 the identity map from the k-construction $k\Omega F(\mathbb{S}^n, G)$ to $\Omega F(\mathbb{S}^n, G)$ is a homotopy equivalence. Combining this fact with [52, Thm. 10.4] we obtain the following result.

6.4.22 **Theorem**. *Let G be a countable abelian group. Then the map $h : F(\mathbb{S}^n, G) \longrightarrow \Omega F(\mathbb{S}^1\wedge\mathbb{S}^n, G)$ given by*

$$h(g_1x_1+\cdots+g_rx_r)(t) = g_1(\bar{t}\wedge x_1)+\cdots+g_r(\bar{t}\wedge x_r)$$

is a homomorphism of H-spaces and also a pointed homotopy equivalence. □

6.4.23 **Corollary**. *Let G be a countable abelian group. Then $F(\mathbb{S}^n, G)$ is an Eilenberg–Mac Lane space of type (G,n).*

Proof: By induction on n. For $n=0$, it is clear that $F(\mathbb{S}^0, G)\approx G$. Assume that the result is true for $F(\mathbb{S}^n, G)$. Then $\pi_{i+1}(F(\mathbb{S}^{n+1}, G)) \cong \pi_i(\Omega F(\mathbb{S}^n, G))$. But by 6.4.22, $\Omega F(\mathbb{S}^{n+1}, G)\simeq F(\mathbb{S}^n, G)$; therefore,

$$\pi_{i+1}(F(\mathbb{S}^{n+1}, G)) \cong \pi_i(F(\mathbb{S}^n, G)) \cong \begin{cases} G & \text{if } i=n,\\ 0 & \text{if } i\neq n. \end{cases}$$

□

Chapter 7

Cohomology Groups and Related Topics

In this chapter we shall use the Eilenberg–Mac Lane spaces introduced in the previous chapter in order to define cohomology groups. Then, using the homotopy properties proved for Moore spaces, we shall introduce a multiplicative structure on cohomology groups.

In order to prove that the homology groups already introduced in the previous chapter, and the cohomology groups, can be obtained using techniques of homological algebra, we introduce cellular homology and cellular cohomology, which then allow us rather simply to calculate the groups for some common spaces. Finally, using concepts from cellular homology, we shall get various exact sequences: the Künneth sequences for calculating homology and cohomology of products of spaces, the universal coefficient sequences for calculating homology and cohomology groups with arbitrary coefficients in terms of simple algebraic constructions involving the corresponding groups with integer coefficients, as well as the Mayer–Vietoris sequences for computing homology and cohomology groups of finite unions of spaces in terms of the groups of the individual spaces.

7.1 Cohomology Groups

In this section we shall define the ordinary cohomology group of a space X as the group of homotopy classes $[X, K(G,n)]$, where $K(G,n)$ is an Eilenberg–Mac Lane space as defined in the previous chapter.

We shall assume from now on that all of the spaces mentioned are pointed CW-complexes whose base point is a 0-cell.

All of the constructions from the previous chapter produce CW-complexes when they operate on CW-complexes. In particular, this has as a consequence that in the class of CW-complexes the homotopy type of a $K(G,n)$ is unique.

7.1.1 NOTE. Since we have

$$\begin{aligned}\pi_q(\Omega K(G,n+1)) &= [\mathbb{S}^q, \Omega K(G,n+1)] = [\Sigma\mathbb{S}^q, K(G,n+1)] \\ &= \pi_{q+1}(K(G,n+1)) = \begin{cases}0 & \text{if } q \neq n, \\ G & \text{if } q = n,\end{cases}\end{aligned}$$

it follows that $\Omega K(G,n+1) \simeq K(G,n)$.

7.1.2 DEFINITION. Let (X,A) be a CW-pair (which means that X is a CW-complex and $A \subset X$ is a subcomplex), and let G be a finitely generated abelian group. We define the *nth cohomology group of* (X,A) *with coefficients in* G as

$$H^n(X,A;G) = [X \cup CA, *; K(G,n), *]\,, \qquad n \geq 1\,,$$

where we are considering pointed homotopy classes (and the base point $*$ of $X \cup CA$ is obvious). If $A = \emptyset$, then $X \cup CA = X^+ = X \sqcup *$. In this case, we write $H^n(X;G) = [X^+, +; K(G,n), *] = [X, K(G,n)]$, where the last expression denotes the free (that is, not pointed) homotopy classes of maps from X to $K(G,n)$.

7.1.3 REMARK. Since $A \hookrightarrow X$ is a cofibration, the quotient map $q : X \cup CA \longrightarrow X/A$ is a homotopy equivalence (see 4.2.3). Therefore, one can define the cohomology groups by

$$H^n(X,A;G) = [X/A, *; K(G,n), *]\,, \qquad n \geq 1\,;$$

(here the base point $*$ of X/A is $\{A\}$).

We can extend this definition to the case $n = 0$ by defining $K(G,0) = G$ (with the discrete topology).

7.1.4 EXERCISE. Prove that $H^0(X,A;G) \cong \prod G$, with as many factors as there are path-connected components C of X satisfying $C \cap A = \emptyset$. In particular, if X is path connected, then $H^0(X;G) \cong G$.

More generally, we have the following **additivity** property.

7.1.5 EXERCISE. Let $(X,A) = \coprod_{\alpha\in\Lambda}(X_\alpha, A_\alpha)$. Prove that

$$H^n(X,A;G) \cong \prod_\alpha H^n(X_\alpha, A_\alpha; G)\,.$$

(Hint: An element $x \in H^n(X,A;G)$ is represented by a pointed map $f : \bigvee_\alpha(X_\alpha/A_\alpha) \longrightarrow K(G,n)$, which in turn, by the universal property of the wedge, corresponds to a family of maps $f_\alpha : X_\alpha/A_\alpha \longrightarrow K(G,n)$, each one of which represents an element $x_\alpha \in H^n(X_\alpha, A_\alpha; G)$.)

Since $K(G,n) \simeq \Omega^2 K(G,n+2)$, it follows from Theorem 2.8.6 that $K(G,n)$ is an H-group. Therefore, $H^n(X,A;G)$ is actually a group, and it is even abelian, since $K(G,n)$ is a double loop space.

If $f : (X,A) \longrightarrow (Y,B)$ is a map of CW-pairs, then the associated map on the quotient spaces $\overline{f} : X/A \longrightarrow Y/B$ induces a homomorphism

$$f^* : H^n(Y,B;G) \longrightarrow H^n(X,A;G).$$

Just as in the case of homology, these cohomology groups and their induced homomorphisms have the following properties.

7.1.6 **Functoriality**. *If $f : (X,A) \longrightarrow (Y,B)$ and $g : (Y,B) \longrightarrow (Z,C)$ are maps of* CW*-pairs, then*

$$(g\circ f)^* = f^* \circ g^* : H^n(Z,C;G) \longrightarrow H^n(X,A;G)\,.$$

Also, if $\mathrm{id}_{(X,A)} : (X,A) \longrightarrow (X,A)$ is the identity, then

$$\mathrm{id}^*_{(X,A)} = 1_{H^n(X,A;G)} : H^n(X,A;G) \longrightarrow H^n(X,A;G)\,.$$

7.1.7 **Homotopy**. *If $f_0 \simeq f_1 : (X,A) \longrightarrow (Y,B)$ (a homotopy of pairs), then*

$$f_0^* = f_1^* : H^n(Y,B;G) \longrightarrow H^n(X,A;G).$$

7.1.8 **Excision**. *Let $(X;X_1,X_2)$ be a* CW*-triad, that is, X_1 and X_2 are subcomplexes of X such that $X = X_1 \cup X_2$. Then the inclusion $j : (X_1, X_1\cap X_2) \longrightarrow (X,X_2)$ induces an isomorphism*

$$j^* : H^n(X,X_2;G) \longrightarrow H^n(X_1, X_1\cap X_2;G), \qquad n \geq 0.$$

7.1.9 **Exactness**. *Suppose that (X, A) is a* CW*-pair. Then we have an exact sequence*

$$\cdots \longrightarrow H^q(A;G) \xrightarrow{\delta} H^{q+1}(X,A;G) \longrightarrow H^{q+1}(X;G) \longrightarrow$$
$$\longrightarrow H^{q+1}(A;G) \xrightarrow{\delta} H^{q+2}(X,A;G) \longrightarrow \cdots .$$

Here δ, called the connecting homomorphism*, is a natural homomorphism, which means that given any map of pairs $f : (Y,B) \longrightarrow (X,A)$ the following diagram is commutative:*

$$\begin{tikzcd} H^q(A;G) \arrow[r, "\delta"] \arrow[d, "(f|B)^*"'] & H^{q+1}(X,A;G) \arrow[d, "f^*"] \\ H^q(B;G) \arrow[r, "\delta"'] & H^{q+1}(Y,B;G) . \end{tikzcd}$$

7.1.10 **Dimension**. *For the space $*$ containing exactly one point we have that*

$$H^i(*;G) = \begin{cases} G & \textit{if } i = 0, \\ 0 & \textit{if } i \neq 0. \end{cases}$$

Proof: Properties 7.1.6 and 7.1.7 follow immediately from the definitions.

In order to prove property 7.1.8 it is enough to note that the conditions imposed on X, X_1, and X_2 imply that

$$X/X_2 \quad \text{and} \quad X_1/X_1 \cap X_2$$

are homeomorphic.

In order to prove property 7.1.9 we first define

$$\delta : H^q(A;G) \longrightarrow H^{q+1}(X,A;G)$$

by using the composite

$$X/A \xrightarrow{p} X^+ \cup CA^+ \xrightarrow{p'} \Sigma A^+ ,$$

where $X^+ \cup CA^+$ is the *unreduced cone* of (X,A) defined alternatively as $X \sqcup A \times I/\sim$, where $X \supset A \ni a \sim (a,0) \in A \times I$ and $(a,1) \sim (a',1)$ in $A \times I$. Analogously, ΣA^+ is the *unreduced suspension* of A. Here p is the homotopy inverse of the homotopy equivalence defined by the composite

$$X^+ \cup CA^+ \longrightarrow X^+ \cup CA^+/CA^+ \approx X/A ,$$

and p' is the quotient map

$$X^+ \cup CA^+ \longrightarrow X^+ \cup CA^+/X^+ \approx \Sigma A^+.$$

So δ is defined by

$$\begin{aligned} H^q(A;G) &= [A^+,+;K(G,q),*] \cong [A^+,+;\Omega K(G,q+1),*] \\ &\cong [\Sigma A^+,*;K(G,q+1),*] \xrightarrow{p^*\circ p'^*} [X/A,*;K(G,q+1),*] \\ &= H^{q+1}(X,A;G). \end{aligned}$$

Some authors include an algebraic sign in the definition of δ in order thereby to get nicer multiplicative properties. Exactness is now obtained by applying the exact sequence of Corollary 3.3.10. Specifically, since we have as above that $H^q(X;G) = [\Sigma X^+,*;K(G,q+1),*]$, it follows that the piece of that sequence corresponding to the inclusion $i : A \hookrightarrow X$ is given as

$$\begin{gathered} [\Sigma X^+, K(G,q+1)] \longrightarrow [\Sigma A^+, K(G;q+1)] \longrightarrow [C_i, K(G,q+1)] \longrightarrow \\ \longrightarrow [X^+, K(G,q+1)] \longrightarrow [A^+, K(G,q+1)]\,, \end{gathered}$$

where we omit the base point for simplicity. This in turn changes into

$$\begin{gathered} H^q(X;G) \longrightarrow H^q(A;G) \longrightarrow H^{q+1}(X,A;G) \longrightarrow \\ \longrightarrow H^{q+1}(X;G) \longrightarrow H^{q+1}(A;G) \end{gathered}$$

by using the isomorphisms proved above and the fact that $C_i \simeq X/A$ (see Corollary 4.2.3).

Grouping together these pieces for $q \geq 0$ we obtain the desired exact sequence.

In order to prove property 7.1.10 it suffices to apply the definition of $K(G,i)$. So we have

$$H^i(*;G) = [\mathbb{S}^0, K(G,i)] = \pi_0(K(G,i)) = \begin{cases} G & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}$$

since $K(G,i)$ is discrete and equal to G if $i = 0$, while it is path connected if $i > 0$. □

All given axioms of functoriality, homotopy, exactness and dimension are the so-called *Eilenberg–Steenrod axioms* for an ordinary (unreduced) cohomology theory.

7.1.11 Definition. We can extend Definition 7.1.2 to arbitrary pairs (X,A) by defining $H^n(X,A;G) = H^n(\widetilde{X},\widetilde{A};G)$, where $(\widetilde{X},\widetilde{A})$ is a CW-approximation of (X,A). If $f:(X,A) \longrightarrow (Y,B)$ is continuous, then we define $f^* = \widetilde{f}^*$. These are well defined due to the approximation theorems 5.1.35 and 5.1.44.

7.1.12 Note. One might also define

$$H^n(X;G) = [X,*;K(G,n),*]$$

for a space X without taking CW-approximations. Let X be a paracompact Hausdorff topological space. If either G is countable or the spaces are compactly generated, then one obtains Čech cohomology groups (see [36]). For polyhedra one can show directly that these *homotopical cohomology groups* are isomorphic to the simplicial cohomology groups (see [68]).

The next result establishes the so-called *wedge axiom* for cohomology (cf. 7.1.5).

7.1.13 **Proposition**. *If* $X = \bigvee_{\lambda\in\Lambda} X_\lambda$, *then*

$$\widetilde{H}_q(X;G) \cong \prod_{\lambda\in\Lambda} \widetilde{H}^q(X_\lambda;G)\,.$$

Proof: This follows immediately from the definition of the reduced cohomology groups and 2.2.9. □

7.1.14 Exercise. Let $(X,A) = \coprod(X_\lambda,A_\lambda)$. Prove that for all q,

$$H^q(X,A;G) \cong \prod_\lambda H^q(X_\lambda,A_\lambda;G)\,.$$

This is the so-called *additivity axiom* for cohomology.

7.1.15 Exercise. Prove that if $f:(X,A) \longrightarrow (Y,B)$ is a weak homotopy equivalence of pairs of topological spaces, then

$$f_*: H^q(Y,B) \longrightarrow H^q(X,A)$$

is an isomorphism for all q. This is the so-called *weak homotopy equivalence axiom* for cohomology.

These cohomology groups defined for arbitrary pairs of topological spaces obviously satisfy the axioms of functoriality, homotopy, exactness, and dimension, which we have introduced above. But in this case we have the following excision axiom.

7.1.16 **Excision**. (For excisive triads) *Let $(X;A,B)$ be an excisive triad; that is, X is a topological space with subspaces A and B such that $\mathring{A} \cup \mathring{B} = X$, where $\mathring{A}$ and $\mathring{B}$ denote the interiors of A and B, respectively. Then the inclusion $j : (A, A\cap B) \longrightarrow (X,B)$ induces an isomorphism*

$$H^n(X,B;G) \longrightarrow H^n(A,A\cap B;G), \qquad n \geq 0.$$

Proof: In order to show that we have this property we take a CW-approximation of $A\cap B$, say $\varphi : \widetilde{A\cap B} \longrightarrow A \cap B$, and extend it to an approximation of A, say $\varphi_1 : \widetilde{A} \longrightarrow A$, and to an approximation of B, say $\varphi_2 : \widetilde{B} \longrightarrow B$, in such a way that $\widetilde{A\cap B} = \widetilde{A} \cap \widetilde{B}$. Thus we can define a map $\widetilde{\varphi} : \widetilde{X} = \widetilde{A} \cup \widetilde{B} \longrightarrow A \cup B = X$ such that $\widetilde{\varphi}|\widetilde{A} = \varphi_1$, $\widetilde{\varphi}|\widetilde{B} = \varphi_2$, and $\widetilde{\varphi}|\widetilde{A\cap B} = \varphi$. Using the hypothesis $\mathring{A} \cup \mathring{B} = X$ we can now prove that $\widetilde{\varphi}$ is a weak homotopy equivalence; that is, $\widetilde{\varphi}$ is a CW-approximation of X (see [31, 16.24]). Using this result it is clear that the excision axiom for excisive triads follows from the excision axiom (7.1.8) for CW-triads. $\square$

7.1.17 EXERCISE. Prove that the excision axiom for excisive triads is equivalent to the following axiom. Suppose that (X,A) is a pair of spaces and that $U \subset A$ satisfies $\overline{U} \subset \mathring{A}$. Then the inclusion $i : (X-U, A-U) \longrightarrow (X,A)$ induces an isomorphism $H^n(X,A;G) \cong H^n(X-U,A-U;G)$ for all $n \geq 0$. (It is precisely this version that gives us the name "excision," because it allows us to "excise" from both X and A a piece "well" contained inside of A without altering the cohomology of the pair.)

Since $[\mathbb{S}^n, K(G,q)] = \pi_n(K(G,q))$ holds, the next result follows.

7.1.18 **Proposition**. *Suppose that $n > 0$. Then we have*

$$H^q(\mathbb{S}^n;G) = \begin{cases} G & \text{if } q = 0, n, \\ 0 & \text{if } q \neq 0, n. \end{cases} \qquad \square$$

Let X be a pointed space with base point x_0. Then for every $n \geq 0$ the inclusion $i : * \longrightarrow X$ defined by $i(*) = x_0$ induces an epimorphism

$$i^* : H^n(X;G) \longrightarrow H^n(*;G),$$

which is split by the monomorphism

$$r^* : H^n(*;G) \longrightarrow H^n(X;G)$$

induced by the unique map $r : X \longrightarrow *$.

7.1.19 DEFINITION. We call $\widetilde{H}^n(X;G) = \ker(i^*)$ the *nth reduced cohomology group* of the pointed space X with coefficients in the group G.

So there is a short exact sequence

$$0 \longrightarrow \widetilde{H}^n(X;G) \longrightarrow H^n(X;G) \longrightarrow H^n(*;G) \longrightarrow 0$$

that splits, and therefore

$$H^n(X;G) = \widetilde{H}^n(X;G) \oplus H^n(*;G).$$

Consequently, by the dimension axiom 7.1.10, we have

$$H^n(X;G) = \begin{cases} \widetilde{H}^0(X;G) \oplus G & \text{if } n = 0, \\ \widetilde{H}^n(X;G) & \text{if } n \neq 0. \end{cases}$$

From now on, if it does not cause confusion, we shall write only $H^n(X)$ (respectively, $\widetilde{H}^n(X)$) instead of $H^n(X;G)$ (respectively, $\widetilde{H}^n(X;G)$).

7.1.20 EXERCISE. Prove that if X is a pointed space with base point x_0, then for every n we have

$$\widetilde{H}^n(X) = H^n(X, x_0).$$

(Hint: The exact sequence of the pair (X, x_0) decomposes into short exact sequences

$$0 \longrightarrow H^n(X, x_0) \longrightarrow H^n(X) \longrightarrow H^n(x_0) \longrightarrow 0$$

that split.)

7.1.21 EXERCISE. Assume that X is contractible. Prove that

$$H^{q-1}(A) \cong H^q(X, A)$$

if $q > 1$, and

$$\widetilde{H}^0(A) \cong H^1(X, A)\,.$$

7.1.22 EXERCISE. Take $A \subset B \subset X$ and assume that the inclusion $A \hookrightarrow B$ is a homotopy equivalence. Prove that the inclusion of pairs $(X, A) \hookrightarrow (X, B)$ induces an isomorphism

$$H^q(X, B) \longrightarrow H^q(X, A)$$

for all q.

The dimension axiom implies that the one-point space, or more generally any contractible space, has trivial reduced cohomology. Specifically, we have the next assertion.

7.1.23 **Proposition**. *Let D be a contractible space. Then we have $\widetilde{H}^n(D) = 0$ for all n.* $\square$

Proposition 7.1.18 can be rewritten in terms of reduced cohomology as follows.

7.1.24 **Proposition**. *Suppose that $n > 0$. Then we have*

$$\widetilde{H}^q(\mathbb{S}^n; G) = \begin{cases} G & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases} \qquad \square$$

7.1.25 EXERCISE. Let X be a pointed space with base point x_0. Prove that $\widetilde{H}^q(X;\mathbb{Z}) = [X, x_0; K(\mathbb{Z}, q), *]$ and thereby conclude that

$$\widetilde{H}^q(X;\mathbb{Z}) \cong \widetilde{H}^{q+1}(\Sigma X;\mathbb{Z}).$$

(Hint: Apply the exact homotopy sequence to $X \xrightarrow{f} * \longrightarrow C_f = \Sigma X$.)

7.1.26 EXERCISE. Suppose that $\alpha_k : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is the map given in Definition 6.1.5. Prove that $\alpha_k^* : \widetilde{H}^n(\mathbb{S}^n;\mathbb{Z}) \longrightarrow \widetilde{H}^n(\mathbb{S}^n;\mathbb{Z})$ corresponds to multiplication by k. (Hint: Prove this by applying the previous exercise and using induction on n.) More generally, verify that the result remains true for any coefficient group G (where multiplication by k is to be understood by viewing G as a module over the integers $\mathbb{Z}$).

7.1.27 EXERCISE. Prove the following assertions:

(a) All the arrows in the sequence

$$H^r(X,A) \longrightarrow H^r(\{1\} \times (X,A)) \xleftarrow{j^*}$$
$$\longleftarrow H^r(\mathbb{S}^0 \times X \cup \mathbb{D}^1 \times A, \{0\} \times X \cup \mathbb{D}^1 \times A) \xrightarrow{\delta}$$
$$\longrightarrow H^{r+1}(\mathbb{D}^1 \times X, \mathbb{S}^0 \times X \cup \mathbb{D}^1 \times A) = H^{r+1}((\mathbb{D}^1, \mathbb{S}^0) \times (X,A))$$

are isomorphisms, where j is the obvious inclusion. We call the composition of these isomorphisms

$$\alpha : H^r(X,A;G) \longrightarrow H^{r+1}((\mathbb{D}^1, \mathbb{S}^0) \times (X,A);G)$$

the *suspension isomorphism*.

(b) The suspension isomorphism defined in part (a) is a natural isomorphism, that is, it commutes with the homomorphisms induced by maps of pairs.

(c) This suspension isomorphism is in a sense another version of the homomorphism of Exercise 7.1.25. Explain.

7.1.28 **Proposition**. *If $X = \mathbb{S}^n \cup_{\alpha_k} e^{n+1}$ is the Moore space of type $(\mathbb{Z}/k, n)$, $n \geq 1$, which has dimension $n+1$, then*

$$H^q(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}/k & \text{if } q = n+1, \\ 0 & \text{if } q \neq 0, n+1. \end{cases}$$

Proof: This is a simple consequence of the exactness property and the fact that

$$\alpha_k^* : H^n(\mathbb{S}^n;\mathbb{Z}) \longrightarrow H^n(\mathbb{S}^n;\mathbb{Z})$$

is multiplication by k. □

7.1.29 EXERCISE. Let X and Y be pointed spaces. Prove that for every n we have

$$\widetilde{H}^n(X \vee Y; G) \cong \widetilde{H}^n(X;G) \oplus \widetilde{H}^n(Y;G).$$

7.1.30 EXERCISE. Suppose that $G_1, G_2, \ldots, G_m$ are finitely generated abelian groups and that $0 < q_1 < q_2 < \cdots < q_m$ are natural numbers. Construct a space X such that

$$H^q(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ G_i & \text{if } q = q_i, \\ 0 & \text{if } q \neq 0, q_i, \qquad i = 1, 2, \ldots, m. \end{cases}$$

7.1.31 EXERCISE. Let X be a space such that $H^q(X;\mathbb{Z}) = 0$ for $q > n$. If $f : \mathbb{S}^{n+1} \longrightarrow X$ is a continuous map, then prove that

$$H^q(C_f;\mathbb{Z}) = \begin{cases} H^q(X;\mathbb{Z}) & \text{if } q \leq n, \\ \mathbb{Z} & \text{if } q = n+2, \\ 0 & \text{if } q \neq i, n+2, \qquad 0 \leq i \leq n. \end{cases}$$

The next exercise illustrates another important application of cohomology. It concerns the existence of tangent vector fields on spheres.

7.1.32 EXERCISE. Prove that the following statements are equivalent:

(a) There exists $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n - 0$ such that $f(x) \perp x$ for all $x \in \mathbb{S}^{n-1}$.

(b) There exists $g : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ such that g has no fixed points and $|g(x) - x| < 2$ for all $x \in \mathbb{S}^{n-1}$.

(c) If $\alpha : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ is the antipodal map (namely, $\alpha(x) = -x$ for $x \in \mathbb{S}^{n-1}$), then $\alpha \simeq \mathrm{id}_{\mathbb{S}^{n-1}}$.

Show that (c), and therefore (a) and (b), can be true only if n is even. In particular, it is not possible to construct a nontrivial tangent vector field on $\mathbb{S}^2$. (We say that one cannot "comb a tennis ball.") (Hints:

(a) $\Rightarrow$ (b) Define

$$g(x) = \frac{x + f(x)}{|x + f(x)|}.$$

(b) $\Rightarrow$ (a) Define

$$f(x) = g(x) - \langle g(x), x\rangle x,$$

where $\langle -, - \rangle$ denotes the usual scalar product on $\mathbb{R}^n$.

(a) $\Rightarrow$ (c) Use the homotopy

$$H(x,t) = (1-2t)x + \sqrt{1-(1-2t)^2}(f(x)/|f(x)|).$$

Finally, for $n = 2k$ and $x = (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) \in \mathbb{S}^{n-1}$, define f by

$$f(x) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$$

and note that f satifies (a). For $n = 2k+1$, note that α cannot be homotopic to the identity. To see this, write

$$\alpha = r_1 \circ r_2 \circ \cdots \circ r_n : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1},$$

where r_i denotes the reflection in the plane $x_i = 0$. Then by using 6.1.7 we get that $\alpha^* = (-1)^n : H^{n-1}(\mathbb{S}^{n-1}) \longrightarrow H^{n-1}(\mathbb{S}^{n-1})$, and so α^* is not the identity, which implies that $\alpha \not\simeq \mathrm{id}$.)

7.1.33 EXERCISE. Suppose that X is a topological space and that $B \subset A \subset X$ are subspaces. Prove that for any group of coefficients we have a long exact sequence

$$\cdots \longrightarrow H^{n-1}(A,B) \xrightarrow{\bar{\delta}} H^n(X,A) \longrightarrow H^n(X,B)H^q(X;G) \longrightarrow$$
$$\longrightarrow H^n(A,B) \longrightarrow \cdots,$$

where the homomorphisms are induced by the inclusions, except for $\overline{\delta}$, which is defined as the composite

$$\overline{\delta} : H^{n-1}(A,B) \longrightarrow H^{n-1}(A) \xrightarrow{\delta} H^n(X,A)\,.$$

This exact sequence is the so-called *exact sequence of the triple* (X,A,B). It generalizes 7.1.9 (just take $B=\emptyset$). (Hint: See 3.5.10.)

7.2 Multiplication in Cohomology

In this section we shall introduce a multiplication in the cohomology of a space that changes the *graded group* $H^*(X)=\{H^n(X)\}$ into a *graded ring.* This structure will be obtained by defining the so-called *cup* product on the cohomology groups, which allows us to distinguish spaces with the same additive structure (see 11.8.31). We start with the next definition, which arises from Definition 6.4.10.

7.2.1 Definition. Suppose that R is a commutative ring with unit that has a countable family of generators as an abelian group. Then for any $r,s\geq 0$ we define the map

$$\mu_{r,s} : K(R,r)\wedge K(R,s) \longrightarrow K(R,r+s)$$

by the triangle

$$\begin{tikzcd}
 & K(R\otimes R, r+s) = \mathrm{SP}\, M(R\otimes R, r+s) \arrow[d, "\widehat{\nu}"] \\
K(R,r)\wedge K(R,s) \arrow[ur, "\gamma_{r,s}"] \arrow[r, "\mu_{r,s}"'] & K(R,r+s) = \mathrm{SP}\, M(R,r+s),
\end{tikzcd}$$

where $\gamma_{r,s}$ is the map defined in Definition 6.4.10 of the previous chapter and where $\nu : M(R\otimes R, r+s) \longrightarrow M(R,r+s)$ is the map induced by the homomorphism $R\otimes R \longrightarrow R$ (which is essentially the ring multiplication map) as in Proposition 6.3.5.

Using the maps $\mu_{r,s}$ defined above, we can now define the multiplication of cohomology groups as follows.

7.2.2 Definition. Let X be a CW-complex with CW-subcomplexes A and A'. The *cup product* (or *interior product*) is the group homomorphism

$$H^r(X,A;R)\otimes H^s(X,A';R) \longrightarrow H^{r+s}(X,A\cup A';R)$$

that associates to the classes

$$x = [\alpha] \in H^r(X, A; R) \quad \text{and} \quad y = [\beta] \in H^s(X, A'; R)$$

the homotopy class of the map

$$X/A \cup A' \xrightarrow{\Delta} X/A \wedge X/A' \xrightarrow{\alpha\wedge\beta} K(R, r) \wedge K(R, s) \xrightarrow{\mu_{r,s}} \longrightarrow K(R, r + s)\,,$$

where $\Delta : X/A \cup A' \longrightarrow X/A \wedge X/A'$ is the map induced by the diagonal $X \longrightarrow X \times X$. This class is denoted by $x \smile y$.

From now on we shall assume that we are always dealing with cohomology that has coefficients in a commutative ring R with unit. For simplicity we shall also omit R from the notation. The cup product gives cohomology a multiplicative structure with the following properties.

7.2.3 **Naturality**. *If $f : (X; A, A') \longrightarrow (Y; B, B')$ is a map of triads (which means that $f(A) \subset B$ and $f(A') \subset B'$), then for all $y \in H^r(Y, B)$ and all $y' \in H^s(Y, B')$ we have that*

$$f^*(y \smile y') = f^*(y) \smile f^*(y') \in H^{r+s}(X, A \cup A')\,.$$

7.2.4 **Associativity**. *For all*

$$x \in H^q(X, A)\,, \quad x' \in H^r(X, A')\,, \quad \textit{and} \quad x'' \in H^s(X, A'')$$

we have that

$$x \smile (x' \smile x'') = (x \smile x') \smile x'' \in H^{q+r+s}(X, A \cup A' \cup A'')\,.$$

7.2.5 **Units**. *Suppose that $1_X \in H^0(X)$ is the element represented by the constant map $X \longrightarrow K(R, 0) = R$ that sends the entire space X to the element $1 \in R$. Then for all $x \in H^q(X, A)$ we have that*

$$1_X \smile x = x \smile 1_X = x \in H^q(X, A).$$

7.2.6 **Stabilility**. *The following diagram is commutative:*

$$\begin{tikzcd}
H^r(A) \otimes H^s(X, A') \arrow[r, "\mathrm{id}\otimes i^*"] \arrow[ddd, "\delta\otimes\mathrm{id}"'] & H^r(A) \otimes H^s(A, A \cap A') \arrow[d, "\smile"] \\
& H^{r+s}(A, A \cap A') \\
& H^{r+s}(A \cup A', A') \arrow[u, "j^*"'] \arrow[d, "\delta"] \\
H^{r+1}(X, A) \otimes H^s(X, A') \arrow[r, "\smile"] & H^{r+s+1}(X, A \cup A').
\end{tikzcd}$$

Here i and j are inclusions. Moreover, j^* actually turns out to be an excision isomorphism.

In particular, for the case $A' = \emptyset$, we obtain the formula

$$\delta(a \smile i^*x) = \delta a \smile x \in H^{r+s+1}(X, A)$$

for $a \in H^r(A)$ and $x \in H^s(X)$.

7.2.7 **Commutativity**. *For all*

$$x \in H^r(X, A) \quad \text{and} \quad x' \in H^s(X, A')$$

we have that

$$x \smile x' = (-1)^{rs} x' \smile x \in H^{r+s}(X, A \cup A')\,.$$

The *proof* of these properties, except commutativity, basically reduces to the uniqueness up to homotopy of the maps between Moore spaces that realize the given group homomorphisms. We leave the details of the proof to the reader in the following exercise. □

7.2.8 Exercise. Establish the properties of naturality, associativity, units, and stability of the cup product in cohomology.

In analogy to the interior or cup product, we can define an exterior or cross product as follows.

7.2.9 Definition. Suppose that X and Y are CW-complexes and that A and B are subcomplexes of X and Y, respectively. The *cross product* (or *exterior product*) is the group homomorphism

$$H^r(X, A; R) \otimes H^s(Y, B; R) \longrightarrow H^{r+s}((X, A) \times (Y, B); R),$$

where $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$, that associates to the classes $x = [\alpha] \in H^r(X, A; R)$ and $y = [\beta] \in H^s(Y, B; R)$ the homotopy class of the map

$$\begin{aligned} X \times Y/A \times Y \cup X \times B \approx (X/A) \wedge (Y/B) &\xrightarrow{\alpha\wedge\beta} \\ \longrightarrow K(R, r) \wedge K(R, s) &\xrightarrow{\mu_{r,s}} K(R, r+s)\,. \end{aligned}$$

This class is denoted by $x \times y$.

The cross product has properties that correspond to those of the cup product due to the fact that these two products are intimately related.

7.2.10 EXERCISE. Suppose that $x \in H^r(X,A)$, $x' \in H^s(X,A')$, and $y \in H^s(Y,B)$. Prove the following two formulas:

(a) $x \times y = p^*(x) \smile q^*(y)$,
where $p\ : (X,A) \times Y \longrightarrow (X,A)$ and $q\ : X \times (Y,B) \longrightarrow (Y,B)$ are the obvious projections.

(b) $x \smile x' = \Delta^*(x \times x')$,
where $\Delta : (X, A \cup A') \longrightarrow (X,A) \times (X,A')$ is the diagonal map.

Using the previous exercise and the properties of the cup product, it is possible to prove the following properties of the cross product. However, they can also be proved directly.

7.2.11 **Naturality**. *If*

$$f : (X',A') \longrightarrow (X,A) \quad \text{and} \quad g : (Y',B') \longrightarrow (Y,B)$$

are maps of pairs, then for all $x \in H^r(X,A)$ and all $y \in H^s(Y,B)$ we have that

$$(f \times g)^*(x \times y) = f^*(x) \times g^*(y) \in H^{r+s}((X',A') \times (Y',B'))\,.$$

7.2.12 **Associativity**. *For all*

$$x \in H^q(X,A)\,, \quad y \in H^r(Y,B)\,, \quad \text{and} \quad z \in H^s(Z,C)$$

we have that

$$x \times (y \times z) = (x \times y) \times z \in H^{q+r+s}((X,A) \times (Y,B) \times (Z,C))\,.$$

7.2.13 **Units**. *Suppose that $1 \in H^0(*) \cong R$ is the element represented by the map $\{*\} \longrightarrow K(R,0) = R$ that sends $\{*\}$ to the element $1 \in R$. Then for all $x \in H^q(X,A)$ we have that*

$$1 \times x = x \times 1 = x \in H^q(\{*\} \times (X,A)) = H^q(X,A)\,.$$

7.2.14 **Stabilility**. *The following diagram is commutative:*

$$\begin{tikzcd}
H^r(A,A')\otimes H^s(Y,B) \arrow[r, "\times"] \arrow[dd, "\delta\otimes\mathrm{id}"'] & H^{r+s}(A\times Y, A\times B\cup A'\times Y) \\
& H^{r+s}(A\times Y\cup X\times B, A'\times Y\cup X\times B) \arrow[u, "\cong"', "j^*"] \arrow[d, "\delta"] \\
H^{r+1}(X,A)\otimes H^s(Y,B) \arrow[r, "\times"] & H^{r+s+1}(X\times Y, A\times Y\cup X\times B)\,.
\end{tikzcd}$$

Here j is the obvious inclusion, and so j^* is actually an excision isomorphism.

In the particular case $B=\emptyset$ we have the formula

$$\delta(a\times y) = (\delta a)\times y \in H^{r+s+1}((X,A)\times Y)\,,$$

where $a\in H^r(A,A')$ and $y\in H^s(Y)$.

7.2.15 **Commutativity**. *For all $x\in H^r(X,A)$ and $y\in H^s(Y,B)$ we have that*

$$T^*(x\times y) = (-1)^{rs} y\times x \in H^{r+s}((Y,B)\times(X,A))\,,$$

where $T:(Y,B)\times(X,A)\longrightarrow(X,A)\times(Y,B)$ interchanges the factors. □

7.2.16 Exercise. Prove the properties of the cross product in cohomology by starting from the properties of the cup product in cohomology.

7.2.17 Note. Conversely, it is also possible to prove the properties of the cup product by starting from the properties of the cross product. That is, both are equivalent structures in conveniently different disguises.

The following exercises can be solved by directly applying the properties of the products and the formulas that they satisfy.

7.2.18 Exercise. Suppose that $x\in H^q(X,A)$, $y\in H^r(Y,B)$, and $y'\in H^s(Y,B')$. Prove that we have the formula

$$x\times(y\smile y') = (x\times y)\smile q^*(y')\in H^{q+r+s}(X\times Y, X\times(B\cup B')\cup A\times Y)\,,$$

where $q\ :X\times Y\longrightarrow Y$ denotes the projection.

7.2.19 Exercise. Let $\sigma\in H^1(\mathbb{D}^1,\mathbb{S}^0;R)$ be the element represented by the composite map $(\mathbb{D}^1,\mathbb{S}^0)\longrightarrow\mathbb{S}^1=K(\mathbb{Z},1)\longrightarrow K(R,1)$, where the first map is the natural identification and the second map is that induced by the group homomorphism $\mathbb{Z} \longrightarrow R$ satisfying $1 \mapsto 1$. Prove that there is an isomorphism $\alpha : H^r(X) \longrightarrow H^{r+1}((\mathbb{D}^1, \mathbb{S}^0) \times X; R)$ defined by

$$\alpha(x) = \sigma \times x.$$

This is precisely the suspension isomorphism defined in 7.1.27. (Hint: Prove that the image of $1 \in H^0(*) = R$ under the suspension isomorphism is precisely σ and then use the properties of the cross product.)

7.2.20 Exercise.

(i) Prove that the inclusion

$$(\mathbb{D}^1, \mathbb{S}^0) \hookrightarrow (\mathbb{R}, \mathbb{R} - 0)$$

induces an isomorphism in cohomology

$$H^*(\mathbb{D}^1, \mathbb{S}^0) \cong H^*(\mathbb{R}, \mathbb{R} - 0)\,.$$

(Hint: The inclusions

$$(\mathbb{D}^1, \mathbb{S}^0) \hookrightarrow (\mathbb{D}^1, \mathbb{D}^1 - 0) \quad \text{and} \quad (\mathbb{D}^1, \mathbb{D}^1 - 0) \hookrightarrow (\mathbb{R}, \mathbb{R} - 0)$$

are respectively an excision and a homotopy equivalence in the second term, and therefore both of them induce isomorphisms. Then use the exact sequence of a pair in the second case.)

(ii) Let $g_1 \in H^1(\mathbb{R}, \mathbb{R} - 0)$ be the element corresponding to σ (from the previous exercise) under the isomorphism from part (i). Prove that the homomorphism $g_1 \times : H^q(X, A) \longrightarrow H^{q+1}((\mathbb{R}, \mathbb{R} - 0) \times (X, A))$ is actually an isomorphism. (Hint: Modulo the isomorphism defined in the hint for part (i), the homomorphism here is the suspension isomorphism from the previous exercise.)

(iii) For each n define $g_n \in H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ inductively as $g_n = g_1 \times g_{n-1}$, where we use $(\mathbb{R}, \mathbb{R} - 0) \times (\mathbb{R}^{n-1}, \mathbb{R}^{n-1} - 0) = (\mathbb{R}^n, \mathbb{R}^n - 0)$. Prove that g_n is a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ as an infinite cyclic group; we call it the *canonical generator*. (Hint: Apply part (ii) and use induction.)

7.3 Cellular Homology and Cohomology

Up to now we have presented homology and cohomology groups from the point of view of homotopy theory, that is, as sets of homotopy classes. Historically, however, (algebraic) homological methods were first used to define these groups. Even though this does not reveal the homotopic nature of the subject, it does allow one to carry out calculations more systematically. In this section we shall present a treatment of these matters that relies on the homological algebra of homology and cohomology groups. This is called cellular homology and cohomology. Besides using this theory for calculating, we also shall use it in the next section to establish the Künneth formula and the universal coefficients theorem. From now on we shall assume that X is a CW-complex, and we shall denote by $H_m(X,A)$ the homology group of X modulo a subcomplex A with coefficients in $\mathbb{Z}$. We start with a theorem.

7.3.1 **Theorem**. *Let $\{*\} = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X$ be the filtration of a* CW*-complex X by its skeletons. Then we have*

$$H_m(X^n, X^{n-1}) \cong \begin{cases} \bigoplus_{i\in J^n} \mathbb{Z} & \textit{if } m = n, \\ 0 & \textit{if } m \neq n, \end{cases}$$

where $\{e_i^n \mid i \in J^n\}$ is the set of all the n-cells of X.

Proof: Consider the following sequence of isomorphisms:

$$\begin{aligned} H_m(X^n, X^{n-1}) &\cong \widetilde{H}_m(X^n/X^{n-1}) \cong \widetilde{H}_m(\bigvee_{i\in J^n} \mathbb{S}^n) \\ &\cong \bigoplus_{i\in J^n} \widetilde{H}_m(\mathbb{S}^n) \cong \begin{cases} \bigoplus_{i\in J^n} \mathbb{Z} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \end{aligned}$$

The first map is an isomorphism because of 5.3.18, since the pair (X^n, X^{n-1}) is a CW-pair. The second map is an isomorphism because the quotient is exactly a wedge of spheres. And for the third map one uses 5.3.31, while for the fourth map one just applies 5.3.29. $\square$

And we get a corollary from this theorem.

7.3.2 **Corollary**. *Under the same hypotheses as above we have the following statements:*

(a) $\widetilde{H}_m(X^n) = 0$ *for* $m > n$.

(b) $H_m(X^n) \cong H_m(X^{n+1}) \cong H_m(X)$ *for* $m < n$.

(c) *The map* $H_n(X^n) \longrightarrow H_n(X^{n+1})$ *induced by the inclusion is an epimorphism.*

Proof: Consider the following portion of the long exact homology sequence of the pair (X^{n+1}, X^n):

$$H_{m+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_m(X^n) \longrightarrow H_m(X^{n+1}) \longrightarrow H_m(X^{n+1}, X^n).$$

Notice that the first group is trivial if $m \neq n$, and the last is trivial if $m \neq n+1$. So part (c) clearly follows, as does the first isomorphism in part (b). To prove part (a) we observe that $\widetilde{H}_m(X^n) \cong \widetilde{H}_m(X^{n-1}) \cong \cdots \cong \widetilde{H}_m(X^{-1}) = 0$ for $m > n$. For $m > 0$ notice that these groups coincide with the corresponding unreduced groups.

Lastly, the second isomorphism in part (b) is obtained from the diagram

$$\begin{tikzcd}
H_m(X^n) \arrow[r, "\cong"] \arrow[dr] & \operatorname{colim} H_m(X^k) \arrow[d, "\{i_{k*}\}"] \\
& H_m(X),
\end{tikzcd}$$

where $i_k : X^k \hookrightarrow X$ denotes the inclusion and $\{i_{k*}\}$ is an isomorphism by Proposition 5.3.30. □

In the following we are going to be using the basic concepts of homological algebra. This material can be found in any introductory book on the subject such as Mac Lane's text [47]. So for any finite CW-complex X let us consider the chain complex

(7.3.3)
$$\begin{aligned}\cdots \xrightarrow{\partial_{n+2}} H_{n+1}(X^{n+1}, X^n) &\xrightarrow{\partial_{n+1}} H_n(X^n, X^{n-1}) \\ &\xrightarrow{\partial_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots,\end{aligned}$$

where $\partial_{n+1} : H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_n(X^n) \to H_n(X^n, X^{n-1})$ defines the maps here.

7.3.4 **Theorem**. *The chain complex (7.3.3) has $H_*(X)$ as its homology.*

Proof: Consider the decomposition

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "\partial"] & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_n(X^n) \arrow[ur, "(1)"'] \arrow[d, two heads] & H_{n-1}(X^{n-1}) \arrow[ur, "(2)"'] & \\
& H_n(X^{n+1}) & &
\end{tikzcd}$$

of the above chain complex, where the diagonal arrows (1) and (2) are monomorphisms and the lower vertical arrow is an epimorphism, as we showed in Corollary 7.3.2. Also, both the two vertical arrows on the left as well as the diagonal arrows (1) and (2) form exact sequences. It follows that

$$\begin{aligned}\ker \partial_n &= \ker \partial \cong H_n(X^n)\,,\\ \operatorname{im} \partial_{n+1} &\cong \operatorname{im}\, \partial \subset H_n(X^n)\,.\end{aligned}$$

Thus we have ker $\partial_n/\text{im } \partial_{n+1} \cong H_n(X^n)/\text{im } \partial \cong H_n(X^{n+1}) \cong H_n(X)$ by Corollary 7.3.2 (b). ☐

7.3.5 DEFINITION. We call the chain complex $\{H_n(X^n, X^{n-1}), \partial_n\}$ in (7.3.3) the *cellular chain complex* of X, and we denote it by

$$C(X) = \{C_n(X), d_n\},$$

where from now on we shall identify $C_n(X)$ with the free group generated by the n-cells of X.

7.3.6 NOTE. One can prove that under this identification of $C_n(X)$ (with the free group generated by the n-cells of X) the operator d_n satisfies

$$d_n(e_i^n) = \sum_{j\in J^{n-1}} \alpha_i^j e_j^{n-1}\,,$$

where $\alpha_j^i \in \mathbb{Z}$ is the degree of the composite

$$\begin{aligned}\mathbb{S}^{n-1} \approx \partial e_i^n &\xrightarrow{\varphi^i} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2}\\ &\approx \bigvee_{j\in J^{n-1}} \mathbb{S}_j^{n-1} \xrightarrow{k_j} \mathbb{S}^{n-1}\,.\end{aligned}$$

Here φ^i is the characteristic map of the cell e_i^n, q is the quotient map, and k_j identifies to a point all of the summands $\mathbb{S}_{j'}^{n-1}$ satisfying $j' \neq j$. (Bredon's book [19] develops all of this material in full detail.)

7.3.7 DEFINITION. Let X be a CW-complex. We define its *nth homology group with coefficients in an abelian group* G as the nth homology group of its *cellular chain complex with coefficients in* G, which is itself defined by

$$C(X;G) = \{C_n(X)\otimes G, d_n \otimes 1_G\}.$$

We denote this homology group by $H_n(X;G)$.

7.3.8 Exercise. Let X be a pointed CW-complex. Define

$$\widetilde{H}_0(X;G) = \ker(H_0(X;G) \longrightarrow H_0(*;G))$$

and $\widetilde{H}_n(X;G) = H_n(X;G)$ for $n \neq 0$. Moreover, for any CW-pair (X, A) define

$$H_n(X, A; G) = \widetilde{H}_n(X \cup CA; G)\,.$$

Prove that the groups $H_n(X, A; G)$ satisfy axioms that correspond to 5.3.13–5.3.17.

7.3.9 Note. In particular, if $G = \mathbb{Z}/k$, then the groups $\widetilde{H}_n(X;G)$ coincide with the groups already described in 5.3.33 (see the comparison theorem 12.1.19).

There is a relative version of all this as well. Theorem 7.3.1 can be proved in the case where we have a filtration $A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X$ of a pair of CW-complexes $A \subset X$. Now, however, X^n represents the *relative n-skeleton*; that is, the union of A with the absolute n-skeleton. In this case, the version of Theorem 7.3.4 corresponding to a *relative cellular chain complex* $C_*(X, A)$ asserts that the homology of this complex is $H_*(X, A)$. There is another point of view, as we see from the next exercise.

7.3.10 Exercise. Suppose that X is a CW-complex with a subcomplex A. Then the quotients $C_n(X)/C_n(A)$ determine a chain complex. Prove that this chain complex is isomorphic to $C_*(X, A)$.

7.3.11 Exercise. Prove that the relative groups $H_n(X, A; G)$ can be defined, in terms of what we said before, by using the chain complex $C_*(X,A;G)$ whose groups are $C_n(X, A) \otimes G$.

As an application of the previous results we now analyze an example.

7.3.12 Example. The *Klein bottle* K is obtained from the square $I \times I$ by identifying $(0, t)$ with $(1, 1-t)$ and $(s, 0)$ with $(s, 1)$ for all $s, t \in I$. We shall calculate its homology and so shall see that this space is not homeomorphic to the *torus* $T = \mathbb{S}^1 \times \mathbb{S}^1$.

As we see in Figure 7.1, one can decompose K as a CW-complex with one 0-cell e^0, two 1-cells e^1 and $\widetilde{e}^1$, and one 2-cell e^2. From the way in which

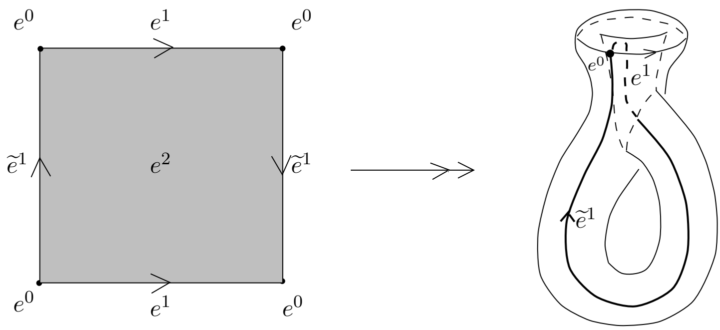

Figure 7.1

these cells are glued together and from 7.3.6 we have in the cellular chain complex of K that

$$d_2(e^2) = 2\widetilde{e}^1\,,$$
$$d_1(e^1) = d_1(\widetilde{e}^1) = 0\,,$$
$$d_0(e^0) = 0\,,$$

implying that

$$H_2(K) = 0, \qquad H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

On the other hand, for the torus T we can similarly prove that its homology is

$$H_2(T) = \mathbb{Z}, \qquad H_1(T) = \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore, the Klein bottle K and the torus T cannot be homeomorphic. In fact, they cannot even have the same homotopy type.

The next example will be of interest in the last chapter of the book.

7.3.13 Example. Consider the complex projective space $\mathbb{CP}^k$, which has one 0-cell, one 2-cell, one 4-cell, and so forth up to one $2k$-cell and which has no odd-dimensional cells. Consequently, its cellular chain complex has the form

$$C_n(\mathbb{CP}^k) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even and } n \le 2k, \\ 0 & \text{if } n \text{ is odd or } n > 2k, \end{cases}$$

and so $d_n = 0$ for all n. Since the homology of the space is equal to that of the cellular chain complex, we get that

$$H_n(\mathbb{CP}^k) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even and } n \leq 2k, \\ 0 & \text{if } n \text{ is odd or } n > 2k. \end{cases}$$

Obviously, we get an analogous result when we calculate the homology with coefficients in a group. (Compare this example with 11.7.29.)

The next example is also rather interesting.

7.3.14 Example. Consider the real projective space $\mathbb{RP}^k$, which has one 0-cell, one 1-cell, one 2-cell, and so forth up to one k-cell. In this way we see that its cellular chain complex with coefficients in G has the form

$$C_n(\mathbb{RP}^k; G) = G$$

for all $n \leq k$ and is trivial for $n > k$. However, the way in which these cells are put together implies either that

$$d_n(g) = 2g \quad \text{if } n \text{ is odd}$$

or that

$$d_n(g) = 0 \quad \text{if } n \text{ is even}$$

for all $g \in G$ (see Exercise 7.3.15). Therefore, if k is even, then we have

$$H_n(\mathbb{RP}^k; G) = \begin{cases} G & \text{if } n = 0, \\ G/2G & \text{if } n \text{ is odd and } n < k, \\ 0 & \text{otherwise.} \end{cases}$$

Now, if k is odd, then

$$H_n(\mathbb{RP}^k; G) = \begin{cases} G & \text{if } n = 0, k, \\ G/2G & \text{if } n \text{ is odd and } n < k, \\ G_{(2)} & \text{otherwise,} \end{cases}$$

where $G_{(2)} = \{g \in G \mid 2g = 0\}$ is the so-called 2-*torsion subgroup* of G. Since for $G = \mathbb{Z}/2$ we have $2G = 0$ and $G_{(2)} = \mathbb{Z}/2$, it follows that

$$H_n(\mathbb{RP}^k; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } n \leq k, \\ 0 & \text{if } n > k. \end{cases}$$

(Compare this result with 11.7.26.) On the other hand, for $G = \mathbb{Z}$ we have $G_{(2)} = 0$. Therefore, for k even,

$$H_n(\mathbb{RP}^k) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n \text{ is odd and } n < k, \\ 0 & \text{otherwise,} \end{cases}$$

and for k odd,

$$H_n(\mathbb{RP}^k) = \begin{cases} \mathbb{Z} & \text{if } n = 0, k, \\ \mathbb{Z}/2 & \text{if } n \text{ is odd and } n < k, \\ 0 & \text{otherwise.} \end{cases}$$

7.3.15 Exercise. Using the way that cells are attached in the real projective space $\mathbb{RP}^n$ and taking into account 7.3.6, check that in the example above, d_n is multiplication by 2 if n is odd, and zero if n is even. (Hint: The number $\alpha \in \mathbb{Z}$ by which we multiply to obtain d_n is the degree of the composite

$$\mathbb{S}^{n-1} \approx \partial e^n \xrightarrow{\ \varphi\ } X^{n-1} \xrightarrow{\ q\ } X^{n-1}/X^{n-2} \approx \mathbb{S}^{n-1}.$$

This map factors as a composite $\mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$, where the first map collapses the equator sphere $\mathbb{S}^{n-2}$ onto the base point and the second one maps the first sphere as the identity and the second sphere as the reflection on the equator. The first of these has degree 1, and the second has degree $(-1)^{n-1}$. Take a look at [19].)

7.3.16 Exercise. Using the cellular decomposition of the Moore space of type $(\mathbb{Z}/k, n)$, namely $X = \mathbb{S}^n \cup_{\alpha_k} e^{n+1}$, calculate $H_q(X;\mathbb{Z})$. (Compare with Proposition 7.1.28.)

In much the same way as above it is possible to discuss cohomology with coefficients. Specifically, we have the next definition.

7.3.17 Definition. Suppose that G is an abelian group. Put $C^n(X;G) = \operatorname{Hom}(C_n(X), G)$ and put $d^n = (d_n)^{\#} : C^{n-1}(X;G) \longrightarrow C^n(X;G)$. We call the cochain complex

$$C^*(X;G) = \{C^n(X;G), d^n\}$$

the *cellular cochain complex of X with coefficients in G.*

The next result for cohomology is dual to Theorem 7.3.4.

7.3.18 **Theorem**. *The cochain complex $C^*(X;G)$ has $H^*(X;G)$ as its cohomology.*

The *proof* of this theorem is based on Milnor's comparison theorem 12.1.19. □

7.3.19 Exercise. Suppose that X is a pointed CW-complex and that the group $H^*_C(X;G)$ is the cohomology of $C^*(X;G)$. Define

$$\widetilde{H}^*_C(X;G) = \ker(i^*)\,,$$

where $i\ :*\hookrightarrow X$ is the inclusion into the base point. Moreover, define

$$H^n_C(X,A;G) = \widetilde{H}^n_C(X/A;G)$$

whenever A is a subcomplex of X.

Prove that the groups $H^n_C(X,A;G)$ so defined satisfy axioms 7.1.6 to 7.1.10.

This exercise allows us to apply the comparision theorem to which we referred above to prove Theorem 7.3.18.

7.3.20 Exercise. Prove that the relative groups $H^n(X,A;G)$ can be defined by using the cochain complex $C^*(X,A;G)$ whose groups are

$$\operatorname{Hom}\left(C_n(X,A),G\right),$$

where $C_n(X,A)$ is described in Exercise 7.3.10.

7.3.21 Exercise. Recall the construction of the oriented and nonorientable closed surfaces of genus g given in 3.2.12(c) and (d). Using it, compute their cellular homology and cohomology groups with coefficients both in $\mathbb{Z}$ and in $\mathbb{Z}/2$.

7.3.22 Exercise. Using the cellular complexes with coefficients in G of the real and complex projective spaces given in 7.3.14 and 7.3.13, compute their cohomology groups with coefficients in G.

7.3.23 Exercise. Let X be a CW-complex of dimension n. Prove that

$$H_m(X;G) = 0 \quad \text{and} \quad H^m(X;G) = 0 \quad \text{for} \quad m > n\,.$$

7.3.24 REMARK. There is an example due to Barratt and Milnor [16] of an $(r-1)$-connected, compact space X, $r > 1$, with its homology and cohomology groups with coefficients in the group of rational numbers such that

$$H_m(X; \mathbb{Q}) \neq 0 \quad \text{and} \quad H^m(X; \mathbb{Q}) \neq 0$$

for an infinite number of values m. This space X is an infinite "wedge" of copies of $\mathbb{S}^r$, but with the topology as a subspace of their product (see note 2.9.2).

7.4 EXACT SEQUENCES IN HOMOLOGY AND COHOMOLOGY

We end this chapter with this section, where we shall present some exact sequences giving the homology and the cohomology of a product of spaces and then, as a consequence, some formulas for changing coefficient groups in homology and cohomology. Likewise, with similar techniques we shall construct the Mayer–Vietoris sequences in homology and cohomology for CW-complexes.

Suppose that X and Y are CW-complexes with countably many cells or suppose that at least one of them is locally compact. It follows in either case that their product $X \times Y$ is again a CW-complex (see 5.1.46). Given all this and that $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{e'_\beta\}_{\beta \in \mathcal{B}}$ are the cells of X and Y, respectively, then $\{e_\alpha \times e'_\beta\}_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}}$ are the cells of $X \times Y$. According to Definition 7.3.5, we know that $C_k(X)$ and $C_l(Y)$ are the abelian groups freely generated by the k-cells of X and the l-cells of Y, respectively. Also, the boundary operators of these chain complexes are given in 7.3.6.

7.4.1 DEFINITION. We define the *product* of the chain complexes $C_*(X)$ and $C_*(Y)$, denoted by $C_*(X) \otimes C_*(Y)$, to be given in dimension m by

$$[C_*(X) \otimes C_*(Y)]_m = \bigoplus_{k+l=m} C_k(X) \otimes C_l(Y),$$

together with the boundary operator d defined by

$$d(a \otimes b) = d(a) \otimes b + (-1)^k a \otimes d(b)$$

for $a \in C_k(X)$ and $b \in C_l(Y)$.

We then have that the function $e_\alpha \otimes e'_\beta \mapsto e_\alpha \times e'_\beta$, being a bijection between generators, determines an isomorphism

$$C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)\,.$$

Furthermore, we can prove using 7.3.6 that the boundary operator in $C_*(X \times Y)$ is given by $d(e_\alpha \times e'_\beta) = d(e_\alpha) \otimes e'_\beta + (-1)^k e_\alpha \otimes d(e'_\beta)$, where k is the dimension of e_α. So we obtain the next result.

7.4.2 **Theorem**. *Suppose either that X and Y are* CW-*complexes with countably many cells or that at least one of them is locally compact. Then there exists an isomorphism of chain complexes*

$$C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$$

defined by $e_\alpha \otimes e'_\beta \mapsto e_\alpha \times e'_\beta$, where $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{e'_\beta\}_{\beta\in\mathcal{B}}$ are the cells of X and Y, respectively. □

Using Definition 7.3.7, we get from Theorem 7.4.2 that

$$H_*(X \times Y; R) \cong H_*(C_*(X \times Y) \otimes R)\,,$$

where R is a commutative ring with unit. But $C_*(X \times Y) \otimes R \cong (C_*(X) \otimes_{\mathbb{Z}} C_*(Y)) \otimes R) = (C_*(X) \otimes R) \otimes_R (C_*(Y) \otimes R)$ holds, and so we have that

$$H_*(X \times Y; R) \cong H_*((C_*(X) \otimes R) \otimes_R (C_*(Y) \otimes R))\,.$$

Analogously, for the case of cohomology with coefficients in R, according to Theorem 7.3.18 we have that

$$H^*(X \times Y; R) \cong H_*(C^*(X; R) \otimes_R C_*(Y; R))\,.$$

We give in the following a general result, and its dual, from homological algebra. These give rise to the *Künneth formula* in homology and cohomology. This material can be found, for example, in Spanier's text [67, 5.3.1, 5.5.11] as well as in Mac Lane's [47, V.10]. In the case of cohomology we require that the chain complexes be of *finite type*, that is, that they have a finite number of generators in each dimension. This will always be the case for the cellular chain complex of a compact CW-complex. We say that a CW-complex is of *finite type* if it has a finite number of cells in each dimension. Therefore, *the cellular chain complex of a* CW-*complex of finite type is of finite type.*

7.4.3 **Theorem**. *Suppose that C and D are free chain complexes over a principal ideal domain R. Put $C^* = \operatorname{Hom}_R(C; R)$ and $D^* = \operatorname{Hom}_R(D; R)$. Then there is a natural short exact sequence*

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{k+l=m} H_k(C) \otimes_R H_l(D) \xrightarrow{\ p\ } H_m(C \otimes_R D) \longrightarrow \\ \longrightarrow \textstyle\bigoplus_{k+l=m-1} \operatorname{Tor}_R(H_k(C), H_l(D)) \longrightarrow 0\,, \end{aligned}$$

where p is given by $[c] \otimes [d] \mapsto [c \otimes d]$. Also, for the cohomology of C^* and D^*, provided, moreover, that C_* and D_* are of finite type, we have a natural short exact sequence

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{k+l=m} H^k(C) \otimes_R H^l(D) &\xrightarrow{\ p^*\ } H^m(C \otimes_R D) \longrightarrow \\ &\longrightarrow \textstyle\bigoplus_{k+l=m+1} \mathrm{Tor}_R(H^k(C), H^l(D)) \longrightarrow 0\,, \end{aligned}$$

where p^* is defined analogously to p.

Furthermore, these exact sequences split, though not naturally. □

From the previous theorem we now get the Künneth formulas.

7.4.4 **Theorem**. (Künneth formula) *Suppose either that X and Y are* CW-*complexes with countably many cells or that one of them is locally compact. Let R be a principal ideal domain. Then we have a natural short exact sequence in homology with coefficients in R,*

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{k+l=m} H_k(X) \otimes_R H_l(Y) &\xrightarrow{\ p\ } H_m(X \times Y) \longrightarrow \\ &\longrightarrow \textstyle\bigoplus_{k+l=m-1} \mathrm{Tor}_R(H_k(X), H_l(Y)) \longrightarrow 0\,, \end{aligned}$$

where p is the homology product defined by $[e] \otimes [e'] \mapsto [e \times e']$. Furthermore, provided that X and Y are of finite type, we have a natural short exact sequence in cohomology with coefficients in R,

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{k+l=m} H^k(X) \otimes_R H^l(Y) &\xrightarrow{\ \times\ } H^m(X \times Y) \longrightarrow \\ &\longrightarrow \textstyle\bigoplus_{k+l=m+1} \mathrm{Tor}_R(H^k(X), H^l(Y)) \longrightarrow 0\,, \end{aligned}$$

where $\times$ is the cross product in cohomology.

In addition, both of these exact sequences split, although not naturally. □

If one of the R-modules appearing in the previous formulas is free, say, for example, that R is a field, then the torsion products given by the functor Tor_R vanish. So we have the following consequence.

7.4.5 **Corollary**. *If R is a field or, more generally, if the R-modules*

$$H_*(X;R) \quad \text{and} \quad H^*(X;R)$$

are free with the latter being of finite type, then there exist natural isomorphisms

$$p : \bigoplus_{k+l=m} H_k(X;R) \otimes_R H_l(Y;R) \xrightarrow{\ \cong\ } H_m(X \times Y;R)\,,$$

$$\times : \bigoplus_{k+l=m} H^k(X;R) \otimes_R H^l(Y;R) \xrightarrow{\ \cong\ } H^m(X \times Y;R)\,.$$

□

7.4.6 NOTE. We should note here that the condition that a CW-complex is of finite type implies that it has countably many cells, so that this one condition actually implies the various general conditions of Theorem 7.4.4, namely, the condition that each CW-complex have countably many cells in the homology case and the condition that the CW-complexes be of finite type for the cohomology case. It follows that *the product of two* CW-*complexes of finite type is a* CW-*complex of finite type.*

On the other hand, in the same theorem for the case of cohomology, it is enough to require that $H^*(X)$ and $H^*(Y)$ be of finite type, which always happens when $C_*(X)$ and $C_*(Y)$ are of finite type. Nonetheless, it is often easier to verify the condition on the cohomology groups than on the chain complexes, and in many cases the latter cannot be of finite type even though their cohomology groups will indeed be of finite type.

7.4.7 REMARK. The Künneth formulas are true for arbitrary spaces X and Y. One can show this using Theorem 7.4.4 and cellular approximations. However, we must stress that in this case we get this result either when both spaces are of the same weak homotopy type as CW-complexes with countably many cells or when one of them is locally compact. To prove the Künneth formula in its full generality requires, instead of Theorem 7.4.2, the Eilenberg–Zilber theorem, which establishes a *chain homotopy equivalence* between the *singular chain complexes* $S_*(X \times Y)$ and $S_*(X) \otimes S_*(Y)$.

The next result is true for any space X, but since we want to derive it as a consequence of Theorem 7.4.3, we shall assume that X is a CW-complex.

7.4.8 **Theorem**. (Universal coefficients theorem) *Let R be a principal ideal domain and let A be an R-module. Then there are natural short exact sequences*

$$0 \longrightarrow H_m(X;R) \otimes_R A \longrightarrow H_m(X;A) \longrightarrow \mathrm{Tor}_R(H_{m-1}(X;R),A) \longrightarrow 0$$

and

$$0 \longrightarrow H^m(X;R) \otimes_R A \longrightarrow H^m(X;A) \longrightarrow \mathrm{Tor}_R(H^{m+1}(X;R),A) \longrightarrow 0\,,$$

where both exact sequences split, although not naturally.

Proof: Suppose that $C = C_*(X) \otimes R$ is the cellular chain complex of X with coefficients in R. Also suppose that D is the chain complex defined by $D_0 = A$ and $D_l = 0$ for $l \neq 0$ with all of its boundary operators defined to be zero. It follows that $C \otimes_R D = C_*(X) \otimes A$. Moreover, we have that $H_0(D) = H^0(D) = A$ and that $H_l(D) = H^l(D) = 0$ for $l \neq 0$. Applying Theorem 7.4.3, we get the desired exact sequences. $\square$

7.4.9 Note. Similar methods of homological algebra allow us to relate homology and cohomology, as can be found in Spanier's text [67, 5.5.12, 5.5.3], and so to get, for any principal ideal domain R and any R-module A, a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}_R(H^{m+1}(X;R),A) \longrightarrow H_m(X;A) \longrightarrow \mathrm{Hom}_R(H^m(X;R),A) \longrightarrow 0$$

and dually, provided that $H_*(X;R)$ is of finite type, a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}_R(H_{m-1}(X;R),A) \longrightarrow H^m(X;A) \longrightarrow \mathrm{Hom}_R(H_m(X;R),A) \longrightarrow 0.$$

As usual, these split, though not naturally.

In analogy to the case of the Künneth formula, for the construction of the *Mayer–Vietoris sequence* we shall need a result from homological algebra, which we state next. We shall not prove this result, but we shall instead refer the reader again to Spanier's book [67, 5.1.13, 5.4.8].

7.4.10 **Theorem**. *Suppose that*

$$0 \longrightarrow D \longrightarrow C \longrightarrow E \longrightarrow 0\,,$$

is a short exact sequence of chain complexes that splits and that G is an abelian group. Then there exist natural long exact sequences in homology

$$\cdots \longrightarrow H_q(D;G) \longrightarrow H_q(C;G) \longrightarrow H_q(E;G) \xrightarrow{\ \partial_*\ } H_{q-1}(D;G) \longrightarrow \cdots$$

and in cohomology

$$\cdots \longrightarrow H^q(E;G) \longrightarrow H^q(C;G) \longrightarrow H^q(D;G) \xrightarrow{\ \delta^*\ } H^{q+1}(E;G) \longrightarrow \cdots\,.$$

□

This theorem is a consequence of the following fundamental theorem.

7.4.11 **Theorem**. *A short exact sequence of chain complexes, say*

$$0 \longrightarrow D \xrightarrow{\alpha} C \xrightarrow{\beta} E \longrightarrow 0\,,$$

determines a natural long exact sequence in homology

$$\cdots \longrightarrow H_q(D;G) \longrightarrow H_q(C;G) \longrightarrow H_q(E;G) \xrightarrow{\ \partial_*\ } H_{q-1}(D;G) \longrightarrow \cdots\,.$$

The main part of the *proof* of this theorem consists in defining the homomorphism ∂_*, which is done as follows. For any $[e] \in H_q(E; G)$ we define $\partial_*([e]) = [\alpha^{-1}\partial\beta^{-1}(e)] \in H_{q-1}(D; G)$, where ∂ is the connecting homomorphism of the complex C. It is now an element-chasing *exercise* to prove that this homomorphism is well defined and that the sequence it determines is indeed exact. □

The *proof* of Theorem 7.4.10 is obtained from this fundamental theorem. This is so, since when we split the given short exact sequence, the sequences that we get by applying the tensor product with G or the functor $\mathrm{Hom}(-, G)$ continue to be short exact sequences, whose homologies yield the desired long exact sequences. □

7.4.12 **Proposition**. *Suppose that $(X; A, B)$ is a CW-triad, that is, $A, B \subset X$ are subcomplexes and $A \cup B = X$, and suppose that $D \subset A \cap B$ is a subcomplex. Then there exists a short exact sequence of free cellular complexes that splits,*

$$0 \longrightarrow C_*(A \cap B)/C_*(D) \longrightarrow C_*(A)/C_*(D) \oplus C_*(B)/C_*(D) \longrightarrow \\ \longrightarrow C_*(X)/C_*(D) \longrightarrow 0\,,$$

where the first homomorphism is given by $[c] \mapsto (i'_*[c], -j'_*[c])$ and the second one is given by $([a], [b]) \mapsto i_*[a] + j_*[b]$. Here i, i', j, and j' are the respective inclusions.

Proof: It is enough to check that the cells that freely generate the complex in the middle either come exactly from the cells that freely generate the complex on the left or, if not, go exactly to the cells that freely generate the complex on the right. □

Consequently, by applying Theorem 7.4.10 we now get the desired Mayer–Vietoris sequences.

7.4.13 **Theorem**. *Suppose that $(X; A, B)$ is a* CW*-triad and $D \subset A \cap B$ is a subcomplex. If G is an abelian group, then there is an exact sequence in homology*

$$\cdots \longrightarrow H_q(A \cap B, D; G) \longrightarrow H_q(A, D; G) \oplus H_q(B, D; G) \longrightarrow \\ \longrightarrow H_q(A \cup B, D; G) \longrightarrow H_{q-1}(A \cap B, D; G) \longrightarrow \cdots\,,$$

where the first homomorphism is defined by

$$[c] \mapsto (i'_*[c], -j'_*[c])$$

and the second one is defined by

$$([a],[b]) \mapsto i_*[a] + j_*[b]\,.$$

Also, there is an exact sequence in cohomology

$$\begin{aligned}\cdots \longrightarrow H^{q-1}(A\cap B, D; G) \longrightarrow H^q(X, D; G) \longrightarrow \\ \longrightarrow H^q(A, D; G) \oplus H^q(B, D; G) \longrightarrow H^q(A\cap B, D; G) \longrightarrow \cdots,\end{aligned}$$

where the second homomorphism is defined by

$$[c] \mapsto (i^*[c], j^*[c])$$

and the third one is defined by

$$([a],[b]) \mapsto i'^*[a] - j'^*[b]\,.$$

Here i, i', j, and j' are the respective inclusions. □

These sequences are known as the *Mayer–Vietoris sequences* for homology and cohomology. In the last chapter these sequences are deduced from the formal properties of homology and cohomology (see 12.1.22).

7.4.14 Remark. There exists a version of Theorem 7.4.13 for *excisive triads*, that is, for triads $(X; A, B)$ that satisfy $X = \overset{\circ}{A} \cup \overset{\circ}{B}$ and $\overline{D} \subset \overset{\circ}{A} \cap \overset{\circ}{B}$, where $\overset{\circ}{U}$ denotes the interior of $U = A, B$ in X. The exact sequences in this new version are just like those in Theorem 7.4.13 itself and can be obtained by appropriately substituting the couples of excisive pairs with couples of CW-pairs. (See Spanier's book [67] for a systematic discussion of this case.)

Chapter 8

Vector Bundles

In this chapter we shall define and study vector bundles, including their classification. We also examine Grassmann manifolds and universal bundles. Our presentation partly follows Dupont [28].

8.1 Vector Bundles

In this section we shall introduce vector bundles. These form a special class of locally trivial bundles, which in turn we already have introduced in Chapter 4.

8.1.1 Definition. We say that a locally trivial bundle $p : E \longrightarrow B$ is a *real (respectively, complex) vector bundle* of *dimension* n or, more briefly, a *real (respectively, complex) n-bundle*, if it has $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) as its fiber and if it satisfies the following compatibility condition. Given any two trivializations $\varphi_U : p^{-1}U \longrightarrow U \times F$ and $\varphi_V : p^{-1}V \longrightarrow V \times F$, where $F = \mathbb{R}^n$ (respectively, $F = \mathbb{C}^n$), over any two neighborhoods U and V of any $b \in B$ (such that p_U and p_V are in fact trivial), it follows that the map

$$\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times F \longrightarrow (U \cap V) \times F,$$

which always has the form $\varphi_U \circ \varphi_V^{-1}(x, y) = (x, \widehat{g}_{UV}(x, y))$ for $(x, y) \in (U \cap V) \times F$, satisfies the compatibility condition that $\widehat{g}_{UV}(x, y)$ is linear in $y \in F$ for each fixed $x \in U \cap V$. (See also 8.5.18.)

This compatibility condition is equivalent to the existence of continuous functions $g_{UV} : U \cap V \longrightarrow \mathrm{GL}_n(\mathbb{R})$ (respectively, $g_{UV} : U \cap V \longrightarrow \mathrm{GL}_n(\mathbb{C})$) such that $\widehat{g}_{UV}(x, y) = g_{UV}(x)y$ for $(x, y) \in (U \cap V) \times F$, where $\mathrm{GL}_n(\mathbb{R})$ (respectively, $\mathrm{GL}_n(\mathbb{C})$) denotes the real (respectively, complex) *general linear group* of $n \times n$ invertible matrices.

In other words, each *change of coordinates* $\varphi_V \circ \varphi_U^{-1}$ is a linear isomorphism on the fibers. This condition allows us to endow each fiber $p^{-1}(x)$ for $x \in B$ with a unique vector space structure over the real (respectively, complex) numbers in such a way that the restriction of each φ_U to any fiber $p^{-1}(x)$, where $x \in U$, is a linear isomorphism from $p^{-1}(x)$ to $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$). It is because of this property of the fibers that these locally trivial bundles are called vector bundles.

Conversely, if we are given an open cover $\mathcal{U}$ of B such that for every pair $U, V \in \mathcal{U}$ there is a map $g_{UV} : U \cap V \longrightarrow \mathrm{GL}_n(\mathbb{R})$ satisfying

(8.1.2) $$g_{UV}(x)g_{VW}(x) = g_{UW}(x), \quad x \in U \cap V \cap W\,,$$

then we can construct a vector bundle using this family of functions, known as a *cocycle*, as if it were a set of "assembly instructions." Specifically, this means that we take the disjoint union

$$\coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n$$

and identify $(x, y) \in U \times \mathbb{R}^n$ with $(x, g_{UV}(x)y) \in V \times \mathbb{R}^n$ whenever $x \in U \cap V$ and $y \in \mathbb{R}^n$. Equation (8.1.2) then guarantees that the quotient $E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n/\!\sim$ under this identification is the total space of a well-defined real vector bundle, where one defines the bundle map itself $p : E \longrightarrow B$ to be locally projection onto the first coordinate. (Notice that the same construction also works in the complex case.) The resulting vector bundle is called the real (respectively, complex) *vector bundle determined by the cocycle* $\{g_{UV} \mid U, V \in \mathcal{U}\}$.

From now on, we shall discuss only the real case. However, the complex case is entirely analogous.

8.1.3 EXERCISE. Prove that every cocycle satisfies the following identities:

$$\begin{aligned} g_{UU}(x) &= 1 \in \mathrm{GL}_n(\mathbb{R}), \quad x \in U, \\ g_{UV}(x) &= g_{VU}(x)^{-1} \in \mathrm{GL}_n(\mathbb{R}), \quad x \in U \cap V. \end{aligned}$$

(Hint: Use (8.1.2).)

8.1.4 DEFINITION. Given two vector bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ we can clearly find an open cover $\mathcal{U}$ of B such that both p and p' are trivial over each $U \in \mathcal{U}$. If the corresponding cocycles are

$$\{g_{UV} : U \cap V \longrightarrow \mathrm{GL}_n(\mathbb{R})\}, \quad \{g'_{UV} : U \cap V \longrightarrow \mathrm{GL}_{n'}(\mathbb{R})\}\,,$$

where $U, V \in \mathcal{U}$, we can then consider operations such as

(i) $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R}) \xrightarrow{\;\oplus\;} \mathrm{GL}_{n+n'}(\mathbb{R})$;

(ii) $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R}) \xrightarrow{\;\otimes\;} \mathrm{GL}_{nn'}(\mathbb{R})$;

(iii) $\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\;()^*\;} \mathrm{GL}_n(\mathbb{R})$;

(iv) $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R}) \xrightarrow{\mathrm{Hom}\,(-^{-1},-)} \mathrm{GL}_{nn'}(\mathbb{R})$;

(v) $\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\;\otimes^k\;} \mathrm{GL}_{n^k}(\mathbb{R})$;

(vi) $\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\;\bigwedge^k\;} \mathrm{GL}_{\binom{n}{k}}(\mathbb{R})$;

which are given for matrices $A \in \mathrm{GL}_n(\mathbb{R})$ and $B \in \mathrm{GL}_{n'}(\mathbb{R})$ as follows:

(i) $A \oplus B = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is the direct sum of A and B.

(ii) $A \otimes B$ is the tensor product of A and B.

(iii) $(A^*)^{-1}$ is the inverse of the adjoint matrix of A.

(iv) $\mathrm{Hom}\,(A^{-1}, B) = (A^*)^{-1} \otimes B$.

(v) $\bigotimes^k A = A \otimes \cdots \otimes A$ (with k factors).

(vi) $\bigwedge^k A$ is the kth exterior power of A.

By composing these operations with the given cocycles, we can define new cocycles

(i) $x \longmapsto g_{UV}(x) \oplus g'_{UV}(x)$,

(ii) $x \longmapsto g_{UV}(x) \otimes g'_{UV}(x)$,

(iii) $x \longmapsto ((g_{UV}(x))^*)^{-1}$,

(iv) $x \longmapsto \mathrm{Hom}\,((g_{UV}(x))^{-1}, g'_{UV}(x))$,

(v) $x \longmapsto \bigotimes^k g_{UV}(x)$,

(vi) $x \longmapsto \bigwedge^k g_{UV}(x)$,

for $x \in U \cap V$, thereby obtaining new "assembly instructions" for constructing vector bundles over the base space B with the corresponding total spaces denoted by

(i) $E \oplus E'$,

(ii) $E \otimes E'$,

(iii) E^*,

(iv) $\mathrm{Hom}\,(E, E')$,

(v) $\bigotimes^k E$,

(vi) $\bigwedge^k E$.

Since vector spaces can obviously be identified with vector bundles over a one-point base space, we can see that these constructions extend to vector bundles the corresponding operations for vector spaces.

8.1.5 NOTE. The bundle $E \oplus E'$ is often called the *Whitney sum* of the bundles E and E'.

8.1.6 EXERCISE. Prove that the Whitney sum of two vector bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ can be obtained as the bundle induced by the diagonal map $\Delta : B \longrightarrow B \times B$, defined by $\Delta(x) = (x, x)$ for $x \in B$, from the product bundle $p \times p' : E \times E' \longrightarrow B \times B$. This means that

$$E \oplus E' \cong \Delta^*(E \times E')\,.$$

8.1.7 EXERCISE. Let $p_1 : E_1 \longrightarrow B_1$ and $p_2 : E_2 \longrightarrow B_2$ be vector bundles. Prove that the product bundle $p_1 \times p_2 : E_1 \times E_2 \longrightarrow B_1 \times B_2$ satisfies a natural identification

$$E_1 \times E_2 \cong \pi_1^*(E_1) \oplus \pi_2^*(E_2)\,,$$

where $\pi_\nu : B_1 \times B_2 \longrightarrow B_\nu$ is the projection for $\nu = 1, 2$.

8.1.8 EXERCISE. Given vector bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$, prove that the fiber over $x \in B$ of each one of the bundles constructed above is given as follows, where $F = p^{-1}(x)$ and $F' = p'^{-1}(x)$ are the fibers over x of p and p', respectively:

(i) $F \oplus F'$,

(ii) $F \otimes F'$,

(iii) F^*,

(iv) $\mathrm{Hom}\,(F, F')$,

(v) $\bigotimes^k F$,

(vi) $\bigwedge^k F$.

8.1.9 DEFINITION. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be vector bundles. A fiber map $\widetilde{f} : E \longrightarrow E'$ that covers a continuous map $f : B \longrightarrow B'$ is called a *vector bundle homomorphism over* f, or more briefly a *bundle homomorphism*, if for each $x \in B$ the restriction of $\widetilde{f}$ to the fiber over x, namely $\widetilde{f}_x : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$, is a linear homomorphism. In other words, this means that $\widetilde{f}$ maps each fiber of p linearly into the corresponding fiber of p' with respect to the linear structures on the fibers. A bundle homomorphism such that fiberwise it is a linear monomorphism (epimorphism) is called a *vector bundle monomorphism* (*epimorphism*). It will be called simply a *vector bundle morphism*, or more briefly a *bundle morphism*, if fiberwise it is a linear isomorphism.

In particular, given vector bundles with the same base space, $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$, we say that a map $\widetilde{f} : E \longrightarrow E'$ that covers the identity map id_B, that is, such that $p' \circ f = p$, is a *vector bundle homomorphism over* B if for each $x \in B$, the restriction to the fiber $\widetilde{f}_x : p^{-1}(x) \longrightarrow p'^{-1}(x)$ is linear. It is a *vector bundle monomorphism* (*epimorphism*) *over* B if $\widetilde{f}_x$ is a linear monomorphism (epimorphism). The map $\widetilde{f} : E \longrightarrow E'$ is a *vector bundle isomorphism* if for each x, $\widetilde{f}_x : p^{-1}(x) \longrightarrow p'^{-1}(x)$ is a linear isomorphism.

A subspace $E_1 \subset E$ of a vector bundle $p : E \longrightarrow B$ is called a *subbundle* if the restriction $p_1 = p|E_1 : E_1 \longrightarrow B$ is a vector bundle and for each $x \in B$, $E_1 \cap p^{-1}(x) \subset p^{-1}(x)$ is a linear subspace. Then the inclusion $E_1 \hookrightarrow E$ is a vector bundle monomorphism.

8.1.10 NOTE. The previous definition of a vector bundle morphism can be formulated as saying that *if* $\widetilde{f} : E \longrightarrow E'$ *is a continuous map that sends fibers linearly and isomorphically to fibers, then* $\widetilde{f}$ *is a vector bundle morphism.* Specifically, since $p : E \longrightarrow B$ is an identification map (because it is both surjective and open) and since the composite $p' \circ \widetilde{f}$ is compatible with the identification map (i.e., $\widetilde{f}$ sends fibers into fibers), it follows that there exists a continuous map $f : B \longrightarrow B'$ that makes the diagram

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, dashed, "f"'] & B' \end{tikzcd}$$

commute. Sometimes when we speak of a vector bundle morphism we mean a diagram such as this.

When one considers the category $\mathcal{V}ect$ of vector bundles, there the morphisms from $p : E \longrightarrow B$ to $p' : E' \longrightarrow B'$ are pairs $(\widetilde{f}, f)$, where $f : B \longrightarrow B'$ is continuous and $\widetilde{f} : E \longrightarrow E'$ is a bundle homomorphism over f, with the obvious composition. There should be no confusion with the widespread notion of a (vector) bundle morphism, which refers only to a homomorphism that fiberwise is an isomorphism.

8.1.11 EXAMPLES.

(a) If M and M' are differentiable manifolds and $f : M' \longrightarrow M$ is a differentiable map, then the derivative of f determines a bundle homomorphism $Df : TM' \longrightarrow TM$ between the tangent (vector) bundles of the given manifolds, which covers f.

(b) A sequence of bundle homomorphisms

$$E' \xrightarrow{i} E \xrightarrow{q} E'',$$

where E', E, and E'' are vector bundles over B, is said to be *exact* if for each $b \in B$ the sequence of fibers

$$E'_b \xrightarrow{i_b} E_b \xrightarrow{q_b} E''_b$$

is exact.

(c) If $\widetilde{f} : E' \longrightarrow E$ is a bundle homomorphism that covers $f : B' \longrightarrow B$, then we define $\ker(\widetilde{f}) = \{e' \in E' \mid \widetilde{f}(e') = 0\}$ and $\mathrm{im}(\widetilde{f}) = \{\widetilde{f}(e') \mid e' \in E'\}$. In general, the restricted maps $\ker(\widetilde{f}) \longrightarrow B'$ and $\mathrm{im}(\widetilde{f}) \longrightarrow B$ are not vector bundles (they are not locally trivial).

8.1.12 EXERCISE. Prove that if $\widetilde{f} : E' \longrightarrow E$ is a bundle epimorphism that covers $f : B' \longrightarrow B$, then $\ker(\widetilde{f}) \longrightarrow B'$ is a vector bundle.

8.1.13 EXERCISE. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be vector bundles over the same base space B. If $\varphi : E \longrightarrow E'$ is a vector bundle isomorphism (see 8.1.9), prove that φ is a homeomorphism. Hence φ is an isomorphism in the category of vector bundles. (Hint: Use the fact that the self map $a \mapsto a^{-1}$ is continuous as a map from $\mathrm{GL}_n(\mathbb{C})$ to itself.)

8.1.14 Exercise. Prove that if $\widetilde{f} : E \longrightarrow E'$ is a vector bundle morphism, then $E \cong f^*E'$, provided that $f : B \longrightarrow B'$ satisfies $f \circ p = p' \circ \widetilde{f}$. (Hint: Apply the previous exercise to E and f^*E'.)

By carefully applying to vector bundles the corresponding results for vector spaces we get the next proposition.

8.1.15 **Proposition**. *Let E, E', and E'' be (the total spaces of) three vector bundles. We have the following natural isomorphisms of vector bundles:*

(a) $E \oplus E' \cong E' \oplus E$.

(b) $(E \oplus E') \oplus E'' \cong E \oplus (E' \oplus E'')$.

(c) $E \otimes E' \cong E' \otimes E$.

(d) $(E \otimes E') \otimes E'' \cong E \otimes (E' \otimes E'')$.

(e) $E \otimes (E' \oplus E'') \cong (E \otimes E') \oplus (E \otimes E'')$.

(f) $\mathrm{Hom}\,(E, E') \cong E^* \otimes E'$.

(g) $\bigwedge^k (E \oplus E') \cong \bigoplus_{i+j=k} (\bigwedge^i E \otimes \bigwedge^j E')$. □

8.1.16 Exercise. Suppose that $p : E \longrightarrow B$ is a vector bundle defined by a cocycle $\{g_{UV} \mid U, V \in \mathcal{U}\}$, where $\mathcal{U}$ is an open cover of B, and that $f : B' \longrightarrow B$ is a continuous map. Show that

$$g'_{f^{-1}Uf^{-1}V} = g_{UV} f | f^{-1}(U) \cap f^{-1}(V) : f^{-1}(U) \cap f^{-1}(V) \longrightarrow \mathrm{GL}_n(\mathbb{R})$$

defines a cocycle for the open cover $\{f^{-1}U \mid U \in \mathcal{U}\}$ of B' induced by f. Moreover, prove that the vector bundle determined by this new cocycle is canonically isomorphic to the vector bundle induced by f, namely $p' : f^*E \longrightarrow B'$.

8.1.17 Exercise. Consider the *trivial bundle* $B \times \mathbb{R}^n \longrightarrow B$, which we shall denote by ε^n (just as in the complex case). Find a minimal cocycle that determines ε^n. (A cocycle is *minimal* if no proper subfamily of elements of it is a cocycle.)

8.1.18 Exercise. Using Exercise 8.1.16 prove again the assertions of Exercise 4.3.10, namely that the induced bundle is a functor.

8.1.19 EXERCISE. Prove the following implications:

(a) $E_1 \cong E_2$ and $E_1' \cong E_2' \Rightarrow E_1 \oplus E_1' \cong E_2 \oplus E_2'$.

(b) $E_1 \cong E_2$ and $E_1' \cong E_2' \Rightarrow E_1 \otimes E_1' \cong E_2 \otimes E_2'$.

(c) $E_1 \cong E_2 \Rightarrow E_1^* \cong E_2^*$.

(d) $E_1 \cong E_2$ and $E_1' \cong E_2' \Rightarrow \operatorname{Hom}(E_1, E_1') \cong \operatorname{Hom}(E_2, E_2')$.

(e) $E_1 \cong E_2 \Rightarrow \bigwedge^k E_1 \cong \bigwedge^k E_2$.

To finish this section on general matters concerning vector bundles, we shall now introduce a concept that will be quite useful in Chapter 11.

8.1.20 DEFINITION. Given a vector bundle $p : E \longrightarrow B$ we say that a continuous family of scalar products $\langle -, - \rangle_x : p^{-1}(x) \times p^{-1}(x) \longrightarrow \mathbb{R}$ for $x \in B$, that is, a continuous map

$$\rho : E \times_B E = \{(e, e') \in E \times E' \mid p(e) = p(e')\} \longrightarrow \mathbb{R}$$

whose restriction $\langle e, e' \rangle = \rho(e, e')$, $e,\ e' \in p^{-1}(x)$, determines a scalar product, is a *Riemannian metric* on the bundle. In the complex case, if $\langle -, - \rangle_x : p^{-1}(x) \times p^{-1}(x) \longrightarrow \mathbb{C}$ is a Hermitian product, then it is called a *Hermitian metric*

8.1.21 NOTE. Strictly speaking, a Riemannian (Hermitian) metric of a vector bundle $p : E \longrightarrow B$ is a section $s : B \longrightarrow (E \otimes E)^*$ (see 8.3.10) of the bundle $(E \otimes E)^* \longrightarrow B$ such that $s(x)$ is a scalar (Hermitian) product on the vector space $p^{-1}(x)$ for every $x \in B$.

8.1.22 **Theorem**. *Let B be paracompact. Then every vector bundle $p : E \longrightarrow B$ admits a Riemannian metric.*

Proof: Suppose that $p : E \longrightarrow B$ is a vector bundle over a paracompact space B. If $\{U_\lambda\}$ is an open cover of B that trivilizes p, so that we have $\varphi_\lambda : E|U_\lambda \xrightarrow{\approx} U_\lambda \times \mathbb{R}^n$ for every λ, then we can use the usual scalar product in $\mathbb{R}^n$ in order to define a scalar product in each fiber. Namely, for each $x \in U_\lambda$, let $\langle -, - \rangle_{\lambda,x} : p^{-1}(x) \times p^{-1}(x) \longrightarrow \mathbb{R}$ be defined by $\langle u, v \rangle_{\lambda,x} = \langle \varphi_{\lambda,x}(u), \varphi_{\lambda,x}(v) \rangle$, where $\varphi_{\lambda,x} = \varphi_\lambda | p^{-1}(x)$ and $\langle -, - \rangle$ represents the usual scalar product in $\mathbb{R}^n$.

Since B is paracompact, there exists a partition of unity $\{\mu_\lambda\}$ subordinate to the cover $\{U_\lambda\}$ (see Basic Concepts and Notation). Then we define

$$\rho(u,v) = \langle u,v\rangle = \langle u,v\rangle_x = \sum_\lambda \mu_\lambda(x)\langle u,v\rangle_{\lambda,x}\,.$$

This clearly defines a Riemannian metric on $p : E \longrightarrow B$. □

8.1.23 **Proposition**. *Let $p : E \longrightarrow B$ be a vector bundle over a paracompact space B, and let $E_1 \subset E$ be a subbundle. Then there exists a subbundle $E_2 \subset E$ such that $E = E_1 \oplus E_2$.* The bundle E_2 is called the *orthogonal complement* of E_1 in E and is denoted by $E_1^\perp$.

Proof: Let $\langle -,-\rangle$ be a Riemannian metric on the bundle $p : E \longrightarrow B$. We then define $E_2 = \{e \in E \mid \langle e,e'\rangle = 0 \text{ for } e' \in E_1 \text{ and } p(e') = p(e)\}$. It is straightforward to show that E_2 actually is a subbundle of E and that $E = E_1 \oplus E_2$. □

We have the following consequence of 8.1.23.

8.1.24 **Corollary**. *Suppose that*

$$0 \longrightarrow E' \xrightarrow{\quad i \quad} E \xrightarrow{\quad q \quad} E'' \longrightarrow 0$$

is a short exact sequence of vector bundles over a paracompact space B. Then the sequence splits. In particular, we have

$$E \cong E' \oplus E''\,.$$

Proof: Let $E_1 \subset E$ be the isomorphic image of E' under i. Then take E_2 to be the orthogonal complement of E_1 as in 8.1.23. Then $q|E_2 : E_2 \longrightarrow E''$ is an isomorphism whose inverse composed with the inclusion into E, namely $j : E'' \cong E_2 \hookrightarrow E$, determines the splitting of the exact sequence. □

8.1.25 Exercise. Prove that every complex vector bundle $p : E \longrightarrow B$ over a paracompact space B admits a Hermitian metric; that is, a family of Hermitian products on each fiber $p^{-1}(x)$ that depend continuously on $x \in B$. (Hint: See the proof of 8.1.22.)

8.1.26 Exercise. Formulate and prove Proposition 8.1.23 and Corollary 8.1.24 in the complex case.

8.2 PROJECTIONS AND VECTOR BUNDLES

Let us suppose that V is a finite-dimensional vector space over $\mathbb{R}$ (respectively, $\mathbb{C}$), and let us consider the space of all linear homomorphisms of V to itself, namely $\operatorname{Hom}(V, V)$.

Letting n denote the dimension of V, we endow $\operatorname{Hom}(V, V)$ with the topology of $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) by means of the canonical bijection $V \cong \mathbb{R}^n$ (respectively, $V \cong \mathbb{C}^n$), which is an isomorphism of vector spaces.

8.2.1 DEFINITION. An element $\pi \in \operatorname{Hom}(V, V)$ is called a *projection* if it is *idempotent*, that is, if $\pi^2 = \pi$. We let $\operatorname{Pr}(V)$ denote the subspace of $\operatorname{Hom}(V, V)$ of all the projections.

For any topological space B, let us consider the function space

$$M(B, \operatorname{Pr}(V))$$

of continuous maps from B to $\operatorname{Pr}(V)$. To each $\varphi \in M(B, \operatorname{Pr}(V))$ we can associate a subspace $E_\varphi \subset B \times V$ defined as

$$E_\varphi = \{(x, v) \in B \times V \mid \varphi(x)v = v\}. \tag{8.2.2}$$

Also define $p : E_\varphi \longrightarrow B$ to be the restriction of $B \times V \longrightarrow B$, the projection onto B, to the subspace E_φ.

8.2.3 **Proposition** and DEFINITION. *The map $p : E_\varphi \longrightarrow B$ is locally trivial and so is a vector bundle.* This bundle is called the *vector bundle associated* to φ; it is a subbundle of the trivial bundle $\operatorname{proj}_B : B \times V \longrightarrow B$

In order to prove this we shall first prepare ourselves with a few remarks and a lemma.

8.2.4 NOTE. It is well known that any topology on a real (or complex) finite-dimensional vector space for which vector addition and scalar multiplication are continuous is precisely the ordinary topology (see [65], for example). On the space $\operatorname{Hom}(V, V)$ we introduce the topology induced by the *norm* $\|\cdot\| : \operatorname{Hom}(V, V) \longrightarrow \mathbb{R}^+$ defined by $\|\alpha\| = \max\{|\alpha(v)| \mid v \in V \quad \text{and} \quad |v| = 1\}$, where $|\cdot| : V \longrightarrow \mathbb{R}^+$ is any norm on V.

Although the norm $\|\cdot\|$ itself depends on the choice of the norm $|\cdot|$ on V, the resulting topology on $\operatorname{Hom}(V, V)$ given by $\|\cdot\|$ is always the same, since $\operatorname{Hom}(V, V)$ is (linearly) homeomorphic to $\mathbb{R}^{n^2}$ for any choice of the norm on V, where $\dim V = n$.

8.2.5 **Lemma**. *Suppose that V is a finite-dimensional vector space and that $\rho, \sigma \in \mathrm{Pr}(V)$ are projections with ranges $R = \rho(V)$ and $S = \sigma(V)$. If $\|\rho - \sigma\| < 1$, then*

$$\rho : S \longrightarrow R$$

is an isomorphism.

Proof: Put $\alpha = \rho - \sigma$. Then we claim that $1+\alpha$ is invertible, where 1 denotes the identity map of V. And this is because if there were a nonzero vector $v \in V$ satisfying $(1+\alpha)v = 0$, then we would have $(1+\alpha)v/|v| = 0$ and therefore $\alpha(v/|v|) = -v/|v|$, which would contradict $\|\alpha\| < 1$.

Now note that we have

$$(1+\alpha)\sigma = (1+\rho-\sigma)\sigma = \sigma + \rho\sigma - \sigma = \rho\sigma.$$

Consequently, we have $(1+\alpha)|S = \rho|S$, which implies that $\rho|S : S \longrightarrow R$ is a monomorphism, and so $\dim S \leq \dim R$.

Similarly, we can prove that $\dim R \leq \dim S$, and so we get the desired conclusion. □

Proof of 8.2.3: The set $U = \varphi^{-1}\{\gamma \in \mathrm{Pr}(V) \mid \|\gamma - \varphi(b)\| < 1\}$ is an open neighborhood of b for any $b \in B$. The maps $\widetilde{\rho} : U \times V \longrightarrow U \times V$ and $\widetilde{\sigma} : U \times V \longrightarrow U \times V$ defined by $\widetilde{\rho}(x,v) = (x, \varphi(b)v)$ and $\widetilde{\sigma}(x,v) = (x, \varphi(x)v)$ are continuous, where $x \in U$ and $v \in V$.

Keeping fixed x, $\widetilde{\rho}$, and $\widetilde{\sigma}$ we see that the hypotheses of Lemma 8.2.5 are satisfied on each fiber, that implies that $\widetilde{\rho}$ induces a homeomorphism

$$\widetilde{\rho} : p^{-1}U \longrightarrow U \times (\varphi(b)V)$$

that is linear on each fiber, because $p^{-1}(x) = \varphi(x)V$ for $x \in U$. Clearly, the inverse of this homeomorphism is the restriction to $U \times (\varphi(b)V)$ of $(1+\varphi(x) - \varphi(b))^{-1}$, which depends continuously on $x \in U$, since the map $\mathrm{Iso}\,(V,V) \longrightarrow \mathrm{Iso}\,(V,V)$ that sends each isomorphism on V to its inverse is continuous. □

8.2.6 EXAMPLES.

(i) The constant function $\kappa : B \longrightarrow \mathrm{Pr}(V)$ defined by $\kappa(x) = 1_V$ for all $x \in B$ has as its associated vector bundle the trivial bundle $B \times V \longrightarrow B$.

(ii) The function $\tau : \mathbb{S}^{n-1} \longrightarrow \mathrm{Pr}(\mathbb{R}^n)$ defined by $\tau(x)v = v - \langle x, v\rangle x$ for $x \in \mathbb{S}^{n-1}$ and $v \in \mathbb{R}^n$ has an associated vector bundle $T(\mathbb{S}^{n-1}) \longrightarrow \mathbb{S}^{n-1}$, which is called the *tangent bundle* and is a bundle of dimension $n-1$. (Here $\langle -, - \rangle$ denotes the usual scalar product on $\mathbb{R}^n$.)

(iii) The function $\varphi : \mathbb{CP}^n \longrightarrow \Pr(\mathbb{C}^{n+1})$ defined by $\varphi([z])v = \langle v, z\rangle z/\langle z, z\rangle$, where $z, v \in \mathbb{C}^{n+1}$ with $z \neq 0$, determines an associated vector bundle $H^* \longrightarrow \mathbb{CP}^n$, which is known as the *dual of the Hopf bundle*, and is a bundle of (complex) dimension one. (Now $\langle -, -\rangle$ denotes the usual Hermitian product on $\mathbb{C}^{n+1}$.) By definition the *Hopf bundle* $H \longrightarrow \mathbb{CP}^n$ is the dual of $H^* \longrightarrow \mathbb{CP}^n$.

(iv) For the real projective space $\mathbb{RP}^n$ we have a situation similar to that of part (iii), and so we get in the same way a bundle of (real) dimension one over the base space $\mathbb{RP}^n$.

(v) In the case $n = 1$ of part (iv) we have $\mathbb{RP}^1 \approx \mathbb{S}^1$. A specific choice of homeomorphism

$$\mathbb{S}^1 \longrightarrow \mathbb{RP}^1$$

is given by $(\cos\theta, \sin\theta) \mapsto [(\cos(\theta/2), \sin(\theta/2))]$, where $-\pi \leq \theta \leq \pi$ (and the square brackets denote the equivalence class in $\mathbb{RP}^1$ of an element in $\mathbb{S}^1$ after identifying antipodes). The associated bundle $M \longrightarrow \mathbb{RP}^1$ is the *Moebius bundle*. If we set $\mathbb{RP}^1$ equal to $\mathbb{S}^1$, then the fiber of the Moebius bundle over the point $(\cos t\pi, \sin t\pi)$ is the line in $\mathbb{R}^3$ generated by $(\sqrt{1-t^2}\cos t\pi, \sqrt{1-t^2}\sin t\pi, t)$, where $-1 \leq t \leq 1$, and $M \subset \mathbb{S}^1 \times \mathbb{R}^3$ (see Figure 8.1).

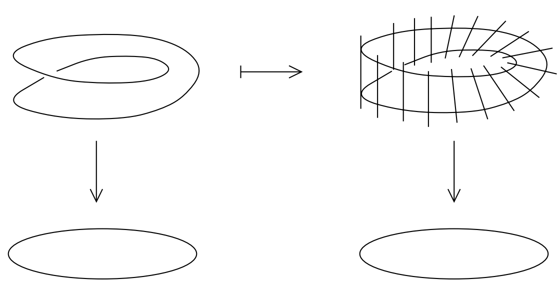

Figure 8.1

8.2.7 EXERCISE.

(a) In a similar manner to the treatment in 8.2.6(ii) give a description of the *normal bundle* $N(\mathbb{S}^{n-1}) \longrightarrow \mathbb{S}^{n-1}$, and prove that it is a trivial bundle of dimension one.

(b) Prove that $T(\mathbb{S}^{n-1}) \oplus N(\mathbb{S}^{n-1}) \longrightarrow \mathbb{S}^{n-1}$ is a trivial bundle.

8.2.8 Exercise. Write out in detail Example 8.2.6(iv).

8.2.9 Remark. It is illuminating to consider a vector bundle over B as a continuous family of vector spaces parametrized by a point in B. In this context the map $\varphi : B \longrightarrow \mathrm{Pr}(V)$ determines such a family by means of the map $b \mapsto \varphi(b)(V) \subset V$.

8.2.10 **Proposition**. *Suppose that $p : E \longrightarrow B$ is the associated bundle of $\varphi \in M(B, \mathrm{Pr}(V))$, where B is a topological space and V is a finite-dimensional vector space. Let $f^\# : M(B, \mathrm{Pr}(V)) \longrightarrow M(B', \mathrm{Pr}(V))$ be the map induced by a map $f : B' \longrightarrow B$, where B' is also a topological space. Then the vector bundle associated to $f^\#(\varphi) \in M(B', \mathrm{Pr}(V))$ is the induced bundle $q : f^*E \longrightarrow B'$* (see 4.3.9).

Proof: The induced bundle is $f^*E = \{(b', e) \in B' \times E \mid \ p(e) = f(b')\}$, and the bundle associated to $f^\#(\varphi)$ is $E' = \{(b', v) \in B' \times V \mid \varphi f(b')v = v\}$.

A vector bundle isomorphism

$$\begin{tikzcd} E' \arrow[rr] \arrow[dr] & & f^*E \arrow[dl] \\ & B' & \end{tikzcd}$$

is given by $(b', v) \mapsto (b', (f(b'), v)) \in B' \times E \subset B' \times B \times V$ for $(b', v) \in E'$.

So we have a bundle morphism

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

defined by $\widetilde{f}(b', v) = (f(b'), v)$ for $(b', v) \in E'$. □

8.3 Grassmann Manifolds and Universal Bundles

Grassmann manifolds, which we shall introduce in this section, allow us to classify bundles. These topological spaces (as well as the Stiefel manifolds, which we also shall construct here) have the structure of CW-complexes and even the structure of differentiable manifolds. Using the Grassmann manifolds as base spaces, we shall construct vector bundles for every natural number k, which we call universal k-bundles. These universal bundles have the property that any k-vector bundle can be expressed as a bundle induced from the universal bundle by means of an appropriate continuous map.

8.3.1 DEFINITION. Suppose that V is a real (or complex) vector space. We define $\mathrm{G}_k(V) = \{W \subset V \mid W \text{ is a linear subspace and } \dim W = k\}$, where $\dim W$ is the real (or complex) dimension of W. Let us define $\mathrm{Mon}(\mathbb{R}^k, V)$ to be the subset of $\mathrm{Hom}\,(\mathbb{R}^k, V)$ consisting of the monomorphisms, and let us equip it with the relative topology. Then we have a surjective map

$$q : \mathrm{Mon}(\mathbb{R}^k, V) \longrightarrow \mathrm{G}_k(V)$$

defined by $\alpha \mapsto \alpha(\mathbb{R}^k)$. Now we give $\mathrm{G}_k(V)$ the quotient topology. We call $\mathrm{G}_k(V)$ the *real (or complex) Grassmann manifold of k-planes in V*. The *Grassmann manifold of k-planes in* $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denoted by $\mathrm{G}_k(\mathbb{R}^n)$ (respectively, $\mathrm{G}_k(\mathbb{C}^n)$) will be of special interest to us in what follows.

In our discussion here we shall focus specifically on the complex case, although our results are in general true also in the real case.

Given $\gamma \in \mathrm{GL}_k(\mathbb{C})$ and $\alpha \in \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)$, we then have that $q(\alpha) = q(\alpha \circ \gamma)$, where q was defined above. Moreover, if $q(\alpha) = q(\beta)$ for $\alpha, \beta \in \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)$, then using any basis $\{v_1, \cdots, v_k\}$ of $\alpha(\mathbb{C}^k) = \beta(\mathbb{C}^k)$ we define $\gamma \in \mathrm{GL}_k(\mathbb{C})$ by $\gamma(\beta^{-1}v_i) = \alpha^{-1}v_i$. This then implies that $\beta = \alpha \circ \gamma$. Thus we have the next result.

8.3.2 **Proposition**. *The map*

$$\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n) \times \mathrm{GL}_k(\mathbb{C}) \longrightarrow \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)$$

given by $(\alpha, \gamma) \mapsto \alpha \circ \gamma$ *is a group action, and the orbit space*

$$\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)/\mathrm{GL}_k(\mathbb{C}),$$

obtained by identifying α *with* $\alpha \circ \gamma$ *for* $\alpha \in \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)$ *and* $\gamma \in \mathrm{GL}_k(\mathbb{C})$, *is homeomorphic to* $\mathrm{G}_k(\mathbb{C}^n)$. □

8.3.3 DEFINITION. There exists a canonical map

$$\mathrm{G}_k(\mathbb{C}^n) \longrightarrow \mathrm{Pr}(\mathbb{C}^n),$$

which is defined by sending a subspace $W \subset \mathbb{C}^n$ of dimension k to the orthogonal projection $\mathbb{C}^n \longrightarrow W \subset \mathbb{C}^n$. The associated vector bundle $\mathrm{E}_k(\mathbb{C}^n) \longrightarrow \mathrm{G}_k(\mathbb{C}^n)$ is called the *n-universal k-vector bundle.*

8.3.4 Definition. Let us define

$$\mathrm{V}_k(\mathbb{C}^n) = \{(v_1, \dots, v_k) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n \mid \langle v_i, v_j \rangle = \delta_{ij}\}\,,$$

where $\langle -, - \rangle$ is the standard Hermitian product on $\mathbb{C}^n$ and δ_{ij} is the Kronecker symbol. Then $\mathrm{V}_k(\mathbb{C}^n)$, equipped with the subspace topology coming from $\mathbb{C}^{nk}$, is the (complex) *Stiefel manifold* of orthonormal k-frames in $\mathbb{C}^n$. We also define an equivalence relation in $\mathrm{V}_k(\mathbb{C}^n) \times \mathbb{C}^k$ by $((v_1, \dots, v_k)A, w) \sim ((v_1, \dots, v_k), Aw)$, where A is an element of U_k, the (topological) group of (complex) unitary $k \times k$ matrices, and where $(v_1, \dots, v_k)A$ is the k-frame we get by considering $(v_1, \dots, v_k)$ as a $1 \times k$ matrix and taking its product with the matrix A.

From Definition 8.3.3 it immediately follows that $\mathrm{E}_k(\mathbb{C}^n) = \{(W, w) \in \mathrm{G}_k(\mathbb{C}^n) \times \mathbb{C}^n \mid w \in W\}$.

8.3.5 Exercise. Show that there is a homeomorphism

$$h : \mathrm{V}_k(\mathbb{C}^n) \times \mathbb{C}^k/{\sim} \longrightarrow \mathrm{E}_k(\mathbb{C}^n)$$

such that the diagram

$$\begin{tikzcd} V_k(\mathbb{C}^n) \times \mathbb{C}^k/\sim \arrow[rr, "h"] \arrow[dr, "p'"'] & & E_k(\mathbb{C}^n) \arrow[dl, "p"] \\ & \mathrm{G}_k(\mathbb{C}^n) & \end{tikzcd}$$

commutes, where $p'[(v_1, \dots, v_k), w]$ is defined to be the subspace of $\mathbb{C}^n$ generated by $\{v_1, \dots, v_k\}$ and where p is the universal bundle defined in Definition 8.3.3.

From this exercise we obtain another description of the n-universal k-vector bundle. Moreover, we have $\mathrm{V}_k(\mathbb{C}^n) \subset \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)$, and then the action of $\mathrm{GL}_n(\mathbb{C})$ on the second term restricts to an action of U_k on the first term, so that $\mathrm{V}_k(\mathbb{C}^n)/\mathrm{U}_k \approx \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^n)/\mathrm{GL}_k(\mathbb{C})$.

Suppose that we have a map $\varphi : B \longrightarrow \mathrm{Pr}(\mathbb{C}^n)$ where B is connected and let $p : E \longrightarrow B$ be its associated vector bundle. The function $B \longrightarrow \mathbb{Z}$ defined by $b \mapsto \dim_{\mathbb{C}}(\varphi(b)\mathbb{C}^n)$ is continuous and therefore constant, say with value k. We also have a map

$$f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^n) \tag{8.3.6}$$

defined by $f(b) = \varphi(b)\mathbb{C}^n$ for $b \in B$.

The name "n-universal k-vector bundle" for the bundle

$$\mathrm{E}_k(\mathbb{C}^n) \longrightarrow \mathrm{G}_k(\mathbb{C}^n)$$

is justified by the next proposition.

8.3.7 **Proposition**. *With the notation established above, we have*

$$E \cong f^*\mathrm{E}_k(\mathbb{C}^n)\,.$$

Proof: Directly from the definitions, we have that $\mathrm{E}_k(\mathbb{C}^n) = \{(W,w) \in \mathrm{G}_k(\mathbb{C}^n) \times \mathbb{C}^n \mid w \in W\}$ and that $E = \{(b,w) \in B \times \mathbb{C}^n \mid w \in \varphi(b)\mathbb{C}^n\}$. The isomorphism asserted to exist in the proposition, namely

$$\widetilde{f} : E \longrightarrow f^*\mathrm{E}_k(\mathbb{C}^n),$$

is then defined for $(b,w) \in E$ by

$$\widetilde{f}(b,w) = (b, (\varphi(b)\mathbb{C}^n, w)) \in f^*\mathrm{E}_k(\mathbb{C}^n) \subset B \times \mathrm{E}_k(\mathbb{C}^n). \qquad \square$$

Notice that the canonical inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ induces a map

$$i : \mathrm{G}_k(\mathbb{C}^n) \longrightarrow \mathrm{G}_k(\mathbb{C}^{n+1})\,.$$

8.3.8 EXERCISE. Prove that $\mathrm{E}_k(\mathbb{C}^n) \cong i^*\mathrm{E}_k(\mathbb{C}^{n+1})$, where i is the map we just defined.

8.3.9 DEFINITION. The colimit (or direct limit) of the sequence

$$\mathrm{G}_k(\mathbb{C}^n) \hookrightarrow \mathrm{G}_k(\mathbb{C}^{n+1}) \hookrightarrow \cdots$$

is the union of these sets with the weak topology. We usually denote this by BU_k, which simply can be described as the space of k-planes in $\mathbb{C}^\infty$. (Recall that $\mathbb{C}^\infty$ can be considered as the colimit of $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \cdots$; see Basic Concepts and Notation presented at the beginning of the book). There is a bundle over BU_k given by $\mathrm{E}_k(\mathbb{C}^\infty) \cong \mathrm{V}_k(\mathbb{C}^\infty) \times \mathbb{C}^k/\sim$, where $\mathrm{V}_k(\mathbb{C}^\infty)$ is the manifold of k-frames, say $(v_1, \dots, v_k)$, in $\mathbb{C}^\infty$ and the equivalence relation $\sim$ is as before. The bundle $\mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow \mathrm{BU}_k$ clearly has the property that

$$\mathrm{E}_k(\mathbb{C}^n) = j^*\mathrm{E}_k(\mathbb{C}^\infty) \quad \text{for} \quad j : \mathrm{G}_k(\mathbb{C}^n) \hookrightarrow \mathrm{BU}_k.$$

8.3.10 Definition. Given a vector bundle $p : E \longrightarrow B$, a *section* is a map $s : B \longrightarrow E$ such that $p \circ s = \mathrm{id}_B$. Given sections $s, t : B \longrightarrow E$ we can define a new section

$$s + t : B \longrightarrow E$$

by $(s+t)(b) = s(b) + t(b)$ for each $b \in B$, where the sum on the right-hand side in the fiber $p^{-1}(b)$ is given by any isomorphism (since they all give the same vector space structure to the fiber) $p^{-1}(b) \subset p^{-1}U \cong U \times V \longrightarrow V$ for any neighborhood U of b over which E is trivial. Similarly, given a section $s : B \longrightarrow E$ and a scalar $\lambda \in \mathbb{C}$, we can define a new section $\lambda s : B \longrightarrow E$.

Therefore, $\Gamma(E) = \{s : B \longrightarrow E \mid p \circ s = \mathrm{id}_B\}$ is a vector space, which is called the *space of sections* of the vector bundle $p : E \longrightarrow B$.

8.3.11 Exercise. Prove that if B is compact, then there exists a finite-dimensional subspace $W \subset \Gamma(E)$ such that the map $B \times W \longrightarrow E$ defined by $(b, s) \mapsto s(b)$ for $(b, s) \in B \times W$ is surjective. (Hint: There is a finite open cover $\{U_1, \ldots, U_l\}$ of B such that $p^{-1}U_i \cong U_i \times \mathbb{C}^k$ for $i = 1, \ldots, l$. Let $s_{i,j} : U_i \longrightarrow p^{-1}U_i$ for $j = 1, \ldots, k$ be sections such that $\{s_{i,1}(u), \ldots, s_{i,k}(u)\}$ is a basis of $p^{-1}(u) \cong \mathbb{C}^k$ for every $u \in U_i$. If $\{\eta_1, \ldots, \eta_l\}$ is a partition of unity subordinate to the cover, then the finite set $\{\widetilde{s}_{ij} \mid \widetilde{s}_{ij}(x) = \eta_i(x)s_{i,j}(x)$ for $x \in U_i, \widetilde{s}_{ij}(x) = 0$ if $x \notin U_i$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k\} \subset \Gamma(E)$ generates a subspace W with the desired property.)

8.3.12 Remark. If B is paracompact, then the statement of this exercise and its proof are still true for vector bundles $p : E \longrightarrow B$ of *finite type*, that is, for those that have a finite open cover of B with trivializations over each open set in the cover. This remark follows directly from the hint given above.

8.3.13 **Corollary**. *Take B paracompact and let $p : E \longrightarrow B$ be a bundle of finite type.* (This holds, for instance, if B is compact.) *Then there exists a finite-dimensional vector space W and a map $\varphi : B \longrightarrow \mathrm{Pr}(W)$ such that the associated vector bundle E_φ is isomorphic to E.*

Proof: Choose $W \subset \Gamma(E)$ such that $\Phi : B \times W \longrightarrow E$, defined by $\Phi(b, w) = w(b)$ for $(b, w) \in B \times W$, is surjective. Next define $\varphi : B \longrightarrow \mathrm{Pr}(W)$ by letting $\varphi(b)$ for $b \in B$ be the orthogonal projection onto $\ker(\Phi_b)^\perp = \{w \in W \mid \langle w, v\rangle = 0 \,\forall\, v$ satisfying $\Phi(b, v) = 0\}$, where $\langle -, -\rangle$ is some Hermitian product on W. □

8.3.14 Note. In fact, just as in 8.3.7, if we put $n = \dim W$, then the map $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^n)$ defined by $f(b) = \ker(\Phi_b)^\perp$ is a continuous map that satisfies $f^*\mathrm{E}_k(\mathbb{C}^n) \cong E$.

For B paracompact, let us define $\mathcal{K}_k(B) = \{[E] \mid p : E \longrightarrow B$ is a vector bundle of finite type and dimension $k\}$, where $[\cdot]$ denotes the isomorphism class of a vector bundle over B. Then we have the next result.

8.3.15 **Proposition**. *Let B be paracompact. Then the function*

$$M(B, \mathrm{BU}_k) \longrightarrow \mathcal{K}_k(B)$$

that sends $f : B \longrightarrow \mathrm{BU}_k$ to $[f^*\mathrm{E}_k(\mathbb{C}^\infty)] \in \mathcal{K}_k(B)$ is surjective.

Proof: Let an arbitrary isomorphism class in $\mathcal{K}_k(B)$ be represented by a bundle $p : E \longrightarrow B$ of finite type. Using 8.3.14 there exists $f_1 : B \longrightarrow \mathrm{G}_k(\mathbb{C}^n)$ such that $f_1^*\mathrm{E}_k(\mathbb{C}^n) \cong E$. Then $f = j \circ f_1 : B \longrightarrow \mathrm{G}_k(\mathbb{C}^n) \hookrightarrow \mathrm{BU}_k$ is an element in $M(B, \mathrm{BU}_k)$ that maps to the isomorphism class of p, as desired. □

8.3.16 Exercise. Prove that there is a homeomorphism $\mathrm{Pr}(\mathbb{C}^m) \approx \mathrm{G}(\mathbb{C}^m) = \bigcup_k \mathrm{G}_k(\mathbb{C}^m)$. Also establish the relation between the previous proposition and Corollary 8.3.13. (Hint: The map $(\pi : \mathbb{C}^m \longrightarrow \mathbb{C}^m) \mapsto \pi(\mathbb{C}^m)$ defines the homeomorphism.)

8.4 Classification of Vector Bundles of Finite Type

We have proved that every k-vector bundle of finite type over paracompact B is induced by means of a map $f : B \longrightarrow \mathrm{BU}_k$. We shall show in what follows that the mapping that assigns the bundle $f^*\mathrm{E}_k(\mathbb{C}^\infty)$ to each $f : B \longrightarrow \mathrm{BU}_k$ gives a classification of the isomorphism classes of vector bundles over B.

First we shall examine the relationship between bundles induced by two homotopic maps. And in order to do that we shall use three preliminary results, which are special cases of 4.6.1, 4.6.2, and 4.6.3.

8.4.1 **Lemma**. *Suppose that $p : E \longrightarrow B \times I$ is a vector bundle whose restrictions to $B \times [0, a]$ and to $B \times [a, 1]$ are trivial for some $a \in I$. Then $p : E \longrightarrow B \times I$ itself is a trivial bundle.* □

8.4.2 **Lemma**. *Let $p : E \longrightarrow B \times I$ be a vector bundle. Then there exists an open cover $\{U\}$ of B such that $p^{-1}(U \times I) \longrightarrow U \times I$ is trivial for every U in the cover.* □

8.4.3 **Proposition**. *Let $p : E \longrightarrow B \times I$ be a vector bundle, where B is a paracompact space. Let $r : B \times I \longrightarrow B \times I$ be the retraction defined by $r(b,t) = (b,1)$ for $(b,t) \in B \times I$. Then there exists a bundle morphism*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "r"'] & B \times I. \end{tikzcd}$$

Therefore, $E \cong r^*E$. □

From the previous results, as in 4.6.4, we have the following consequence.

8.4.4 **Theorem**. *Let $p' : E' \longrightarrow B'$ be a vector bundle and B a paracompact space, and suppose that we have two homotopic maps $f, g : B \longrightarrow B'$. Then we have a bundle isomorphism $f^*E' \cong g^*E'$.* □

8.4.5 **Corollary**. *Let B be paracompact. Then we have a natural surjective function*

$$[B, \mathrm{BU}_k] \longrightarrow \mathcal{K}_k(B)$$

defined by $[f] \mapsto [f^*\mathrm{E}_k(\mathbb{C}^\infty)]$. □

8.4.6 **Corollary**. *Let B, B' be paracompact. If $h : B' \longrightarrow B$ is a homotopy equivalence, then the function $h^* : \mathcal{K}_k(B) \longrightarrow \mathcal{K}_k(B')$ defined by $[E] \mapsto [h^*E]$ is bijective.*

Proof: If $h' : B \longrightarrow B'$ is a homotopy inverse for h, then $h \circ h' \simeq \mathrm{id}_B$, and therefore $h'^*h^*[E] = [E]$ holds. Similarly, we have $h^*h'^*[E'] = [E']$. □

We shall now show that for B compact the function in 8.4.5 is injective. And to do that we shall need the next lemma.

8.4.7 **Lemma**. *Let $p : E \longrightarrow B$ be a k-dimensional vector bundle. Then for each m we have a bijective correspondence between maps $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ such that $f^*(\mathrm{E}_k(\mathbb{C}^m)) \cong E$ and epimorphisms of bundles $\varphi : B \times \mathbb{C}^m \longrightarrow E$, that is, bundle homomorphisms φ covering id_B, such that for every $b \in B$ the restriction over b of φ, namely $\varphi_b : \mathbb{C}^m \longrightarrow p^{-1}(b)$, is a linear epimorphism. In a diagram,*

$$\begin{tikzcd} B \times \mathbb{C}^m \arrow[rr, two heads, "\varphi"] \arrow[dr] & & E \arrow[dl] \\ & B\,. & \end{tikzcd}$$

Proof: First let $\varphi : B \times \mathbb{C}^m \longrightarrow E$ be an epimorphism. Then we define $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ by $f(b) = \ker(\varphi_b : \mathbb{C}^m \longrightarrow p^{-1}(b))^\perp$ for $b \in B$. It is then easy to prove that $f^*(\mathrm{E}_k(\mathbb{C}^m)) \cong E$.

If we now start with $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$, such that we have an isomorphism $f^*(\mathrm{E}_k(\mathbb{C}^m)) = \{(b, W, w) \in B \times \mathrm{G}_k(\mathbb{C}^m) \times \mathbb{C}^m \mid f(b) = W \text{ and } w \in W\} \cong E$, then it suffices to associate to f an epimorphism $\varphi : B \times \mathbb{C}^m \longrightarrow f^*(\mathrm{E}_k(\mathbb{C}^m))$. So we define $\varphi(b, v) = (b, f(b), \mathrm{proj}_{f(b)}(v))$ for $(b, v) \in B \times \mathbb{C}^m$, where $\mathrm{proj}_{f(b)} : \mathbb{C}^m \longrightarrow f(b)$ is the orthogonal projection onto the subpace $f(b)$ of $\mathbb{C}^m$.

It is a straightforward *exercise* to prove that these assignments are well defined and are inverses of each other. □

8.4.8 **Theorem**. *Let B be compact. The function*

$$[B, \mathrm{BU}_k] \longrightarrow \mathcal{K}_k(B)$$

is a natural bijection, which by definition sends a class $[f]$ of a map $f : B \longrightarrow \mathrm{BU}_k$ to the vector bundle $f^*\mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow B$.

Proof: By what we have shown above, it is enough to prove that $[B, \mathrm{BU}_k] \longrightarrow \mathcal{K}_k(B)$ is injective. So suppose that we are given $[f], [g] \in [B, \mathrm{BU}_k]$ such that $f^*\mathrm{E}_k(\mathbb{C}^m) \cong g^*\mathrm{E}_k(\mathbb{C}^n)$. Since B is compact, any class $[f] \in [B, \mathrm{BU}_k]$ has a representative $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ for some integer m. So we can assume that the classes $[f], [g]$ are represented by maps $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ and $g : B \longrightarrow \mathrm{G}_k(\mathbb{C}^n)$ for some integers m and n. Choose $E \in [f^*\mathrm{E}_k(\mathbb{C}^m)] = [g^*\mathrm{E}_k(\mathbb{C}^n)]$ and suppose that f and g correspond to epimorphisms

$$\varphi : B \times \mathbb{C}^m \longrightarrow E \quad \text{and} \quad \psi : B \times \mathbb{C}^n \longrightarrow E\,,$$

as in Lemma 8.4.7. Now for $t \in I$ let us define $\gamma_t : B \times \mathbb{C}^m \times \mathbb{C}^n \longrightarrow E$ by $\gamma_t(b, u, v) = (1 - t)\varphi(b, u) + t\psi(b, v)$, where $(b, u, v) \in B \times \mathbb{C}^m \times \mathbb{C}^n$. This also is an epimorphism for every $t \in I$. Let $h_t : B \longrightarrow \mathrm{G}_k(\mathbb{C}^{m+n})$ be the map that corresponds to γ_t according to Note 8.3.14. Next, let $i_{m+n,m} : \mathrm{G}_k(\mathbb{C}^m) \hookrightarrow \mathrm{G}_k(\mathbb{C}^{m+n})$ be induced by $\mathbb{C}^m \hookrightarrow \mathbb{C}^{m+n}$ and, respectively, let $i_{m+n,n} : \mathrm{G}_k(\mathbb{C}^n) \hookrightarrow \mathrm{G}_k(\mathbb{C}^{m+n})$ be induced by $\mathbb{C}^n \hookrightarrow \mathbb{C}^{m+n}$, where the vector space inclusions are into the first m, respectively first n, coordinates. Then we have that

$$h_0 = i_{m+n,m} \circ f \quad \text{and} \quad h_1 = T \circ i_{m+n,n} \circ g,$$

where $T : \mathrm{G}_k(\mathbb{C}^{m+n}) \longrightarrow \mathrm{G}_k(\mathbb{C}^{m+n})$ is induced by the map

$$\mathbb{C}^{m+n} \longrightarrow \mathbb{C}^{m+n}$$

defined by

$$(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) \mapsto (z_{n+1}, \ldots, z_{n+m}, z_1, \ldots, z_n),$$

which is homotopic to the identity by a homotopy that passes all the way through isomorphisms. And therefore it follows that $T \simeq \mathrm{id}_{\mathrm{G}_k(\mathbb{C}^{m+n})}$. So we obtain that $i_{m+n,m} \circ f \simeq i_{m+n,n} \circ g$.

Now let us consider the diagram

$$\begin{tikzcd}
 & \mathrm{G}_k(\mathbb{C}^m) \arrow[dr, "i_{m+n,m}"] \arrow[drr, bend left, "i_m"] & & \\
B \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathrm{G}_k(\mathbb{C}^{m+n}) \arrow[r, "i_{m+n}"] & BU_k, \\
 & \mathrm{G}_k(\mathbb{C}^n) \arrow[ur, "i_{m+n,n}"'] \arrow[urr, bend right, "i_n"'] & &
\end{tikzcd}$$

where the maps i_m and i_n exist by the definition of colimit, because $\mathrm{BU}_k = \operatorname{colim}_n \mathrm{G}_k(\mathbb{C}^n)$. Moreover, these maps satisfy $i_m = i_{m+n} \circ i_{m+n,m}$, $i_n = i_{m+n} \circ i_{m+n,n}$, as the diagram indicates. Therefore we conclude that $i_m \circ f = i_{m+n} \circ i_{m+n,m} \circ f \simeq i_{m+n} \circ i_{m+n,n} \circ g = i_n \circ g$.

From these considerations we have that the maps f and g represent the same element in $[B, \mathrm{BU}_k]$, which is just what we wanted to show. $\square$

Let us note that over a compact space every vector bundle is of finite type. So the previous theorem gives us a classification of all vector bundles over any compact space.

In fact, using our results about bundles over paracompact spaces, we can also classify all bundles over that class of spaces, which includes all CW-complexes (cf. [59]).

However, we shall achieve this extension of Theorem 8.4.7 in the next section by using Gauss maps instead of projections, since this allows us to present another (dual) point of view for classifying bundles.

8.5 Classification of Vector Bundles over Paracompact Spaces

The hypothesis of paracompactness of the base space of a vector bundle is satisfied by very important classes of spaces, such as CW-complexes and metric spaces. It shall replace the condition used before that the bundle be of finite type. In this section we shall classify vector bundles over paracompact spaces.

8.5.1 DEFINITION. Given a space B we denote by $\mathrm{Vect}_k(B)$ the isomorphism classes of complex vector bundles of dimension k over B.

As we have mentioned before, if B is compact, then we have $\mathrm{Vect}_k(B) = \mathcal{K}_k(B)$.

8.5.2 DEFINITION. Let $p : E \longrightarrow B$ be a vector bundle of dimension k. A map $g : E \longrightarrow \mathbb{C}^m$, where $k \leq m \leq \infty$, is called a *Gauss map* if g restricted to each fiber is a (linear) monomorphism of vector spaces.

8.5.3 NOTE. Given a Gauss map $g : E \longrightarrow \mathbb{C}^m$ of a vector bundle $p : E \longrightarrow B$, there is an induced bundle homomorphism

$$G : E \longrightarrow B \times \mathbb{C}^m$$

covering the identity map id_B such that $G(e) = (p(e), g(e))$, which is, in fact, a monomorphism. Conversely, given a vector bundle monomorphism $G : E \longrightarrow B \times \mathbb{C}^m$, then $g = \mathrm{proj}_{\mathbb{C}^m} \circ G : E \longrightarrow \mathbb{C}^m$ is a Gauss map. Therefore, a Gauss map for a vector bundle $p : E \longrightarrow B$ might also be considered as a bundle monomorphism $G : E \longrightarrow \varepsilon^m$ covering id_B, where ε^m is the trivial bundle of dimension m over B.

8.5.4 **Proposition**. *Let $p : E \longrightarrow B$ be a k-vector bundle. Then there exists a Gauss map $g : E \longrightarrow \mathbb{C}^m$ if and only if there exists a map $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ such that $f^*(\mathrm{E}_k(\mathbb{C}^m)) \cong E$.* The map f is called a *classifying map.*

Proof: First let $g : E \longrightarrow \mathbb{C}^m$ be a Gauss map. We shall define a bundle morphism

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathrm{E}_k(\mathbb{C}^m) \arrow[d] \\ B \arrow[r, "f"'] & \mathrm{G}_k(\mathbb{C}^m) \end{tikzcd}$$

in the following discussion. We define the map f of the base spaces in terms of the given map g by $f(b) = g(p^{-1}(b)) \in \mathrm{G}_k(\mathbb{C}^m)$ for $b \in B$. In order to prove that f is continuous it is enough to prove that $f|U_\alpha$ is continuous for each $\alpha \in \Lambda$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of B for which $p^{-1}(U_\alpha)$ is trivial for each $\alpha \in \Lambda$.

Recall that $\mathrm{G}_k(\mathbb{C}^m)$ has the quotient topology induced by the map $\rho : \mathrm{V}_k(\mathbb{C}^m) \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$, where $\mathrm{V}_k(\mathbb{C}^m)$ is the Stiefel manifold of k-frames in $\mathbb{C}^m$ (see 8.3.4) and where ρ sends a k-frame to the subspace it generates. For each $\alpha \in \Lambda$ choose a trivialization $\varphi_\alpha : U_\alpha \times \mathbb{C}^k \longrightarrow p^{-1}U_\alpha$. Also, let $\{v_1, \dots, v_k\}$ be a basis of $\mathbb{C}^k$. If $b \in U_\alpha$, then $\{g\varphi_\alpha(b, v_1), \dots, g\varphi_\alpha(b, v_k)\}$ is a basis of $f(b)$, and so $f|U_\alpha = \rho \circ F_\alpha$, where we define $F_\alpha : B \longrightarrow \mathrm{V}_k(\mathbb{C}^m)$ by $F_\alpha(b) = (g\varphi_\alpha(b, v_1), \dots, g\varphi_\alpha(b, v_k))$ for $b \in B$. Since F_α is clearly continuous, it follows that $f|U_\alpha$ is also continuous.

Next we define the map $\widetilde{f}$ of the total spaces by

$$\widetilde{f}(e) = (fp(e), g(e))$$

for $e \in E$. This map is also manifestly continuous.

We leave it to the reader to verify that $\widetilde{f}$ is a bundle morphism. According to Exercise 8.1.14 this is equivalent to saying that $f^*(\mathrm{E}_k(\mathbb{C}^m)) \cong E$.

Conversely, suppose that we have a bundle morphism

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathrm{E}_k(\mathbb{C}^m) \arrow[d] \\ B \arrow[r, "f"'] & \mathrm{G}_k(\mathbb{C}^m). \end{tikzcd}$$

Now we always have a map $q : \mathrm{E}_k(\mathbb{C}^m) \longrightarrow \mathbb{C}^m$ defined by $q(V, v) = v$ for $(V, v) \in \mathrm{E}_k(\mathbb{C}^m) \subset \mathrm{G}_k(\mathbb{C}^m) \times \mathbb{C}^m$. If we then define $g = q \circ \widetilde{f}$, it is easy to check that g is a Gauss map (see Remark 8.5.3). $\square$

8.5.5 REMARK. Given a Gauss map $G : E \longrightarrow \mathbb{C}^m$ and its induced bundle monomorphism $G : E \longrightarrow B \times \mathbb{C}^m$, $m < \infty$, then $G(E) \subset B \times \mathbb{C}^m$ is a subbundle that is isomorphic to E. There is a bundle epimorphism $\varphi : B \times \mathbb{C}^m \longrightarrow E$ given by taking fiberwise the orthogonal projection (with respect to the usual Hermitian product on $\mathbb{C}^m$) $B \times \mathbb{C}^m \longrightarrow G(E)$ and then composing with the isomorphism $G^{-1} : G(E) \longrightarrow E$.

8.5.6 EXERCISE. Prove that the map $f : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ associated to the φ of the previous remark according to 8.4.7 is the same as the one associated to g according to Proposition 8.5.4.

8.5.7 EXERCISE. Prove that there is a one-to-one correspondence between Gauss maps $g : E \longrightarrow \mathbb{C}^m$ and maps $\varphi : B \longrightarrow \mathrm{Pr}(\mathbb{C}^m)$ such that $E_\varphi \cong E$ (see 8.2.1).

8.5.8 EXERCISE. Let $p : E \longrightarrow B$ be a complex k-vector bundle.

(a) Prove that the construction in the proof of 8.5.4 establishes a bijection between the set of bundle morphisms

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & \mathrm{E}_k(\mathbb{C}^m) \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & \mathrm{G}_k(\mathbb{C}^m) \end{array}$$

and the set of Gauss maps $g : E \longrightarrow \mathbb{C}^m$.

(b) Prove that if $G : E \times I \longrightarrow \mathbb{C}^m$ is a homotopy such that $G_t : E \longrightarrow \mathbb{C}^m$ is a Gauss map for every $t \in I$, where we define $G_t(e) = G(t, e)$ for $e \in E$, then we can use the above construction in order to obtain a bundle morphism

$$\begin{array}{ccc} E \times I & \xrightarrow{\widetilde{F}} & \mathrm{E}_k(\mathbb{C}^m) \\ {\scriptstyle p\times \mathrm{id}}\downarrow & & \downarrow \\ B \times I & \xrightarrow[F]{} & \mathrm{G}_k(\mathbb{C}^m) \end{array}$$

with the following property. If $f_\nu : B \longrightarrow \mathrm{G}_k(\mathbb{C}^m)$ for $\nu = 0, 1$ are the functions associated to G_ν for $\nu = 0, 1$, then F is a homotopy between f_0 and f_1.

In order to prove that every bundle over a paracompact space has a Gauss map we shall have the next important lemma, which is a special case of 4.6.12.

8.5.9 **Lemma**. *Let $p : E \longrightarrow B$ be a vector bundle over a paracompact space B. Then there exists a countable open cover of B, say $\{W_n\}$ with $n \geq 1$, such that $p^{-1}W_n$ is trivial for all $n \geq 1$.*

Proof: Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of B such that $p^{-1}(U_\alpha) \longrightarrow U_\alpha$ is trivial for all $\alpha \in \Lambda$. Since B is paracompact, there exists a partition of unity $\{\eta_\alpha\}_{\alpha\in\Lambda}$ subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$. For each $b \in B$ let us define $S(b)$ to be the finite set of those $\alpha \in \Lambda$ that satisfy $\eta_\alpha(b) > 0$. Also, for each finite subset $S \subset \Lambda$, let us define $W(S) = \{b \in B \mid \eta_\alpha(b) > \eta_\beta(b)$ whenever $\alpha \in S$ and $\beta \notin S\}$.

We claim that $W(S)$ is open in B. In fact, $B_{\alpha,\beta} = \{b \in B \mid \eta_\alpha(b) > \eta_\beta(b)\}$ is open, since $B_{\alpha,\beta} = (\eta_\alpha - \eta_\beta)^{-1}(0, 1]$. Now for any given $b_0 \in W(S)$ there exists a neighborhood $V(b_0)$ of b_0 such that only $\eta_{\beta_1}, \eta_{\beta_2}, \ldots, \eta_{\beta_r}$, $\beta_i \notin S$, are different from zero in $V(b_0)$ for some finite integer r. We put $N = \bigcap_{\alpha\in S}(B_{\alpha,\beta_1} \cap B_{\alpha,\beta_2} \cap \cdots \cap B_{\alpha,\beta_r})$, which is open, being a finite intersection of open sets. We then have $b_0 \in N \cap V(b_0) \subset W(S)$, and therefore $W(S)$ is open.

If S and S' are two distinct subsets of Λ each having m elements, then $W(S) \cap W(S') = \emptyset$. This is so, since there exists $\alpha \in S$ such that $\alpha \notin S'$ and there exists $\beta \in S'$ such that $\beta \notin S$, and therefore $b \in W(S) \cap W(S')$ would imply $\eta_\alpha(b) > \eta_\beta(b)$ and $\eta_\beta(b) > \eta_\alpha(b)$, a patent contradiction.

Now we define $W_n = \bigcup\{W(S(b)) \mid |S(b)| = n\}$ for every integer n, where here $|\cdot|$ denotes the cardinality of a set.

If $\alpha \in S(b)$, then $W(S(b)) \subset \eta_\alpha^{-1}(0,1] \subset U_\alpha$, and therefore we have that $p^{-1}W(S(b)) \longrightarrow W(S(b))$ is trivial. Since for each n the open set W_n is a disjoint union of sets of the form $W(S(b))$, it follows that $p^{-1}W_n \longrightarrow W_n$ is also trivial. □

8.5.10 Note. From the proof it is clear that any vector bundle $p : E \longrightarrow B$ is a bundle of finite type whenever B is also finite-dimensional and the dimensions of the fibers are bounded. This is because each $b \in B$ belongs to at most m subsets U_α, and so we have that $W_i = \emptyset$ for $i > m$. Therefore, there exists a finite open cover $\{W_i\}$ for $i = 1, \ldots, m$ such that $p^{-1}W_i \longrightarrow W_i$ is trivial. And this proves the claim.

8.5.11 **Proposition**. *Every vector bundle over a paracompact space has a Gauss map.*

Proof: Let $p : E \longrightarrow B$ be a k-vector bundle. Using Lemma 8.5.9 and the hypothesis that B is paracompact, there exists a countable open cover $\{W_n\}_{n=1}^\infty$ of B such that $p^{-1}W_n \longrightarrow W_n$ is trivial for each $n \geq 1$. Choose a trivialization $h_n : p^{-1}W_n \longrightarrow W_n \times \mathbb{C}^k$ for each $n \geq 1$. Next let $\{\eta_n\}_{n=1}^\infty$ be a partition of unity subordinate to $\{W_n\}_{n=1}^\infty$. For each $n \geq 1$, we define $g_n : E \longrightarrow \mathbb{C}^k$ by

$$g_n(e) = \begin{cases} [\eta_n p(e)]\mathrm{proj} h_n(e) & \text{if } e \in p^{-1}(W_n), \\ 0 & \text{if } e \notin p^{-1}(W_n), \end{cases}$$

where $\mathrm{proj} : W_n \times \mathbb{C}^k \longrightarrow \mathbb{C}^k$ is the projection onto the second factor and $\eta_n \circ p(e)$ is a (real) scalar that multiplies the vector $\mathrm{proj} \circ h_n(e)$. Using the properties of a partition of unity we see that each g_n is continuous. Then we can define a function of sets $g : E \longrightarrow \mathbb{C}^\infty$ by $g(e) = (g_1(e), g_2(e), \ldots, g_n(e), \ldots)$ for $e \in E$, since for each $e \in E$ only a finite number of the values $g_n(e)$ are different from zero. Again by using the properties of a partition of unity, we see that g is continuous. And of course, it is easy to show that g is the desired Gauss map. $\square$

From the previous proof we get the following conclusion in the case of bundles of finite type.

8.5.12 **Corollary**. *Let B be paracompact. Then, every vector bundle $p : E \longrightarrow B$ of finite type has a Gauss map.* $\square$

The following is a generalization of Theorem 8.4.8.

8.5.13 **Theorem**. *Let B be a paracompact space. Then there exists a natural bijection $[B, \mathrm{BU}_k] \longrightarrow \mathrm{Vect}_k(B)$, which sends the homotopy class of $f : B \longrightarrow \mathrm{BU}_k$ to the isomorphism class of $f^*\mathrm{E}_k(\mathbb{C}^\infty)$.* This function is called the *classifying map.*

Proof: By Theorem 8.4.4 this function is well defined. And then using Propositions 8.5.4 and 8.5.11 we deduce that the function is surjective. So it remains to show that the function is injective. But before doing that we prove some auxiliary results.

First we define $\mathbb{C}_1^\infty = \{(z_i) \in \mathbb{C}^\infty \mid z_{2i} = 0, i = 1, 2, 0, \ldots\}$ and $\mathbb{C}_2^\infty = \{(z_i) \in \mathbb{C}^\infty \mid z_{2i+1} = 0,\ i = 0, 1, 2, \ldots\}$. Then we clearly have that $\mathbb{C}^\infty = \mathbb{C}_1^\infty \oplus \mathbb{C}_2^\infty$. Next we define two homotopies $h^1, h^2 : \mathbb{C}^\infty \times I \longrightarrow \mathbb{C}^\infty$ by

$$h^1((z_1, z_2, z_3, \ldots), t) = (1-t)(z_1, z_2, z_3, \ldots) + t(z_1, 0, z_2, 0, z_3, \ldots)\,,$$

$$h^2((z_1, z_2, z_3, \ldots), t) = (1-t)(z_1, z_2, z_3, \ldots) + t(0, z_1, 0, z_2, 0, z_3, \ldots)\,,$$

where $(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty$ and $t \in I$. These homotopies start with the identity and end with maps that we denote by

$$h_1^1 : \mathbb{C}^\infty \longrightarrow \mathbb{C}_1^\infty \subset \mathbb{C}^\infty \quad \text{and} \quad h_1^2 : \mathbb{C}^\infty \longrightarrow \mathbb{C}_2^\infty \subset \mathbb{C}^\infty.$$

The composites $h_1^\nu \circ q : \mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow \mathbb{C}^\infty$ for $\nu = 1, 2$ are Gauss maps, where $q : \mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow \mathbb{C}^\infty$ is the projection. According to 8.5.8(a), these maps induce two bundle morphisms, namely,

$$\begin{tikzcd}
\mathrm{E}_k(\mathbb{C}^\infty) \arrow[r, "\widetilde{\varphi}_\nu"] \arrow[d] & \mathrm{E}_k(\mathbb{C}^\infty) \arrow[d] \\
\mathrm{BU}_k \arrow[r, "\varphi_\nu"'] & \mathrm{BU}_k,
\end{tikzcd} \qquad \nu = 1, 2.$$

The composites $h^\nu \circ (q \times \mathrm{id}) : \mathrm{E}_k(\mathbb{C}^\infty) \times I \longrightarrow \mathbb{C}^\infty$ for $\nu = 1, 2$ are homotopies that start with q, since $h^\nu(q \times \mathrm{id})(e, 0) = h^\nu(q(e), 0) = q(e)$ for $e \in \mathrm{E}_k(\mathbb{C}^\infty)$, and end with $h_1^\nu \circ q$. Moreover, the restrictions of these homotopies to the slices at each fixed $t \in I$ are Gauss maps. Using 8.5.8(b) we then have that φ_ν for $\nu = 1, 2$ is homotopic to the map induced by q, which is obviously the identity. So we have shown that $\varphi_\nu \simeq \mathrm{id}$ for $\nu = 1, 2$.

We are now ready to show that the function is injective. Suppose that we are given $f_\nu : B \longrightarrow \mathrm{BU}_k = \mathrm{G}_k(\mathbb{C}^\infty)$ for $\nu = 1, 2$ satisfying $f_1^*\mathrm{E}_k(\mathbb{C}^\infty) \cong f_2^*\mathrm{E}_k(\mathbb{C}^\infty)$. So to prove injectivity we must show that f_1 and f_2 are homotopic.

Denoting $f_1^*\mathrm{E}_k(\mathbb{C}^\infty)$ by E and using the above isomorphism, we get two bundle morphisms

$$\begin{tikzcd} E \arrow[r, "\widetilde{f_\nu}"] \arrow[d] & \mathrm{E}_k(\mathbb{C}^\infty) \arrow[d] \\ B \arrow[r, "f_\nu"'] & \mathrm{BU}_k\,, \end{tikzcd} \qquad \nu = 1, 2.$$

Let $g_\nu : E \longrightarrow \mathbb{C}^\infty$ for $\nu = 1, 2$ be the associated Gauss maps; that is, $g_\nu = q \circ \widetilde{f_\nu}$.

Consider the composites $h_1^\nu \circ g_\nu : E \longrightarrow \mathbb{C}^\infty$ for $\nu = 1, 2$. These are Gauss maps, and according to 8.5.8(a) they induce two bundle morphisms of the form

$$\begin{tikzcd} E \arrow[r, "\widetilde{f_\nu}"] \arrow[d] & \mathrm{E}_k(\mathbb{C}^\infty) \arrow[r, "\widetilde{\varphi_\nu}"] \arrow[d] & \mathrm{E}_k(\mathbb{C}^\infty) \arrow[d] \\ B \arrow[r, "f_\nu"'] & \mathrm{BU}_k \arrow[r, "\varphi_\nu"'] & \mathrm{BU}_k\,, \end{tikzcd} \qquad \nu = 1, 2.$$

We then define $G : E \times I \longrightarrow \mathbb{C}^\infty$ by $G(e, t) = (1 - t)h_1^1 g_1(e) + t h_1^2 g_2(e)$ for $(e, t) \in E \times I$. This is a homotopy between $h_1^1 \circ g_1$ and $h_1^2 \circ g_2$. Since $h_1^1(\mathbb{C}^\infty) \cap h_1^2(\mathbb{C}^\infty) = 0$, it follows that G_t is a Gauss map for each $t \in I$. Therefore, using 8.5.8(b) we have that $\varphi_1 \circ f_1 \simeq \varphi_2 \circ f_2$. But we have already seen that $\varphi_\nu \simeq \mathrm{id}$ for $\nu = 1, 2$, and so $f_1 \simeq f_2$ follows. $\square$

8.5.14 Note. The previous theorem is still true if instead of assuming that B is paracompact, we assume only that the vector bundles that we wish to classify have the property that the base space has an open cover with an associated subordinate partition of unity so that over each open set of the cover we have a trivialization of the bundle (see [24]). These are the so-called *numerable bundles.*

To end this chapter we shall present a theorem that relates the concepts of bundle of finite type, orthogonal complement, and classifying map.

8.5.15 **Theorem**. *Let $p : E \longrightarrow B$ be a vector bundle of dimension n over a paracompact space. Then the following are equivalent:*

(i) *The bundle $p : E \longrightarrow B$ is of finite type.*

(ii) *There exists a map $f_E : B \longrightarrow \mathrm{G}_n(K^m)$ that classifies E for some integer $m < \infty$, where $K = \mathbb{R}$ or $\mathbb{C}$.*

(iii) *There exists a vector bundle $\overline{p} : \overline{E} \longrightarrow B$ such that $E \oplus \overline{E}$ is trivial.*

Proof: (i)⇒(ii). By Corollary 8.5.12 the bundle $p : E \longrightarrow B$ has a Gauss map, and by Proposition 8.5.4 it has a classifying map into $\mathrm{G}_n(K^m)$ for some m.

(ii)⇒(iii). Let $f_E : B \longrightarrow \mathrm{G}_n(K^m)$ be a classifying map. If

$$\overline{\mathrm{E}}_{m-n}(K^m) \longrightarrow \mathrm{G}_n(K^m)$$

is the orthogonal complement of the bundle

$$\mathrm{E}_n(K^m) \longrightarrow \mathrm{G}_n(K^m)$$

given by $\overline{\mathrm{E}}_{m-n}(K^m) = \{(W, v) \in \mathrm{G}_n(K^m) \times K^m \mid v \perp W\}$, then we have that $\mathrm{E}_n(K^m) \oplus \overline{\mathrm{E}}_{m-n}(K^m) \cong \mathrm{G}_n(K^m) \times K^m$. So putting $\overline{E} = f_E^* \overline{\mathrm{E}}_{m-n}(K^m)$, it follows that $E \oplus \overline{E} = \varepsilon^m$.

(iii)⇒(i). If $E \oplus \overline{E} \cong \varepsilon^m$, then the composite

$$E \longrightarrow E \oplus \overline{E} \cong B \times K^m \longrightarrow K^m ,$$

where the last map is the projection onto the second factor, is a Gauss map for E. By Proposition 8.5.4, there exists a classifying map $f : B \longrightarrow \mathrm{G}_n(K^m)$. However, the bundle $\mathrm{E}_n(K^m) \longrightarrow \mathrm{G}_n(K^m)$ is of finite type because $\mathrm{G}_n(K^m)$ is compact. Letting $\{V_i\}_{i=1,\dots,k}$ be a finite trivializing open cover of $\mathrm{G}_n(K^m)$ for this last bundle, it follows that $\{f^{-1}V_i\}_{i=1,\dots,k}$ is a finite trivializing open cover of B for $E \longrightarrow B$. □

8.5.16 EXERCISE. Prove that (i), (ii), and (iii) in the previous theorem are also equivalent to the following:

(iv) *There exists a Gauss map $g : E \longrightarrow K^m$ (or equivalently a vector bundle monomorphism $G : E \longrightarrow B \times K^m$; see 8.5.3) for some $m < \infty$, where $K = \mathbb{R}$ or $\mathbb{C}$.*

(v) *There exists a bundle epimorphism* $\Phi : B \times K^m \longrightarrow E$ *for some* $m < \infty$, *where* $K = \mathbb{R}$ *or* $\mathbb{C}$.

8.5.17 EXERCISE. Let $K = \mathbb{C}$ or $K = \mathbb{R}$.

(a) Consider the canonical embedding

$$j_m : \mathrm{G}_n(K^m) \longrightarrow \mathrm{G}_{n+1}(K^{m+1}),$$

and let

$$\mathrm{E}_n(K^m) \longrightarrow \mathrm{G}_n(K^m) \quad \text{and} \quad \mathrm{E}_{n+1}(K^{m+1}) \longrightarrow \mathrm{G}_{n+1}(K^{m+1})$$

be the corresponding canonical vector bundles. Prove that

$$j_m^* \mathrm{E}_{n+1}(K^{m+1}) \cong \mathrm{E}_n(K^m) \oplus \varepsilon^1,$$

where ε^1 represents the trivial line bundle.

(b) Given the canonical embedding

$$j : \mathrm{G}_n(K^\infty) \longrightarrow \mathrm{G}_{n+1}(K^\infty)$$

and the corresponding universal bundles

$$\mathrm{E}_n(K^\infty) \longrightarrow \mathrm{G}_n(K^\infty) \quad \text{and} \quad \mathrm{E}_{n+1}(K^\infty) \longrightarrow \mathrm{G}_{n+1}(K^\infty),$$

conclude that

$$j^* \mathrm{E}_{n+1}(K^\infty) \cong \mathrm{E}_n(K^\infty) \oplus \varepsilon^1,$$

where ε^1 represents again the trivial line bundle.

The following exercise provides us with an equivalent definition of a vector bundle.

8.5.18 EXERCISE. Prove that a locally trivial bundle $p : E \longrightarrow B$ is a vector bundle if and only if $p^{-1}(x)$ is a (real or complex) vector space and for each trivialization $\varphi_U : p^{-1}(U) \longrightarrow U \times F$ ($F = \mathbb{R}$ or $\mathbb{C}$), the restriction $\varphi_{U,x} : p^{-1}(x) \longrightarrow F$ is a linear isomorphism, $x \in U$.

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Chapter 9

K-Theory

Based on considerations made in the last chapter, we shall now introduce a functor, called the K-functor or K-theory, that has characteristics analogous to those of cohomology as was studied in Chapter 7, but with particularly useful properties, as we shall see in Chapter 10. The foundation for the construction of K-theory is the abelian semigroup $\mathrm{Vect}(B)$ of isomorphism classes of vector bundles over B. In the course of the chapter we shall give various interpretations to $K(B)$, one of these based precisely on the classification results of the previous chapter. Finally, we state the Bott periodicity theorem, whose proof is postponed to Appendix B, and analyze some of its consequences.

9.1 Grothendieck Construction

In this short section we describe a basic construction, known as the Grothendieck construction. This assigns a group to a semigroup in a universal way and generalizes in some sense the construction of the integers from the natural numbers as well as the construction of the rationals from the integers. This construction allows us to define K-theory from the abelian semigroup $\mathrm{Vect}(B)$.

9.1.1 **Proposition** and Definition. *If A is any abelian semigroup, we can associate to it an abelian group A', unique up to isomorphism, and a homomorphism of semigroups $\alpha : A \longrightarrow A'$ such that we have the following universal property:*

If G is any abelian group and $\gamma : A \longrightarrow G$ is any homomorphism of semigroups, then there exists a unique homomorphism of groups $\gamma' : A' \longrightarrow G$

such that this diagram of semigroups commutes:

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \arrow[dr, "\gamma"] & \\ A' \arrow[r, "\gamma'"'] & G. \end{tikzcd}$$

The pair (A', α) is called the *Grothendieck construction* associated to the semigroup A.

Proof: We define A' by adding to A the inverses of its elements. This is done as follows: We define an equivalence relation in $A \times A$ by $(a_1, b_1) \sim (a_2, b_2)$ if there exists $c \in A$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. Then we put $A' = A \times A/\sim$. If we denote the equivalence class of (a, b) by $\langle a, b \rangle$, then the sum in A' is defined by $\langle a, b \rangle + \langle a', b' \rangle = \langle a+a', b+b' \rangle$. Therefore, the negative of $\langle a, b \rangle$ is $\langle b, a \rangle$. Since A is abelian, clearly A' is an abelian group. We define $\alpha : A \longrightarrow A'$ by $\alpha(a) = \langle a, 0 \rangle$. This construction is due to Grothendieck (see [13]). $\square$

9.1.2 EXERCISE. (a) Prove that $\alpha : A \longrightarrow A'$ has the desired universal property.

(b) Abusing notation, for any $a \in A$ we also use a to denote its image $\alpha(a) \in A'$. Clearly, we have $\langle a, b \rangle = a - b \in A'$. Prove that $a_1 = a_2 \in A'$ if and only if there exists $a \in A$ such that $a_1 + a = a_2 + a \in A$.

(c) Prove that $\alpha : A \longrightarrow A'$ is injective if and only if the cancellation law holds in A. In this case the c in the definition and the a in part (b) can be taken to be 0.

(d) Prove that the property that A' and α have characterizes them uniquely. That is, if A'' is another abelian group and $\alpha' : A \longrightarrow A''$ is a homomorphism of semigroups such that they have the universal property described in 9.1.1, that is, such that for any abelian group G and any homomorphism of semigroups $\gamma : A \longrightarrow G$ there exists a unique homomorphism of groups $\gamma'' : A'' \longrightarrow G$ that makes the diagram

$$\begin{tikzcd} A \arrow[d, "\alpha'"'] \arrow[dr, "\gamma"] & \\ A'' \arrow[r, "\gamma''"'] & G \end{tikzcd}$$

commute, then there exists a (unique) isomorphism of groups $\varphi : A' \longrightarrow A''$ that makes the triangle

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ A' \arrow[rr, "\varphi"'] & & A'' \end{tikzcd}$$

commute.

9.1.3 EXERCISE. Given an abelian semigroup A, prove that the following abelian group A'' and the homomorphism of semigroups $\alpha' : A \longrightarrow A''$ given below have the universal property of 9.1.1. That is, they constitute an alternative to the Grothendieck construction.

Namely, let $L(A)$ be the free abelian group generated by the elements of A and let $M(A)$ be the subgroup of $L(A)$ generated by the elements of the form $a \oplus a' - (a + a')$, where $+$ is the sum in A and $\oplus$ is the sum in $L(A)$ and where $a, a' \in A$. Then $A'' = L(A)/M(A)$ and $\alpha' : A \longrightarrow A''$, the obvious function, have the desired universal property.

9.1.4 EXERCISE. Prove that if A is a semiring, that is, a semigroup with a multiplication distributive over the sum, then the Grothendieck construction (A', α) (or (A'', α') of 9.1.3) gives us a ring. (Hint: Define the product $\langle a, b\rangle\langle c, d\rangle$ in A' as $\langle ac + bd, ad + bc\rangle$. What would be the definition of the multiplication in A'' of 9.1.3 above?)

9.1.5 EXERCISE. Prove that the Grothendieck construction has the following functorial properties:

(a) If $f : A \longrightarrow B$ is a homomorphism of semigroups and (A', α) and (B', β) are the corresponding abelian groups and semigroup homomorphisms given by the Grothendieck construction, then there exists a unique homomorphism of groups $f' : A' \longrightarrow B'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

commutes.

(b) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are homomorphisms of semigroups, then $(g \circ f)' = g' \circ f'$, where $(g \circ f)'$, g' y f', are the homomorphisms corresponding to $g \circ f$, g, and f as in part (a).

(c) If $f = \mathbf{1}_A : A \longrightarrow A$, then $f' = \mathbf{1}_{A'} : A' \longrightarrow A'$.

9.2 DEFINITION OF $K(B)$

In this section we shall apply the results of Section 9.1 to the abelian semigroup $\mathrm{Vect}(B)$ of isomorphism classes of complex vector bundles over a paracompact space B. For this we need a slightly more general definition of a vector bundle.

9.2.1 DEFINITION. A *vector bundle over* B is a map $p : E \longrightarrow B$ such that each fiber is a finite-dimensional vector space satisfying the following condition. For each $b \in B$, there is a neighborhood U of b, an integer $n \geq 0$, and a homeomorphism $\varphi_U : p^{-1}(U) \longrightarrow U \times F^n$ ($F = \mathbb{R}$ or $\mathbb{C}$) such that for each $b' \in U$, φ_U maps $p^{-1}(b')$ isomorphically onto $\{b'\} \times F^n$. $\mathrm{Vect}(B)$ will denote the set of isomorphism classes of (complex) vector bundles over B. The direct sum of bundles (Whitney sum), as we know from Exercise 8.1.4(i), gives $\mathrm{Vect}(B)$ the structure of an abelian semigroup. Specifically, the sum is given by

$$[E] + [E'] := [E \oplus E'] \,.$$

The Grothendieck construction applied to $\mathrm{Vect}(B)$ gives rise to an abelian group $K(B)$, called the (complex) K*-theory* of B.

The tensor product of vector bundles, by Exercise 8.1.4(ii), induces a multiplication in $\mathrm{Vect}(B)$ such that

$$[E] \cdot [E'] = [E \otimes E'],$$

and gives $\mathrm{Vect}(B)$ the structure of a semiring. Therefore, by Exercise 9.1.4, $K(B)$ acquires the structure of a ring.

Notice that there is a locally constant function $d_E : B \longrightarrow \mathbb{N} \cup \{0\}$ given by $d_E(b) = \dim p^{-1}(b)$. Therefore, d_E is constant on each connected component of B. When this function is constant with value n, then the vector bundle is an n-vector bundle as defined in 8.1.1 (cf. 8.5.18).

9.2.2 EXERCISE. Prove that $K(B)$ is actually a commutative ring with 1, such that the element 1 is represented by the product bundle $B \times \mathbb{C} \longrightarrow B$ and the element 0 by the bundle $\mathrm{id} : B \longrightarrow B$ whose fiber is $\{*\} \cong \{0\}$.

Given a map $f : B' \longrightarrow B$ we have a homomorphism of semigroups (or of semirings) $f^* : \mathrm{Vect}(B) \longrightarrow \mathrm{Vect}(B')$ that associates to the class of a bundle $p : E \longrightarrow B$ the class of the induced bundle $p' : f^*E \longrightarrow B'$. Using the universal property of the Grothendieck construction, we can define a homomorphism of abelian groups $f^* : K(B) \longrightarrow K(B')$ that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Vect}(B) \arrow[r, "f^*"] \arrow[d, "\alpha"'] & \mathrm{Vect}(B') \arrow[d, "\alpha"] \\ K(B) \arrow[r, "f^*"'] & K(B')\,. \end{tikzcd}$$

9.2.3 EXERCISE. Prove that K is a functor from the category of topological spaces to the category of commutative rings with 1.

9.2.4 NOTE. We can see easily that if $f : B' \longrightarrow B$ is continuous, then the homomorphism of abelian groups $f^* : K(B) \longrightarrow K(B')$, as defined above, is also a homomorphism of rings.

9.2.5 **Corollary**. *$K(B)$ is a ring, whose sum is induced by $[E] + [E'] = [E \oplus E']$ and whose product is given by $[E] \cdot [E'] = [E \otimes E']$. Moreover, given $f : B' \longrightarrow B$, we have a homomorphism of rings $f^* : K(B) \longrightarrow K(B)$ such that $f^*([E]) = [f^*E]$.* □

9.2.6 **Proposition**. *If $f_0 \simeq f_1 : B' \longrightarrow B$, then*

$$f_0^* = f_1^* : K(B) \longrightarrow K(B')\,.$$

Proof: If $f_0 \simeq f_1$ and $p : E \longrightarrow B$ is a vector bundle, then by 8.4.4, $f_0^*E \cong f_1^*E$. So $f_0^* = f_1^* : \mathrm{Vect}(B) \longrightarrow \mathrm{Vect}(B')$, and so $f_0^* = f_1^* : K(B) \longrightarrow K(B')$. □

9.2.7 NOTE. It is possible to give to $\mathrm{BU}_k = \mathrm{colim}_n\, \mathrm{G}_k(\mathbb{C}^n)$ the structure of a CW-complex so that each $\mathrm{G}_k(\mathbb{C}^n)$ is a subcomplex with a finite number of cells (see [58]), and in such a way that each BU_k is paracompact. If we consider the bundle $\mathrm{E}_k(\mathbb{C}^\infty) \oplus \varepsilon^1$ over BU_k, then by 8.5.13 there exists a map $i_k : \mathrm{BU}_k \longrightarrow \mathrm{BU}_{k+1}$, unique up to homotopy, such that $i_k^*(\mathrm{E}_{k+1}(\mathbb{C}^\infty)) \cong \mathrm{E}_k(\mathbb{C}^\infty) \oplus \varepsilon^1$, where ε^1 represents the trivial vector bundle over B, $B \times \mathbb{C} \longrightarrow B$, of complex dimension 1.

In fact, it is possible to give an explicit i_k as follows. The Stiefel manifolds $\mathrm{V}_k(\mathbb{C}^n)$ and the Grassmann mainfolds $\mathrm{G}_k(\mathbb{C}^n)$ can be expressed as homogeneous spaces; that is, we have a homeomorphism $U_n/\mathrm{U}_{n-k} \approx \mathrm{V}_k(\mathbb{C}^n)$, given by $[A] \mapsto \{Ae_{n-k+1}, \ldots, Ae_n\}$, where U_{n-k} is the subgroup of U_n consisting of the matrices of the form

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & \mathrm{I}_k \end{array}\right),$$

with $M \in \mathrm{U}_{n-k}$ and $\mathrm{I}_k \in \mathrm{U}_k$, the identity matrix. We also have a homeomorphism $\mathrm{U}_n/\mathrm{U}_{n-k} \times \mathrm{U}_k \approx \mathrm{G}_k(\mathbb{C}^n)$, given by

$$[A] \mapsto \langle Ae_{n-k+1}, \ldots, Ae_n \rangle \,,$$

where $\langle \cdot \rangle$ indicates the subspace generated, and $\mathrm{U}_{n-k} \times \mathrm{U}_k$ is the subgroup of U_n consisting of the matrices of the form

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)$$

with $M \in \mathrm{U}_{n-k}$ and $N \in \mathrm{U}_k$. With these identifications,

$$\mathrm{BU}_k = \operatorname{colim}\,(\cdots \longrightarrow \mathrm{U}_n/\mathrm{U}_{n-k} \times \mathrm{U}_k \longrightarrow \mathrm{U}_{n+1}/\mathrm{U}_{n-k+1} \times \mathrm{U}_k \longrightarrow \cdots),$$

where the homomorphisms in each level are given by

$$[A] \mapsto \left[\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)\right] .$$

Then $i_k : \mathrm{BU}_k \hookrightarrow \mathrm{BU}_{k+1}$ is the map induced in the colimit by the maps

$$\mathrm{U}_n/\mathrm{U}_{n-k} \times \mathrm{U}_k \longrightarrow \mathrm{U}_{n+1}/\mathrm{U}_{n-k} \times \mathrm{U}_{n+1}$$

such that

$$[A] \mapsto \left[\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)\right] .$$

9.2.8 DEFINITION. Let BU be the colimit

$$\mathrm{BU} = \operatorname{colim}\,\{\mathrm{BU}_k, i_k\}_{k\geq 0} \,.$$

Since each BU_k is a CW-complex with a countable number of cells, the product $\mathrm{BU}_k \times \mathrm{BU}_l$, $k, l \geq 0$, is also a CW-complex and so is paracompact. If we consider the product bundle $\mathrm{E}_k(\mathbb{C}^\infty) \times \mathrm{E}_l(\mathbb{C}^\infty)$ over $\mathrm{BU}_k \times \mathrm{BU}_l$, which is a bundle of dimension $k + l$, then, using 8.5.13, there exists a map $w_{k,l} : \mathrm{BU}_k \times \mathrm{BU}_l \longrightarrow \mathrm{BU}_{k+l}$, unique up to homotopy, such that $w_{k,l}^*(\mathrm{E}_{k+l}(\mathbb{C}^\infty)) \cong \mathrm{E}_k(\mathbb{C}^\infty) \times \mathrm{E}_l(\mathbb{C}^\infty)$.

It is possible to give an explicit description of $w_{k,l}$, using homogeneous spaces, in a way similar to what we did earlier with i_k. Nevertheless, in this case, the details are more complicated. These maps $w_{k,l}$ in the colimit define a map $w : \mathrm{BU} \times \mathrm{BU} \longrightarrow \mathrm{BU}$. One can prove that w gives to BU the structure of an H-group, commutative up to homotopy, in such a way that the following diagram commutes up to homotopy:

$$\begin{tikzcd} \mathrm{BU}_k \times \mathrm{BU}_l \arrow[r, "w_{k,l}"] \arrow[d, hook] & \mathrm{BU}_{k+l} \arrow[d, hook] \\ \mathrm{BU} \times \mathrm{BU} \arrow[r, "w"'] & \mathrm{BU}. \end{tikzcd}$$

9.2.9 **Corollary**. $[B, \mathrm{BU}]$ *is an abelian group.* □

9.2.10 Exercise. Prove that there is an isomorphism $K(*) \cong \mathbb{Z}$, given by $\operatorname{Vect}(*) \longrightarrow \mathbb{N}$, $V \mapsto \dim V$.

9.3 $\widetilde{K}(B)$ and Stable Equivalence of Vector Bundles

By the Grothendieck construction, we have seen that the elements of $K(B)$ are essentially differences of isomorphism classes of vector bundles over B. In this section we shall define the reduced K-theory of B, $\widetilde{K}(B)$, of differences of classes of bundles of the same dimension. We also shall introduce the concept of stably equivalent vector bundles over B. And we shall prove that these stable classes represent all of the elements of $\widetilde{K}(B)$, so that here we do not need to take differences.

Abusing notation, we shall denote the image of the isomorphism class of a vector bundle $E \longrightarrow B$ in $K(B)$ again by $[E]$. So every element of $K(B)$ is of the form $[E] - [E']$. However, we should make it clear that $[E] - [E'] = 0 \in K(B)$ does not mean that E and E' are isomorphic, but rather that there exists another bundle E'' such that $E \oplus E'' \cong E' \oplus E''$ (see 9.1.2(b)).

9.3.1 Definition. Let B be a pointed space and $i : \{*\} \longrightarrow B$ the inclusion of the base point. Consider the induced homomorphism

$$i^* : K(B) \longrightarrow K(*) \cong \mathbb{Z}\,.$$

We define the subgroup $\widetilde{K}(B) = \ker(i^* : K(B) \longrightarrow \mathbb{Z})$ of $K(B)$, which is called the *reduced K-theory* of the pointed space B.

From the definitions it is clear that $i^* : K(B) \longrightarrow K(*) \cong \mathbb{Z}$ is induced by the function that associates to each vector bundle over B the dimension of the bundle over the component containing the base point.

Let $c : B \longrightarrow \{*\}$ be the constant map. Then $c \circ i = \mathrm{id}$. By the functoriality of K we have that $(c \circ i)^* = i^* \circ c^* = \mathbf{1}$, and therefore the exact sequence of abelian groups

$$0 \longrightarrow \widetilde{K}(B) \longrightarrow K(B) \xrightarrow{\quad i^* \quad} K(*) \longrightarrow 0 \tag{9.3.2}$$

splits. And so we have $K(B) \cong \widetilde{K}(B) \oplus K(*) \cong \widetilde{K}(B) \oplus \mathbb{Z}$.

9.3.3 EXERCISE. Prove that $\widetilde{K}$ is a functor from the category of pointed paracompact spaces and pointed maps to the category of abelian groups and homomorphisms such that

$$\text{if } f_0 \simeq f_1 : (B', b_0') \longrightarrow (B, b_0)\,, \text{ then } f_0^* = f_1^* : \widetilde{K}(B) \longrightarrow \widetilde{K}(B')\,.$$

9.3.4 EXERCISE. Prove that the isomorphism $K(B) \cong \widetilde{K}(B) \oplus \mathbb{Z}$ is given by $\langle E, E' \rangle \mapsto (\langle E \oplus \varepsilon^n, E' \oplus \varepsilon^m \rangle,\ m - n)$, where m is the dimension of E and n is the dimension of E' (over the component containing the base point.

Now we shall give another interpretation of the groups $\widetilde{K}(B)$. To do this, we shall need the following lemma, which, even though it is a special case of 8.5.15, we can prove without having to appeal to a Riemannian metric.

9.3.5 **Lemma**. *Let $p : E \longrightarrow B$ be a k-vector bundle, where B is compact. Then there exists a bundle $\overline{p} : \overline{E} \longrightarrow B$ such that $E \oplus \overline{E}$ is isomorphic to a trivial bundle.*

Proof: By 8.1.14 there exists a bundle morphism

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{E}_k(\mathbb{C}^m) \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & \mathrm{G}_k(\mathbb{C}^m) \end{array}$$

such that $E \cong f^*\mathrm{E}_k(\mathbb{C}^m)$. Since B is compact, we can take $m < \infty$. We define an $(m-k)$-vector bundle

$$\overline{\mathrm{E}}_k(\mathbb{C}^m) \xrightarrow{\quad \overline{\pi} \quad} \mathrm{G}_k(\mathbb{C}^m)$$

in the following way:

$$\overline{\mathrm{E}}_k(\mathbb{C}^m) = \{(V, w) \in \mathrm{G}_k(\mathbb{C}^m) \times \mathbb{C}^m \mid w \in V^\perp\} \text{ and } \overline{\pi}(V, w) = V.$$

This is the bundle defined by the map $\overline{\varphi} : \mathrm{G}_k(\mathbb{C}^m) \longrightarrow \mathrm{Pr}(\mathbb{C}^m)$ such that $\overline{\varphi}(V)$ is the orthogonal projection onto the orthogonal complement $V^\perp$ of V in $\mathbb{C}^m$.

Let us consider the following bundle morphisms:

$$\begin{tikzcd}
\mathrm{G}_k(\mathbb{C}^m) \times \mathbb{C}^m \arrow[r, "\overline{\Delta}"] \arrow[d] & \mathrm{E}_k(\mathbb{C}^m) \times \overline{\mathrm{E}}_k(\mathbb{C}^m) \arrow[d, "\pi\times\overline{\pi}"] \\
\mathrm{G}_k(\mathbb{C}^m) \arrow[r, "\Delta"'] & \mathrm{G}_k(\mathbb{C}^m) \times \mathrm{G}_k(\mathbb{C}^m) ,
\end{tikzcd}$$

where Δ is the diagonal map and $\overline{\Delta}(V,z) = ((V,v),(V,w))$ for $z = v + w$, where $v \in V$ and $w \in V^\perp$.

From this we deduce that $\mathrm{E}_k(\mathbb{C}^m) \oplus \overline{\mathrm{E}}_k(\mathbb{C}^m) \cong \mathrm{G}_k(\mathbb{C}^m) \times \mathbb{C}^m = \varepsilon^m$, where, as before, ε^m represents the trivial complex vector bundle of dimension m.

If we define $\overline{E} = f^*\overline{\mathrm{E}}_k(\mathbb{C}^m)$, then

$$\begin{aligned}
E \oplus \overline{E} &\cong f^*\mathrm{E}_k(\mathbb{C}^m) \oplus f^*\overline{\mathrm{E}}_k(\mathbb{C}^m) \\
&\cong f^*(\mathrm{E}_k(\mathbb{C}^m) \oplus \overline{\mathrm{E}}_k(\mathbb{C}^m)) \cong f^*(\varepsilon^m) \cong \varepsilon^m .
\end{aligned}$$

□

Let us recall that a function $f : B \longrightarrow S$, where S is a set, is locally constant if each point $x \in B$ has a neighborhood V such that $f|V$ is constant. If we give S the discrete topology, then $f : B \longrightarrow S$ is locally constant if and only if it is continuous.

If B is compact, then $d_E(B)$ is finite, where d_E is as after Definition 9.2.1. That is,

$$d_E(B) = \{n_1, n_2, \dots, n_r\} ,$$

and B is the disjoint union of subsets B_i that are simultaneously open and closed, and therefore compact. So $B_i = d_E^{-1}(n_i), i = 1, 2, \dots, r$. In this way we can apply the previous lemma to each restriction $p^{-1}(B_i) \longrightarrow B_i$ and obtain a bundle $\overline{p}_i : \overline{E}_i \longrightarrow B_i$ such that $p^{-1}(B_i) \oplus \overline{E}_i$ is trivial. Moreover, adding appropriate trivial bundles ε_i, we can arrange that all of the bundles $p^{-1}(B_i) \oplus \overline{E}_i \oplus \varepsilon_i$, $1 \le i \le r$, have the same dimension. If we define $\overline{p} : \overline{E} \longrightarrow X$ such that $\overline{p}^{-1}(B_i) = \overline{E}_i \oplus \varepsilon_i$, then $E \oplus \overline{E} \cong \varepsilon$, where ε is a trivial bundle. And so we have proved the following result.

9.3.6 **Proposition**. *Let $E \longrightarrow B$ be a vector bundle, where B is compact. Then there exists a bundle $\overline{E} \longrightarrow B$ such that $E \oplus \overline{E}$ is isomorphic to a trivial bundle.* □

9.3.7 DEFINITION. We say that the vector bundles $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ are *stably equivalent* if there exist trivial bundles ε and ε' such that $E \oplus \varepsilon \cong E' \oplus \varepsilon'$.

This is clearly an equivalence relation, and we denote by $\mathcal{S}(B)$ the set of stable classes of bundles over B. Denote by $\{E\}$ the stable class of E. We can give $\mathcal{S}(B)$ the structure of an abelian semigroup by defining $\{E\} + \{E'\} = \{E \oplus E'\}$. The zero is the class of any trivial bundle ε over B. By proposition 9.3.6 we have that each element of $\mathcal{S}(B)$ has an inverse, and so $\mathcal{S}(B)$ is an abelian group.

9.3.8 **Theorem**. *Let B be a pointed compact space. Then $\widetilde{K}(B) \cong \mathcal{S}(B)$.*

Proof: Let $[E]$ be the isomorphism class of a bundle over B. We define a homomorphism of semigroups $\rho : \mathrm{Vect}(B) \longrightarrow \mathcal{S}(B)$ by $\rho[E] = \{E\}$. Since $\mathcal{S}(B)$ is an abelian group, using the universal property of the Grothendieck construction there exists a homomorphism $\overline{\rho} : K(B) \longrightarrow \mathcal{S}(B)$ that makes the diagram

$$\begin{tikzcd}
\mathrm{Vect}(B) \arrow[r, "\rho"] \arrow[d, "\alpha"'] & \mathcal{S}(B) \\
K(B) \arrow[ur, "\overline{\rho}"'] &
\end{tikzcd}$$

commute.

We shall show that $\overline{\rho}|\widetilde{K}(B)$ is an isomorphism. In fact, take $\{E\} \in \mathcal{S}(B)$ and let us suppose that over the component containing the base point, E has dimension k. Let ε^k be the trivial bundle of dimension k. Then we have $[E] - [\varepsilon^k] \in \widetilde{K}(B)$ and $\overline{\rho}([E] - [\varepsilon^k]) = \rho[E] - \rho[\varepsilon^k] = \{E\} - \{\varepsilon^k\} = \{E\}$, and therefore $\overline{\rho}|\widetilde{K}(B)$ is an epimorphism. Now let $[E] - [E'] \in \widetilde{K}(B)$ be an element whose image under $\overline{\rho}$ is 0. Then it follows that $0 = \overline{\rho}([E] - [E']) = \rho[E] - \rho[E'] = \{E\} - \{E'\}$; that is, $\{E\} = \{E'\}$. Hence, there exist trivial bundles $\varepsilon^m, \varepsilon^n$ of dimensions m and n, respectively, such that $E \oplus \varepsilon^m \cong E' \oplus \varepsilon^n$. But the dimensions of E and E' coincide over the component of the base point, and so $m = n$. Finally, by the Grothendieck construction (see 9.1.2(b)), it follows that $[E] = [E'] \in \widetilde{K}(B)$ and $[E] - [E'] = 0$. □

9.3.9 EXERCISE. (a) Prove that if B is a disjoint union of open subspaces $B_1 \sqcup B_2 \sqcup \cdots \sqcup B_r$, then $K(B) \cong K(B_1) \oplus K(B_2) \oplus \cdots \oplus K(B_r)$.

(b) The previous statement is not true for $\widetilde{K}(B)$. Give a counterexample. What would be the correct formulation in the reduced case?

9.3.10 NOTE. When B is not connected one might imagine that one could study $K(B)$ in terms of the K-theory of its connected components. However, the connected components in general are not open in B (unless, for example, B is locally connected).

9.4 REPRESENTATIONS OF $K(B)$ AND $\widetilde{K}(B)$

In the following we shall see how to express $K(B)$ and $\widetilde{K}(B)$ in terms of homotopy, when B is compact. In order to do this we shall give another decomposition of $K(B)$, which will coincide with $K(B) \cong \widetilde{K}(B) \oplus \mathbb{Z}$ when B is connected.

As we mentioned in the proof of 9.3.6, we have that $\{f : B \longrightarrow \mathbb{N} \mid f$ is locally constant$\} = M(B, \mathbb{N})$, where $\mathbb{N}$ has the discrete topology. Moreover, it is clear that $M(B, \mathbb{N}) = [B, \mathbb{N}]$.

9.4.1 DEFINITION. Let $d : \mathrm{Vect}(B) \longrightarrow [B, \mathbb{N}]$ be the function defined by $d[E] = d_E$ for any vector bundle $p : E \longrightarrow B$, where $d_E(x)$ is the dimension of the fiber $p^{-1}(x)$ over $x \in B$. Since $\mathbb{N}$ is a semigroup, $[B, \mathbb{N}]$ has the structure of a semigroup in such a way that d is a homomorphism of semigroups. Let $\alpha : [B, \mathbb{N}] \longrightarrow [B, \mathbb{Z}]$ be the canonical inclusion. By the universal property of the Grothendieck construction we get a homomorphism $\overline{d} : K(B) \longrightarrow [B, \mathbb{Z}]$ that makes the diagram

$$\begin{tikzcd} \mathrm{Vect}(B) \arrow[r, "d"] \arrow[d, "\alpha"'] & {[B,\mathbb{N}]} \arrow[d, "\alpha"] \\ K(B) \arrow[r, "\overline{d}"'] & {[B,\mathbb{Z}]} \end{tikzcd}$$

commute. Notice that $\alpha : [B, \mathbb{N}] \longrightarrow [B, \mathbb{Z}]$ is, of course, the Grothendieck construction for the semigroup $[B, \mathbb{N}]$. We shall denote $\ker(\overline{d})$ by $\widehat{K}(B)$.

9.4.2 **Proposition**. *The sequence*

$$0 \longrightarrow \widehat{K}(B) \hookrightarrow K(B) \xrightarrow{\overline{d}} [B, \mathbb{Z}] \longrightarrow 0$$

is exact and splits. Consequently, we have $K(B) \cong \widehat{K}(B) \oplus [B, \mathbb{Z}]$.

Proof: Take $f : B \longrightarrow \mathbb{N}$. Since B is compact, $f(B)$ is finite. Then $f(B) = \{n_1, n_2, \ldots, n_r\}$, and B can be expressed as a disjoint union of open sets $B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_r$, where $B_i = f^{-1}(n_i)$. We define a bundle over B by taking the trivial bundle ε^{n_i} over each B_i. This defines a homomorphism of semigroups $\varphi : [B, \mathbb{N}] \longrightarrow \mathrm{Vect}(B)$, and clearly $d \circ \varphi = \mathrm{id}$. By the universal property of the Grothendieck construction there exists a homomorphism $\overline{\varphi} : [B, \mathbb{Z}] \longrightarrow K(B)$ such that $\overline{d} \circ \overline{\varphi} = \mathrm{id}$. $\square$

9.4.3 **Corollary**. *If B is connected, then $\widetilde{K}(B) \cong \widehat{K}(B)$.*

Proof: An element $\langle [E], [E'] \rangle \in K(B)$ is in $\widetilde{K}(B)$ if and only if $\dim p^{-1}(*) = \dim p'^{-1}(*)$, where $* \in B$ is the base point. On the other hand, $\langle [E], [E'] \rangle$ is in $\widehat{K}(X)$ if and only if $\dim p^{-1}(x) = \dim p'^{-1}(x)$ for all $x \in B$. Using the exact sequences from (9.3.2) and 9.4.2, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{K}(B) \arrow[r] \arrow[d] & K(B) \arrow[r, "\overline{d}"] \arrow[d, "\mathrm{id}"] & {[B, \mathbb{Z}]} \arrow[r] \arrow[d, "i^{\#}"] & 0 \\
0 \arrow[r] & \widetilde{K}(B) \arrow[r] & K(B) \arrow[r, "\delta"'] & {[*, \mathbb{Z}]} \arrow[r] & 0,
\end{tikzcd}$$

where δ associates to each bundle the dimension of the fiber over $*$ and $i : \{*\} \hookrightarrow B$. If B is connected, then $i^{\#}$ is an isomorphism, and so $\widehat{K}(B) \cong \widetilde{K}(B)$. $\square$

In the following we shall describe $\widehat{K}$ in terms of homotopy, and using this, we shall obtain the desired expressions for K and $\widetilde{K}$.

9.4.4 DEFINITION. Let us consider the sets $\mathrm{Vect}_k(B)$, $k \geq 0$, of complex vector bundles of dimension k. By adding a trivial bundle of dimension one, we can define functions $t_k : \mathrm{Vect}_k(B) \longrightarrow \mathrm{Vect}_{k+1}(B)$, namely, $t_k[E] = [E \oplus \varepsilon^1]$, $k \geq 0$.

Let us denote by $\mathrm{Vect}^s(B)$ the colimit

$$\mathrm{Vect}^s(B) = \operatorname{colim} \{\mathrm{Vect}_k(B), t_k\}_{k \geq 0} \,.$$

Using the Whitney sum we define

$$\mathrm{Vect}_k(B) \times \mathrm{Vect}_l(B) \longrightarrow \mathrm{Vect}_{k+l}(B)$$

by $([E], [E']) \mapsto [E \oplus E']$, $k, l \geq 0$. This allows us to define a sum $\mathrm{Vect}^s(B) \times \mathrm{Vect}^s(B) \longrightarrow \mathrm{Vect}^s(B)$ that gives $\mathrm{Vect}^s(B)$ the structure of an abelian semigroup.

9.4.5 EXERCISE. Prove that if B is compact, then $[E_1] - [F_1] = [E_2] - [F_2]$ in $K(B)$ if and only if there exists a trivial bundle ε such that $E_1 \oplus F_2 \oplus \varepsilon \cong E_2 \oplus F_1 \oplus \varepsilon$ (cf. Definition 9.1.1).

9.4.6 **Proposition**. *Let B be a compact space. Then we have* $\mathrm{Vect}^s(B) \cong \widehat{K}(B)$.

Proof: For each $k \geq 0$, we define $\varphi_k : \mathrm{Vect}_k(B) \longrightarrow \widehat{K}(B)$ by $\varphi_k[E] = [E] - [\varepsilon^k] \in \widehat{K}(B)$. We then have $\varphi_{k+1}t_k[E] = \varphi_{k+1}[E \oplus \varepsilon^1] = [E \oplus \varepsilon^1] - [\varepsilon^{k+1}] = [E] + [\varepsilon^1] - [\varepsilon^k] - [\varepsilon^1] = [E] - [\varepsilon^k] = \varphi_k[E]$. Therefore, by the universal property of colimits, there exists $\varphi : \mathrm{Vect}^s(B) \longrightarrow \widehat{K}(B)$ that makes the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k(B) \arrow[r] \arrow[dr, "\varphi_k"'] & \mathrm{Vect}^s(B) \arrow[d, "\varphi"] \\
& \widehat{K}(B)
\end{tikzcd}$$

commute for every k.

We shall prove that φ, which is a homomorphism of semigroups, is an epimorphism and a monomorphism. In particular, this will show that $\mathrm{Vect}^s(B)$ is a group. Take $[E]-[E'] \in \widehat{K}(B)$. Using 9.3.5, there exists a bundle $\overline{E}'$ such that $E' \oplus \overline{E}' \cong \varepsilon^n$ for some n. Then we have $[E]-[E'] = [E]+[\overline{E}'-[E' \oplus \overline{E}'] = [E] + [\overline{E}'] - [\varepsilon^n] = [E \oplus \overline{E}'] - [\varepsilon^n]$. Since $[E] - [E'] \in \widehat{K}(B) = \ker \overline{d}$, it follows that $d[E \oplus \overline{E}'] = d[\varepsilon^n]$; that is, $E \oplus \overline{E}'$ has constant dimension equal to n. From this we obtain $\varphi_n[E \oplus \overline{E}'] = [E] - [E']$, and so we have proved that φ is surjective.

Next let us suppose that $[E] - [\varepsilon^k] = [E'] - [\varepsilon^l]$ in $\widehat{K}(B)$. Then, using 9.4.5, we have that $E \oplus \varepsilon^{l+n} \cong E' \oplus \varepsilon^{k+n}$ for some n. Therefore, $[E]$ and $[E']$ represent the same element in $\mathrm{Vect}^s(B)$, and so φ is injective. □

9.4.7 EXERCISE. Prove the following statements:

(a) Take $X = \mathrm{colim}\, X_n$, where the maps $X_n \longrightarrow X_{n+1}$ are embeddings. Then the maps $X_n \longrightarrow X$ are embeddings.

(b) Let $X = \bigcup_{i\geq 0} X_i$ be a Hausdorff space, where $X_i \subset X_{i+1}$ is closed, $i \geq 0$. If X has the topology induced by the family $\{X_i\}_{i\geq 0}$ (that is, $F \subset X$ is closed $\Leftrightarrow$ $F \cap X_i$ is closed in X_i, for each i), then for every compact $C \subset X$ there exists an $n \geq 0$ such that $C \subset X_n$.

9.4.8 **Theorem**. *Let B be a compact space. Then it follows that* $\widehat{K}(B) \cong [B, \mathrm{BU}]$.

Proof: According to 9.4.6, we have

$$\widehat{K}(B) \cong \mathrm{Vect}^s(B) = \mathrm{colim}\, \mathrm{Vect}_k(B)\,,$$

where the colimit is taken with respect to the maps

$$t_k : \mathrm{Vect}_k(B) \longrightarrow \mathrm{Vect}_{k+1}(B)$$

given by $t_k[E] = [E \oplus \varepsilon^1]$.

On the other hand, we have $\mathrm{BU} = \operatorname{colim} \mathrm{BU}_k$, where the colimit is taken with respect to the embeddings $i_k : \mathrm{BU}_k \longrightarrow \mathrm{BU}_{k+1}$ that satisfy $i_k^*(\mathrm{E}_{k+1}(\mathbb{C}^\infty)) \cong \mathrm{E}_k(\mathbb{C}^\infty) \oplus \varepsilon^1$. Since B is compact, we then have by Exercise 9.4.7(b) that $[B, \mathrm{BU}] \cong \operatorname{colim}[B, \mathrm{BU}_k]$, where $i_{k\#} : [B, \mathrm{BU}_k] \longrightarrow [B, \mathrm{BU}_{k+1}]$ is induced by i_k.

Using 8.5.13 we have that $\mathrm{Vect}_k(B) \cong [B, \mathrm{BU}_k]$. Clearly, these equivalences are compatible with the functions t_k and $i_{k\#}$, $k \geq 0$. So they induce an isomorphism $\operatorname{colim}[B, \mathrm{BU}_k] \cong \operatorname{colim} \mathrm{Vect}_k(B)$. □

9.4.9 **Corollary**. *Let B be a compact space. Then:*

(a) $K(B) \cong [B, \mathrm{BU} \times \mathbb{Z}]$.

(b) $\widetilde{K}(B) \cong [B, \mathrm{BU}]$, *provided that B is connected.*

Proof: (a) Using 9.4.2, we have $K(B) \cong \widehat{K}(B) \oplus [B, \mathbb{Z}]$. Also, by 9.4.8, we obtain $\widehat{K}(B) \cong [B, \mathrm{BU}]$. Then it follows that $K(B) \cong [B, \mathrm{BU}] \oplus [B, \mathbb{Z}] = [B, \mathrm{BU} \times \mathbb{Z}]$.

(b) Since B is connected, using 9.4.3 we get $\widetilde{K}(B) \cong \widehat{K}(B)$. And since $\widehat{K}(B) \cong [B, \mathrm{BU}]$, we obtain the desired result. □

9.4.10 Remark. The results of the previous corollary are equally true if one assumes B to be a finite-dimensional CW-complex. This follows from the fact that then every path component of B can be covered with a finite number of open sets that are contractible in B (see 5.1.30).

9.5 Bott Periodicity and Applications

The following theorem, known as the Bott periodicity theorem, is the central result of K-theory. The original proof due to Bott uses Morse theory to analyze the loop space of a Lie group. Even though there are other methods for proving it, all of the proofs are rather difficult. See, for example, [32], which also appears in the collection of articles compiled by J. Frank Adams [4]. A quite complete list of proofs of the Bott theorem is given in [7].

We shall postpone our proof until Appendix B, since the methods that we use, even though only topological and linear in character, are intrincate and would pull us away from the main line of our presentation. Nevertheless, we shall use the version of the theorem that we are about to present in order to calculate the homotopy groups of BU and therefore the K-theory of spheres.

9.5.1 **Theorem**. (Bott periodicity) *There exists a homotopy equivalence*

$$\mathrm{BU} \times \mathbb{Z} \simeq \Omega^2 \mathrm{BU}\,. \qquad \square$$

From this we deduce that

$$\pi_{i+2}(\mathrm{BU}) \cong \pi_i(\Omega^2\mathrm{BU}) \cong \pi_i(\mathrm{BU}\times\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \pi_i(\mathrm{BU}) & \text{if } i \geq 1. \end{cases}$$

This means that the homotopy groups of BU repeat with period two. And this is the reason for the name "periodicity theorem."

From the above we obtain $\pi_2(\mathrm{BU}) \cong \mathbb{Z}$, and from the periodicity we get $\pi_{2n}(\mathrm{BU}) \cong \mathbb{Z}, n \geq 1$. Moreover, since BU is connected, we have $\pi_0(\mathrm{BU}) = 0$. In order to obtain the odd groups we use the following equality.

9.5.2 **Proposition**. $\pi_i(\mathrm{BU}_k) \cong \pi_i(\mathrm{BU}_{k+1})$ *if* $i < 2k+1$.

This result is proved by applying the exact homotopy sequence of a certain fibration $p : \mathrm{BU}_k \longrightarrow \mathrm{BU}_{k+1}$ with fiber $\mathbb{S}^{2k+1}$. Here we are using the notation BU_k to denote a space with the same homotopy type as $\mathrm{G}_k(\mathbb{C}^\infty)$. $\square$

Using 9.5.2 we obtain $\pi_i(\mathrm{BU}_k) \cong \pi_i(\mathrm{BU})$ if $i < 2k+1$. In particular, $\pi_1(\mathrm{BU}) \cong \pi_1(\mathrm{BU}_1)$. But $\mathrm{BU}_1 = \mathrm{G}_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$, and so $\pi_1(\mathrm{BU}) = 0$, and then, by periodicity, $\pi_{2n+1}(\mathrm{BU}) = 0$, $n \geq 0$.

Therefore, we have the following statement.

9.5.3 **Theorem**.

$$\pi_i(\mathrm{BU}) = \begin{cases} 0 & \textit{if } i = 0, \\ \mathbb{Z} & \textit{if } i > 0 \textit{ is even}, \\ 0 & \textit{if } i > 0 \textit{ is odd}. \end{cases} \qquad \square$$

9.5.4 **Corollary**.

$$\widetilde{K}(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & \textit{if } n \textit{ is even}, \\ 0 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof: If $n = 0$, then $K(\mathbb{S}^0) \cong \widetilde{K}(\mathbb{S}^0) \oplus \mathbb{Z}$ by using (9.3.2). And using 9.3.9(a), we have $K(\mathbb{S}^0) \cong K(*) \oplus K(*) \cong \mathbb{Z} \oplus \mathbb{Z}$. Consequently, $\widetilde{K}(\mathbb{S}^0) \cong \mathbb{Z}$.

If $n > 0$, then $\mathbb{S}^n$ is connected, and according to 9.4.9(b) we have $\widetilde{K}(\mathbb{S}^n) = [\mathbb{S}^n, \mathrm{BU}]$. Since BU is an H-space, we get $[\mathbb{S}^n, \mathrm{BU}] = \pi_n(\mathrm{BU})$. So the result now follows from 9.5.3. □

9.5.5 Note. Combining 9.5.2 and 9.5.3 we obtain the following isomorphisms:

$$\pi_i(\mathrm{BU}_k) \cong \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i \text{ is even and positive, for } i < 2k+1, \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

One can prove that $\Omega \mathrm{BU}_k \simeq \mathrm{U}_k$. So the previous result gives us the homotopy groups U_k in the appropriate range, and then 9.5.3 gives us the homotopy groups of U.

With the help of periodicity, we can extend the functor K to a whole family of functors K^n, $n \in \mathbb{Z}$, which will combine to form a generalized cohomology theory (see 12.1) satisfying all axioms that cohomology satisfies (see 7.1) except dimension. Although periodicity implies that there are essentially only two functors in this theory, viewing the whole family of them as a generalized cohomology theory often facilitates matters.

9.5.6 Definition. Let X be a pointed CW-complex. Then we define

$$\widetilde{K}^0(X) = [X, \mathrm{BU} \times \mathbb{Z}]_*$$

and

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X), \quad n \in \mathbb{N} \cup \{0\}.$$

If $A \subset X$ is closed, we define

$$K^{-n}(X, A) = \widetilde{K}^{-n}(X \cup CA).$$

9.5.7 Note. Applying Corollary 9.4.9 one has $\widetilde{K}(X) \cong \widetilde{K}^0(X)$ if X is compact.

From (3.3.11) we obtain the long exact sequence

$$\begin{aligned} \cdots \longrightarrow [\Sigma^n(X \cup CA), \mathrm{BU}] \longrightarrow [\Sigma^n X, \mathrm{BU}] \longrightarrow [\Sigma^n A, \mathrm{BU}] \longrightarrow \\ \cdots \longrightarrow [X \cup CA, \mathrm{BU}] \longrightarrow [X, \mathrm{BU}] \longrightarrow [A, \mathrm{BU}]. \end{aligned} \tag{9.5.7}$$

The previous sequence can be rewritten as the exact sequence

(9.5.8)
$$\begin{aligned}\cdots \xrightarrow{\delta^{q-1}} K^{-n}(X,A) \xrightarrow{i^*} K^{-n}(X) \xrightarrow{i^*} K^{-n}(A) \xrightarrow{\delta^q} \\ \longrightarrow K^{-n+1}(X,A) \longrightarrow \cdots \,,\end{aligned}$$

known as the *long exact sequence in K-theory of the pair* (X,A).

9.5.9 EXERCISE. Prove that the assignment $(X,A) \mapsto K^{-n}(X,A)$ is a functor from the category whose objects are pairs of paracompact spaces and closed subspaces and whose morphisms are maps of pairs to the category of abelian groups and homomorphisms such that if $f_0 \simeq f_1 : (Y,B) \longrightarrow (X,A)$, then $f_0^* = f_1^* : K^{-n}(X,A) \longrightarrow K^{-n}(Y,B)$ for all n. That is, K^{-n} is *homotopy invariant.*

9.5.10 EXERCISE. Let X be paracompact and let $U \subset X$ be open and $A \subset X$ be closed such that $\overline{U} \subset \overset{\circ}{A}$. Prove that the inclusion map $i : (X - U, A - U) \hookrightarrow (X,A)$ induces an isomorphism

$$i^* : K^{-n}(X,A) \xrightarrow{\cong} K^{-n}(X-U, A-U)\,.$$

That is, K^{-n} has an *excision* property.

9.5.11 REMARK. Let $A \subset X$ be a closed subspace of the paracompact space X. The last portion of the long exact sequence (9.5.7) transforms into the exact sequence

$$\widetilde{K}(X \cup CA) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)$$

and the other portions into the exact sequences

$$\widetilde{K}^{-n}(X \cup CA) \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow \widetilde{K}^{-n}(A)\,.$$

These imply the *exactness* property of the reduced K-theory.

Another immediate consequence of the Bott periodicity theorem is the following result.

9.5.12 **Theorem**. *$K^{-n}(X,A) \cong K^{-n+2}(X,A)$ if $n \geq 2$.* □

This result allows us to extend the notation $K^n(X,A)$ to every integer n. From (9.5.8) and 9.5.12 we deduce the following proposition.

9.5.13 **Proposition**. *If X is compact and $A \subset X$ is closed, then we have the following exact hexagon:*

$$\begin{tikzcd}
 & K^0(X) \arrow[dr, "i^*"] & \\
K^0(X,A) \arrow[ur, "j^*"] & & K^0(A) \arrow[d] \\
K^{-1}(A) \arrow[u, "\delta"] & & K^{-1}(X,A) \arrow[dl, "j^*"] \\
 & K^{-1}(X) \arrow[ul, "i^*"] &
\end{tikzcd}$$

$\square$

9.5.14 Exercise. Deduce from the Bott periodicity theorem that $K(X \times \mathbb{S}^2)$ has the structure of a free module over the ring $K(X)$ with two generators. These are $\mathbf{1}$, the class of the trivial bundle of dimension 1, and $[L] - \mathbf{1}$. Here L is the bundle induced by $\mathrm{proj}_2 : X \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ from the canonical bundle $H^* \longrightarrow \mathbb{S}^2$, considering $\mathbb{S}^2$ as $\mathbb{CP}^1$, the Riemann sphere. The module structure is given by

$$\begin{aligned} K(X) \otimes K(X \times \mathbb{S}^2) &\longrightarrow K(X \times \mathbb{S}^2)\,, \\ \xi \otimes \rho &\longmapsto \mathrm{proj}_2^*(\xi) \cdot \rho\,, \end{aligned}$$

where, as we have noted, the product $\cdot$ in $K(X \times \mathbb{S}^2)$ is given by the tensor product of vector bundles.

We have treated in this chapter only the complex case, using complex vector bundles, complex Grassmann manifolds, unitary groups U_n, etc. We can repeat the analysis for the real case (real vector bundles, real Grassmann manifolds, orthogonal groups O_n, etc.) and so obtain real K-theory of a space B, usually denoted by $KO(B)$. Its representation is obtained in terms of the spaces BO_k (instead of BU_k) and BO (instead of BU). Nonetheless, the periodicity results are very different. The periodicity in the complex case is of period 2, while in the real case it is of period 8.

9.5.15 **Theorem**. (Real Bott periodicity) *There exists a homotopy equivalence* $\mathrm{BO} \times \mathbb{Z} \simeq \Omega^8\mathrm{BO}$. $\square$

For the *proof* of this theorem, we refer to [15], where similar methods to ours are used.

9.5.16 Note. Using some homotopic properties of the groups O_n, corresponding to Theorem 9.5.3, one can prove that

$$\pi_{i+8}(\mathrm{BO}) \cong \pi_i(\Omega^8\mathrm{BO}) \cong \pi_i(\mathrm{BO}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } i = 1, 2, \\ 0 & \text{if } i = 3, 5, 6, 7, \\ \mathbb{Z} & \text{if } i = 4, 8. \end{cases}$$

This means, in particular, that the homotopy groups of BO repeat with period eight.

9.5.17 Exercise. Define $\widetilde{KO}^{-n}(X) = [\Sigma^n X, \mathrm{BO} \times \mathbb{Z}]_*$, so that for any compact pointed space X, $\widetilde{KO}(X) \cong \widetilde{KO}^0(X)$. Prove that $\widetilde{KO}^{-n}(X) \cong \widetilde{KO}^{-n+8}(X)$ for every pointed CW-complex X. Compute $\widetilde{KO}^{-n}(\mathbb{S}^q)$ for all $q \geq 0$.

9.5.18 Note. Among the major achievements of (topological) K-theory we have the following: the solution of the vector field problem of spheres by Adams, where he computes the maximal number of linearly independent sections in the tangent bundle of a sphere (see [2]), the short proof of the Hopf conjecture that we shall present in Chapter 10 (see 10.6.15), and the index theorem for elliptic differential operators by Atiyah and Singer (see [14]).

In another direction it is possible to define K-theory for the so-called C^*-algebras. By analyzing noncommutative C^*-algebras and their K-theory, Connes [23] studied important aspects of what is now known as noncommutative geometry. This K-theory has been generalized by Kasparov [38], who defined groups $KK(A,B)$ for each pair of C^*-algebras A, B. He used this theory in his work on the Novikov conjecture concerning the homotopy invariants of higher signatures.

Other applications will be mentioned at the end of Chapter 12.

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Chapter 10

Adams Operations and Applications

In this chapter we shall define the important Adams operations in complex K-theory and see how they are applied to prove a central theorem of mathematics, namely, to determine the dimensions n for which $\mathbb{R}^n$ admits the structure of a division algebra.

10.1 Definition of the Adams Operations

Making use of the concept of a formal power series and its properties, which are identical to those of a Taylor series, we introduce in this section the Adams operations in complex K-theory.

10.1.1 Definition. An *operation* θ in K-theory assigns a function (in general, not a homomorphism) $\theta_X : K(X) \longrightarrow K(X)$ to each space X in such a way that for every map $f : X \longrightarrow Y$, the diagram (of sets)

$$\begin{tikzcd} K(Y) \arrow[r, "\theta_Y"] \arrow[d, "f^*"'] & K(Y) \arrow[d, "f^*"] \\ K(X) \arrow[r, "\theta_X"'] & K(X) \end{tikzcd}$$

is commutative; that is, an operation θ is a *natural transformation.*

10.1.2 Note. In order to simplify notation we shall suppress the subindex that represents the space. So we shall denote θ_X simply by θ.

In what follows we shall construct certain operations that will be the basis for the applications that we make of K-theory. In order to do this, we need the following definitions.

10.1.3 Definition. Let R be a commutative ring with 1. We shall denote by $R[[t]]$ the *ring of formal power series* with coefficients in R. That is, the elements of $R[[t]]$ are expressions of the form $\sum_{i\geq 0} r_i t^i$, where $r_i \in R$, $i \geq 0$. The sum is defined by

$$\left(\sum_{i\geq 0} r_i t^i\right) + \left(\sum_{i\geq 0} r_i' t^i\right) = \sum_{i\geq 0} (r_i + r_i') t^i$$

and the product by

$$\left(\sum_{i\geq 0} r_i t^i\right)\left(\sum_{j\geq 0} r_j' t^j\right) = \sum_{k\geq 0} r_k'' t^k,$$

where

$$r_k'' = \sum_{i+j=k} r_i r_j' .$$

The element $1 \in R$ is clearly the unit of $R[[t]]$ when we take this to mean the series with $r_0 = 1$ and $r_i = 0$ for $i > 0$.

Put

$$1 + tR[[t]] = \left\{ \sum_{i\geq 0} r_i t^i \in R[[t]] \mid r_0 = 1 \right\}.$$

Clearly, the product in $R[[t]]$ can be restricted to $1 + tR[[t]]$, and moreover, every element in $1+tR[[t]]$ has an inverse. Namely, if $1+\sum_{i\geq 1} r_i t^i \in 1+tR[[t]]$, then its multiplicative inverse is $1 + \sum_{i\geq 1} \overline{r}_i t^i$, where $\overline{r}_1 = -r_1$, $\overline{r}_2 = r_1^2 - r_2$, $\overline{r}_3 = -r_1^3 + 2r_1r_2 - r_3$, and in general,

$$\overline{r}_n = \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!} (-r_1)^{i_1} \cdots (-r_n)^{i_n}.$$

This shows that $1 + tR[[t]]$ is an abelian group under multiplication.

The ring $R[[t]]$ of formal power series behaves like the ring of power series with real or complex coefficients in analysis. We can differentiate formal power series term by term; namely,

$$\frac{d}{dt} \sum_{i=0}^{\infty} r_i t^i = \sum_{i=1}^{\infty} i r_i t^{i-1}.$$

We can define the standard analytic functions sin, cos, log, exp, and so forth, by the usual Taylor series formulas. They then will satisfy equations analogous to those of analysis. As an example, if $x(t) = \sum_{i=0}^{\infty} r_i t^i$, we can define $\log x(t)$. And we can calculate its derivative and thereby get the formula

$$\frac{d}{dt}(\log x(t)) = x'(t)(x(t))^{-1},$$

which is defined, for example, if the constant term in $x(t)$ is 1.

10.1.4 DEFINITION. Let $E \longrightarrow X$ be a vector bundle with X compact. We define the formal power series $\lambda_t[E] \in K(X)[[t]]$ by

$$\lambda_t[E] = \sum_{i=0}^{\infty} \left[\bigwedge^i E\right] t^i,$$

where $\bigwedge^i E$ is the ith exterior power of E (see 8.1.4). Using the isomorphism mentioned in 8.1.15(g),

$$\bigwedge^k (E \oplus E') \cong \bigoplus_{i+j=k} \left(\bigwedge^i E \otimes \bigwedge^j E'\right),$$

we obtain the formula

$$\lambda_t[E \oplus E'] = \lambda_t[E]\lambda_t[E']. \tag{10.1.5}$$

Because the constant term in $\lambda_t[E]$ is 1, we have that $\lambda_t[E] \in 1 + tK(X)[[t]]$, and so $\lambda_t[E]$ is invertible.

So we have a homomorphism

$$\lambda_t : \mathrm{Vect}(X) \longrightarrow 1 + tK(X)[[t]]$$

from the additive semigroup $\mathrm{Vect}(X)$ of the isomorphism classes of complex vector bundles over X to the multiplicative group of formal power series over $K(X)$ with constant term 1. By the universal property of the Grothendieck construction, this homomorphism can be extended to

$$\lambda_t : K(X) \longrightarrow 1 + tK(X)[[t]].$$

Taking the coefficient of t^i in $\lambda_t(x)$, $x \in K(X)$, we get *operations*

$$\lambda^i : K(X) \longrightarrow K(X),$$

such that $\lambda_t(x) = 1 + \sum_{i \geq 1} \lambda^i(x) t^i$. Explicitly, since the elements of $K(X)$ can be expressed as differences $[E] - [E']$, we have

$$\lambda_t([E] - [E']) = \lambda_t[E]\lambda_t[E']^{-1}.$$

10.1.6 Definition. The *rank operation*

$$\text{rank} : K(X) \longrightarrow K(X)$$

is defined as follows. As in the proof of 9.3.6 we know that if $E \longrightarrow X$ is a vector bundle, then $X = \bigcup_{i=1}^r X_i$, where each X_i is open and $E|X_i$ has constant dimension n_i. We define a bundle $r(E) \longrightarrow X$ such that $r(E)|X_i = \varepsilon^{n_i}$, i.e., the product bundle on X_i of dimension n_i. This defines a homomorphism of semirings $r : \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X)$, $r([E]) = [r(E)]$, which, by the universal property of the Grothendieck construction, induces the operation $\text{rank} : K(X) \longrightarrow K(X)$. For the sake of clarity, let us note that if X is locally connected, its connected components are both open and closed and the bundle $r(E) \longrightarrow X$ is trivial over each component with dimension equal to that of E over said component.

10.1.7 Definition. We define the *Adams operations*

$$\psi^i : K(X) \longrightarrow K(X)$$

as follows. First we define

$$\psi^0(x) = \text{rank } (x) \,.$$

Then in the ring $K(X)[[t]]$ we define $\psi_t(x) = \sum_{i=0}^{\infty} \psi^i(x)t^i$ by

$$\psi_t(x) = \psi^0(x) - t\frac{d}{dt}(\log \lambda_{-t}(x)) \,,$$

where the second term is t times the formal derivative of the formal logarithm of the series $\lambda_{-t}(x)$, that is

$$\psi_t(x) = \psi^0(x) + \frac{\lambda'_{-t}(x)}{\lambda_{-t}(x)} t \,.$$

Using the formal properties of the logarithm we can prove the following result.

10.1.8 **Proposition**. *For all $x, y \in K(X)$ the following are true:*

(a) $\psi^k(x+y) = \psi^k(x) + \psi^k(y), \quad k = 0, 1, 2, \ldots$.

(b) *If $x = [L]$, where $L \longrightarrow X$ is a bundle of dimension* 1*, then $\psi^k(x) = x^k$.*

(c) *The properties* (a) *and* (b) *characterize the operations ψ^k.*

Proof: Using (10.1.5) we deduce that $\lambda_{-t}(x+y) = \lambda_{-t}(x)\lambda_{-t}(y)$. Consequently,

$$\begin{aligned}\psi_t(x+y) &= \psi^0(x+y) - t\frac{d}{dt}(\log\,\lambda_{-t}(x+y))\\ &= \text{rank}\,(x+y) - t\frac{d}{dt}(\log\,(\lambda_{-t}(x)\lambda_{-t}(y)))\\ &= \text{rank}\,(x) + \text{rank}\,(y) - t\frac{d}{dt}(\log\,(\lambda_{-t}(x)) + \log\,(\lambda_{-t}(y))\\ &= \psi_t(x) + \psi_t(y)\,.\end{aligned}$$

This proves (a).

To prove (b), we note that if $x = [L]$ is the class of a line bundle L (i.e., of dimension 1), then $\lambda_{-t}(x) = 1 - tx$, because $\bigwedge^i(L) = 0$ if $i > 1$. Therefore,

$$\frac{d}{dt}(\log\,(1-tx)) = \frac{-x}{1-tx} = -x - tx^2 - t^2x^3 - \cdots\,.$$

So $\psi_t(x) = 1 + tx + t^2x^2 + \cdots$, and from this we get the desired equality.

Statement (c) is obtained from the "splitting principle," which we shall encounter later on (see 10.2.5). □

The following theorem will be very important in the present chapter.

10.1.9 **Theorem**. *For all $x, y \in K(X)$ the following properties hold:*

(a) $\psi^k(xy) = \psi^k(x)\psi^k(y), \quad k = 0, 1, 2, \ldots\,.$

(b) $\psi^k(\psi^l(x)) = \psi^{kl}(x), \quad k, l = 0, 1, 2, \ldots\,.$

(c) p prime $\Rightarrow \psi^p(x) \equiv x^p \bmod p\,.$

(d) *If $b \in \widetilde{K}^0(\mathbb{S}^{2n})$ is a generator, then $\psi^k(b) = k^n b, \quad k = 0, 1, 2, \ldots\,.$*

The proof is an application of 10.1.8 and of the splitting principle, the latter of which we shall study in the following section. □

10.2 The Splitting Principle

The splitting principle is a process that transforms an arbitrary vector bundle to a Whitney sum of line bundles, these being bundles of dimension 1. This thereby permits the simplification of various calculations involving vector bundles. The following definition is fundamental for the splitting principle.

10.2.1 DEFINITION. Let $p : E \longrightarrow X$ be a vector bundle. We define its *associated projective bundle* as the map

$$q : P(E) \longrightarrow X.$$

Here $P(E) = (E - E^0)/\sim$, where E^0 is the zero section of the bundle E and $e \sim e'$ if $p(e) = p(e') \in X$ and there exists $\lambda \in \mathbb{C}$ such that $\lambda e = e'$. If $[e]$ denotes the class of e in $P(E)$, then $q([e]) = p(e)$ is continuous.

10.2.2 EXERCISE. Prove that the projective bundle $q : P(E) \longrightarrow X$ is a locally trivial bundle with fiber $q^{-1}(x)$ homeomorphic, for every $x \in X$, to the complex projective space associated to the vector space $p^{-1}(x)$. (Hint: Over each open subset of X over which $p : E \longrightarrow X$ is trivial, q is trivial as well.)

10.2.3 DEFINITION. We define the *tautological line bundle* or the *canonical bundle* $\pi : L \longrightarrow P(E)$ as follows. Define

$$L = \{(e', [e]) \in E \times P(E) \mid p(e') = p(e), e' = \lambda e, \lambda \in \mathbb{C}\}$$

and let π be the projection onto the second coordinate. This is clearly a vector bundle of dimension 1, that is, a *line bundle*. Actually, if $\varphi : X \longrightarrow \mathrm{Pr}(\mathbb{C}^m)$ is the map that defines E, namely, if $E = \{(x, v) \in X \times \mathbb{C}^m \mid \varphi(x)v = v\}$, then $L \longrightarrow P(E)$ is the subbundle of $q^*(E)$ associated to

$$\begin{aligned} \psi : P(E) &\longrightarrow \mathrm{Pr}(\mathbb{C}^m), \\ [e] &\longmapsto (\mathbb{C}^m \xrightarrow{\varphi(x)} \mathbb{C}^m \xrightarrow{\pi_v} \mathbb{C}^m), \end{aligned}$$

where $e = (x, v) \in X \times \mathbb{C}^m$, $\varphi(x)v = v$, and π_v is the orthogonal projection onto the line $\langle v \rangle$ generated by v ($v \neq 0$).

10.2.4 **Proposition**. *Let $p : E \longrightarrow X$ be a vector bundle and $q : P(E) \longrightarrow X$ its associated projective bundle. Then $q^*(E) = E' \oplus L$, where $L \longrightarrow P(E)$ is the tautological bundle.*

Proof: Let $E' \longrightarrow P(E)$ be the vector bundle associated to

$$\begin{aligned} \psi' : P(E) &\longrightarrow \mathrm{Pr}(\mathbb{C}^m), \\ [e] &\longmapsto (\mathbb{C}^m \xrightarrow{\varphi(x)} \mathbb{C}^m \xrightarrow{\pi'_v} \mathbb{C}^m), \end{aligned}$$

where, as before, $e = (x, v) \in X \times \mathbb{C}^m$ and $\varphi(x)v = v$, and now π'_v is the orthogonal projection onto the orthogonal complement of $\langle v \rangle$ in $\varphi(x)\mathbb{C}^m$.

Since any element in $\varphi(x)\mathbb{C}^m$ has a unique expression of the form $w + w'$ with $w \in \langle v \rangle = \psi(x)\mathbb{C}^m$ and $w' \in \pi'_v\varphi(x)\mathbb{C}^m = \psi'(x)\mathbb{C}^m$, we have the desired splitting. $\square$

Using the periodicity theorem one can prove [13, 2.7.9] that $K(P(E))$ is a free module over the ring $K(X)$ with generators $1, 1-[L], (1-[L])^2, \ldots, (1-[L])^{k-1}$, where $k = \dim E$, with respect to the $K(X)$-module structure given by $K(X) \otimes K(P(E)) \longrightarrow K(P(E))$ such that $\xi \otimes \rho \mapsto q^*(\xi) \cdot \rho$. In particular, we deduce from this that $q^* : K(X) \longrightarrow K(P(E))$ is a monomorphism (which includes $K(X)$ as the part generated by $1 \in K(P(E))$).

10.2.5 **Theorem**. (Splitting principle) *Given a vector bundle $p : E \longrightarrow X$ of dimension k there exists a map $f : F \longrightarrow X$ such that*

(a) *$f^* : K(X) \longrightarrow K(F)$ is a monomorphism, and*

(b) *the induced bundle satisfies $f^*(E) = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, where $L_i \longrightarrow F$ is a line bundle, $i = 1, 2, \ldots, k$.*

Proof: According to 10.2.4, $q^*(E) = E' \oplus L$. Put $L_k = L$ and apply 10.2.4 once more, only now to $E' \longrightarrow P(E)$. Then $q_1 : P(E') \longrightarrow P(E)$ is such that $q_1^*(E') = E'' \oplus L'$. Now put $L_{k-1} = L'$.

Repeating this process we get $q_{k-2} : P(E^{(k-2)}) \longrightarrow P(E^{(k-3)})$ such that $q_{k-2}^*(E^{(k-2)}) = E^{(k-1)} \oplus L_2$. Defining

$$f = q_{k-1} \circ q_{k-2} \circ \cdots \circ q_1 \circ q : F = P(E^{(k-1)}) \longrightarrow X,$$

we then obtain the desired result by the comments after the proof of 10.2.4. This construction can be visualized in the following diagram:

$$\begin{array}{ccccccc}
E^{(k-1)} \oplus L_2 \oplus \cdots \oplus L_k & \to \cdots \to & E'' \oplus L_{k-1} \oplus L_k & \to & E' \oplus L_k & \to & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P(E^{(k-2)}) & \longrightarrow \cdots \longrightarrow & P(E') & \longrightarrow & P(E) & \longrightarrow & X.
\end{array}$$

Let us note that $L_1 = E^{(k-1)}$ is already a line bundle. □

10.3 NORMED ALGEBRAS

As an example of an application of K-theory, in what follows we shall study a classical theorem of linear algebra. We are going to analyze which of the spaces $\mathbb{R}^n$ admits the structure of a normed algebra.

Even though we have already used the following concept, it is better that we give a precise definition now because of its essential role in this section.

10.3.1 Definition. Let A be a real vector space of finite dimension. A *norm* in A is a function

$$\begin{aligned} A &\longrightarrow \mathbb{R}^+ = [0,\infty)\,, \\ x &\longmapsto \|x\|\,, \end{aligned}$$

such that

$$\begin{aligned} \|x+y\| &\leq \|x\| + \|y\|\,, \quad x,y \in A\,, \\ \|\lambda x\| &= |\lambda|\|x\|\,, \quad \lambda \in \mathbb{R}\,, \quad x \in A\,, \\ \|x\| &= 0 \Leftrightarrow x = 0\,. \end{aligned}$$

A *normed algebra* is a real vector space of finite dimension equipped with a bilinear multiplication

$$\begin{aligned} A \times A &\longrightarrow A\,, \\ (x,y) &\longmapsto xy, \end{aligned}$$

with unit $1 \in A$ such that $1x = x1 = x$ (which makes it an *algebra*) and equipped with a norm such that

$$\|xy\| = \|x\|\|y\|$$

(which makes it *normed*).

10.3.2 Examples. The following are normed algebras:

(a) $A = \mathbb{R}$, $\|x\| = |x|$, $x \in \mathbb{R}$, with the usual multiplication on $\mathbb{R}$.

(b) $A = \mathbb{R}^2$, $\|z\| = \sqrt{x_1^2 + x_2^2}$, $z = x_1 + x_2\mathrm{i}$, $x_i \in \mathbb{R}$, $1 = (1,0)$, $\mathrm{i} = (0,1)$, with the multiplication of complex numbers on $\mathbb{R}^2 = \mathbb{C}$. If $\overline{z} = x_1 - x_2\mathrm{i}$, then $\|z\|^2 = z\overline{z}$.

(c) $A = \mathbb{R}^4$, $\|q\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$, $q = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}$, $x_i \in \mathbb{R}$, $1 = (1,0,0,0)$, $\mathrm{i} = (0,1,0,0)$, $\mathrm{j} = (0,0,1,0)$, $\mathrm{k} = (0,0,0,1)$, with the multiplication of the *quaternions* on $\mathbb{R}^4 = \mathbb{H}$. The multiplication is determined by $\mathrm{ij} = \mathrm{k}$, $\mathrm{jk} = \mathrm{i}$, $\mathrm{ki} = \mathrm{j}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$. If $\overline{q} = x_1 - x_2\mathrm{i} - x_3\mathrm{j} - x_4\mathrm{k}$, then $\|q\|^2 = q\overline{q}$. We can see that $q = z_1 + z_2\mathrm{j}$, with $z_1 = x_1 + x_2\mathrm{i}$, $z_2 = x_3 + x_4\mathrm{i} \in \mathbb{C}$. So $\overline{q} = \overline{z}_1 - z_2\mathrm{j}$ and the multiplication rules in $\mathbb{H}$ are obtained from those of $\mathbb{C}$, provided that we carefully mind the order of the factors. ($\mathbb{H}$ is an associative algebra, but it is not commutative.)

(d) $A = \mathbb{R}^8$, $\|c\| = \sqrt{x_1^2 + \cdots + x_8^2}$, $c = (x_1, \ldots, x_8)$, with the multiplication of the *Cayley numbers* (or *octonians*) on $\mathbb{R}^8 = \mathbb{O}$. This multiplication is obtained by considering $c = (q_1, q_2)$ with $q_1 = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}$, $q_2 = x_5 + x_6\mathrm{i} + x_7\mathrm{j} + x_8\mathrm{k} \in \mathbb{H}$, and by then defining $cc' = (q_1, q_2)(q_1', q_2') = (q_1 q_1' - \overline{q}_2' q_2, q_2' q_1 + q_2 \overline{q}_1')$. This multiplication has $(1, 0) \in \mathbb{H} \times \mathbb{H} = \mathbb{O}$ as unit. We define $\overline{c} = \overline{(q_1, q_2)} = (\overline{q}_1, -q_2)$, and so $\|c\|^2 = c\overline{c}$. ($\mathbb{O}$ is a nonassociative algebra.)

10.3.3 EXERCISE. Write out in coordinates the multiplication of $\mathbb{O} = \mathbb{R}^8$.

10.3.4 EXERCISE. Prove that the canonical inclusions

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

(the last being $q \mapsto (q, 0)$) are multiplicative and send 1 to 1; that is, the product of $\mathbb{O}$ restricts to those of $\mathbb{H}$, $\mathbb{C}$, and $\mathbb{R}$. In other words, these inclusions are *algebra homomorphisms.*

10.3.5 EXERCISE. Verify that the multiplication rule for the complex numbers in terms of the real numbers is the same as that of the quaternions in terms of the complex numbers and that of the octonians in terms of the quaternions. (Use $x = \overline{x}$, if $x \in \mathbb{R}$.)

10.3.6 EXERCISE. Starting with the multiplication on $\mathbb{O}$, can we define a multiplication on $\mathbb{R}^{16}$ such that it becomes a normed algebra?

10.3.7 EXERCISE. Show that the multiplications on $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ actually turn them into normed algebras.

So we have the following result.

10.3.8 **Theorem**. *If $n = 1, 2, 4, 8$, then $\mathbb{R}^n$ has the structure of a normed algebra.* $\square$

10.4 DIVISION ALGEBRAS

In 1900 A. Hurwitz proved algebraically the converse of Theorem 10.3.8; namely, the only values of n for which $\mathbb{R}^n$ admits the structure of a normed algebra are precisely $n = 1, 2, 4, 8$. We shall prove this converse in what follows. As part of this we shall give some definitions, make some historical comments, and present other equivalent results.

10.4.1 DEFINITION. A *division algebra* is an algebra A over $\mathbb{R}$ such that

$$xy = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0.$$

10.4.2 **Proposition**. *Let A be an associative algebra of finite dimension. Then A is a division algebra if and only if for all $x \neq 0$ in A there exists a unique x' in A such that $xx' = x'x = 1$, in other words, if and only if the elements different from zero in A form a group under multiplication.*

Proof: Assume that $x \neq 0$ and that there exists x' such that $xx' = x'x = 1$ and moreover that $xy = 0$. Then we have $x'(xy) = (x'x)y = y = 0$. The symmetric case follows similarly.

Conversely, suppose that $x \neq 0$. Since A has finite dimension, the sequence $\{1, x, x^2, x^3, \ldots, x^m, \ldots\}$ does not form a linearly independent set. So for some m we have

$$x^m + \sum_{i=0}^{m-1} \alpha_i x^i = 0.$$

Let m be the smallest integer with this property. This polynomial of minimal degree is clearly unique, since if there were two such, we would be able to decrease m. If $\alpha_0 = 0$ were true, then we would have

$$x\left(x^{m-1} + \sum_{i=1}^{m-1} \alpha_i x^{i-1}\right) = 0\,,$$

which would contradict the minimality of m, since A is a division algebra. So $x' = -\alpha_0^{-1}\left(x^{m-1} + \sum_{i=1}^{m-1} \alpha_i x^{i-1}\right)$ is an inverse for x. □

10.4.3 EXERCISE. Prove that in an algebra A, if $a \in A$ satisfies $ax = 0 \Rightarrow x = 0$, then there exists a unique $a' \in A$ such that $aa' = 1$.

10.4.4 **Theorem**. *If $\mathbb{R}^n$ has the structure of a normed algebra, then $\mathbb{R}^n$ with this structure is a division algebra.*

Proof: $xy = 0 \Rightarrow 0 = \|xy\| = \|x\|\|y\| \Rightarrow \|x\| = 0$ or $\|y\| = 0 \Rightarrow x = 0$ or $y = 0$. □

10.5 Multiplicative Structures on $\mathbb{R}^n$ and on $\mathbb{S}^{n-1}$

Around 1900 the following question was posed: For which values of n is $\mathbb{R}^n$ a division algebra? In 1960 J.F. Adams [1], making heavy use of the machinery of homology theory, proved that the values of n are precisely those of Hurwitz, that is, $n = 1, 2, 4, 8$. What we shall present here are essentially results due to Adams and M.F. Atiyah in [5], where Adams' original proof is simplified.

Recall that an *H-space* is a space X equipped with a map $\mu : X \times X \longrightarrow X$, called the *multiplication*, and an element $e \in X$, called the *unit*, such that $\mu(e, x) = x = \mu(x, e)$. (Cf. 2.7.2. Here we are requiring that the unit be *strict*, namely that the relations $\mu(e, x) = x = \mu(x, e)$ hold as strict equalities and not just as relations up to homotopy. This is not a big restriction, since when the pointed space (X, e) is *well pointed*, which means that the inclusion $\{e\} \hookrightarrow X$ is a cofibration, then this definition is equivalent to 2.7.2; *exercise*. The condition of being well pointed holds in many important examples as well as in all of those that we are going to consider from now on.)

10.5.1 **Proposition**. *If $\mathbb{R}^n$ has the structure of a normed algebra, then $\mathbb{S}^{n-1}$ inherits the structure of an H-space.*

The *proof* is an immediate consequence of the following lemma. $\square$

10.5.2 **Lemma**. *Assume that $\mathbb{R}^n$ has the structure of a normed algebra with norm $\|\cdot\|$. Then $\Sigma = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is homeomorphic to $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, where $|\cdot|$ is the usual norm.*

Proof: The map $\varphi : \mathbb{S}^{n-1} \longrightarrow \Sigma$, defined by $\varphi(x) = x/\|x\|$, is continuous, since $x \mapsto \|x\|$ is continuous. Its inverse is $\psi : \Sigma \longrightarrow \mathbb{S}^{n-1}$, $\psi(x) = x/|x|$. $\square$

10.5.3 Exercise. Prove that the map $x \mapsto \|x\|$ in the previous proof is actually continuous.

10.5.4 Exercise. Prove that if $\mathbb{R}^n$ has the structure of a division algebra, then $\mathbb{S}^{n-1}$ inherits the structure of an H-space. (Hint: First prove that $\mathbb{R}^n - 0$ with the restriction of the multiplication on $\mathbb{R}^n$ is an H-space.)

10.5.5 Definition. The sphere $\mathbb{S}^{n-1}$ is *parallelizable* if its tangent bundle $T(\mathbb{S}^{n-1}) = \{(x, y) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \mid \langle x, y\rangle = 0\} \longrightarrow \mathbb{S}^{n-1}$ is trivial, where $\langle -, -\rangle$

represents the usual scalar product in $\mathbb{R}^n$. (This means that this bundle is isomorphic to the bundle $\mathbb{S}^{n-1} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{S}^{n-1}$.)

This definition is equivalent to saying that there exist $n-1$ tangent *vector fields* on $\mathbb{S}^{n-1}$ that are linearly independent.

10.5.6 **Theorem**. *If $\mathbb{R}^n$ has the structure of a division algebra, then $\mathbb{S}^{n-1}$ is parallelizable.*

Proof: Choose a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that $e_1 = 1$. Take $x \in \mathbb{S}^{n-1}$ and define

$$v_i(x) = xe_i - \langle x, xe_i \rangle x, \quad i \geq 2.$$

Then we have $\langle x, v_i(x) \rangle = 0$, and so $(x, v_i(x)) \in T(\mathbb{S}^{n-1})$. (This means that v_i is a tangent vector field on $\mathbb{S}^{n-1}$.) Since

$$\{1, e_2, \ldots, e_n\}$$

is a linearly independent set, so also is

$$\{x, xe_2, \ldots, xe_n\}.$$

Thus the vectors $v_2(x), \ldots, v_n(x)$ are linearly independent. Consequently, $\varphi : \mathbb{S}^{n-1} \times \mathbb{R}^{n-1} \longrightarrow T(\mathbb{S}^{n-1})$ given by

$$\varphi(x, (t_2, \ldots, t_n)) = (x, t_2 v_2(x) + \cdots + t_n v_n(x))$$

is the isomorphism we are seeking. $\square$

10.5.7 **Theorem**. *If $\mathbb{S}^{n-1}$ is parallelizable, then it has the structure of an H-space.*

Proof: Consider the composite

$$\nu : \mathbb{S}^{n-1} \times \mathbb{R}^{n-1} \xrightarrow{\varphi} T(\mathbb{S}^{n-1}) \xrightarrow{\psi} \mathbb{S}^{n-1} ,$$

where φ is a trivialization of the tangent bundle and $\psi(x, y)$ is defined for $(x, y) \in T(\mathbb{S}^{n-1})$ by

$$\psi(x, y) = \frac{4}{4 + |y|^2}(2x + y) - x \,.$$

It is easy to check that $\psi(x, y) \in \mathbb{S}^{n-1}$. Figure 10.1 depicts the definition of ψ.

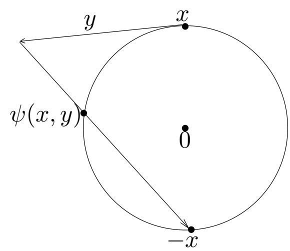

Figure 10.1

Clearly, if $y \to \infty$, then $\psi(x, y) \to -x$. So if $\Sigma^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^{n-1}$, then ν can be extended to a map

$$\nu^* : \mathbb{S}^{n-1} \times \Sigma^{n-1} \longrightarrow \mathbb{S}^{n-1}$$

such that $\nu^*(x, \infty) = -x$. Taking a fixed element e in $\mathbb{S}^{n-1}$, we get a homeomorphism $\eta : \Sigma^{n-1} \longrightarrow \mathbb{S}^{n-1}$ such that $\eta(y) = \nu^*(e, y)$ and $\eta(\infty) = -e$. Then η^{-1} is the stereographic projection from $-x$. The composite

$$\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \xrightarrow{\mathrm{id} \times \eta^{-1}} \mathbb{S}^{n-1} \times \Sigma^{n-1} \xrightarrow{\nu^*} \mathbb{S}^{n-1}$$

is a multiplication with unit e that converts $\mathbb{S}^{n-1}$ into an H-space. (Note that $\nu^*(x, 0) = x$.) $\square$

10.6 The Hopf Invariant

In the following we are going to associate an integer, known as the Hopf invariant, to each element in the homotopy group $\pi_{2n-1}(\mathbb{S}^n)$. The role that this invariant will play in the present chapter is illustrated by the following diagram of implications, which gives a historical outline of the problem we are treating as well as all of the various interrelationships that it has to other properties. This diagram appeared in the article by J.F. Adams [1] mentioned earlier.

Assume that $n > 1$. Then the following holds.

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{R}^n \text{ is a normed algebra over the reals} \arrow[r, Leftrightarrow, "(1)"] \arrow[d, Rightarrow, "(8)"] & n = 2, 4, \text{ or } 8 \\
 & \mathbb{R}^n \text{ is a division algebra over the reals} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "(9)"] \arrow[ddd, Rightarrow, bend right] & n = 2^m \\
 & \left.\begin{array}{c} \mathbb{S}^{n-1}, \text{ with its usual differentiable} \\ \text{structure, is parallelizable} \end{array}\right\} \arrow[r, Rightarrow, "(7)"] \arrow[d, Rightarrow] & n = 2, 4, \text{ or } 8 \\
 & \begin{array}{c} \mathbb{S}^{n-1}, \text{ with some differentiable structure} \\ \text{(perhaps extraordinary), is parallelizable} \end{array} \arrow[d, Rightarrow, "(6)"] & \\
 & \mathbb{S}^{n-1} \text{ is an } H\text{-space} \arrow[d, Leftrightarrow, "(10)"] & \\
 & \text{There is an element in } \pi_{2n-1}(\mathbb{S}^n) \text{ with Hopf invariant } 1 &
\end{tikzcd}
$$

$$
\begin{array}{cccc}
\Downarrow (3) & \Downarrow (4) & \Downarrow (5) & \Downarrow (2) \\
n = 2 \text{ or } 4r & n = 2^m & n \neq 16 & n = 2, 4, \text{ or } 8\,.
\end{array}
$$

As we have already mentioned, the equivalence (1) was proved in 1900 by Hurwitz using algebraic methods. (We have already proved the trivial implication $\Leftarrow$.)

Implication (2), which closes the circle and makes all the statements equivalent, was proved by Adams in [1]. Implications (3), (4), and (5) are particular cases proved by G.W. Whitehead [81], J. Adem [6], and H. Toda [77], respectively. Adem used the Adem relations in his proof, while Toda used in his proof an elegant lemma from homotopy theory as well as extensive calculations of homotopy groups of spheres.

Implication (6) is due to A. Dold and answers a question posed by A. Borel. (It is worth mentioning that Theorem 10.6.11, which we shall prove later, implies strong results about the nonparallelizability of manifolds, as M. Kervaire has proved in [41].)

Implication (7) was independently proved by M. Kervaire [40] and by R. Bott and J. Milnor [18]. In both cases it was deduced from deep results due to Bott [17] concerning the orthogonal groups O_n.

Besides the left implication in (1) (which is Theorem 10.3.8), implication (8) (which is 10.4.4), implication (9) (which is 10.5.6), and implication (6) for the case of the usual differentiable structure (which is Theorem 10.5.5), the program that we have followed here consists in proving Theorem 10.6.10, which is the fundamental result for closing the circle, since it proves equivalence (10).

10.6.1 Definition. The *join* of two topological spaces X and Y, denoted by $X * Y$, is defined by

$$X * Y = X \times I \times Y/\!\sim,$$

where $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$ for every $x, x' \in X$ and $y, y' \in Y$.

10.6.2 Exercise. (a) Prove that $X*Y \approx CX \times Y \cup X \times CY \subset CX \times CY$, where we define here $CZ = Z \times I/Z \times \{1\}$ for any space Z.

(b) Conclude that $\mathbb{S}^{m-1} * \mathbb{S}^{n-1} \approx \mathbb{S}^{m+n-1}$.

10.6.3 Definition. Let $f : X \times Y \longrightarrow Z$ be continuous. The map

$$H(f) : X * Y \longrightarrow \Sigma Z = CZ/Z \times \{0\}$$

given by $H(f)[x, t, y] = [f(x, y), t]$ is called the *Hopf construction* applied to f.

If $\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ is a multiplication, then the Hopf construction induces a map

$$\widetilde{\mu} = H(\mu) : \mathbb{S}^{n-1} * \mathbb{S}^{n-1} = \mathbb{S}^{2n-1} \longrightarrow \Sigma\mathbb{S}^{n-1} = \mathbb{S}^{n}\,.$$

10.6.4 Definition. Given $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ we define an integer $h(f)$, called the *Hopf invariant* of f, as follows. In the case that n is odd, we define

$$h(f) = \begin{cases} 0 & \text{if } n = 2m + 1,\ m > 0, \\ 1 & \text{if } n = 1. \end{cases}$$

When n is even, we consider the short exact sequence

$$0 \longrightarrow \widetilde{K}^0(\mathbb{S}^{2n}) \xrightarrow{\ \rho^*\ } \widetilde{K}^0(C_f) \xrightarrow{\ i^*\ } \widetilde{K}^0(\mathbb{S}^n) \longrightarrow 0\,,$$

which we obtain by applying (3.3.13), where $i : \mathbb{S}^n \hookrightarrow C_f$ is the canonical inclusion and $\rho : C_f \twoheadrightarrow \mathbb{S}^{2n} = \Sigma\mathbb{S}^{2n-1}$ is the canonical quotient map, since according to 9.4.9(b), $\widetilde{K}(X) = [X, BU]$ for every compact, connected pointed space X. Since $\widetilde{K}(\Sigma X) = \widetilde{K}^{-1}(X)$, it follows from $\widetilde{K}^{-1}(\mathbb{S}^{2n}) = 0 = \widetilde{K}^{-1}(\mathbb{S}^n)$ that the exact sequence is indeed short.

On the other hand, $\widetilde{K}^0(\mathbb{S}^{2n}) \cong \mathbb{Z} \cong \widetilde{K}^0(\mathbb{S}^n)$. Let $b_n \in \widetilde{K}^0(\mathbb{S}^n)$ be a generator. Then there exists $u \in \widetilde{K}^0(C_f)$ that is a generator satisfying $i^*(u) = b_n$. However, $i^*(u^2) = (i^*(u))^2 = 0$, since all the squares in $\widetilde{K}^0(\mathbb{S}^n)$ are zero. Therefore, there exists a unique $y \in \widetilde{K}^0(\mathbb{S}^{2n})$ such that $\rho^*(y) = u^2$. If $v = \rho^*(b_{2n})$, then we define $h(f)$ by

$$u^2 = h(f)v \qquad (\text{or} \quad y = h(f)b_{2n})\,,$$

where $b_{2n} \in \widetilde{K}^0(\mathbb{S}^{2n})$ is the generator that satisfies $b_{2n} = b_n \times b_n$. We claim that $h(f)$ does not depend on u. To see this, let u' be such that $i^*(u') = b_n$. Then $i^*(u' - u) = 0$, and so $u' - u = \rho^*(\lambda b_{2n})$ for some $\lambda \in \mathbb{Z}$. Consequently,

$$u' = u + \rho^*(\lambda b_{2n}) = u + \lambda v, \quad v = \rho^*(b_{2n})\,,$$

and

$$(u')^2 = u^2 + 2\lambda uv + \lambda^2 v^2 = u^2,$$

since $v^2 = i^*(b_{2n}^2) = 0$ and $uv = 0$.

10.6.5 Exercise. Fill in the details in the definition of $h(f)$. In particular, prove that all of the squares x^2 for $x \in \widetilde{K}^0(\mathbb{S}^n)$ are zero.

10.6.6 Exercise. Show that $f \simeq g \Rightarrow h(f) = h(g)$.

10.6.7 Definition. Let $\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ be a continuous map, $n > 1$. By choosing $e \in \mathbb{S}^{n-1}$ we have maps given as follows:

$$\begin{aligned} \mu_1 &: \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}, \quad \mu_1(x) = \mu(x, e), \\ \mu_2 &: \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}, \quad \mu_2(x) = \mu(e, x)\,. \end{aligned}$$

These maps are independent of e, up to homotopy, since $\mathbb{S}^{n-1}$ is path connected. We define the *bidegree* of μ as

$$\text{bidegree } (\mu) = (\text{degree } (\mu_1), \text{degree } (\mu_2)),$$

where the *degree* of μ_i is the integer that corresponds to $[\mu_i] \in \pi_{n-1}(\mathbb{S}^{n-1})$ under the isomorphism $\pi_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ given by the correspondence $[\mathrm{id}_{\mathbb{S}^{n-1}}] \leftrightarrow 1$. In other words, the homomorphism $\mu_i^* : \pi_{n-1}(\mathbb{S}^{n-1}) \longrightarrow \pi_{n-1}(\mathbb{S}^{n-1})$ is multiplication by degree (μ_i).

10.6.8 Remark. If $g : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ has degree p and $n > 1$ is odd, then $g^* : \widetilde{K}(\mathbb{S}^{n-1}) \longrightarrow \widetilde{K}(\mathbb{S}^{n-1})$ is multiplication by p. If n is even, then $(\Sigma g)^* : \widetilde{K}(\mathbb{S}^n) \longrightarrow \widetilde{K}(\mathbb{S}^n)$ is also multiplication by p.

10.6.9 **Theorem**. *Let n be even. If $\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ has bidegree (p, q), then the Hopf invariant of $f = H(\mu) : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ is equal to $p \cdot q$.*

Proof: Let us consider S_1 and S_2, each one of the factors of the product $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, as the boundaries of the n-dimensional balls B_1 and B_2, respectively. We can take B_i to be the quotient of $S_i \times I$ by the relation that identifies $S_i \times \{1\}$ to a point.

Let S^n_+ and S^n_- be the upper and lower hemispheres of $\mathbb{S}^n$. These consist of the points $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$ such that $x_{n+1} \geq 0$ and $x_{n+1} \leq 0$, respectively.

From μ we obtain maps $f_1 : S_1 \times B_2 \longrightarrow S^n_+$ given by $(x, y, t) \mapsto ((\sqrt{1-t^2})\mu(x, y), t)$ for $t \in I$ and $f_2 : B_1 \times S_2 \longrightarrow S^n_-$ given by $(x, t, y) \mapsto ((\sqrt{1-t^2})\mu(x, y), -t)$. Clearly, $S_1 \times B_2 \cup B_1 \times S_2$ is homeomorphic to $S_1 * S_2 = \mathbb{S}^{2n-1}$. Also, f_1 and f_2 determine $f : S_1 \times B_2 \cup B_1 \times S_2 \longrightarrow S^n_+ \cup S^n_-$, which coincides with $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ under the homeomorphism.

With this description of f, the mapping cone C_f is the quotient of $Z = (B_1 \times B_2) \cup \mathbb{S}^n$ by the relation that identifies $(x, y) \in \partial(B_1 \times B_2) = S_1 \times B_2 \cup B_1 \times S_2$ with $g(x, y) \in \mathbb{S}^n$. We denote by $f_0 : B_1 \times B_2 \longrightarrow C_f$ the restriction of the quotient map. Note that $\mathbb{S}^n$ (and thus S^n_+ and S^n_-) are subspaces of C_f in a natural way. Let

$$g = (f_0, f_1, f_2) : (B_1 \times B_2, S_1 \times B_2, B_1 \times S_2) \longrightarrow (C_f, S^n_+, S^n_-)$$

be the corresponding map of triples.

Therefore, we have an isomorphism

$$g^* : K(C_f, S^n_+ \cup S^n_-) \longrightarrow K((B_1, S_1) \times (B_2, S_2)),$$

since the corresponding restriction of g is a relative homeomorphism (that is, it defines a homeomorphism of the complements).

Now, if

$$g_1 : (B_1 \times B_2, S_1 \times B_2) \longrightarrow (C_f, S^n_+),$$
$$g_2 : (B_1 \times B_2, B_1 \times S_2) \longrightarrow (C_f, S^n_-),$$

are restrictions of g, then we have that the composite

$$\varphi_1 : \widetilde{K}(C_f) \cong K(C_f, *) \cong K(C_f, S^n_+) \xrightarrow{g_1^*} K((B_1, S_1) \times B_2) \cong K(B_1, S_1) \cong \widetilde{K}(\mathbb{S}^n)$$

has the property that if $u \in \widetilde{K}(C_f)$ is the generator such that $i^*(u) = b_n \in \widetilde{K}(\mathbb{S}^n)$ (see 10.6.4), then $\varphi_1(u) = pb_n$. Analogously, the composite

$$\begin{aligned}\varphi_2 : \widetilde{K}(C_f) &\cong K(C_f,*) \cong K(C_f, S^n_-)\\ &\xrightarrow{g_2^*} K(B_1 \times (B_2, S_2)) \cong K(B_2, S_2) \cong \widetilde{K}(\mathbb{S}^n)\end{aligned}$$

satisfies $\varphi_2(u) = qb_n$.

We can take generators

$$b'_n \in K((B_1,S_1) \times B_2) \quad \text{and} \quad b''_n \in K(B_1 \times (B_2 \times S_2))$$

such that they correspond to b_n under the isomorphisms and such that $b'_n \smile b''_n$ corresponds to b_{2n} under the respective isomorphism. We have the commutative diagram

$$\begin{tikzcd}
\widetilde{K}(C_f)\otimes\widetilde{K}(C_f) \arrow[r, "\smile"] & \widetilde{K}(C_f) & \\
K(C_f,S^n_+)\otimes K(C_f,S^n_-) \arrow[u, "\cong"] \arrow[r, "\smile"] \arrow[d, "\varphi'_1\otimes\varphi'_2"'] & K(C_f,\mathbb{S}^n) \arrow[u] \arrow[d, "\cong"] & \\
K((B_1,S_1)\times B_2)\otimes K(B_1\times(B_2,S_2)) \arrow[r, "\smile"] & K(B_1\times B_2, S_1\times B_2\cup B_1\times S_2) & \\
 & \widetilde{K}(\mathbb{S}^{2n}), \arrow[u, "\cong"'] \arrow[uuu, bend right=60, "\rho^*"'] &
\end{tikzcd}$$

where $\smile$ denotes the (interior) product in $\widetilde{K}$ induced by $\otimes$ in Vect (that is, by the tensor product of vector bundles) and φ'_1 and φ'_2 correspond to φ_1 and φ_2 under the isomorphisms. So, chasing through the diagram starting with $u \otimes u$, we have

$$\begin{tikzcd}
u \otimes u \arrow[r, mapsto] \arrow[d, mapsto] & u^2 & = & pqv \\
(pb_n) \otimes (qb_n) \arrow[rrr, mapsto] & & & pqb_{2n}\,, \arrow[u, mapsto]
\end{tikzcd}$$

which yields $u^2 = pqv$ and consequently $h(f) = pq$. $\square$

10.6.10 **Proposition**. *Let $n > 1$ be odd and let*

$$\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$$

have bidegree (p,q). Then $pq = 0$.

Proof: We know that in K-theory we have

$$K(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \cong K(\mathbb{S}^{n-1}) \otimes K(\mathbb{S}^{n-1}) \cong (\mathbb{Z} \oplus \mathbb{Z}u) \otimes (\mathbb{Z} \oplus \mathbb{Z}v)\,,$$

where u and v are generators of $\widetilde{K}(\mathbb{S}^{n-1})$ in the first and second factors, respectively. If we write $K(\mathbb{S}^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}w$, then

$$\mu^* : \mathbb{Z} \oplus \mathbb{Z}w \longrightarrow (\mathbb{Z} \oplus \mathbb{Z}u) \otimes (\mathbb{Z} \oplus \mathbb{Z}v)$$

sends w to an element of the form $pu \otimes 1 + 1 \otimes qv + s(u \otimes v)$. Because μ^* is a homomorphism of rings, we have that $0 = w^2$ leads to $(pu \otimes 1 + 1 \otimes qv + s(u \otimes v))^2 = 2pq(u \otimes v)$, since squares are zero. Therefore $pq = 0$. □

From 10.6.9 and 10.6.10 we get the next result.

10.6.11 **Theorem**. *If $\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ is an H-space multiplication, then $f = H(\mu)$ has Hopf invariant $h(f) = 1$.*

Proof: Note that bidegree $(\mu) = (1, 1)$ and so n is even according to 10.6.10. Then using 10.6.9 we have $h(f) = 1$. □

Now we shall prove the theorem that closes the circle of implications described at the beginning of this section.

10.6.12 **Theorem**. *Suppose that $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ has odd Hopf invariant. Then $n = 2, 4,$ or 8.*

Proof: Assume that $n = 2r$. (Note that n cannot be odd by definition). Let b_{2n}, b_n, u, and v be as before, which can be expressed in a diagram as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}(\mathbb{S}^{2n}) & \xrightarrow{p^*} & \widetilde{K}(C_f) & \xrightarrow{i^*} & \widetilde{K}(\mathbb{S}^n) & \longrightarrow & 0\,, \\ & & b_{2n} & \longmapsto & v, \quad u & \longmapsto & b_n\,. & & \end{array}$$

Using the naturality of the Adams operations we see that

$$\begin{aligned} \psi^k(v) &= p^*\psi^k(b_{2n}) \\ &= p^*(k^{2r}b_{2n}) \qquad \text{(by 10.1.9(d))} \\ &= k^{2r}v\,. \end{aligned} \tag{10.6.13}$$

On the other hand, we also have that

$$\begin{aligned} i^*(\psi^k(u) - k^r u) &= \psi^k(b_n) - k^r b_n \\ &= k^r b_n - k^r b_n \qquad \text{(by 10.1.9(d))} \\ &= 0\,. \end{aligned}$$

So we obtain

$$\psi^k(u) - k^r u = \sigma(k)v, \quad \sigma(k) \in \mathbb{Z}\,. \tag{10.6.14}$$

However, using 10.1.9(c) we have

$$\psi^2(u) \equiv u^2 \bmod 2 \equiv h(f)v \bmod 2\,.$$

Then from (10.6.14) we get

$$\psi^2(u) = 2^r u + \sigma(2)v \equiv h(f)v \bmod 2\,.$$

Consequently, $\sigma(2)$ and $h(f)$ have the same parity, which means that $\sigma(2)$ is odd.

But by 10.1.9(b) we know that $\psi^k\psi^l = \psi^l\psi^k$, and so

$$\begin{aligned} \psi^k\psi^l(u) &= \psi^k(l^r u + \sigma(l)v) \\ &= l^r(k^r u + \sigma(k)v) + \sigma(l)k^{2r}v \\ &= k^r l^r u + (l^r\sigma(k) + k^{2r}\sigma(l))v\,. \end{aligned}$$

Analogously, we obtain

$$\psi^l\psi^k(u) = k^r l^r u + (k^r\sigma(l) + l^{2r}\sigma(k))v\,.$$

Thus we get $k^r\sigma(l) + l^{2r}\sigma(k) = l^r\sigma(k) + k^{2r}\sigma(l)$, which in turn implies $l^r(l^r - 1)\sigma(k) = k^r(k^r - 1)\sigma(l)$.

In particular, if we take $l = 2$ and k odd, we have that

$$2^r(2^r - 1)\sigma(k) = k^r(k^r - 1)\sigma(2)\,.$$

Therefore, since $\sigma(2)$ is odd, $2^r | k^r - 1$ for all odd k. In particular, this holds then for $r = 1$.

Assume that $r > 1$ and consider the group of units $(\mathbb{Z}/2^r)^*$, which has even order. So the congruence $k^r \equiv 1 \bmod 2^r$ implies that r is even, since the order of $(\mathbb{Z}/2^r)^*$ has to divide r. Therefore, $r = 2, 4, 6, 8, \ldots$. If we now take

$$k = 1 + 2^{r/2},$$

then we have that $k^r \equiv 1 + r2^{r/2} \bmod 2^r$, which implies $2^{r/2}|r$, since $2^r|k^r - 1$ and so $2^r|r2^{r/2}$. But this can happen only if $r = 2, 4$, since $r > 4 \Rightarrow 2^{r/2} > r$.

So by the preceding we have $n = 2, 4$, or 8. $\square$

We can summarize all of our results in the next theorem.

10.6.15 **Theorem**. *The following statements are equivalent:*

(a) $n = 1, 2, 4,$ *or* 8.

(b) $\mathbb{R}^n$ *has the structure of a normed algebra.*

(c) $\mathbb{R}^n$ *has the structure of a division algebra.*

(d) $\mathbb{S}^{n-1}$ *is parallelizable or* $n = 1$.

(e) $\mathbb{S}^{n-1}$ *is an* H*-space. (Recall that* $\mathbb{S}^0 = \mathbb{Z}_2$.*)*

(f) *There exists a map* $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ *with Hopf invariant equal to* 1. $\square$

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Chapter 11

Relations Between Cohomology and Vector Bundles

In the present chapter we shall establish some relations between vector bundles over a space and the cohomology of the space. These relations are determined by the *characteristic classes*, which are called the Stiefel–Whitney classes in the case of real vector bundles and are called Chern classes in the complex case. To be more precise, we shall first rely on the fact that $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ are simultaneously Eilenberg–Mac Lane spaces (of type $K(\mathbb{Z}/2, 1)$ and $K(\mathbb{Z}, 2)$, respectively) and Grassmann manifolds (namely, $\mathrm{G}_1(\mathbb{R}^\infty)$ and $\mathrm{G}_1(\mathbb{C}^\infty)$, respectively). Here $\mathrm{G}_1(\mathbb{R}^\infty) = \mathrm{G}_1^{\mathbb{R}}(\mathbb{R}^\infty)$ denotes the Grassmann manifold of **real** one-dimensional subspaces of $\mathbb{R}^\infty$, while $\mathrm{G}_1(\mathbb{C}^\infty) = \mathrm{G}_1^{\mathbb{C}}(\mathbb{C}^\infty)$ denotes the Grassmann manifold of **complex** one-dimensional subspaces of $\mathbb{C}^\infty$. This means that on the one hand these two spaces determine the cohomology functors $H^1(-;\mathbb{Z}/2)$ and $H^2(-;\mathbb{Z})$, while on the other hand they classify real and complex line bundles, denoted functorially by $\mathrm{Vect}_1^{\mathbb{R}}$ and $\mathrm{Vect}_1^{\mathbb{C}}$. In this way we shall define the first Stiefel–Whitney class and the first Chern class.

Later on we shall introduce the Thom class together with the Thom isomorphism theorem and then construct the absolute and relative Gysin sequences for real and complex bundles. These sequences will be the fundamental tool for constructing the Stiefel–Whitney and Chern classes in dimensions bigger than one.

We shall end the chapter by proving the famous Borsuk–Ulam theorem.

11.1 Contractibility of $\mathbb{S}^\infty$

An important fact in the understanding of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$, the infinite-dimensional projective spaces, is that each of them is obtained as a quotient space of a contractible space, namely the infinite-dimensional sphere $\mathbb{S}^\infty$. In this section we shall prove this.

First recall that $\mathbb{S}^\infty = \operatorname{colim} \mathbb{S}^{n-1} \subset \operatorname{colim} \mathbb{R}^n = \mathbb{R}^\infty$. More precisely, we can describe $\mathbb{R}^\infty$ as the set of sequences of real numbers that are eventually zero, that is, those sequences

$$(x_1, x_2, x_3, \dots, x_k, x_{k+1}, \dots)$$

for which there exists some n such that $x_k = 0$ for all $k > n$. We shall be using the next definition in the following.

11.1.1 Definition. The *infinite-dimensional sphere* $\mathbb{S}^\infty$ is the subspace of $\mathbb{R}^\infty$ containing the sequences $(x_1, x_2, x_3, \dots, x_k, x_{k+1}, \dots)$ satisfying $x_1^2 + x_2^2 + \cdots = 1$. Note that this is a finite sum, since all but finitely many of the x_k are zero.

11.1.2 Note. Topologically speaking, just as $\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$, so we have that $\mathbb{C}^\infty$ is homeomorphic to $\mathbb{R}^\infty$. The difference is that $\mathbb{C}^\infty$ has the structure of a complex vector space, while $\mathbb{R}^\infty$ has the structure of a real vector space. Since there is a commutative diagram

$$\begin{tikzcd} \mathbb{S}^{2n-1} \arrow[r, hook] \arrow[d, hook] & \mathbb{C}^n \arrow[d, hook] \\ \mathbb{S}^{2n+1} \arrow[r, hook] & \mathbb{C}^{n+1}, \end{tikzcd}$$

we can view $\mathbb{S}^\infty$ as the subspace of $\mathbb{C}^\infty$ of eventually zero sequences of complex numbers $(z_1, z_2, \dots)$ satisfying $|z_1|^2 + |z_2|^2 + \cdots = 1$.

11.1.3 **Theorem**. *The infinite-dimensional sphere $\mathbb{S}^\infty$ is contractible.*

Proof: First, consider the map $H : \mathbb{S}^\infty \times I \longrightarrow \mathbb{S}^\infty$ defined for

$$(x_1, x_2, x_3, \dots) \in \mathbb{S}^\infty \quad \text{and} \quad t \in I$$

by

$$H(x_1, x_2, x_3, \dots, t) = ((1-t)x_1, tx_1 + (1-t)x_2, tx_2 + (1-t)x_3, \dots)/N,$$

where the denominator N is the norm of the (nonzero) vector in the numerator, namely,

$$N = \sqrt{((1-t)x_1)^2 + (tx_1 + (1-t)x_2)^2 + (tx_2 + (1-t)x_3)^2 + \cdots}\,.$$

This homotopy clearly starts with the identity $\mathrm{id} : \mathbb{S}^\infty \longrightarrow \mathbb{S}^\infty$ and ends with the map $H_1 : \mathbb{S}^\infty \longrightarrow \mathbb{S}^\infty$ defined by $H_1(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$, whose image is the set $A = \{x \in \mathbb{S}^\infty \mid x_1 = 0\}$.

Let us now define a new homotopy $H' : A \times I \longrightarrow \mathbb{S}^\infty$ by

$$H'(0, x_2, x_3, \dots, t) = (t, (1-t)x_2, (1-t)x_3, \dots)/N',$$

where the denominator N' plays the same role as N did before, namely, it normalizes the (nonzero) vector in the numertor. For $t = 0$ the homotopy H' is the inclusion $A \hookrightarrow \mathbb{S}^\infty$, while for $t = 1$ it is a constant map. The composition of these two homotopies defines the desired contraction. □

11.1.4 EXERCISE.

(a) Prove that the homotopies in the previous proof are well defined and continuous.

(b) Compose those homotopies in order to obtain an explicit homotopy from the identity $\mathrm{id}_{\mathbb{S}^\infty}$ to the constant map $\mathbb{S}^\infty \longrightarrow \mathbb{S}^\infty$ whose value is $(1, 0, 0, \dots)$.

11.1.5 EXERCISE.

(a) Prove that the inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ is nullhomotopic. (Hint: Adapt the homotopies H and H' from the proof of Theorem 11.1.3 to this situation.)

(b) Conclude from part (a) that any map $f : \mathbb{S}^k \longrightarrow \mathbb{S}^n$ is nullhomotopic, provided that $k < n$. (Hint: According to the cellular approximation theorem 5.1.44, f factors up to homotopy through the inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$.)

From the last exercise we get the following important result.

11.1.6 **Corollary**. *$\pi_k(\mathbb{S}^n) = 0$ for $k < n$.* □

11.2 Description of $K(\mathbb{Z}/2,1)$

We shall prove in this section that $\mathbb{RP}^\infty$ is simultaneously homeomorphic to $\mathrm{G}_1(\mathbb{R}^\infty)$ and has the homotopy type of a $K(\mathbb{Z}/2,1)$.

Before starting, it is worthwhile mentioning that the following is a description of $\mathbb{RP}^n$, the real projective space of dimension n.

11.2.1 Definition. Consider the equivalence relation on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ generated by pairs of antipodal points; namely, take the equivalence relation given by $x \sim -x$ for all $x \in \mathbb{S}^n$. Then we define $\mathbb{RP}^n = \mathbb{S}^n/\sim$. Consequently, there is a quotient map

$$p : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$$

whose inverse image of any point in $\mathbb{RP}^n$ is a copy of $\mathbb{S}^0$.

11.2.2 Exercise. Prove that the map $p : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$ defined above is a locally trivial bundle. (Hint: Define $U_i = \{[x] \in \mathbb{RP}^n \mid x_i \neq 0\}$, for $i = 1, 2, \ldots, n+1$. Then $\{U_i\}$ is an open cover of $\mathbb{RP}^n$ and $p|p^{-1}U_i$ is trivial.)

11.2.3 **Proposition**. *There exists a Serre fibration*

$$p : \mathbb{S}^\infty \longrightarrow \mathbb{RP}^\infty$$

with fiber $\mathbb{S}^0$.

Proof: For each n there is a commutative diagram

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, hook] \arrow[d] & \mathbb{S}^{n+1} \arrow[d] \\ \mathbb{RP}^n \arrow[r, hook] & \mathbb{RP}^{n+1} \end{tikzcd}$$

such that the upper horizontal inclusion is a homeomorphism on the fibers $\mathbb{S}^0$.

In the colimit, these inclusions determine a map $p : \mathbb{S}^\infty \longrightarrow \mathbb{RP}^\infty$ whose fibers are $\mathbb{S}^0$. To prove that p is a Serre fibration we have to show that it has the HLP for the cubes I^k. Specifically, we have to show that for any given commutative square

$$\begin{tikzcd} I^k \arrow[r, "h"] \arrow[d, hook, "j_0"'] & \mathbb{S}^\infty \arrow[d, "p"] \\ I^k \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & \mathbb{RP}^\infty \end{tikzcd}$$

there exists a lift $\widetilde{H}$. However, since both I^k and $I^k \times I$ are compact, the images of h and H lie in $\mathbb{S}^n$ and $\mathbb{RP}^n$, respectively, for some n. And this means that we have a commutative diagram

$$\begin{tikzcd} I^k \arrow[r, "h"] \arrow[d, hook, "j_0"'] & \mathbb{S}^n \arrow[d, "p"] \\ I^k \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & \mathbb{RP}^n . \end{tikzcd}$$

Clearly, there exists $\widetilde{H}$ that makes the triangles commute in the last diagram, since $\mathbb{S}^n \longrightarrow \mathbb{RP}^n$ is locally trivial by 11.2.2 and so is a Serre fibration. Then $\widetilde{H} : I^k \times I \longrightarrow \mathbb{S}^n \hookrightarrow \mathbb{S}^\infty$ makes the triangles commute in the first diagram, which proves that p is a Serre fibration. □

For what we shall need in the following it is enough to know that $p : \mathbb{S}^\infty \longrightarrow \mathbb{RP}^\infty$ is a quasifibration, which is true because it is a Serre fibration. Actually, it is even more than a Serre fibration, as we now shall show.

11.2.4 Exercise. Prove that $p : \mathbb{S}^\infty \longrightarrow \mathbb{RP}^\infty$ is a locally trivial bundle and, using the fact that $\mathbb{RP}^\infty$ is paracompact (since it is a CW-complex), deduce that p is really a Hurewicz fibration.

From Proposition 11.2.3 we get the long exact sequence

$$\cdots \longrightarrow \pi_n(\mathbb{S}^\infty) \longrightarrow \pi_n(\mathbb{RP}^\infty) \longrightarrow \pi_{n-1}(\mathbb{S}^0) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_1(\mathbb{S}^\infty) \longrightarrow \pi_1(\mathbb{RP}^\infty) \longrightarrow \pi_0(\mathbb{S}^0) \longrightarrow 0 .$$

Since $\pi_n(\mathbb{S}^\infty) = 0$ for all n (because $\mathbb{S}^\infty$ is contractible) and since $\pi_n(\mathbb{S}^0) = 0$ for all $n \neq 0$ (because $\mathbb{S}^0$ is discrete), we obtain from the previous exact sequence that $\pi_n(\mathbb{RP}^\infty) = 0$ for $n \neq 1$ and that $\pi_1(\mathbb{RP}^\infty) \cong \pi_0(\mathbb{S}^0)$. Since $\pi_0(\mathbb{S}^0)$ contains two elements, it follows that $\pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2$. So we have proved the next result.

11.2.5 **Theorem**. $\mathbb{RP}^\infty$ *is an Eilenberg–Mac Lane space of type* $K(\mathbb{Z}/2,1)$. □

Using Definition 7.1.2 and Theorem 11.2.5 we get the following immediate consequence.

11.2.6 **Corollary**. *For any* CW-*complex* B,

$$[B, \mathbb{RP}^\infty] = H^1(B;\mathbb{Z}/2) .$$ □

The elements of the Grassmann manifold $\mathrm{G}_1(\mathbb{R}^{n+1})$ are the one-dimensional subspaces of $\mathbb{R}^{n+1}$. So we have a bijection between the elements of $\mathrm{G}_1(\mathbb{R}^{n+1})$ and pairs of antipodal points of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. In other words, the map

$$q : \mathbb{S}^n \longrightarrow \mathrm{G}_1(\mathbb{R}^{n+1})$$

defined by $x \mapsto \langle x \rangle$ (where $\langle x \rangle$ denotes as above the real one-dimensional subspace of $\mathbb{R}^{n+1}$ generated by x) is surjective, and for every line $l \in \mathrm{G}_1(\mathbb{R}^{n+1})$ we have $q^{-1}(l) = l \cap \mathbb{S}^n$, which means that $q^{-1}(l)$ consists of a pair of antipodal points of $\mathbb{S}^n$. Since $\mathbb{S}^n$ is compact and $\mathrm{G}_1(\mathbb{R}^{n+1})$ is Hausdorff, q is an identification map, and so there exists a homeomorphism $\varphi : \mathbb{RP}^n \longrightarrow \mathrm{G}_1(\mathbb{R}^{n+1})$ that gives us a commutative triangle

$$\begin{tikzcd} & \mathbb{S}^n \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{RP}^n \arrow[rr, "\varphi"'] & & G_1(\mathbb{R}^{n+1}) \,. \end{tikzcd}$$

So we have proved the following.

11.2.7 **Proposition**. *There is a canonical homeomorphism*

$$\mathbb{RP}^n \approx \mathrm{G}_1(\mathbb{R}^{n+1})\,. \qquad \square$$

As a consequence of Proposition 11.2.7 we now can prove the next theorem.

11.2.8 **Theorem**. *There is a canonical homeomorphism*

$$\mathbb{RP}^\infty \approx \mathrm{G}_1(\mathbb{R}^\infty)\,.$$

Proof: The inclusions $\cdots \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2} \hookrightarrow \cdots$ induce inclusions

$$\cdots \hookrightarrow \mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \cdots\,,$$

$$\cdots \hookrightarrow \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1} \hookrightarrow \cdots\,,$$

$$\cdots \hookrightarrow \mathrm{G}_1(\mathbb{R}^{n+1}) \hookrightarrow \mathrm{G}_1(\mathbb{R}^{n+2}) \hookrightarrow \cdots\,,$$

so that we have commutative squares

$$\begin{tikzcd} \mathbb{RP}^n \arrow[r, hook] \arrow[d, "\approx"'] & \mathbb{RP}^{n+1} \arrow[d, "\approx"] \\ \mathrm{G}_1(\mathbb{R}^{n+1}) \arrow[r, hook] & \mathrm{G}_1(\mathbb{R}^{n+2}) \end{tikzcd}$$

for every n. Therefore, in the colimit we obtain the desired homeomorphism. $\square$

If we let $\mathrm{Vect}_1^{\mathbb{R}}(B)$ denote the set of isomorphism classes of real line bundles over B, then we have the following consequence of the previous theorem.

11.2.9 **Corollary**. *There is an isomorphism*

$$[B, \mathbb{RP}^\infty] \cong \mathrm{Vect}_1^{\mathbb{R}}(B)\,.$$

□

11.3 Classification of Real Line Bundles

The work for this section has essentially been done in the previous one. By combining Corollaries 11.2.6 and 11.2.9 we obtain the classification theorem of real line bundles.

11.3.1 **Theorem**. $\mathrm{Vect}_1^{\mathbb{R}}(B) \cong H^1(B;\mathbb{Z}/2)$. □

11.3.2 Definition. Let $p : E \longrightarrow B$ be a real line bundle. We define its *first Stiefel–Whitney class* $w_1(E) \in H^1(B;\mathbb{Z}/2)$ to be the image of $[E] \in \mathrm{Vect}_1^{\mathbb{R}}(B)$ under the isomorphism of Theorem 11.3.1. This element is also called the *Euler class* of the line bundle p. (Cf. Definition 11.7.13.)

By definition, $w_1(E)$ is an invariant of the isomorphism class of E. One of the important properties of w_1 is naturality, which we now shall discuss.

11.3.3 **Proposition**. *Suppose that $f : B' \longrightarrow B$ is a continuous map and that $E \longrightarrow B$ is a real line bundle. Then we have the naturality property*

$$w_1(f^*E) = f^*w_1(E) \in H^1(B';\mathbb{Z}/2),$$

where $f^*E \longrightarrow B'$ is the bundle induced by f from $E \longrightarrow B$, and $f^*w_1(E)$ is the image of $w_1(E) \in H^1(B;\mathbb{Z}/2)$ under the homomorphism induced by f in cohomology, namely $f^* : H^1(B;\mathbb{Z}/2) \longrightarrow H^1(B';\mathbb{Z}/2)$.

Proof: It is enough to note that by the naturality of the classifying isomorphism of $\mathrm{Vect}_1^{\mathbb{R}}(B)$ (see 8.5.13) we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Vect}_1^{\mathbb{R}}(B) \arrow[r, "\cong"] \arrow[d, "f^*"'] & H^1(B;\mathbb{Z}/2) \arrow[d, "f^*"] \\
\mathrm{Vect}_1^{\mathbb{R}}(B') \arrow[r, "\cong"'] & H^1(B';\mathbb{Z}/2).
\end{tikzcd}$$

□

11.3.4 **Corollary**. *If $p : E \longrightarrow B$ is a trivial real line bundle, then $w_1(E) = 0$; that is, $w_1(\varepsilon^1) = 0$.*

Proof: Since $p : E \longrightarrow B$ is trivial, it is isomorphic to the bundle $f^*\mathbb{R}$ induced from the bundle $\mathbb{R} \longrightarrow *$ over a one-point space by the unique map $f : B \longrightarrow *$. Consequently, we have that

$$w_1(E) = w_1(f^*\mathbb{R}) = f^*w_1(\mathbb{R}) = 0\,.$$

Here we have used $w_1(\mathbb{R}) \in H^1(*;\mathbb{Z}/2) = 0$, which holds because $H^1(*;\mathbb{Z}/2) = [*, \mathbb{RP}^\infty]$ and $\mathbb{RP}^\infty$ is path connected. □

11.3.5 Definition. The *canonical line bundle*, or *Hopf bundle*, $L \longrightarrow \mathbb{RP}^n$ is defined as follows. We consider $\mathbb{RP}^n$ to be the space of lines $l \subset \mathbb{R}^{n+1}$ and define

$$L = \{(x, l) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n \mid x \in l\} \xrightarrow{\text{proj}} \mathbb{RP}^n.$$

This means that this is the bundle whose fiber over each point $l \in \mathbb{RP}^n$ in the base space is the very same line l. Or in other words, if we consider $\mathbb{RP}^n$ to be the quotient space of the sphere $\mathbb{S}^n$ (which we get by identifying each pair of antipodal points x, $-x$ to a single point $\langle x \rangle$), then the fiber of the Hopf bundle over a point $\langle x \rangle \in \mathbb{RP}^n$ is the line containing the pair of antipodal points x, $-x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$.

11.3.6 Note. Obviously, the Hopf bundle $L \longrightarrow \mathbb{RP}^1 \approx \mathbb{S}^1$ is homeomorphic to the open Moebius strip (see Figure 11.1).

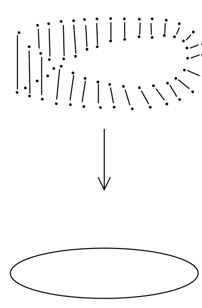

Figure 11.1

11.3.7 **Proposition**. *We have $w_1(L) \neq 0$, where $L \longrightarrow \mathbb{RP}^1$ is the Hopf bundle.*

Proof: There are isomorphisms

$$\mathrm{Vect}_1^{\mathbb{R}}(\mathbb{RP}^1) \cong [\mathbb{RP}^1, \mathbb{RP}^\infty] \cong H^1(\mathbb{S}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

which imply together with Corollary 11.3.4 that $w_1(E) = 0$ if and only if $E \longrightarrow \mathbb{RP}^1$ is a trivial line bundle. Since $L \longrightarrow \mathbb{RP}^1$ is nontrivial, it follows that $w_1(L) \neq 0$. $\square$

11.3.8 Exercise. Prove that the Hopf bundle $q : L \longrightarrow \mathbb{RP}^1$ is in fact nontrivial. (Cf. Exercise 11.3.11.) (Hint: The trivial bundle $p : E \longrightarrow \mathbb{RP}^1$ has the topological property that when we remove from E the zero section, that is, when we consider the fiber space

$$E_0 = \{x \in E \mid x \neq 0 \text{ in } p^{-1}p(x)\},$$

we obtain a space with two connected components. However, for $q : L \longrightarrow \mathbb{RP}^1$, the fiber space

$$L_0 = \{x \in L \mid x \neq 0 \in q^{-1}q(x)\}$$

has only one component. See Figure 11.2.)

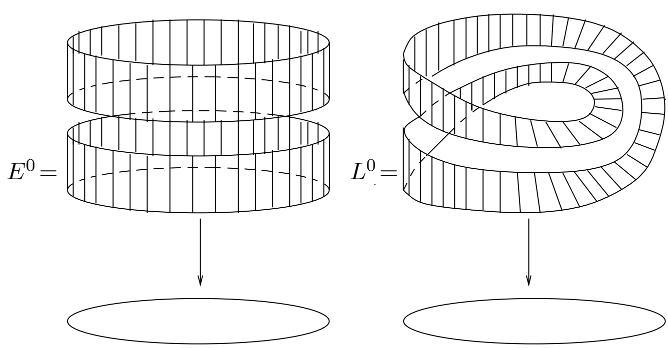

Figure 11.2

11.3.9 Exercise. Prove that $\mathbb{RP}^1 \approx \mathbb{S}^1$. (Hint: Consider $\mathbb{S}^1 = \{e^{2\pi it} \mid t \in [0,1]\} \subset \mathbb{C}$. Then a homeomorphism $\varphi : \mathbb{RP}^1 \longrightarrow \mathbb{S}^1$ is given by the commutative triangle

$$\begin{tikzcd} \mathbb{S}^1 \arrow[d] \arrow[dr, "\varphi'"] & \\ \mathbb{RP}^1 \arrow[r, "\varphi"'] & \mathbb{S}^1, \end{tikzcd}$$

where we define $\varphi'(e^{2\pi it}) = e^{4\pi it}$ for $t \in [0,1]$, or in other words, $\varphi'(\zeta) = \zeta^2$ for $\zeta \in \mathbb{S}^1$.)

Using the previous exercise, Proposition 11.3.7 is really a statement about line bundles over the circle $\mathbb{S}^1$. And so we have the following consequence.

11.3.10 **Corollary**. $\mathrm{Vect}_1^{\mathbb{R}}(\mathbb{S}^1)$ *has two elements, namely, the isomorphism class of the trivial line bundle and the isomorphism class of the Hopf bundle (which is also known as the open Moebius strip).* $\square$

11.3.11 EXERCISE. Recall from Definition 8.3.10 that a section of a vector bundle $p : E \longrightarrow B$ is a map $s : B \longrightarrow E$ satisfying $p \circ s = \mathrm{id}_B$. We say that a section s is *nowhere zero* if $s(b) \neq 0$ in $p^{-1}(b)$ for every point $b \in B$.

(a) Prove that every trivial bundle of nonzero dimension admits a nowhere-zero section.

(b) Prove that the Hopf bundle $L \longrightarrow \mathbb{RP}^1$ does not admit a nowhere-zero section. (Hint: Use the intermediate value theorem.)

(c) Deduce from parts (a) and (b) that the Hopf bundle $L \longrightarrow \mathbb{RP}^1$ is nontrivial.

11.4 DESCRIPTION OF $K(\mathbb{Z}, 2)$

In this section we shall essentially repeat what was done in Section 11.2, only now in the complex case. We shall prove that $\mathbb{CP}^\infty$ is simultaneously homeomorphic to $\mathrm{G}_1(\mathbb{C}^\infty)$ and has the homotopy type of a $K(\mathbb{Z}, 2)$.

Consider the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$, namely,

$$\mathbb{S}^{2n+1} = \left\{ (z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i \overline{z}_i = 1 \right\}.$$

Much as in the real case, we have the following description of $\mathbb{CP}^n$, the complex projective space of (complex) dimension n.

11.4.1 DEFINITION. For $z \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ and $\zeta \in \mathbb{S}^1 \subset \mathbb{C}$ we have that $\zeta z \in \mathbb{S}^{2n+1}$. We define the *complex projective space* $\mathbb{CP}^n$ to be the space we get by identifying in $\mathbb{S}^{2n+1}$ the points z and ζz for all $z \in \mathbb{S}^{2n+1}$ and all $\zeta \in \mathbb{S}^1$. This means that $\mathbb{CP}^n = \mathbb{S}^{2n+1}/\!\sim$, where the equivalence relation $\sim$ is defined for z and z' in $\mathbb{S}^{2n+1}$ by $z \sim z'$ if and only if there exists $\zeta \in \mathbb{S}^1$ such that $z' = \zeta z$. So there is a map

$$p : \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$$

whose inverse image of every point in $\mathbb{CP}^n$ is a copy of $\mathbb{S}^1$.

11.4.2 Exercise. Prove that the map $p : \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$ defined above is a locally trivial bundle. (Hint: For $i = 1, 2, \ldots, n+1$ define $U_i = \{[z] = [(z_1, z_2, \ldots, z_{n+1})] \in \mathbb{CP}^n \mid z_i \neq 0\}$ and show that $p|p^{-1}U_i$ is trivial.)

11.4.3 **Proposition**. *There exists a Serre fibration*

$$\mathbb{S}^\infty \longrightarrow \mathbb{CP}^\infty$$

with fiber $\mathbb{S}^1$.

Proof: For every n we have a commutative diagram

$$\begin{tikzcd} \mathbb{S}^{2n-1} \arrow[r, hook] \arrow[d] & \mathbb{S}^{2n+1} \arrow[d] \\ \mathbb{CP}^{n-1} \arrow[r, hook] & \mathbb{CP}^n \end{tikzcd}$$

such that the upper horizontal inclusion is a homeomorphism on the fibers $\mathbb{S}^1$.

In the colimit these inclusions determine a map $p : \mathbb{S}^\infty \longrightarrow \mathbb{CP}^\infty$ whose fibers are $\mathbb{S}^1$. To prove that p is a Serre fibration, we have to show that it has the HLP for the cubes I^k. This means that we have to show that given any commutative square

$$\begin{tikzcd} I^k \arrow[r, "h"] \arrow[d, hook, "j_0"'] & \mathbb{S}^\infty \arrow[d, "p"] \\ I^k \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & \mathbb{CP}^\infty \end{tikzcd}$$

there exists a lift $\widetilde{H}$. However, since both I^k and $I^k \times I$ are compact, there exists some n such that the images of h and H lie respectively in $\mathbb{S}^{2n+1}$ and $\mathbb{CP}^n$. This says that we have a commutative diagram

$$\begin{tikzcd} I^k \arrow[r, "h"] \arrow[d, hook, "j_0"'] & \mathbb{S}^{2n+1} \arrow[d, "p"] \\ I^k \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & \mathbb{CP}^n. \end{tikzcd}$$

Clearly, there exists $\widetilde{H}$ that makes the triangles commute in the last diagram, since $\mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$ is a Serre fibration because it is locally trivial by 11.4.2. Then $\widetilde{H} : I^k \times I \longrightarrow \mathbb{S}^{2n+1} \hookrightarrow \mathbb{S}^\infty$ makes the triangles commute in the first diagram, which proves that p is a Serre fibration. $\square$

In the following it will be sufficient to know that $p : \mathbb{S}^\infty \longrightarrow \mathbb{CP}^\infty$ is a quasifibration, which is true because it is a Serre fibration. Nonetheless, it really is more than a Serre fibration, as we now shall see.

11.4.4 Exercise. Prove that $p : \mathbb{S}^\infty \longrightarrow \mathbb{CP}^\infty$ is a locally trivial bundle and, using the fact that $\mathbb{CP}^\infty$ is paracompact (since it is a CW-complex), deduce that p is really a Hurewicz fibration.

From Proposition 11.4.3 we get the long exact sequence

$$\cdots \longrightarrow \pi_n(\mathbb{S}^\infty) \longrightarrow \pi_n(\mathbb{CP}^\infty) \longrightarrow \pi_{n-1}(\mathbb{S}^1) \longrightarrow$$
$$\cdots \longrightarrow \pi_2(\mathbb{S}^\infty) \longrightarrow \pi_2(\mathbb{CP}^\infty) \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^\infty) = 0\,.$$

Since $\pi_n(\mathbb{S}^\infty) = 0$ for all n (because $\mathbb{S}^\infty$ is contractible) and $\pi_n(\mathbb{S}^1) = 0$ for all $n \neq 1$, we get from the previous exact sequence that $\pi_n(\mathbb{CP}^\infty) = 0$ for all $n \neq 2$ and that $\pi_2(\mathbb{CP}^\infty) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. So we have proved the next result.

11.4.5 **Theorem**. $\mathbb{CP}^\infty$ *is an Eilenberg–Mac Lane space of type* $K(\mathbb{Z}, 2)$. □

Definition 7.1.2 and Theorem 11.4.5 have the following consequence.

11.4.6 **Corollary**. *For any* CW*-complex* B*, there is a natural isomorphism*

$$[B, \mathbb{CP}^\infty] \cong H^2(B; \mathbb{Z})\,. \qquad \square$$

The elements of the Grassmann manifold $\mathrm{G}_1(\mathbb{C}^{n+1})$ are the one-dimensional (complex) subspaces of $\mathbb{C}^{n+1}$. So we have a bijection between the elements of $\mathrm{G}_1(\mathbb{C}^n)$ and the great circles in $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Here *great circle* means, of course, the intersection of $\mathbb{S}^{2n+1}$ with any one-dimensional (complex) subspace of $\mathbb{C}^{n+1}$, and not the intersection of $\mathbb{S}^{2n+1}$ with an arbitrary two-dimensional (real) subspace of $\mathbb{C}^{n+1}$. In other words, the map

$$q : \mathbb{S}^{2n+1} \longrightarrow \mathrm{G}_1(\mathbb{C}^{n+1}),$$

defined by $z \mapsto \langle z \rangle$ (where $\langle x \rangle$ denotes as above the complex one-dimensional subspace of $\mathbb{C}^{n+1}$ generated by x), is surjective, and for every line $l \in \mathrm{G}_1(\mathbb{C}^n)$ we have $q^{-1}(l) = l \cap \mathbb{S}^{2n+1}$, which means that $q^{-1}(l)$ is a great circle in $\mathbb{S}^{2n+1}$. Since $\mathbb{S}^{2n+1}$ is compact and $\mathrm{G}_1(\mathbb{C}^{n+1})$ is Hausdorff, q is an identification map, and so there exists a homeomorphism $\varphi : \mathbb{CP}^n \longrightarrow \mathrm{G}_1(\mathbb{C}^{n+1})$ that gives us a commutative triangle

$$\begin{tikzcd} & \mathbb{S}^{2n+1} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{CP}^n \arrow[rr, "\varphi"'] & & G_1(\mathbb{C}^{n+1}). \end{tikzcd}$$

So we have proved the next result.

11.4.7 **Proposition**. *There is a homeomorphism*

$$\mathbb{CP}^n \approx \mathrm{G}_1(\mathbb{C}^{n+1})\,. \qquad \square$$

As a consequence of Proposition 11.4.7 we now prove the following theorem.

11.4.8 **Theorem**. *There is a homeomorphism* $\mathbb{CP}^\infty \approx \mathrm{G}_1(\mathbb{C}^\infty)$.

Proof: The inclusions $\cdots \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \cdots$ induce the inclusions

$$\begin{gathered} \cdots \hookrightarrow \mathbb{S}^{2n-1} \hookrightarrow \mathbb{S}^{2n+1} \hookrightarrow \cdots\,, \\ \cdots \hookrightarrow \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^{n} \hookrightarrow \cdots\,, \\ \cdots \hookrightarrow \mathrm{G}_1(\mathbb{C}^n) \hookrightarrow \mathrm{G}_1(\mathbb{C}^{n+1}) \hookrightarrow \cdots\,, \end{gathered}$$

so that we have commutative squares

$$\begin{tikzcd} \mathbb{CP}^{n-1} \arrow[r, hook] \arrow[d, "\approx"'] & \mathbb{CP}^n \arrow[d, "\approx"] \\ \mathrm{G}_1(\mathbb{C}^n) \arrow[r, hook] & \mathrm{G}_1(\mathbb{C}^{n+1}) \end{tikzcd}$$

for every n. So in the colimit we get the desired homeomorphism. $\square$

If we let $\mathrm{Vect}_1^{\mathbb{C}}(B)$ denote the set of isomorphism classes of complex line bundles over B, then we have the following consequence of the previous theorem.

11.4.9 **Corollary**. *For any* CW-*complex B there is a natural isomorphism*

$$[B, \mathbb{CP}^\infty] \cong \mathrm{Vect}_1^{\mathbb{C}}(B)\,. \qquad \square$$

11.5 Classification of Complex Line Bundles

The work for this section has essentially been done in the previous one. By combining 11.4.6 and 11.4.9 we obtain the classification theorem of complex line bundles.

11.5.1 **Theorem**. *There is a natural isomorphism* $\mathrm{Vect}_1^{\mathbb{C}}(B) \cong H^2(B;\mathbb{Z})$.$\square$

11.5.2 Definition. Let $p : E \longrightarrow B$ be a complex line bundle. We define its *first Chern class* $c_1(E) \in H^2(B;\mathbb{Z})$ to be the image of $[E] \in \mathrm{Vect}_1^{\mathbb{C}}(B)$ under the isomorphism of Theorem 11.5.1. This element is also called the *Euler class* of the vector bundle p. (Cf. Definition 11.7.13.)

By definition, $c_1(E)$ is an invariant of the isomorphism class of E. One of the important properties of c_1 is naturality, which we now shall discuss.

11.5.3 **Proposition**. *Suppose that $f : B' \longrightarrow B$ is a continuous map and that $E \longrightarrow B$ is a complex line bundle. Then we have the naturality property*

$$c_1(f^*E) = f^*c_1(E) \in H^2(B';\mathbb{Z}),$$

where $f^*E \longrightarrow B'$ is the bundle induced by f from $E \longrightarrow B$ and $f^*c_1(E)$ is the image of $c_1(E) \in H^2(B;\mathbb{Z})$ under the homomorphism induced by f in cohomology, namely $f^* : H^2(B;\mathbb{Z}) \longrightarrow H^2(B';\mathbb{Z})$.

Proof: It is enough to note that by the naturality of the classifying isomorphism of $\mathrm{Vect}_1^{\mathbb{C}}(B)$ (see 8.5.13) we have a commutative diagram

$$\begin{tikzcd} \mathrm{Vect}_1^{\mathbb{C}}(B) \arrow[r, "\cong"] \arrow[d, "f^*"'] & H^2(B;\mathbb{Z}) \arrow[d, "f^*"] \\ \mathrm{Vect}_1^{\mathbb{C}}(B') \arrow[r, "\cong"'] & H^2(B';\mathbb{Z}). \end{tikzcd}$$

□

11.5.4 **Corollary**. *If $p : E \longrightarrow B$ is a trivial complex line bundle, then $c_1(E) = 0$.*

Proof: Since $p : E \longrightarrow B$ is trivial, it is isomorphic to the bundle $f^*\mathbb{C}$ induced from the bundle $\mathbb{C} \longrightarrow *$ over a one-point space. Consequently, we have that

$$c_1(E) = c_1(f^*\mathbb{C}) = f^*c_1(\mathbb{C}) = 0.$$

Here we have used $c_1(\mathbb{C}) \in H^2(*;\mathbb{Z}) = 0$, which holds because

$$H^2(*;\mathbb{Z}) = [*, \mathbb{CP}^\infty]$$

and $\mathbb{CP}^\infty$ is path connected. □

11.5.5 Definition. The *canonical line bundle*, or *Hopf bundle*,

$$L \longrightarrow \mathbb{CP}^n$$

is defined as

$$L = \{(z,l) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \mid z \in l\} \xrightarrow{\text{proj}} \mathbb{CP}^n.$$

This means that this is the bundle whose fiber over each point $l \in \mathbb{CP}^n$ in the base space is the very same complex line l.

11.5.6 Note. The complex projective space $\mathbb{CP}^1$ (which has complex dimension one) is homeomorphic to the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \infty$. We can give a homeomorphism as follows. We first define an identification map $p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ by $p(z, z') = z/z'$ for $z' \neq 0$ and $p(z, z') = \infty$ for $z' = 0$, where we use $\mathbb{S}^3 \subset \mathbb{C}^2 - 0$ to identify points in $\mathbb{S}^3$ as pairs of complex numbers (z, z') in $\mathbb{C}^2 - 0$. This map p has already been studied in Example 4.5.10, where we showed that it precisely identifies to a point in $\mathbb{S}^2$ each circle in $\mathbb{S}^3$ of the form $\zeta(z, z') \in \mathbb{S}^3 \subset \mathbb{C}^2$ for some fixed (z, z') in $\mathbb{S}^3$ and all $\zeta \in \mathbb{S}^1$. In this way p induces a homeomorphism from the quotient space of $\mathbb{S}^3$ that results from identifying these circles to a point (this being exactly the projective space $\mathbb{CP}^1$) to $\mathbb{S}^2$.

Therefore, we have $H^2(\mathbb{CP}^1; \mathbb{Z}) \cong H^2(\mathbb{S}^2; \mathbb{Z})$, and so we get the next result, which will be proved later on in Corollary 11.7.29 in more generality.

11.5.7 **Proposition**. *Let $L \longrightarrow \mathbb{CP}^1$ be the Hopf bundle. It follows that $c_1(L)$ generates $H^2(\mathbb{CP}^1; \mathbb{Z})$ as an infinite cyclic group. In particular, $c_1(L) \neq 0$.* $\square$

11.6 Characteristic Classes

In Sections 11.3 and 11.5 we introduced the first Stiefel–Whitney class w_1 and the first Chern class c_1 for real and complex line bundles, respectively. In this section we shall define the Stiefel–Whitney and Chern classes for arbitrary real and complex bundles and shall analyze many of their properties. However, the proof of their existence, which we shall present in Section 11.8, requires some special preparatory material, which will be given in Section 11.7.

11.6.1 Definition. Suppose that $p : E \longrightarrow B$ is any real vector bundle. Cohomology classes

$$w_i(E) \in H^i(B; \mathbb{Z}/2), \quad i = 0, 1, 2, \dots ,$$

are called *Stiefel–Whitney classes* (for the bundle $p : E \longrightarrow B$) if they are invariants of the isomorphism class of the bundle and satisfy the following axioms:

(i) The class $w_0(E)$ is the unit element

$$1 \in H^0(B; \mathbb{Z}/2)$$

and $w_i(E) = 0$ for $i > \dim_{\mathbb{R}}(E)$, that is, for $i > n$, where E is a real n-vector bundle.

(ii) **Naturality.** If $f : B' \longrightarrow B$ is continuous and $p : E \longrightarrow B$ is a real vector bundle, then we have for every i that

$$w_i(f^*E) = f^*w_i(E) \in H^i(B'; \mathbb{Z}/2),$$

where $f^*E \longrightarrow B'$ is the bundle induced by f from $p : E \longrightarrow B$.

(iii) **Whitney Formula.** If $E \longrightarrow B$ and $E' \longrightarrow B$ are real vector bundles over the same base space, then

$$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \smile w_{k-i}(E').$$

In particular, we have

$$w_1(E \oplus E') = w_1(E) + w_1(E'),$$
$$w_2(E \oplus E') = w_2(E) + w_1(E) \smile w_1(E') + w_2(E'),$$

and so on. Here the symbol $\smile$ denotes the interior (or cup) product in cohomology. (See Definition 7.2.2.)

(iv) For the Hopf bundle $L \longrightarrow \mathbb{RP}^1$ over $\mathbb{RP}^1$ (the circle) the first Stiefel–Whitney class $w_1(L)$ is nonzero.

11.6.2 **Proposition**. *Suppose that $E \longrightarrow B$ and $E' \longrightarrow B'$ are real vector bundles and that $\widetilde{f} : E' \longrightarrow E$ is a bundle morphism covering a map $f : B' \longrightarrow B$. Then we have $w_i(E') = f^*(w_i(E))$ for every i.*

Proof: Since we have $f^*E' \cong E$ by using Exercise 8.1.14, the desired result follows immediately from the naturality and isomorphism-class invariance of the Stiefel–Whitney classes. □

11.6.3 Note. Actually, Proposition 11.6.2 is equivalent to naturality and isomorphism-class invariance. Specifically, if $E \longrightarrow B$ is a bundle and $f : B' \longrightarrow B$ is continuous, then $\widetilde{f} : f^*E \longrightarrow E$ is a bundle morphism, so that Proposition 11.6.2 implies naturality. And moreover, if we have an isomorphism $E' \cong E$, then this isomorphism is a bundle morphism over id_B, so that again by Proposition 11.6.2 we get $w_i(E') = w_i(E)$, which is precisely the property of isomorphism-class invariance.

Without having to prove the existence of the Stiefel–Whitney classes, we can draw some consequences from the axioms.

11.6.4 **Proposition**. *For each $n \geq 0$ let ε^n be a trivial real vector bundle of dimension n over the space B. Then we have $w_i(\varepsilon^n) = 0$ for every $i > 0$.*

Proof: The proof is carried out in essentially the same way as in Corollary 11.3.4, namely, by applying naturality and using $H_i(*; \mathbb{Z}/2) = 0$ for $i > 0$. □

The following is an important property of characteristic classes; it is sometimes known as **stability**.

11.6.5 **Proposition**. *Suppose that ε^n is a trivial real vector bundle of dimension n over the space B for some $n \geq 0$ and that $E \longrightarrow B$ is any real vector bundle. Then we have $w_i(\varepsilon^n \oplus E) = w_i(E)$ for every $i > 0$.*

Proof: This is an immediate consequence of Proposition 11.6.4 and the Whitney formula. □

It is worthwhile to introduce the next formal definition, which allows us to treat all of the Stiefel–Whitney classes with one fell swoop.

11.6.6 Definition. We use the notation $H^{\Pi}(B; \mathbb{Z}/2)$ for the ring of infinite formal series

$$a = a_0 + a_1 + a_2 + \cdots$$

satisfying $a_i \in H^i(B; \mathbb{Z}/2)$ for every i. The product in this ring is naturally defined by using the multiplicative structure in cohomology given by the cup product. Specifically, for any pair of elements $a = (a_0 + a_1 + a_2 + \cdots)$ and $b = (b_0 + b_1 + b_2 + \cdots)$ we define their product by

$$\begin{aligned} ab &= (a_0 \smile b_0) + (a_1 \smile b_0 + a_0 \smile b_1) + (a_2 \smile b_0 + a_1 \smile b_1 + a_0 \smile b_2) + \cdots \\ &= \sum_{i,j \geq 0} a_i \smile b_j \,. \end{aligned}$$

This multiplicative structure converts $H^{\Pi}(B; \mathbb{Z}/2)$ into a commutative and associative ring with unit. The additive structure is, of course, just that of the direct product the the abelian groups $H^i(B; \mathbb{Z}/2)$. Now we define the *total Stiefel–Whitney class* of a real n-vector bundle $E \longrightarrow B$ to be

$$w(E) = 1 + w_1(E) + w_2(E) + \cdots + w_n(E) + 0 + \cdots \in H^{\Pi}(B; \mathbb{Z}/2).$$

Using this definition, the Whitney formula reduces to the simple expression

$$w(E \oplus E') = w(E)w(E').$$

Analogous to the Stiefel–Whitney classes, we have the following.

11.6.7 Definition. Suppose that $p : E \longrightarrow B$ is any complex vector bundle. Cohomology classes

$$c_i(E) \in H^{2i}(B;\mathbb{Z}), \quad i = 0, 1, 2, \dots ,$$

are called the *Chern classes* for the bundle $p : E \longrightarrow B$ if they are invariant under vector bundle isomorphisms and satisfy the following axioms.

(i) The class $c_0(E)$ is the unit element

$$1 \in H^0(B;\mathbb{Z})\,,$$

and $c_i(E) = 0$ for $i > \dim_{\mathbb{C}}(E)$, that is, for $i > n$, where E is a complex n-vector bundle.

(ii) **Naturality.** If $f : B' \longrightarrow B$ is continuous and $p : E \longrightarrow B$ is a complex vector bundle then we have for every i that

$$c_i(f^*E) = f^*c_i(E) \in H^{2i}(B';\mathbb{Z}),$$

where $f^*E \longrightarrow B'$ is the bundle induced by f from $p : E \longrightarrow B$.

(iii) **Whitney Formula.** If $E \longrightarrow B$ and $E' \longrightarrow B$ are complex vector bundles over the same base space, then

$$c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(E').$$

In particular, we have

$$c_1(E \oplus E') = c_1(E) + c_1(E')\,,$$

$$c_2(E \oplus E') = c_2(E) + c_1(E) \smile c_1(E') + c_2(E')\,,$$

and so on.

(iv) For the Hopf bundle $L \longrightarrow \mathbb{CP}^1$ over $\mathbb{CP}^1$ (the 2-sphere) the first Chern class $c_1(L)$ is nonzero.

Analogously to the real case, we can deduce corresponding properties of the Chern classes from the axioms. Since this is formally the same, we leave it to the reader as an *exercise*. When we have occasion to refer to one of these properties in the complex case, we shall do it by mentioning the complex version of the corresponding real property.

11.7 Thom Isomorphism and Gysin Sequence

In order to construct the Stiefel–Whitney and Chern classes we shall need two important tools: the *Thom isomorphism* and the *Gysin sequence*. This section will be devoted to developing these tools. Some of the results used here will not be proved, and so we refer the reader to the text of Milnor and Stasheff [58] for their proofs.

Consider the exact cohomology sequence with coefficients in a ring R of the pair $(\mathbb{R}^n, \mathbb{R}^n - 0)$. In view of the fact that $\mathbb{R}^n$ is contractible and that $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n - 0$ we get the isomorphisms

$$\widetilde{H}^{i-1}(\mathbb{S}^{n-1}; R) \xrightarrow[\cong]{r^*} \widetilde{H}^{i-1}(\mathbb{R}^n - 0; R) \xrightarrow[\cong]{\delta} H^i(\mathbb{R}^n, \mathbb{R}^n - 0; R).$$

Using Proposition 7.1.24 we have that

$$H^i(\mathbb{R}^n, \mathbb{R}^n - 0; R) \cong \begin{cases} R & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

Moreover, according to 7.2.20(iii), $H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R)$ is generated by a canonical generator g_n. In more generality, if V is a real or complex vector space, we can find an $\mathbb{R}$-linear isomorphism $\mathbb{R}^n \cong V$ (that is, we can choose a basis of V as a real vector space), and thereby get that

$$H^i(V, V - 0; R) \cong \begin{cases} R & \text{if } i = \dim_{\mathbb{R}}(V), \\ 0 & \text{if } i \neq \dim_{\mathbb{R}}(V), \end{cases}$$

and that $H^n(V, V - 0; R)$ is generated by an element g_V that corresponds to g_n under the isomorphism.

11.7.1 Definition. Let $p : E \longrightarrow B$ be a vector bundle whose dimension over the reals is n. Let $E_0 \subset E$ be the complement of the zero section. We say that the bundle is *orientable with respect to* R if there exists an element $t_E \in H^n(E, E_0; R)$ such that for every $x \in B$ the homomorphism $j_x^* : H^n(E, E_0; R) \longrightarrow H^n(p^{-1}(x), p^{-1}(x) - 0; R)$ sends t_E to a generator, where $j_x : (p^{-1}(x), p^{-1}(x) - 0) \longrightarrow (E, E_0)$ is the inclusion. The element t_E is called the *Thom class* of the bundle for the ring R.

In particular, if $n = 0$, then $p : E \longrightarrow B$ is nothing other than $\mathrm{id} : B \longrightarrow B$, and so $E_0 = \emptyset$, which implies that the bundle is orientable. Specifically, we can take $t_E = 1 \in H^0(E, E_0; R) = H^0(B; R)$, whose restriction to $\{b\} \subset B$ is the generator $1 \in H^0(b)$ for every $b \in B$.

For simplicity, in what follows we shall sometimes omit the coefficient ring R in the cohomology.

11.7.2 NOTE. Assume that $p : E \longrightarrow B$ is a vector bundle provided with a Riemannian (or Hermitian) metric. Let E'_0 denote the set of vectors in E with norm ≥ 1. Then the inclusion $(E, E'_0) \hookrightarrow (E, E_0)$ induces isomorphisms in cohomology (as one deduces after comparing the exact sequences of both pairs). Since $E'_0 \hookrightarrow E$ is a cofibration, it follows that the quotient map $(E, E'_0) \twoheadrightarrow (E/E'_0, *)$ induces an isomorphism in cohomology; namely, there is an isomorphism

$$H^n(E, E_0) \cong \widetilde{H}^n(E/E'_0)\,.$$

Given a Thom class t_E, one has a corresponding element $t'_E \in \widetilde{H}^n(E/E'_0)$, which is also called the *Thom class*. The space $T(E) = E/E'_0$ is the so-called *Thom space* of the given bundle.

11.7.3 EXERCISE. Given a fiber $F \subset E$ of a vector bundle $p : E \longrightarrow B$ of real dimension n, let F'_0 be the subset $F \cap E'_0$ of F. Then $F/F'_0 \approx \mathbb{S}^n$. Assuming that t'_E is a Thom class for the bundle, prove that $i^*(t'_E) \in \widetilde{H}^n(\mathbb{S}^n)$ is a generator if $i : \mathbb{S}^n \hookrightarrow E/E'_0 = T(E)$ is the corresponding embedding.

11.7.4 EXERCISE. Prove the following properties of the Thom space. Let $p : E \longrightarrow B$, $p' : E' \longrightarrow B$ be vector bundles and denote by $\varepsilon^n \longrightarrow B$ the trivial bundle of (real) dimension n over B.

(a) $T(\varepsilon^n) \approx \Sigma^n(B^+)$, where $B^+ = B \sqcup \{*\}$.

(b) $T(E \oplus \varepsilon^1) \approx \Sigma T(E)$.

(c) $T(E \oplus \varepsilon^n) \approx \Sigma^n T(E)$.

(d) $T(E \times E') \approx T(E) \wedge T(E')$.

Here Σ denotes the (reduced) suspension (see 2.10.1), and $\wedge$ denotes the smash product of pointed spaces (see 5.1.49).

11.7.5 DEFINITION. Let V be a real vector space of dimension n. An *orientation* of V is an equivalence class of ordered bases, where we say that two ordered bases $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ are *equivalent* if the *change of basis matrix* (a_i^j), which is defined by the relation $w_i = \sum_1^n a_i^j v_j$, has a positive determinant. Obviously, every real vector space V has exactly two orientations. In particular, $\mathbb{R}^n$ has a *canonical orientation* corresponding to its canonical ordered basis $(e_1, e_2, \ldots, e_n)$ defined by $e_i = (0, \ldots, 1, \ldots, 0)$, where 1 appears in the ith position. Any given ordered basis $(v_1, v_2, \ldots, v_n)$ of a real vector space V determines an isomorphism $\mathbb{R}^n \cong V$ and thereby a generator $g_V \in H^n(V, V - 0; R)$. Two ordered bases are in the same equivalence class if and only if their corresponding isomorphisms determine homeomorphisms of pairs $(\mathbb{R}^n, \mathbb{R}^n - 0) \longrightarrow (V, V - 0)$ that are homotopic. For $R = \mathbb{Z}$ this is the case if and only if $g_V = g'_V$, where g_V and g'_V are the respective generators for each isomorphism. Consequently, this element determines an orientation of V, and unambiguously we also call it an *orientation* of V with respect to R. For $R = \mathbb{Z}/2$ this orientation is unique, while for $R = \mathbb{Z}$ there are two orientations, which correspond to the two generators.

Now we shall generalize the definition of orientation to the case of vector bundles.

11.7.6 Definition. Let $p : E \longrightarrow B$ be a real vector bundle of dimension n. An *orientation* of p is a function μ that assigns to each point $x \in B$ an orientation of the real vector space $p^{-1}(x)$ and that satisfies the following compatibility condition: Each point $x_0 \in B$ in the base space has a neighborhood U_0 together with a family of linearly independent sections $s_1, s_2, \ldots, s_n : U_0 \longrightarrow p^{-1}(U_0)$ such that for every $x \in U_0$ the ordered basis $(s_1(x), s_2(x), \ldots, s_n(x))$ of the fiber $p^{-1}(x)$ defines the orientation $\mu(x)$.

A real vector bundle $p : E \longrightarrow B$ equipped with an orientation μ is called an *oriented bundle.*

11.7.7 **Proposition**. *For a real vector bundle $p : E \longrightarrow B$ of dimension n we have the following statements:*

(i) *The bundle has a unique Thom class $t_E \in H^n(E, E_0; R)$.*

(ii) $H^k(E, E_0; R) = 0$ *for $k < n$.*

Here we take $R = \mathbb{Z}$ if the bundle is oriented, though in general we can take only $R = \mathbb{Z}/2$. Also, E_0 denotes as above the complement of the zero section in E.

Proof: We shall prove this in five steps.

(a) First let us assume that the bundle is trivial, namely that $E = B \times \mathbb{R}^n$. Consider the composite of maps of pairs

$$(\mathbb{R}^n, \mathbb{R}^n - 0) \xrightarrow{\;i_b\;} B \times (\mathbb{R}^n, \mathbb{R}^n - 0) \xrightarrow{\;\text{proj}\;} (\mathbb{R}^n, \mathbb{R}^n - 0)\,,$$

where for each $b \in B$ we define $i_b(y) = (b, y)$ for $y \in \mathbb{R}^n$. Notice that this composite of maps of pairs is the identity for every $b \in B$. Consider the canonical generator (7.2.20(iii)) $g \in H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R)$, which is the unique nonzero element if $R = \mathbb{Z}/2$ and is the generator given by the orientation if $R = \mathbb{Z}$. It follows by functoriality that $\mathrm{proj}^*(g) = 1 \times g \in H^n(B \times (\mathbb{R}^n, \mathbb{R}^n - 0); R)$ is an element satisfying $i_b^*(1 \times g) = g$ for every $b \in B$. But since g is the generator, we have identified the Thom class $t_E = 1 \times g$.

Since $H^*(\mathbb{R}^n, \mathbb{R}^n - 0; R)$ is free, we can use the Künneth formula 7.4.5 to obtain an isomorphism

$$\bigoplus_{i+j=k} H^i(B; R) \otimes_R H^j(\mathbb{R}^n, \mathbb{R}^n - 0; R) \cong H^k(B \times (\mathbb{R}^n, \mathbb{R}^n - 0); R)\,,$$

and consequently

$$H^{k-n}(B; R) \otimes_R H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R) \cong H^k(B \times (\mathbb{R}^n, \mathbb{R}^n - 0); R) = 0\,,$$

for $k < n$, which implies $H^k(E, E_0; R) = 0$ in this case.

(b) Using part (a), we find that (i) and (ii) are true in open sets U for which $E|U$ is trivial. So let us assume that (i) and (ii) hold for $E|U$, $E|V$, and $E|U \cap V$, where $U, V \subset B$ are open. We shall now prove that (i) and (ii) are also true for $E|U \cup V$. Consider the Mayer–Vietoris sequence 7.4.14 for the couple of excisive pairs $(E|U, E_0|U)$ and $(E|V, E_0|V)$, namely

$$\begin{gathered} H^{k-1}(E|U \cap V, E_0|U \cap V) \longrightarrow H^k(E|U \cup V, E_0|U \cup V) \longrightarrow \\ \longrightarrow H^k(E|U, E_0|U) \oplus H^k(E|V, E_0|V) \longrightarrow H^k(E|U \cap V, E_0|U \cap V)\,. \end{gathered}$$

For $k < n$ the sequence collapses to $0 \longrightarrow H^k(E|U \cup V, E_0|U \cup V) \longrightarrow 0$, and so (ii) holds for $E|U \cup V$. For $k = n$ the sequence becomes

$$\begin{gathered} H^k(E|U \cup V, E_0|U \cup V) \rightarrowtail H^k(E|U, E_0|U) \oplus H^k(E|V, E_0|V) \xrightarrow{\alpha} \\ \longrightarrow H^k(E|U \cap V, E_0|U \cap V)\,. \end{gathered}$$

By hypothesis we have Thom classes $t_{E|U}$ and $t_{E|V}$, and then by the uniqueness property of Thom classes we have $\iota_U^*(t_{E|U}) = \iota_V^*(t_{E|V}) \in H^n(E|U \cap V, E_0|U \cap V; R)$, where $\iota_U : E|U \cap V \hookrightarrow E|U$ and $\iota_V : E|U \cap V \hookrightarrow E|V$ are the inclusions. Therefore, $\alpha(t_{E|U}, t_{E|V}) = \iota_U^*(t_{E|U}) - \iota_V^*(t_{E|V}) = 0$, and so by the exactness of the sequence there exists a unique element $t_{E|U \cup V} \in H^n(E|U \cup V, E_0|U \cup V; R)$ that restricts to $t_{E|U}$ as well as to $t_{E|V}$.

(c) If the bundle E is of finite type, it is the union of a finite number N of trivial bundles, and so the result is obtained from part (b) by induction on N.

(d) The case of a CW-complex B follows from part (c) by a limiting argument. Using 5.1.30 we know that each k-skeleton B^k can be covered with a finite number (namely $k+1$) of open sets that are contractible in B^k.

Therefore, the bundle $E^k = E|B^k$ is of finite type, and so by part (c) the theorem is true for each skeleton of B.

Let $t^k \in H^n(E^k, E_0^k)$ be the Thom class. By naturality,

$$(t^0, t^1, t^2, \dots) \in \prod_k H^n(E^k, E_0^k; R)$$

determines an element in $\lim_k H^n(E^k, E_0^k; R)$. As Milnor shows in his article [56], there exists a natural short exact sequence

$$\begin{aligned} 0 \longrightarrow {\lim}^1{}_k\, H^{n-1}(E^k, E_0^k) \longrightarrow H^n(E, E_0) \longrightarrow \\ \longrightarrow \lim_k H^n(E^k, E_0^k) \longrightarrow 0\,. \end{aligned} \tag{11.7.8}$$

Since $H^{n-1}(E^k, E_0^k) = 0$, we have an isomorphism

$$H^n(E, E_0) \longrightarrow \lim_k H^n(E^k, E_0^k)\,,$$

so that to the sequence $(t^0, t^1, t^2, \dots)$ on the right there corresponds an element t_E on the left. Clearly, t_E is the desired Thom class.

(e) The general case now follows immediately from part (d), if we take a CW-approximation of B (see Theorem 5.1.35), say $f : \widetilde{B} \longrightarrow B$, and consider the induced bundle $\widetilde{E} = f^*E$ over $\widetilde{B}$. Then it follows that the Thom class of E is given by $t_E = f^*(t_{\widetilde{E}})$, where $t_{\widetilde{E}}$ is the Thom class of $\widetilde{E}$. □

11.7.9 Note. Let W be a complex vector space of dimension m. If

$$(w_1, w_2, \dots, w_m)$$

is a basis of W, then the vectors

$$w_1, \mathrm{i}w_1, w_2, \mathrm{i}w_2, \dots, w_n, \mathrm{i}w_n$$

form a basis of W as a real vector space. These vectors in this order determine an orientation of W. Since the group $\mathrm{GL}_m(\mathbb{C})$ is connected, we can go continuously from any complex basis to any other complex basis, and so the corresponding orientations of the two bases are equal. In other words, W has a canonical orientation.

Now, if $p : E \longrightarrow B$ is a complex vector bundle, each fiber has a canonical orientation so that the underlying real vector bundle $p_{\mathbb{R}} : E_{\mathbb{R}} \longrightarrow B$ is an oriented bundle. Using Proposition 11.7.7 we then have the next result.

11.7.10 **Proposition**. *Let $p : E \longrightarrow B$ be a complex vector bundle of dimension m. Then its underlying real vector bundle $p_{\mathbb{R}} : E_{\mathbb{R}} \longrightarrow B$ has a unique Thom class $t_E = t_{E_{\mathbb{R}}} \in H^{2m}(E, E_0; \mathbb{Z})$.* □

11.7.11 **Proposition**. *Suppose that $p' : E' \longrightarrow B'$ is a vector bundle of real dimension n that is orientable with respect to a ring R and that $f : B \longrightarrow B'$ is continuous. If $p : E \longrightarrow B$ is the bundle induced from p' by f, namely so that we have a commutative diagram*

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' , \end{tikzcd}$$

then $p : E \longrightarrow B$ is also orientable with respect to R. Moreover, if t_E and $t_{E'}$ are the respective Thom classes, we have that $\widetilde{f}^*(t_{E'}) = t_E \in H^n(E, E_0; R)$.

Proof: For every $x \in B$ there is a commutative diagram

$$\begin{tikzcd} (E, E_0) \arrow[r, "\widetilde{f}"] & (E', E'_0) \\ (p^{-1}(x), p^{-1}(x) - 0) \arrow[r, "\widetilde{f}_x"'] \arrow[u, hook, "j_x"] & (p'^{-1}(f(x)), p'^{-1}(f(x)) - 0) , \arrow[u, hook, "j_{f(x)}"'] \end{tikzcd}$$

where $\widetilde{f}_x$ is the restriction of $\widetilde{f}$ to the fiber over x. By the definition of induced bundle we have that $\widetilde{f}_x$ is a homeomorphism. Applying cohomology with coefficients in R (as a functor) we get the diagram

$$\begin{tikzcd} H^n(E', E'_0) \arrow[r, "\widetilde{f}^*"] \arrow[d, "j^*_{f(x)}"'] & H^n(E, E_0) \arrow[d, "j^*_x"] \\ H^n(p'^{-1}(f(x)), p'^{-1}(f(x)) - 0) \arrow[r, "\cong", "\widetilde{f}^*_x"'] & H^n(p^{-1}(x), p^{-1}(x) - 0) . \end{tikzcd}$$

Since $j^*_{f(x)}(t_{E'})$ is a generator and $\widetilde{f}^*_x$ is an isomorphism, it follows that $\widetilde{f}^*_x j^*_{f(x)}(t_{E'}) = j^*_x \widetilde{f}^*(t_{E'})$ is a generator for all $x \in B$; that is, $\widetilde{f}^*(t_{E'})$ is a Thom class of $p : E \longrightarrow B$. Using uniqueness of the Thom class, we have $\widetilde{f}^*(t_{E'}) = t_E$. $\square$

There also is a property of the Thom class with respect to the Whitney sum of two vector bundles over the same space, say $p : E \longrightarrow B$ of dimension n and $p' : E' \longrightarrow B$ of dimension n'. Recall that if $\Delta : B \longrightarrow B \times B$ is the diagonal map, then the Whitney sum of two bundles is induced from their product by a Δ, namely,

$$E \oplus E' = \Delta^*(E \times E') ,$$

which in turn means that we have a commutative diagram

$$\begin{CD} E \oplus E' @>{\widetilde{\Delta}}>> E \times E' \\ @VVV @VVV \\ B @>>{\Delta}> B \times B\,. \end{CD}$$

11.7.12 **Proposition**. *Suppose that $E \longrightarrow B$ and $E' \longrightarrow B$ are vector bundles of dimensions n and n', respectively. Then the Thom class of their Whitney sum $E \oplus E'$ is the image of $t_E \times t_{E'}$ under the composite γ,*

$$H^n(E, E_0) \otimes H^{n'}(E', E'_0) \xrightarrow{\times} H^{n+n'}(E \times E', E \times E'_0 \cup E_0 \times E')$$

$$= H^{n+n'}(E \times E', (E \times E')_0) \xrightarrow{\widetilde{\Delta}^*} H^{n+n'}(E \oplus E', (E \oplus E')_0)\,,$$

where the first arrow represents an isomorphism. In other words, to calculate this Thom class we have the formula

$$t_{E \oplus E'} = \widetilde{\Delta}^*(t_E \times t_{E'})\,.$$

Proof: First note that the fibers over any $b \in B$ satisfy $p^{-1}(b) \cong \mathbb{R}^n$ and $p'^{-1}(b) \cong \mathbb{R}^{n'}$. Also, the inclusion $\{b\} \hookrightarrow B$ induces inclusions $p^{-1}(b) \hookrightarrow E$ and $p'^{-1}(b) \hookrightarrow E'$. Using these facts we obtain a commutative diagram

$$\begin{CD} H^n(E, E_0) \otimes H^{n'}(E', E'_0) @>>> H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H^{n'}(\mathbb{R}^{n'}, \mathbb{R}^{n'} - 0) \\ @V{\gamma}VV @V{\cong}V{\times}V \\ H^{n+n'}(E \oplus E', (E \oplus E')_0) @>>> H^{n+n'}(\mathbb{R}^{n+n'}, \mathbb{R}^{n+n'} - 0)\,. \end{CD}$$

This diagram shows that $\widetilde{\Delta}^*(t_E \times t_{E'})$ restricts to the generator $e_{n+n'}$ of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H^{n'}(\mathbb{R}^{n'}, \mathbb{R}^{n'} - 0)$, which is the cross product $e_n \times e_{n'}$ of the two generators of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ and $H^{n'}(\mathbb{R}^{n'}, \mathbb{R}^{n'} - 0)$, respectively. And so in fact, we obtain $t_{E \oplus E'} = \widetilde{\Delta}^*(t_E \times t_{E'})$. □

11.7.13 Definition. Suppose that $p : E \longrightarrow B$ is a real vector bundle of dimension n and that $z : B \longrightarrow E \hookrightarrow (E, E_0)$ is the map induced by its zero section. The class $e(E) = z^*(t_E) \in H^n(B; \mathbb{Z}/2)$ is called the *Euler class* of the real vector bundle $p : E \longrightarrow B$.

For $n = 0$ we obtain, in particular, the bundle $\mathrm{id} : B \longrightarrow B$ with zero section $z = \mathrm{id} : B \longrightarrow B$. Since $t_E = 1$, we therefore conclude that $e(E) = 1$.

Analogously, if $p : E \longrightarrow B$ is a complex vector bundle of dimension m and $z : B \longrightarrow E \hookrightarrow (E, E_0)$ is the map induced by the zero section of the bundle, then we call the class $e(E) = z^*(t_E) \in H^{2m}(B; \mathbb{Z})$ the *Euler class* of the complex vector bundle p.

11.7.14 Note. Let $L \longrightarrow \mathbb{RP}^1$ be the canonical bundle. As we have already indicated before, L is topologically the open Moebius strip, and the complement of its zero section L_0 has the same homotopy type of the circle. In other words, the pair (L, L_0) has the same homotopy type of the pair $(M, \partial M)$ of the compact Moebius strip and its boundary. So we have in cohomology that $H^1(L, L_0; \mathbb{Z}/2) = H^1(M, \partial M; \mathbb{Z}/2) = H^1(M/\partial M; \mathbb{Z}/2)$. But we also have $M/\partial M \approx \mathbb{RP}^2$, which then implies

$$H^1(L, L_0; \mathbb{Z}/2) = H^1(\mathbb{RP}^2; \mathbb{Z}/2) = [\mathbb{RP}^2, \mathbb{RP}^\infty] = [\mathbb{RP}^2, \mathbb{RP}^2] = \mathbb{Z}/2\,.$$

Since $t_L \in H^1(L, L_0; \mathbb{Z}/2)$ is nonzero, under the above identifications it corresponds to the class $[\mathrm{id}] \in [\mathbb{RP}^2, \mathbb{RP}^2]$, and so, again under the above identifications as well as by the isomorphism $H^1(\mathbb{RP}^2; \mathbb{Z}/2) \cong H^1(\mathbb{RP}^1; \mathbb{Z}/2)$ that is induced by the inclusion, the Euler class $e(L) \in H^1(\mathbb{RP}^1; \mathbb{Z}/2)$ corresponds to the homotopy class of the inclusion $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$ in $[\mathbb{RP}^1, \mathbb{RP}^2] = \mathbb{Z}/2$.

More generally, since $L \longrightarrow \mathbb{RP}^1$ is the restriction of the canonical bundle $L^1 \longrightarrow \mathbb{RP}^\infty$ and since $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$ induces an isomorphism in cohomology, we have that *the Euler class of the canonical line bundle over* $\mathbb{RP}^\infty$*, namely* $e(L^1) \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2$*, is equal to the generator* (cf. 11.7.26).

In the complex case, we can analogously assert that *the Euler class of the canonical complex line bundle* $L^1 \longrightarrow \mathbb{CP}^\infty$*, namely* $e(L^1) \in H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$*, is equal to one of the generators.*

In the following we shall present some properties of the Euler class.

11.7.15 **Proposition**. *The Euler class is natural. This means that if* $p : E \longrightarrow B$ *is a vector bundle and* $f : B' \longrightarrow B$ *is continuous, then it follows that* $e(f^*E) = f^*e(E)$.

Proof: We have a commutative diagram

$$\begin{CD} f^*E @>{\widetilde{f}}>> E \\ @VVV @VVV \\ B' @>>{f}> B\,. \end{CD}$$

Letting $z_E : B \longrightarrow (E, E_0)$ and $z_{f^*E} : B' \longrightarrow (f^*E, f^*E_0)$ be the maps induced by the zero sections, we can conclude that $\widetilde{f} \circ z_{f^*E} = z_E \circ f$. And so by using Proposition 11.7.11 we obtain $\widetilde{f}^*(t_E) = t_{f^*E}$, which implies that $e(f^*E) = z_{f^*E}^*(t_{f^*E}) = f^*z_E^*(t_E) = f^*(e(E))$. □

11.7.16 EXERCISE. Prove that Definition 11.7.13 of the Euler class is consistent with those given in Definition 11.3.2 if $p : E \longrightarrow B$ is a real line bundle and in Definition 11.5.2 if it is a complex line bundle. (Hint: First, discuss the real case. Since the Euler class $e(L^1) \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is equal to the class $[\mathrm{id}] \in [\mathbb{RP}^\infty, \mathbb{RP}^\infty]$ according to 11.7.14, it follows that the isomorphism $\mathrm{Vect}_1^{\mathbb{R}}(\mathbb{RP}^\infty) \cong H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ identifies the class of the canonical bundle $[L^1]$ with $e(L^1)$. So in the particular case $L^1 \longrightarrow \mathbb{RP}^\infty$ Definitions 11.3.2 and 11.7.13 are consistent. Since any line bundle $E \longrightarrow B$ is induced from $L^1 \longrightarrow \mathbb{RP}^\infty$ by some map, the naturality of the Euler class implies the consistency of these two definitions for any bundle. The complex case is handled similarly.)

11.7.17 **Proposition**. *For the Euler class of the Whitney sum $E \oplus E'$ of two vector bundles $E \longrightarrow B$ and $E' \longrightarrow B$ we have the formula*

$$e(E \oplus E') = e(E) \smile e(E') .$$

Proof: Letting $z : B \longrightarrow E \hookrightarrow (E, E_0)$ and $z' : B \longrightarrow E' \hookrightarrow (E', E'_0)$ be the zero sections of the given bundles, it follows that $(z, z') : B \longrightarrow (E, E_0) \times (E', E'_0)$ is the zero section of their product. Then using 11.7.12, 7.2.16, and 7.2.11 we have

$$\begin{aligned} e(E \oplus E') &= (z, z')^*(t_{E \oplus E'}) \\ &= (z, z')^* \widetilde{\Delta}^*(t_E \times t_{E'}) \\ &= \Delta^*(z^*(t_E) \times z'^*(t_{E'})) \\ &= z^*(t_E) \smile z'^*(t_{E'}) \\ &= e(E) \smile e(E') . \end{aligned}$$

□

The next proposition gives the property of the Euler class that is analogous to the properties expressed in Corollaries 11.3.4 and 11.5.4; moreover, its proof is the same.

11.7.18 **Proposition**. *For any $n > 0$ let ε^n denote the trivial bundle of dimension n. Then its Euler class is given by $e(\varepsilon^n) = 0$.* □

11.7.19 **Proposition**. *If $p : E \longrightarrow B$ is a vector bundle that has a nowhere-zero section, then its Euler class satisfies $e(E) = 0$.*

Proof: Suppose that $i : E_0 \hookrightarrow E$ and $j : E \longrightarrow (E, E_0)$ are the inclusions and that $s : B \longrightarrow E_0 \subset E$ is the nowhere-zero section of E. Here, as usual, E_0 denotes the complement of the zero section in E. Then the composite

$$B \xrightarrow{\ s\ } E_0 \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B$$

is the identity, and therefore in cohomology the composite

$$H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{i^*} H^n(E_0) \xrightarrow{s^*} H^n(B)$$

is also the identity.

Letting $s_0 : B \longrightarrow E$ denote the zero section, we have that $z = j \circ s_0$, and so by definition we get $e(E) = z^*(t_E) = s_0^* j^*(t_E)$. Next we note that $p \circ s_0 = \mathrm{id}_B$ implies $s_0^* \circ p^* = 1$. From the exactness of the long cohomology sequence of the pair (E, E_0) we get that $i^* \circ j^* = 0$. Now, since we have $s_0 \circ p \simeq \mathrm{id}_E$ (*exercise*), it follows that $e(E) = s^* i^* p^*(e(E)) = s^* i^* p^*(s_0^* j^*(t_E)) = s^* i^* j^*(t_E) = 0$. □

We shall now present the Thom isomorphism theorem.

11.7.20 **Theorem**. (Thom isomorphism) *Let $p : E \longrightarrow B$ be a vector bundle of real dimension n. Then for every q the map $b \mapsto p^*(b) \smile t_E$, where $b \in H^q(B; R)$ and t_E is the Thom class of E for the ring R, is an isomorphism $\varphi : H^q(B; R) \cong H^{q+n}(E, E_0; R)$ for the case where $R = \mathbb{Z}/2$ and the bundle is arbitrary and for the case where $R = \mathbb{Z}$ and the bundle is oriented.* We call φ the *Thom isomorphism.*

Note that in the composite

$$\varphi : H^k(B; R) \xrightarrow{p^*} H^k(E; R) \longrightarrow H^{k+n}(E, E_0; R)$$

the first homomorphism p^*, being induced by p, is an isomorphism, since p is a homotopy equivalence. So what Theorem 11.7.20 is really saying is that the second homomorphism, which is defined by taking the cup product with t_E, is in fact an isomorphism.

Proof of 11.7.20*:* As in the proof of Proposition 11.7.7, we shall prove this in five steps.

(a) Suppose that $p : E \longrightarrow B$ is a trivial bundle, that is, $p = \pi_1 : E = B \times \mathbb{R}^n \longrightarrow B$, where π_1 is the projection onto the first factor. By part (a) of the proof of Proposition 11.7.7 we have that $t_E = \pi_2^*(g_n) = 1 \times g_n$, where

$$\pi_2 : B \times (\mathbb{R}^n, \mathbb{R}^n - 0) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n - 0)$$

is the projection onto the second factor and $g_n \in H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R)$ is the canonical generator. Since $H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R)$ is free, we have by the Künneth formula 7.4.3 that there is an isomorphism

$$H^{q-n}(B; R) \otimes_R H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R) \longrightarrow H^q(B \times (\mathbb{R}^n, \mathbb{R}^n - 0); R)$$

defined by $b \otimes y \mapsto b \times y$. On the other hand, we also have an isomorphism

$$H^{q-n}(B;R) \longrightarrow H^{q-n}(B;R) \otimes_R H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R)\,,$$

defined by $a \mapsto a \times g_n$.

When we combine these isomorphisms, we get an isomorphism

$$H^{q-n}(B;R) \longrightarrow H^q(B \times (\mathbb{R}^n, \mathbb{R}^n - 0); R)\,,$$

which satisfies $b \mapsto b \times g_n$. But this isomorphism is precisely the Thom isomorphism, since $b \times g_n = \pi_1^*(b) \smile \pi_2^*(g_n) = p^*(b) \smile t_E$.

(b) We now assume that this theorem is true for the restriction of the bundle $E \longrightarrow B$ to the open sets U, V, and $U \cap V$ in B. We shall prove that the theorem is also true for $U \cup V$. For every subspace $A \subset B$ define $\varphi_A : H^{q-n}(A) \longrightarrow H^q(E|A, E_0|A)$ by $\varphi_A(b) = p_A^*(b) \smile t_{E|A}$. Since $t_{E|A} = i_A^*(t_E)$, we have a commutative diagram

$$\begin{tikzcd}
H^{q-n}(C) \arrow[r, "\varphi_C"] \arrow[d, "i^*"'] & H^q(E|C, E_0|C) \arrow[d, "\tilde{i}^*"] \\
H^{q-n}(A) \arrow[r, "\varphi_A"] & H^q(E|A, E_0|A)\,,
\end{tikzcd}$$

whenever A and C are subsets of B satisfying $A \subset C$. So we get from the Mayer–Vietoris sequences 7.4.14 of the couple of excisive pairs $(U, \emptyset)$ and $(V, \emptyset)$ as well as for the couple of excisive pairs $(E|U, E_0|U)$ and $(E|V, E_0|V)$ the commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q-n-1}(U\cap V) \arrow[r] \arrow[d, "\varphi_{U\cap V}"', "\cong"] & H^{q-n-1}(U\cup V) \arrow[r] \arrow[d, "\varphi_{U\cup V}"'] & {} \\
\cdots \arrow[r] & H^{q-1}(E|(U\cap V), E_0|(U\cap V)) \arrow[r] & H^{q-1}(E|(U\cup V), E_0|(U\cup V)) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^{q-n}(U) \oplus H^{q-n}(V) \arrow[r] \arrow[d, "\varphi_U \oplus \varphi_V"', "\cong"] & H^{q-n}(U\cap V) \arrow[r] \arrow[d, "\varphi_{U\cap V}"', "\cong"] & \cdots \\
{} \arrow[r] & H^q(E|U, E_0|U) \oplus H^q(E|V, E_0|V) \arrow[r] & H^q(E|(U\cap V), E_0|(U\cap V)) \arrow[r] & \cdots\,.
\end{tikzcd}$$

Applying the five lemma, it follows that $\varphi_{U\cup V}$ is an isomorphism.

(c) If $p : E \longrightarrow B$ is of finite type, then B is covered by a finite number N of open sets over each of which E is trivial. By induction on N and part (b), we obtain the isomorphism in this case.

(d) If B is a CW-complex, then, just as in part (d) of the proof of Proposition 11.7.7, the restriction E^k of E to each skeleton B^k of B is of finite type, and so by part (c), we have an isomorphism $\varphi_k : H^{q-n}(B^k;R) \longrightarrow H^q(E^k, E_0^k;R)$ given by $\varphi_k(b) = p_k^*(b) \smile t_k$, where $t_k = t_{E^k}$ and p_k is the restriction of p to E^k. In analogy to the Milnor sequence (11.7.8) we have an exact sequence

$$0 \longrightarrow \lim_k{}^1 H^{q-n-1}(B^k) \longrightarrow H^{q-n}(B) \longrightarrow \lim_k H^{q-n}(B^k) \longrightarrow 0\,,$$

and then, by the naturality of these sorts of exact sequences, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \lim^1{}_k H^{q-n-1}(B^k) \arrow[r] \arrow[d] & H^{q-n}(B) \arrow[r] \arrow[d, "\varphi_E"] & \lim_k H^{q-n}(B^k) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^1{}_k H^{q-1}(E^k, E_0^k) \arrow[r] & H^q(E, E_0) \arrow[r] & \lim_k H^q(E^k, E_0^k) \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows, both on the right and on the left, are the isomorphisms induced by φ_k. So again by the five lemma (or one could say the "three" lemma), we get that φ_E is also an isomorphism.

(e) In the general case, we take a CW-approximation $f : \widetilde{B} \longrightarrow B$. Of course, this means that $\widetilde{B}$ is a CW-complex and that f is a weak homotopy equivalence. Letting $\widetilde{E} \longrightarrow \widetilde{B}$ be the bundle induced by f, it follows from 4.3.37 that $\widetilde{f} : \widetilde{E} \longrightarrow E$ and $\widetilde{f} : \widetilde{E}_0 \longrightarrow E_0$ are also weak homotopy equivalences, and so they induce isomorphisms in cohomology. Comparing the exact sequences of the pairs $(\widetilde{E}, \widetilde{E}_0)$ and (E, E_0), we find that $\widetilde{f}$ also induces isomorphisms in cohomology between these pairs. We then have the commutative diagram

$$\begin{tikzcd}
H^{q-n}(B) \arrow[r, "f^*", "\cong"'] \arrow[d, "\varphi_E"'] & H^{q-n}(\widetilde{B}) \arrow[d, "\cong"', "\varphi_{\widetilde{E}}"] \\
H^q(E, E_0) \arrow[r, "\cong", "\widetilde{f}^*"'] & H^q(\widetilde{E}, \widetilde{E}_0)\,,
\end{tikzcd}$$

from which we conclude that φ_E is an isomorphism. And with this we have finished the proof of the theorem. □

11.7.21 Note. Since any complex vector bundle $p : E \longrightarrow B$ is orientable, it follows from Theorem 11.7.20 that we have a Thom isomorphism in cohomology with integral coefficients

$$\varphi : H^k(B;\mathbb{Z}) \longrightarrow H^{k+2m}(E, E_0;\mathbb{Z})$$

given by $\varphi(b) = p^*(b) \smile t_E$, where m is the complex dimension of the bundle.

11.7.22 **Theorem**. *Suppose that $p : E \longrightarrow B$ is a real vector bundle of dimension n. Then there exists a long exact sequence*

$$\cdots \longrightarrow H^{q+n-1}(E_0) \xrightarrow{\psi} H^q(B) \xrightarrow{\smile e(E)} H^{q+n}(B) \xrightarrow{p_0^*} H^{q+n}(E_0) \longrightarrow \cdots ,$$

where ψ is given by the composite

$$H^{q+n-1}(E_0) \xrightarrow{\delta} H^{q+n}(E, E_0) \xleftarrow[\cong]{\varphi} H^q(B) .$$

Here φ is the Thom isomorphism (11.7.21) and $p_0 = p|E_0$. Also, all of the groups have coefficients in $\mathbb{Z}/2$. This exact sequence is known as the *Gysin sequence* of the real vector bundle.

Proof: Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^{q+n-1}(E_0) \arrow[r, "\psi"] \arrow[d, "1"'] & H^q(B) \arrow[r, "\smile e(E)"] \arrow[d, "\varphi", "\cong"'] & H^{q+n}(B) \arrow[r, "p_0^*"] \arrow[d, "p^*"] & H^{q+n}(E_0) \arrow[r] \arrow[d, "1"] & \cdots \\ \cdots \arrow[r] & H^{q+n-1}(E_0) \arrow[r, "\delta"'] & H^{q+n}(E, E_0) \arrow[r, "j^*"'] & H^{q+n}(E) \arrow[r, "i^*"'] & H^{q+n}(E_0) \arrow[r] & \cdots , \end{tikzcd}$$

where φ is the Thom isomorphism (11.7.20) and the lower sequence is the long exact sequence of the pair (E, E_0). The first square commutes by definition of ψ and the third by definition of p_0. So we only have to verify the commutativity of the second square. But just as in the proof of Proposition 11.7.19, we have that $e(E) = s_0^* j^*(t_E)$ and that $p^* \circ s_0^* = 1$, where $s_0 : B \longrightarrow E$ is the zero section. Then for all $a \in H^q(B)$ it follows that $p^*(a \smile e(E)) = p^*(a) \smile p^*(s_0^* j^*(t_E)) = p^*(a) \smile j^*(t_E) = j^*(p^*(a) \smile t_E) = j^* \varphi(a)$. □

The next theorem is the version of Theorem 11.7.22 for the complex case.

11.7.23 **Theorem**. *Suppose that $p : E \longrightarrow B$ is a complex vector bundle of dimension m. Then there exists a long exact sequence*

$$\cdots \longrightarrow H^{q+2m-1}(E_0) \xrightarrow{\psi} H^q(B) \xrightarrow{\smile e(E)} H^{q+2m}(B) \xrightarrow{p_0^*}$$
$$\longrightarrow H^{q+2m}(E_0) \longrightarrow \cdots ,$$

where ψ is given by the composite

$$H^{q+2m-1}(E_0) \xrightarrow{\delta} H^{q+2m}(E, E_0) \xleftarrow[\cong]{\varphi} H^q(B) .$$

Here φ is the Thom isomorphism (11.7.21) and $p_0 = p|E_0$. Also, all of the groups have coefficients in $\mathbb{Z}$. This exact sequence is known as the *Gysin sequence* of the complex vector bundle.

Proof: Since $p : E \longrightarrow B$ is a complex vector bundle, the underlying real vector bundle is an oriented vector bundle of dimension $2m$. So, in a way similar to the proof of Theorem 11.7.22, we obtain the desired sequence, except that now we use integral coefficients in the long exact cohomology sequence of the pair (E, E_0) and we use the version 11.7.21 of the Thom isomorphism for complex vector bundles. $\square$

An important application of the Euler class is calculating the cohomology ring $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$. We shall need the next lemma.

11.7.24 **Lemma**. *Let $p : L \longrightarrow \mathbb{RP}^\infty$ be the canonical line bundle. Then L_0 is contractible, where L_0 is the complement in L of the zero section.*

Proof: First note that $L = \mathbb{S}^\infty \times \mathbb{R}/\sim$, where $(x,t) \sim (-x,-t)$. (Cf. Definition 11.3.5.) It follows that $L_0 = (\mathbb{S}^\infty \times (\mathbb{R}-0)/\sim) = ((\mathbb{S}^\infty \times \mathbb{R}^+ \cup \mathbb{S}^\infty \times \mathbb{R}^-)/\sim) \approx \mathbb{S}^\infty \times \mathbb{R}^+ \simeq \mathbb{S}^\infty$. But Theorem 11.1.3 says that $\mathbb{S}^\infty$ is contractible. $\square$

11.7.25 **Theorem**. *The cohomology ring $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is generated as a ring by the Euler class $e(L) \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ and as such can be identified as a polynomial ring in one variable.*

Proof: First let us consider the Gysin sequence (11.7.22) of the canonical line bundle $p : L \longrightarrow \mathbb{RP}^\infty$:

$$0 \longrightarrow H^0(\mathbb{RP}^\infty) \xrightarrow{p_0^*} H^0(L_0) \xrightarrow{\psi} H^0(\mathbb{RP}^\infty) \xrightarrow{\smile e(L)} H^1(\mathbb{RP}^\infty) \xrightarrow{p_0^*}$$
$$\longrightarrow H^1(L_0) \longrightarrow \cdots \longrightarrow H^q(L_0) \xrightarrow{\psi} H^q(\mathbb{RP}^\infty) \xrightarrow{\smile e(L)}$$
$$\longrightarrow H^{q+1}(\mathbb{RP}^\infty) \xrightarrow{p_0^*} H^{q+1}(L_0) \longrightarrow \cdots .$$

Using Lemma 11.7.24, we have that $H^q(L_0) = 0$ for $q > 0$, and so the cup product with the Euler class determines an isomorphism $H^q(\mathbb{RP}^\infty) \cong H^{q+1}(\mathbb{RP}^\infty)$ for $q > 0$. On the other hand, since $H^0(\mathbb{RP}^\infty)$ and $H^0(L_0)$ are isomorphic to $\mathbb{Z}/2$, we have that p_0^* is an isomorphism. It follows that $\psi : H^0(L_0) \longrightarrow H^0(\mathbb{RP}^\infty)$ is the zero homomorphism, and then $\smile e(L) : H^0(\mathbb{RP}^\infty) \longrightarrow H^1(\mathbb{RP}^\infty)$ is also an isomorphism. $\square$

As a consequence of this theorem we can calculate the multiplicative structure of the cohomology with coefficients in $\mathbb{Z}/2$ of real projective spaces.

11.7.26 **Corollary**. *As an algebra over the field $\mathbb{Z}/2 = \mathbb{Z}_2$, we have*

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[e(L_n)]/e(L_n)^{n+1},$$

where $L_n \longrightarrow \mathbb{RP}^n$ is the canonical line bundle.

Proof: Let $C_*(\mathbb{RP}^\infty, \mathbb{RP}^n)$ be the cellular chain complex of the pair of spaces $(\mathbb{RP}^\infty, \mathbb{RP}^n)$. Since the cells of $\mathbb{RP}^\infty - \mathbb{RP}^n$ have dimension greater than n, it follows that $C_i(\mathbb{RP}^\infty, \mathbb{RP}^n) = 0$ for $i \leq n$, and so

$$H^i(\mathbb{RP}^\infty, \mathbb{RP}^n; \mathbb{Z}_2) = 0$$

for $i \leq n$. Then using the long exact sequence of the pair, we get that the inclusion $j : \mathbb{RP}^n \longrightarrow \mathbb{RP}^\infty$ induces an isomorphism $j^* : H^i(\mathbb{RP}^\infty; \mathbb{Z}_2) \longrightarrow H^i(\mathbb{RP}^n; \mathbb{Z}_2)$ for $i \leq n-1$. For $i = n$ we have a portion of the exact sequence

$$0 \longrightarrow H^n(\mathbb{RP}^\infty; \mathbb{Z}_2) \xrightarrow{\ j^*\ } H^n(\mathbb{RP}^n; \mathbb{Z}_2)\,.$$

However, according to Theorem 11.7.25 we have that

$$H^n(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2\,,$$

which implies that j^* is also an isomorphism for $i = n$. Now from the naturality of the Euler class, proved in Proposition 11.7.15, we have that $e(L_n) = e(j^*L) = j^*(e(L))$. Also, since j^* is multiplicative, Theorem 11.7.25 implies that the generators of $H^*(\mathbb{RP}^n; \mathbb{Z}_2)$ as an abelian group are the powers $e(L_n)^i$ for $0 \leq i \leq n$. □

The following is a rather interesting consequence.

11.7.27 **Corollary**. *Suppose that $p : T\mathbb{S}^n \longrightarrow \mathbb{S}^n$ is the tangent bundle of the n-sphere. Then we have that $e(T\mathbb{S}^n) \in H^n(\mathbb{S}^n; \mathbb{Z}/2)$ is zero.*

Proof: Let $q : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$ be the quotient map. Since q is a local diffeomorphism, p is the bundle induced from the tangent bundle $p' : T\mathbb{RP}^n \longrightarrow \mathbb{RP}^n$ by q, and so we have a commutative square

$$\begin{array}{ccc} T\mathbb{S}^n & \xrightarrow{\ \widetilde{q}\ } & T\mathbb{RP}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathbb{S}^n & \xrightarrow[\ q\ ]{} & \mathbb{RP}^n\,, \end{array}$$

where $\widetilde{q}$ is the derivative of q. Moreover, $\widetilde{q}$ induces isomorphisms on the fibers. Now let us consider $q^* : H^n(\mathbb{RP}^n; \mathbb{Z}/2) \longrightarrow H^n(\mathbb{S}^n; \mathbb{Z}/2)$ for $n > 1$. According to Corollary 11.7.26, $e(L_n)^n$ is the generator of $H^n(\mathbb{RP}^n; \mathbb{Z}/2)$. Since $q^*(e(L_n)^n) = (q^*e(L_n))^n$ and $q^*e(L_n) \in H^1(\mathbb{S}^n; \mathbb{Z}/2) = 0$, it follows that $q^* : H^n(\mathbb{RP}^n; \mathbb{Z}/2) \longrightarrow H^n(\mathbb{S}^n; \mathbb{Z}/2)$ is the zero homomorphism. Then by the naturality of the Euler class we get $e(T\mathbb{S}^n) = q^*(e(T\mathbb{RP}^n)) = 0$. This proves the result for the case $n > 1$.

For the case $n = 1$ we note that $T\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a trivial bundle and so has a nowhere-zero section. But this implies by Proposition 11.7.19 that $e(T\mathbb{S}^1) = 0$. □

It is an *exercise* to check that we also have complex versions, as follows, of the previous theorems for the cohomology of complex projective spaces.

11.7.28 **Theorem**. *The cohomology ring $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ is generated as a ring by the Euler class $e(L) \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ and as such can be identified as a polynomial ring in one variable.* □

11.7.29 **Corollary**. *As an algebra over $\mathbb{Z}$ we have*

$$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[e(L_n)]/e(L_n)^{n+1}\,,$$

where $L_n \longrightarrow \mathbb{CP}^n$ is the canonical line bundle. □

In order to construct the $(n-1)$st Stiefel–Whitney class of a real n-vector bundle we shall use generalizations of the Thom isomorphism theorem and of the Gysin sequence, which we shall present in the following discussion. Before doing that we present a definition.

11.7.30 Definition. Suppose that $p : E \longrightarrow B$ is a vector bundle over a CW-complex B. Using the discussion just prior to Definition 8.1.20, we know that there exists a Riemannian metric on p that endows each fiber $p^{-1}(x)$ with a scalar product $\langle -,-\rangle_x$ that depends continuously on $x \in B$. The *sphere bundle* associated to the bundle $p : E \longrightarrow B$, which we denote by $S(E) \longrightarrow B$, is the locally trivial bundle whose total space is defined by

$$S(E) = \{y \in E \mid \langle y,y\rangle_x = 1, x = p(y)\}\,.$$

11.7.31 Exercise. Verify that the map $S(E) \longrightarrow B$ (which is the restriction of p to $S(E)$) does actually define a locally trivial bundle. (Hint: Whenever $E \longrightarrow B$ is trivial over some $U \subset B$, then $S(E) \longrightarrow B$ also is trivial over U.)

Suppose that B is a CW-complex and that $C \subset B$ is a subcomplex. For any real vector bundle $p : E \longrightarrow B$ of dimension n, let $p_C : E|C \longrightarrow C$ denote the restriction of the bundle to C and let $E_0|C$ denote the complement of the zero section in $E|C$. Since B is a CW-complex, E also is a CW-complex and both $E|C$ and E_0 are subcomplexes of E. Moreover, the inclusion $S(E) \hookrightarrow E_0$ is a homotopy equivalence. Since $E|C$ and $S(E)$ are subcomplexes of E, the triple $(E|C \cup S(E); E|C, S(E))$ satisfies 7.1.8, and consequently the triple $(E|C \cup E_0; E|C, E_0)$ also does. Therefore, the inclusions induce these isomorphisms in cohomology:

$$H^q(E|C \cup E_0, E_0) \cong H^q(E|C, E_0|C), \qquad (11.7.32)$$

$$(11.7.33) \qquad H^q(E|C \cup E_0, E|C) \cong H^q(E_0, E_0|C).$$

The next theorem not only is the relative version of the Thom isomorphism theorem 11.7.20 but is also is a consequence of it, as we shall now see.

11.7.34 **Theorem**. *Let $p : E \longrightarrow B$ be a real vector bundle of dimension n over a* CW*-complex B with $C \subset B$ a subcomplex. Then for each q we have an isomorphism*

$$\varphi : H^q(B, C; \mathbb{Z}/2) \longrightarrow H^{q+n}(E, E|C \cup E_0; \mathbb{Z}/2)\,.$$

Proof: Consider the commutative diagram in cohomology with $\mathbb{Z}/2$ coefficients

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q+n-1}(E|C,E_0|C) \arrow[r] & H^{q+n}(E,E_0^C) \arrow[r] & H^{q+n}(E,E_0) \arrow[r] & H^{q+n}(E|C,E_0|C) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{q-1}(C) \arrow[r] \arrow[u, "\alpha", "\cong"'] & H^q(B,C) \arrow[r] \arrow[u, "\beta"] & H^q(B) \arrow[r] \arrow[u, "\gamma", "\cong"'] & H^q(C) \arrow[r] \arrow[u, "\cong"'] & \cdots,
\end{tikzcd}$$

where $E_0^C = E|C \cup E_0$. Here the first row is the exact sequence of the triple (E, E_0^C, E_0) (see 7.1.33), where we have substituted $H^*(E|C, E_0|C)$ in place of $H^*(E_0^C, E_0)$ using (11.7.32). And the second row is the exact sequence of the pair (B, C). Finally, the vertical arrows α, β, and γ are given by the Thom classes $t_{E|C}$, t_E, and t_E; namely, they send x to $p^*(x) \smile t_{E|C}$, $p^*(x) \smile t_E$, and $p^*(x) \smile t_E$ for $x \in H^q(C)$, $H^q(B, C)$, and $H^q(B)$, respectively. According to the Thom isomorphism theorem 11.7.20, α and γ are isomorphisms, so that an application of the five lemma gives us that β also is an isomorphism, as we wanted to show. $\square$

11.7.35 Exercise. Suppose that $p : E \longrightarrow B$ is a complex vector bundle of dimension m over a CW-complex B and that $C \subset B$ is a subcomplex. Prove that there is an isomorphism

$$\varphi : H^q(B, C; \mathbb{Z}) \longrightarrow H^{q+2m}(E, E|C \cup E_0; \mathbb{Z})\,.$$

There also is a relative version of the Gysin sequence that, as we shall see in the following, is like the absolute Gysin sequence of Theorem 11.7.22 in that it is a consequence of the Thom isomorphism theorem, although now of the relative Thom isomorphism theorem 11.7.34.

11.7.36 **Theorem**. *Suppose that $p : E \longrightarrow B$ is a real vector bundle of dimension n over a* CW*-complex B and that $C \subset B$ is a subcomplex. Then there exists an exact sequence in cohomology with coefficients in $\mathbb{Z}/2$,*

$$\cdots \longrightarrow H^{q+n-1}(E_0, E_0|C) \xrightarrow{\ \psi\ }$$
$$\longrightarrow H^q(B,C) \xrightarrow{\smile e(E)} H^{q+n}(B,C) \xrightarrow{p_0^*} H^{q+n}(E_0, E_0|C) \longrightarrow \cdots .$$

This exact sequence is known as the *relative Gysin sequence* of the real vector bundle.

Proof: Analogously to the absolute case in Theorem 11.7.22, we consider the commuative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q+n-1}(E_0,E_0|C) \arrow[r, "\psi"] & H^q(B,C) \arrow[r, "\smile e(E)"] \arrow[d, "\cong"', "\varphi"] & H^{q+n}(B,C) \arrow[r, "p_0^*"] \arrow[d, "p^*"] & H^{q+n}(E_0,E_0|C) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{q+n-1}(E_0^C,E|C) \arrow[u, "\cong", "\iota^*"'] \arrow[r, "\delta"'] & H^{q+n}(E,E_0^C) \arrow[r] & H^{q+n}(E,E|C) \arrow[r] & H^{q+n}(E_0^C,E|C) \arrow[u, "\cong", "\iota^*"'] \arrow[r] & \cdots ,
\end{tikzcd}$$

where $E_0^C = E|C \cup E_0$, φ is the relative Thom isomorphism 11.7.34 and the lower sequence is the long exact sequence of the triple $(E, E_0^C, E|C)$. The fact that $p : (E, E|C) \longrightarrow (B, C)$ is a homotopy equivalence implies that p^* is an isomorphism. Then using (11.7.33), we have that ι^* is an isomorphism. Next we define $\psi = \varphi^{-1} \circ \delta \circ (\iota^*)^{-1}$. We then can verify, in a way similar to the proof of Theorem 11.7.22, that the second square is commutative. In the same way we check the commutativity of the third square. Therefore, the exactness of the lower sequence implies the exactness of the upper sequence. □

11.7.37 Exercise. Let $p : E \longrightarrow B$ be a complex vector bundle of dimension m. Prove that there exists an exact sequence in cohomology with integral coefficients

$$\cdots \longrightarrow H^{q+2m-1}(E_0, E_0|C) \xrightarrow{\ \psi\ } H^q(B,C) \longrightarrow$$
$$\xrightarrow{\smile e(E)} H^{q+2m}(B,C) \xrightarrow{p_0^*} H^{q+2m}(E_0, E_0|C) \longrightarrow \cdots .$$

This exact sequence is known as the *relative Gysin sequence* of the complex vector bundle.

11.8 Construction of Characteristic Classes and Applications

In this section we shall use the Gysin sequence studied in the previous section to construct the Stiefel–Whitney classes of a real vector bundle. Then we shall indicate how to realize the corresponding program of constructing the Chern classes of a complex vector bundle. Finally, as an application of the Stiefel–Whitney classes, we shall prove the Borsuk–Ulam theorem in its general form.

11.8.1 Definition. Let $p : E \longrightarrow B$ be a real vector bundle of dimension n. Letting E_0 denote the complement of the zero section in E as usual, we now define a new bundle of dimension $n-1$ over E_0, denoted by $q : \widetilde{E} \longrightarrow E_0$, as follows.

Consider $\widetilde{E}' = \{(v,e) \in E_0 \times E \mid p(v) = p(e)\} \longrightarrow E_0$, which is the bundle over E_0 induced from the bundle p by the map $p|E_0$. Next, take the line subbundle of $\widetilde{E}'$ given by $L = \{(v,e) \in \widetilde{E}' \mid e = \lambda v,\, \lambda \in \mathbb{R}\}$. We then define $q : \widetilde{E} \longrightarrow E_0$ to be the bundle quotient $\widetilde{E} = \widetilde{E}'/L \longrightarrow E_0$. For any $v \in E_0$ the fiber $q^{-1}(v)$ is the vector space quotient $p^{-1}(b)/\langle v\rangle$, where $p(v) = b$ defines $b \in B$ and $\langle v\rangle$ denotes the subspace of $p^{-1}(b)$ generated by the vector $v \in p^{-1}(b)$. It follows that the dimension of the bundle $\widetilde{E} \longrightarrow E_0$ is $n-1$.

Clearly, this construction can also be carried out in the complex case.

11.8.2 Note. Define $p_0 = p|E_0 : E_0 \longrightarrow B$, and then let $i_b : p_0^{-1}(b) \hookrightarrow E_0$ denote the inclusion of the fiber over $b \in B$. Then the restriction $\widetilde{E}|p_0^{-1}(b) = i_b^*(\widetilde{E})$ has total space $\bigcup_{v\in p_0^{-1}(b)}(p^{-1}(b)/\langle v\rangle)$. Since the dimension of $p : E \longrightarrow B$ is n, it follows that $p_0^{-1}(b) \approx \mathbb{R}^n - 0$, which in turn implies that $i_b^*(\widetilde{E})$ is essentially the bundle over $\mathbb{R}^n - 0$ whose fiber over a point v is $\mathbb{R}^n/\langle v\rangle = v^\perp$. In other words, this fiber is the hyperplane in $\mathbb{R}^n$ orthogonal to v, so that restricting even further to $\mathbb{S}^{n-1} \subset \mathbb{R}^n - 0$ we obtain the tangent bundle of the $(n-1)$-sphere.

11.8.3 Exercise. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be two vector bundles. Prove that $\widetilde{E \oplus E'} \cong \widetilde{E} \oplus p_0'^*(E') \cong p_0^*(E) \oplus \widetilde{E'}$.

11.8.4 **Proposition**. *Suppose that $p : E \longrightarrow B$ is a real vector bundle of dimension n over a* CW*-complex B. Then the Euler class $e(\widetilde{E})$ lies in the image of $p_0^* : H^{n-1}(B;\mathbb{Z}/2) \longrightarrow H^{n-1}(E_0;\mathbb{Z}/2)$.*

Proof: First let us prove this in the case where B is path connected. We start by considering the following portion of the Gysin sequence of the pair $(B,\{b\})$ from Theorem 11.7.36:

$$H^{-1}(B,\{b\}) \xrightarrow{\smile e(E)} H^{n-1}(B,\{b\}) \overset{p_0^*}{\to} H^{n-1}(E_0, p^{-1}(b) - 0) \longrightarrow$$
$$\longrightarrow H^0(B,\{b\})\,.$$

But $H^{-1}(B,\{b\}) = 0$, and since B is path connected, we also have that $H^0(B,\{b\}) = 0$. And this implies that

$$p_0^* : H^{n-1}(B,\{b\}) \longrightarrow H^{n-1}(E_0, p^{-1}(b) - 0)$$

is an isomorphism.

Now consider the following portion of the exact sequence of the pair $(E_0, p^{-1}(b) - 0) = (E_0, \mathbb{R}^n - 0)$:

$$H^{n-2}(\mathbb{R}^n - 0) \longrightarrow H^{n-1}(E_0, \mathbb{R}^n - 0) \xrightarrow{j^*} H^{n-1}(E_0) \xrightarrow{i^*} H^{n-1}(\mathbb{R}^n - 0)\,.$$

If $e(\widetilde{E}) \in H^{n-1}(E_0)$ is the Euler class, then using Corollary 11.7.27 and Note 11.8.2, we have that $i^*(e(\widetilde{E})) = e(i^*(\widetilde{E})) = 0$. By the exactness of the sequence there exists a (unique) element $x \in H^{n-1}(E_0, \mathbb{R}^n - 0)$ satisfying $j^*(x) = e(\widetilde{E})$.

Since the case $n = 1$ is trivial, we can assume that $n > 1$ hereafter. So we then have

$$p_0^* : H^{n-1}(B;\mathbb{Z}/2) \cong H^{n-1}(B,\{b\};\mathbb{Z}/2) \cong H^{n-1}(E_0, \mathbb{R}^n - 0;\mathbb{Z}/2)\,,$$

and therefore $e(\widetilde{E}) \in \operatorname{im}(p_0^*)$ follows from $j^*(x) = e(\widetilde{E})$. And this proves the result in the case that B is path connected.

Finally, for the case where B is not path connected, let us consider $B = \bigcup_\alpha B_\alpha$, where each B_α is a path component. Then using 7.1.5 we have $(i_\alpha^*) : H^*(B) \cong \prod_\alpha H^*(B_\alpha)$, where each $i_\alpha : B_\alpha \longrightarrow B$ is an inclusion. Now applying the previous case to each restriction $i_\alpha^*(E) = E|B_\alpha$, we get the result in this case. □

11.8.5 Definition. Let $p : E \longrightarrow B$ be a real vector bundle of dimension n over a CW-complex B. We shall define the *Stiefel–Whitney classes* $w_i(E) \in H^i(B;\mathbb{Z}/2)$ of the bundle inductively on n, as follows. Consider from Theorem 11.7.22 the following portion of the Gysin sequence of E:

$$H^{i-n}(B;\mathbb{Z}/2) \xrightarrow{\smile e(E)} H^i(B;\mathbb{Z}/2) \xrightarrow{p_0^*} H^i(E_0;\mathbb{Z}/2) \xrightarrow{\psi} H^{i-n+1}(B;\mathbb{Z}/2)\,.$$

For $i \leq n-2$ we have that $H^{i-n}(B;\mathbb{Z}/2)$ and $H^{i-n+1}(B;\mathbb{Z}/2)$ are zero, and so p_0^* is an isomorphism. For $i = n-1$ we have that p_0^* is a monomorphism. Also, from Proposition 11.8.4 it follows that $e(\widetilde{E}) \in \operatorname{im}(p_0^*)$. So, by induction on the dimension n, we define

$$w_n(E) = e(E)$$

and, using the fact that the dimension of $\widetilde{E}$ is $n-1$, for $i<n$ we define

$$w_i(E) = (p_0^*)^{-1}(w_i(\widetilde{E}))\,.$$

In particular, if $\dim E = 0$, we have $w_0(\widetilde{E}) = 1$, and therefore for any E with $\dim E \geq 0$ we also have $w_0(E) = 1$. Finally, for $i > n$ we define $w_i(E) = 0$.

11.8.6 Exercise. Prove that this definition is compatible with the definition of w_1 given in Definition 11.3.2. (Hint: Apply Exercise 11.7.16.)

11.8.7 **Theorem**. *The classes $w_i(E) \in H^i(B;\mathbb{Z}/2)$ defined above in 11.8.5 satisfy the axioms 11.6.1(i)–(iv).*

Proof: First, axiom (i) is satisfied by definition.

To prove axiom (ii) it is enough to note that the Euler class is natural by Proposition 11.7.15.

Let $E \longrightarrow B$ and $E' \longrightarrow B$ be two bundles of dimensions n and n', respectively. Then axiom (iii) follows for $k = n+n'$ from Proposition 11.7.17, since $w_{n+n'}(E \oplus E') = e(E \oplus E')$, $w_n(E) = e(E)$ and $w_{n'}(E') = e(E')$. For $k < n + n'$ we argue by induction on the dimension of $E \oplus E'$. The case of dimension one is straightforward. Next, using 11.8.3, it follows that

$$\begin{aligned}
w_k(E\oplus E') &= (p_0^*)^{-1}\left(w_k(\widetilde{E\oplus E'})\right) \\
&= (p_0^*)^{-1}\left(w_k(\widetilde{E}\oplus p_0^*E')\right) \\
&= (p_0^*)^{-1}\left(\sum_{i+j=k} w_i(\widetilde{E}) \smile w_j(p_0^*E')\right) \\
&= \sum_{i+j=k} (p_0^*)^{-1}(w_i(\widetilde{E})) \smile (p_0^*)^{-1}p_0^*(w_j(E')) \\
&= \sum_{i+j=k} w_i(E) \smile w_j(E')\,.
\end{aligned}$$

Finally, axiom (iv) was proved in Proposition 11.3.7. (See also 11.7.14.)

$\square$

11.8.8 Definition. Let $p : E \longrightarrow B$ be a complex vector bundle of dimension m over a CW-complex B. We shall define the *Chern classes* $c_i(E) \in H^2i(B;\mathbb{Z})$ of the bundle inductively on m, as follows. Consider from Theorem 11.7.23 the following portion of the Gysin sequence of E:

$$H^{2i-2m}(B;\mathbb{Z}) \xrightarrow{\smile e(E)} H^{2i}(B;\mathbb{Z}) \xrightarrow{p_0^*} H^{2i}(E_0;\mathbb{Z}) \xrightarrow{\psi} H^{2i-2m+1}(B;\mathbb{Z})\,.$$

For $2i \leq 2m-2$ we have that $H^{2i-2m}(B;\mathbb{Z})$ and $H^{2i-2m+1}(B;\mathbb{Z})$ are zero, and so p_0^* is an isomorphism. So, by induction on the complex dimension m, we define

$$c_m(E) = e(E)\,,$$

and using the fact that the dimension of $\widetilde{E}$ is $m-1$, for $i<m$ we define

$$c_i(E) = (p_0^*)^{-1}(c_i(\widetilde{E}))\,.$$

In particular, if $\dim E = 0$, we have $c_0(\widetilde{E}) = 1$, and therefore for any E with $\dim E \geq 0$, we also have $c_0(E) = 1$. Finally, for $i > m$ we define $c_i(E) = 0$.

11.8.9 Note. Suppose that $\mathrm{E}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ is the real universal n-vector bundle (cf. Definition 8.3.9). We denote its Stiefel–Whitney classes by $w_i = w_i(\mathrm{E}_n(\mathbb{R}^\infty)) \in H^i(\mathrm{G}_n(\mathbb{R}^\infty);\mathbb{Z}/2)$. These classes are *universal* in the following sense. By the real version of Theorem 8.5.13, for any given real n-vector bundle $E \longrightarrow B$ with paracompact base space there exists a map $f: B \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$, unique up to homotopy, such that $E \cong f^*(\mathrm{E}_n(\mathbb{R}^\infty))$. Therefore, by the naturality of characteristic classes we know that $w_i(E) = f^*(w_i)$. So starting with the classes w_i for $i = 0,1,\ldots,n$ we can construct the Stiefel–Whitney classes of any real n-vector bundle over a paracompact space. The complex case is handled similarly.

We shall calculate the cohomology of the Grassmann manifolds $\mathrm{G}_n(\mathbb{R}^\infty)$ with $\mathbb{Z}/2$ coefficients and $\mathrm{G}_n(\mathbb{C}^\infty)$ with $\mathbb{Z}$ coefficients. This will generalize the calculation of the cohomologies of $\mathrm{G}_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$ and $\mathrm{G}_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$ given in 11.7.26 and 11.7.29, respectively. This will allow us to obtain the uniqueness of the Stiefel–Whitney and the Chern classes. We shall discuss only the real case, but everything is true in the complex case. We begin with a definition.

11.8.10 Definition. A *characteristic class* of dimension i for real n-vector bundles is a function c that assigns to each real n-vector bundle $E \longrightarrow B$ over a paracompact base space an element $c(E) \in H^i(B;\mathbb{Z}/2)$, which is an invariant of the isomorphism class of the bundle and which is natural; that is, whenever $f: B' \longrightarrow B$ is continuous, we have $c(f^*(E)) = f^*(c(E))$. We shall let $\mathcal{C}_n^i$ denote the set of these characteristic classes. This set has the structure of an abelian group, where the sum is given by the formula

$$(c+c')(E) = c(E) + c'(E)\,.$$

Moreover, the collection of these groups for fixed n and variable i has the structure of a graded ring with multiplication

$$\mathcal{C}_n^i \times \mathcal{C}_n^j \longrightarrow \mathcal{C}_n^{i+j}$$

given by the formula

$$(c \cdot c')(E) = c(E) \smile c'(E)\,.$$

It is an *exercise* left to the reader to verify the statements made in the prior definition.

11.8.11 **Theorem**. *There exists an isomorphism of graded rings*

$$\varphi : \mathcal{C}_n^* \cong H^*(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}/2)\,,$$

defined by $\varphi(c) = c(\mathrm{E}_n(\mathbb{R}^\infty))$ *for* $c \in \mathcal{C}_n^*$.

Proof: Define $\psi : H^i(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}/2) \longrightarrow \mathcal{C}_n^i$ for $i = 0, 1, \ldots$ by

$$\psi(x)(E) = f_E^*(x)\,,$$

where $x \in H^i(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}/2)$ and $E \longrightarrow B$ is a real n-vector bundle, which has a classifying map $f_E : B \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$. We claim that ψ is the inverse of φ.

First, since

$$\psi\varphi(c)(E) = f_E^*(\varphi(c)) = f_E^*(c(\mathrm{E}_n(\mathbb{R}^\infty))) = c(f_E^*(\mathrm{E}_n(\mathbb{R}^\infty))) = c(E)\,,$$

it follows that $\psi \circ \varphi = 1$.

Next, for all $x \in H^i(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}/2)$ we have that

$$\varphi\psi(x) = \psi(x)(\mathrm{E}_n(\mathbb{R}^\infty)) = \mathrm{id}^*_{\mathrm{G}_n(\mathbb{R}^\infty)}(x) = x\,,$$

which implies that $\varphi \circ \psi = 1$.

Since φ is clearly a ring homomorphism, we have proved the desired result. $\square$

As we shall see later on in Corollary 11.8.16, the previous theorem together with a knowledge of $H^*(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}/2)$ will allow us to identify all real vector bundle characteristic classes having values in cohomology with $\mathbb{Z}/2$ coefficients.

11.8.12 **Proposition**. *Let* $\mathrm{E}_n^0(\mathbb{R}^\infty)$ *be the complement of the zero section of the real universal bundle. Then there exists a homotopy equivalence* $\alpha : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{E}_n^0(\mathbb{R}^\infty)$ *such that the composite*

$$\mathrm{G}_{n-1}(\mathbb{R}^\infty) \xrightarrow{\ \alpha\ } \mathrm{E}_n^0(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$$

is a classifying map for the bundle $\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty)$.

Proof: Let $\mathbb{R}_1^\infty$ be the subspace of $\mathbb{R}^\infty$ consisting of all vectors of the form $(0, a_1, a_2, \dots)$. The map $\tau : \mathbb{R}_1^\infty \longrightarrow \mathbb{R}^\infty$ defined by $\tau(0, a_1, a_2, \dots) = (a_1, a_2, \dots)$ is a homeomorphism, whose inverse σ is defined by $\sigma(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$.

Then τ determines a homeomorphism

$$\widetilde{\tau} : \mathrm{G}_{n-1}(\mathbb{R}_1^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)\,,$$

defined by $\widetilde{\tau}(V) = \tau V$, whose inverse $\widetilde{\sigma}$ is defined similarly.

We now define $\alpha : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{E}_n^0(\mathbb{R}^\infty)$ by $\alpha(V) = (\langle e_0 \rangle \oplus \widetilde{\sigma}(V), e_0)$, where $e_0 = (1, 0, 0, \dots) \in \mathbb{R}^\infty - 0$. Moreover, we define $\beta : \mathrm{E}_n^0(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)$ by $\beta(W, w) = W/\langle w \rangle$, where $0 \neq w \in W$ and W is an n-dimensional subspace of $\mathbb{R}^\infty$. That is, $\beta(W, w)$ is the orthogonal complement in W of the one-dimensional subspace generated by w. We shall now prove that α and β are homotopy inverses.

First, for any $V \in \mathrm{G}_{n-1}(\mathbb{R}^\infty)$ we note that

$$\beta\alpha(V) = \beta(\langle e_0 \rangle \oplus \widetilde{\sigma}(V), e_0) = (\langle e_0 \rangle \oplus \widetilde{\sigma}(V))/\langle e_0 \rangle = \widetilde{\sigma}(V)\,.$$

The homotopy $h_t : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ defined by $h_t(a_1, a_2, a_3, \dots) = (ta_1, (1-t)a_1 + ta_2, (1-t)a_2 + ta_3, \dots)$ is a monomorphism for every t, and it also induces a homotopy $\widetilde{h_t} : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)$ that begins with $\beta \circ \alpha$ and ends with the identity.

On the other hand, for W, w, and e_0 as above we have

$$\alpha\beta(W, w) = \alpha(W/\langle w \rangle) = (\langle e_0 \rangle \oplus \widetilde{\sigma}(W/\langle w \rangle), e_0)\,.$$

In this case, we define a homotopy $\widetilde{k_t} : \mathrm{E}_n^0(\mathbb{R}^\infty) \longrightarrow \mathrm{E}_n^0(\mathbb{R}^\infty)$ by $\widetilde{k_t}(W, w) = (\langle w(t) \rangle \oplus \widetilde{h_t}(W/\langle w \rangle), w(t))$, where $w(t)$ is any path in $\mathbb{R}^\infty - 0$ going from e_0 to w. Then the homotopy $\widetilde{k_t}$ begins with $\alpha \circ \beta$ and ends with the identity.

Finally, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{R} \times \mathrm{E}_{n-1}(\mathbb{R}^\infty) \arrow[r, "\gamma"] \arrow[d, "q"'] & \mathrm{E}_n(\mathbb{R}^\infty) \arrow[d, "p"] \\
\mathrm{G}_{n-1}(\mathbb{R}^\infty) \arrow[r, "p_0 \circ \alpha"'] & \mathrm{G}_n(\mathbb{R}^\infty)\,,
\end{tikzcd}$$

where in the obvious notation we define $p(W, w) = W$, $q(s, (V, v)) = V$, and $\gamma(s, (V, v)) = (\langle e_0 \rangle \oplus \widetilde{\sigma}(V), se_0 + \sigma(v))$. Moreover, γ is an isomorphism on each fiber, since for each $V \in \mathrm{G}_{n-1}(\mathbb{R}^\infty)$ the fiber over V is $\mathbb{R} \times V$ and γ maps it isomorphically by the formula $(s, v) \mapsto se_0 + \sigma(v)$ to the fiber over $p_0\alpha(V) = \langle e_0 \rangle \oplus \widetilde{\sigma}(V)$, where $p_0 : E_n^0 \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ is the restriction of p. Therefore, using 8.1.14, we conclude that $p_0 \circ \alpha$ classifies $\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)$. □

11.8.13 **Proposition**. *Let*

$$\mathrm{E}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty) \quad \text{and} \quad \mathrm{E}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)$$

for $n > 1$ be the universal bundles. Let $f : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ be a classfying map for the bundle $\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_{n-1}(\mathbb{R}^\infty)$. Then there exists a long exact sequence

$$\cdots \xrightarrow{\ \gamma\ } H^q(\mathrm{G}_n(\mathbb{R}^\infty)) \xrightarrow{\smile e(\mathrm{E}_n(\mathbb{R}^\infty))} H^{q+n}(\mathrm{G}_n(\mathbb{R}^\infty)) \xrightarrow{\ f^*\ }$$
$$\xrightarrow{\ f^*\ } H^{q+n}(\mathrm{G}_{n-1}(\mathbb{R}^\infty)) \xrightarrow{\ \gamma\ } H^{q+1}(\mathrm{G}_n(\mathbb{R}^\infty)) \longrightarrow \cdots .$$

Proof: Using Proposition 11.8.12 we know that the composition $p_0 \circ \alpha : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ classifies $\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty)$ and that α is a homotopy equivalence.

From Theorem 11.7.22 we have the Gysin sequence of $\mathrm{E}_n(\mathbb{R}^\infty)$,

$$\begin{tikzcd}
\arrow[r, "\smile e"] & H^{q+n}(\mathrm{G}_n(\mathbb{R}^\infty)) \arrow[r, "p_0^*"] \arrow[dr, "f^*"'] & H^{q+n}(\mathrm{E}_n^0(\mathbb{R}^\infty)) \arrow[r, "\psi"] \arrow[d, "\alpha^*", "\cong"'] & H^{q+1}(\mathrm{G}_n(\mathbb{R}^\infty)) \arrow[r] & {,} \\
& & H^{q+n}(\mathrm{G}_{n-1}(\mathbb{R}^\infty)) \arrow[ur, dashed, "\gamma"'] & &
\end{tikzcd}$$

where $e = e(\mathrm{E}_n(\mathbb{R}^\infty))$. If we take f to be $p_0 \circ \alpha$ and define γ to be $\psi \circ \alpha^{*-1}$, then we get the desired sequence. □

11.8.14 Note. Propositions 11.8.12 and 11.8.13 clearly are also valid in the complex case.

11.8.15 **Theorem**. *As an algebra over $\mathbb{Z}/2 = \mathbb{Z}_2$,*

$$H^*(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_n],$$

where $w_1, w_2, \ldots, w_n$ are the Stiefel–Whitney classes

$$w_i = w_i(\mathrm{E}_n(\mathbb{R}^\infty)) \in H^i(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}_2), \quad i = 1, \ldots, n.$$

Proof: The proof will be by induction on n. For $n = 1$, the result is nothing other than Corollary 11.7.26. So we assume that the theorem holds for $n-1$ for some $n > 1$. Let $f : \mathrm{G}_{n-1}(\mathbb{R}^\infty) \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ be a classifying map for the bundle $\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty)$. By the naturality property 11.6.1(ii) and the stability property 11.6.5 of the Stiefel–Whitney classes, we get

$$\begin{aligned} f^*(w_i(\mathrm{E}_n(\mathbb{R}^\infty))) &= w_i(f^*(\mathrm{E}_n(\mathbb{R}^\infty))) \\ &= w_i(\varepsilon^1 \oplus \mathrm{E}_{n-1}(\mathbb{R}^\infty)) = w_i(\mathrm{E}_{n-1}(\mathbb{R}^\infty)) \end{aligned}$$

for $i = 1, 2, \ldots, n$. Furthermore, since $\dim \mathrm{E}_{n-1}(\mathbb{R}^\infty) = n-1$, we have that $w_n(\mathrm{E}_{n-1}(\mathbb{R}^\infty)) = 0$.

By the induction hypothesis, we have as algebras

$$\begin{aligned}&H^*(\mathrm{G}_{n-1}(\mathbb{R}^\infty))\\ &\qquad = \mathbb{Z}_2[w_1(\mathrm{E}_{n-1}(\mathbb{R}^\infty)), w_2(\mathrm{E}_{n-1}(\mathbb{R}^\infty)), \ldots, w_{n-1}(\mathrm{E}_{n-1}(\mathbb{R}^\infty))]\,,\end{aligned}$$

implying that the ring homomorphism f^* is surjective in cohomology. By definition $e(\mathrm{E}_n(\mathbb{R}^\infty)) = w_n(\mathrm{E}_n(\mathbb{R}^\infty))$, so that the exact sequence of Proposition 11.8.13 yields the exact sequence

$$H^q(\mathrm{G}_n(\mathbb{R}^\infty)) \xrightarrow{\smile w_n} H^{q+n}(\mathrm{G}_n(\mathbb{R}^\infty)) \xrightarrow{f^*} H^{q+n}(\mathrm{G}_{n-1}(\mathbb{R}^\infty))\,.$$

From this short exact sequence we find that every element $a \in H^{q+n}(\mathrm{G}_n(\mathbb{R}^\infty))$ can be written as $a = b + c$, where b comes from $H^q(\mathrm{G}_n(\mathbb{R}^\infty))$ and therefore b is a polynomial in which every term contains w_n. Moreover, c comes from $H^{q+n}(\mathrm{G}_{n-1}(\mathbb{R}^\infty))$, and so by the induction hypothesis c is a polynomial in $w_1, w_2, \ldots, w_{n-1}$. Now an induction on the dimension of a proves the desired result. □

From Theorems 11.8.11 and 11.8.15 we immediately get the following.

11.8.16 **Corollary**. *Let c be a characteristic class of dimension k for real n-dimensional vector bundles. Then we have that*

$$c = \sum_{J \in \mathcal{I}_k} \lambda_J w_1^{i_1} w_2^{i_2} \cdots w_n^{i_n}\,,$$

where $\mathcal{I}_k = \{J = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{\nu=1}^n i_\nu = k\}$ and $\lambda_J \in \mathbb{Z}/2$. That is, for every real n-dimensional vector bundle E we have

$$c(E) = \sum_{J \in \mathcal{I}_k} \lambda_J w_1^{i_1}(E) \smile w_2^{i_2}(E) \smile \cdots \smile w_n^{i_n}(E)\,.$$

□

The previous corollary implies that any characteristic class for real vector bundles can be expressed in terms of the Stiefel–Whitney classes. We shall now see that these latter classes are characterized by axioms 11.6.1(i)–(iv).

11.8.17 **Proposition**. *Suppose that $L \longrightarrow \mathbb{RP}^\infty$ is the canonical bundle over $\mathbb{RP}^\infty$ and that $f : \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ is a map that classifies the bundle $L \times \cdots \times L$ (with n factors). Then the homomorphism*

$$f^* : H^*(\mathrm{G}_n(\mathbb{R}^\infty); \mathbb{Z}_2) \longrightarrow H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty; \mathbb{Z}_2)$$

is a monomorphism.

Before starting the proof, note that if $V_1, V_2, \dots, V_n \in \mathbb{RP}^\infty$ are distinct one-dimensional subspaces of $\mathbb{R}^\infty$, then $f(V_1, V_2, \dots, V_n) = V_1 \oplus V_2 \oplus \cdots \oplus V_n \in \mathrm{G}_n(\mathbb{R}^\infty)$.

Proof: We know from Theorem 11.7.25 that

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[w_1(L)]\,.$$

Using the Künneth formula 7.4.4, which in this case asserts that

$$H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty; \mathbb{Z}_2) = H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \otimes \cdots \otimes H^*(\mathbb{RP}^\infty; \mathbb{Z}_2)\,,$$

we can deduce that $H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, \dots, t_n]$, where we define $t_i = \pi_i^*(w_1(L))$, and $\pi_i : \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$ is the projection onto the ith coordinate. By hypothesis we have $f^*(\mathrm{E}_n(\mathbb{R}^\infty)) = L \times \cdots \times L$. And using Exercise 8.1.7, we have $L \times \cdots \times L = \pi_1^*(L) \oplus \cdots \oplus \pi_n^*(L)$.

Using the naturality axiom 11.6.1(ii) and the Whitney formula axiom 11.6.1(iii), but applied now to the total Stiefel–Whitney class of Definition 11.6.6, we get

$$\begin{aligned} f^*(w(\mathrm{E}_n(\mathbb{R}^\infty))) &= w(f^*(\mathrm{E}_n(\mathbb{R}^\infty))) \\ &= w(L \times \cdots \times L) \\ &= w(\pi_1^*(L) \oplus \cdots \oplus \pi_n^*(L)) \\ &= \prod_{i=1}^n w(\pi_i^*(L)) \\ &= \prod_{i=1}^n (1 + t_i)\,. \end{aligned}$$

Consequently, for each dimension 1 to n we have

$$\begin{aligned} f^*(w_1(\mathrm{E}_n(\mathbb{R}^\infty))) &= t_1 + \cdots + t_n \\ f^*(w_2(\mathrm{E}_n(\mathbb{R}^\infty))) &= t_1t_2 + t_1t_3 + \cdots + t_1t_n + \cdots + t_{n-1}t_n \\ &\;\vdots \\ f^*(w_n(\mathrm{E}_n(\mathbb{R}^\infty))) &= t_1 \cdots t_n\,. \end{aligned}$$

In other words, this says that

$$f^*(w_k(\mathrm{E}_n(\mathbb{R}^\infty))) = \sigma_k(t_1, \dots, t_n), \qquad k = 1, \dots n,$$

where σ_k for $k = 1, \dots, n$ denotes the *kth elementary symmetric function* in n variables, which is defined in general by

$$\sigma_k(a_1, a_2, \dots, a_n) = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1} a_{i_2} \cdots a_{i_k}\,.$$

It is a fundamental result of Artin [10] that the subring of $\mathbb{Z}_2[t_1, \ldots, t_n]$ consisting of the symmetric polynomials is in turn the ring of polynomials generated by the elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Since $H^*(\mathrm{G}_n(\mathbb{R}^\infty)) = \mathbb{Z}_2[w_1(\mathrm{E}_n(\mathbb{R}^\infty)), \ldots, w_n(\mathrm{E}_n(\mathbb{R}^\infty))]$ holds by Theorem 11.8.15, it follows that f^* is injective. In fact, the image of f^* is precisely the subring of the symmetric polynomials. □

Now we have assembled enough machinery to dispose of the proof of the uniqueness of the Stiefel–Whitney classes in short order.

11.8.18 **Theorem**. (Uniqueness of the Stiefel–Whitney classes) *There exists a unique sequence of cohomology classes associated to real vector bundles over paracompact base spaces and satisfying axioms 11.6.1(i)–(iv).*

Proof: Let us assume that for every real vector bundle over a paracompact base space we have a sequence of cohomology classes $\widetilde{w}_i(E)$ that are invariants of the isomorphism class of the bundle and that satisfy 11.6.1(i)–(iv). Consider the canonical line bundle $L_1 \longrightarrow \mathbb{RP}^1$. By axiom 11.6.1(iv) we have that $\widetilde{w}_1(L_1) = w_1(L_1)$, since both coincide with the nonzero element of $H^1(\mathbb{RP}^1;\mathbb{Z}_2) = \mathbb{Z}_2$. Because L_1 is induced from the canonical line bundle $L \longrightarrow \mathbb{RP}^\infty$ by the inclusion $\iota : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$, and $\iota^* : H^1(\mathbb{RP}^\infty;\mathbb{Z}_2) \longrightarrow H^1(\mathbb{RP}^1;\mathbb{Z}_2)$ is an isomorphism (see 11.7.14), the naturality axiom 11.6.1(ii) implies that $\iota^*(\widetilde{w}_1(L)) = \widetilde{w}_1(L_1) = w_1(L_1)$ and therefore $\widetilde{w}_1(L) = w_1(L)$. Consequently, the total class corresponding to the classes $\widetilde{w}_k$, defined again as the sum of all together, satisfies $\widetilde{w}(L) = 1 + w_1(L)$.

Let $f : \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$ be as before the classifying map of the bundle $L \times \cdots \times L \longrightarrow \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty$. Then from the naturality axiom 11.6.1(ii) and the Whitney formula axiom 11.6.1(iii), much as in the proof of Proposition 11.8.17, it follows that

$$\begin{aligned}
f^*(\widetilde{w}(\mathrm{E}_n(\mathbb{R}^\infty))) &= \widetilde{w}(f^*\mathrm{E}_n(\mathbb{R}^\infty)) \\
&= \widetilde{w}(L \times \cdots \times L) \\
&= \widetilde{w}(\pi_1^*(L) \oplus \cdots \oplus \pi_n^*(L)) \\
&= \prod_{i=1}^n \widetilde{w}(\pi_i^*(L)) \\
&= \prod_{i=1}^n (1 + \widetilde{w}_1(\pi_i^*(L))) \\
&= \prod_{i=1}^n (1 + w_1(\pi_i^*(L)))
\end{aligned}$$

$$= \prod_{i=1}^{n}(1+t_i)$$
$$= f^*(w(\mathrm{E}_n(\mathbb{R}^\infty)))\,.$$

Here $t_i = \pi_i^*(w_1(L)) = w_1(\pi_i^*(L))$ is just as in the proof of Proposition 11.8.17.

But again using Proposition 11.8.17, we know that f induces a monomorphism f^* in cohomology. And so we obtain from the previous calculation that $\widetilde{w}(\mathrm{E}_n(\mathbb{R}^\infty)) = w(\mathrm{E}_n(\mathbb{R}^\infty))$.

Now if $E \longrightarrow B$ is any real vector bundle of dimension n over a paracompact space with classifying map $f_E : B \longrightarrow \mathrm{G}_n(\mathbb{R}^\infty)$, then using the naturality axiom 11.6.1(ii) and the result just obtained we find that

$$\begin{aligned}\widetilde{w}(E) &= \widetilde{w}(f_E^*(\mathrm{E}_n(\mathbb{R}^\infty)))\\ &= f_E^*(\widetilde{w}(\mathrm{E}_n(\mathbb{R}^\infty)))\\ &= f_E^*(w(\mathrm{E}_n(\mathbb{R}^\infty)))\\ &= w(f_E^*(\mathrm{E}_n(\mathbb{R}^\infty)))\\ &= w(E)\,.\end{aligned}$$

And this proves that the two sequences of characteristic classes for this bundle are equal term by term. □

We shall now give some interesting applications of characteristic classes. First we shall see that those that are nonzero are *obstructions* to the existence of nowhere-zero sections of a bundle. To do this we start off with a definition.

11.8.19 DEFINITION. Suppose that $p : E \longrightarrow B$ is a vector bundle with sections $s_1, s_2, \ldots, s_k$. We say that these sections are *linearly independent* if for each point $b \in B$ the vectors $s_1(b), s_2(b), \ldots, s_k(b)$ are linearly independent as elements of the vector space $p^{-1}(b)$. In particular, each section s_i is nowhere zero. (See 11.3.11.)

11.8.20 **Lemma**. *Let $p : E \longrightarrow B$ be a real vector bundle over a paracompact space B, for example a* CW*-complex. If the bundle admits linearly independent sections $s_1, s_2, \ldots, s_k$, then the bundle has a decomposition as a sum $E' \oplus \varepsilon^k$, where ε^k is a trivial bundle of dimension k and $E' \longrightarrow B$ is some other bundle.*

Proof: The subbundle E_1 of E defined by $E_1 = \{e = \sum_{i=1}^k \lambda_i s_i(x) \mid \lambda_i \in \mathbb{R} \text{ and } x \in B\}$ is a trivial bundle of dimension k, as can be seen from the explicit trivialization $B \times \mathbb{R}^k \longrightarrow E_1$ defined by $(b, \lambda_1, \ldots, \lambda_k) \mapsto \sum \lambda_i s_i(b)$. Since B is paracompact, the bundle E has a Riemannian metric (see Definition 8.1.20 and the discussion preceeding it), and so by Proposition 8.1.23 there exists a subbundle E_2 of E that is the orthogonal complement of E_1 in E and that, moreover, satisfies $E \cong E_2 \oplus \varepsilon^k$. □

Combining Proposition 11.6.5 with Lemma 11.8.20, we can prove a result that generalizes Proposition 11.7.19. Specifically, from 11.6.5 we get $w_i(E) = w_i(E_2)$, which implies for $i > \dim(E_2) = n - k$ that $w_i(E) = 0$. We then have the next result.

11.8.21 **Proposition**. *Suppose that $E \longrightarrow B$ is a real vector bundle of dimension n and that B is a paracompact space. If the bundle admits a nowhere-zero section, then $w_n(E) = 0$. More generally, if the bundle admits k linearly independent sections, then*

$$w_{n-k+1}(E) = w_{n-k+2}(E) = \cdots = w_n(E) = 0.$$ □

In this way, the last nonzero Stiefel–Whitney class, say w_{n-k}, is an *obstruction* to the existence of more than k linearly independent sections in E. There is a similar statement for complex vector bundles and Chern classes, using the corresponding results for the complex case. They are stated below and are proved in exactly the same way as their counterparts in the real case, and are left to the reader as *exercises*.

11.8.22 **Theorem**. *The classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$ defined in 11.8.8 satisfy the axioms 11.6.7(i)–(iv).* □

Let now $\mathcal{C}_n^i$ denote the set of characteristic classes for complex n-bundles with values in $H^i(B; \mathbb{Z})$, as in 11.8.10.

11.8.23 **Theorem**. *There exists an isomorphism of graded rings*

$$\varphi : \mathcal{C}_n^* \cong H^*(\mathrm{G}_n(\mathbb{C}^\infty); \mathbb{Z}),$$

defined by $\varphi(c) = c(\mathrm{E}_n(\mathbb{C}^\infty))$ for $c \in \mathcal{C}_n^*$. □

11.8.24 **Theorem**. *As an algebra over $\mathbb{Z}$,*

$$H^*(\mathrm{G}_n(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n],$$

where $c_1, c_2, \ldots, c_n$ are the Chern classes

$$c_i = c_i(\mathrm{E}_n(\mathbb{C}^\infty)) \in H^{2i}(\mathrm{G}_n(\mathbb{C}^\infty); \mathbb{Z}), \quad i = 1, \ldots, n.$$ □

11.8.25 **Corollary**. *Let c be a characteristic class of dimension k for complex n-dimensional vector bundles. Then we have that*

$$c = \sum_{J \in \mathcal{I}_k} \lambda_J c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n} \,,$$

where $\mathcal{I}_k = \{J = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n \mid \sum_{\nu=1}^n i_\nu = k\}$ and $\lambda_J \in \mathbb{Z}$. That is, for every complex n-dimensional complex vector bundle E we have

$$c(E) = \sum_{J \in \mathcal{I}_k} \lambda_J c_1^{i_1}(E) \smile c_2^{i_2}(E) \smile \cdots \smile c_n^{i_n}(E) \,.$$

□

11.8.26 **Proposition**. *Suppose that $L \longrightarrow \mathbb{CP}^\infty$ is the canonical bundle over $\mathbb{CP}^\infty$ and that $f : \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \longrightarrow \mathrm{G}_n(\mathbb{C}^\infty)$ is a map that classifies the bundle $L \times \cdots \times L$ (with n factors). Then the homomorphism*

$$f^* : H^*(\mathrm{G}_n(\mathbb{C}^\infty); \mathbb{Z}) \longrightarrow H^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty; \mathbb{Z})$$

is a monomorphism. □

11.8.27 **Theorem**. (Uniqueness of the Chern classes) *There exists a unique sequence of cohomology classes associated to complex vector bundles over paracompact base spaces and satisfying axioms 11.6.7(i)–(iv).* □

To end this chapter we shall now present one more application of the Stiefel–Whitney classes. This will be a proof of the Borsuk–Ulam theorem, whose classical formulation is as follows. Already in Chapter 2, we have given in 2.4.29 the special case where $n = 2$.

11.8.28 **Theorem**. (Borsuk–Ulam) *Suppose that $g : \mathbb{S}^n \longrightarrow \mathbb{R}^n$ is continuous. Then there exists $x \in \mathbb{S}^n$ that satisfies $g(x) = g(-x)$.*

Proof: If there were no such point x, that is, if $g(x) \neq g(-x)$ for every $x \in \mathbb{S}^n$, then the formula

$$f(x) = \frac{g(x) - g(-x)}{\|g(x) - g(-x)\|}$$

would define an *odd* map

$$f : \mathbb{S}^n \longrightarrow \mathbb{S}^{n-1},$$

namely, a map satisfying $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$. However, this would contradict Theorem 11.8.29, which we shall prove later. So the desired point $x \in \mathbb{S}^n$ has to exist. □

As it was in the case $n = 2$ (2.4.31), we have the following.

11.8.29 **Theorem**. *For $m < n$ there does not exist an odd map $f : \mathbb{S}^n \longrightarrow \mathbb{S}^m$, that is, a map satisfying $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$.*

Proof: If there were such a map f, then it would induce a map $\overline{f} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^m$ making the diagram

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{S}^m \arrow[d, "q"] \\
\mathbb{RP}^n \arrow[r, "\overline{f}"'] & \mathbb{RP}^m
\end{tikzcd}$$

commute, where p and q are the usual quotient maps. This is really a diagram of locally trivial bundles, which in turn induces a map $H_n \longrightarrow H_m$ of the canonical line bundles over the projective spaces. More precisely, for every k the canonical line bundle $H_k \longrightarrow \mathbb{RP}^k$, which is given in Definition 11.3.5, is the projection onto the second coordinate restricted to the space of pairs

$$H_k = \{(x, l) \in \mathbb{R}^{k+1} \times \mathbb{RP}^k \mid x \in l\}.$$

Then there is a commutative diagram of vector bundles

$$\begin{tikzcd}
H_n \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & H_m \arrow[d, "q"] \\
\mathbb{RP}^n \arrow[r, "\overline{f}"'] & \mathbb{RP}^m,
\end{tikzcd}$$

where $\widetilde{f}(x, l) = (|x| f(x/|x|), \overline{f}(l))$ for $x \neq 0$ and $\widetilde{f}(0, l) = (0, \overline{f}(l))$. It immediately follows that $\widetilde{f}$ is well defined and is continuous. Moreover, it is linear on the fibers, for which it is enough to show that it commutes with scalar multiplication, namely that

$$\begin{aligned}
\widetilde{f}(\lambda x, l) &= (|\lambda x| f(\frac{\lambda x}{|\lambda x|}), \overline{f}(l)) \\
&= \begin{cases} (\lambda |x| f(\frac{x}{|x|}), \overline{f}(l)) = \lambda \widetilde{f}(x, l) & \text{if } \lambda \geq 0, \\ (-\lambda |x| f(\frac{-x}{|x|}), \overline{f}(l)) = \lambda \widetilde{f}(x, l) & \text{if } \lambda < 0, \end{cases}
\end{aligned}$$

where the second case follows from the first case and the fact that f is odd.

Using Proposition 11.3.3, we find that the homomorphism induced in cohomology $\overline{f}^* : H^*(\mathbb{RP}^m; \mathbb{Z}/2) \longrightarrow H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ satisfies $\overline{f}^*(x_m) = x_n$, where $x_k = w_1(H_k) \in H^1(\mathbb{RP}^k;\mathbb{Z}/2)$ is the Euler class of the bundle $H_k \longrightarrow \mathbb{RP}^k$ for $k = m, n$. In particular, using Proposition 11.3.7 and $m+1 \leq n$, we have that $0 = \overline{f}^*(x_m^{m+1}) = x_n^{m+1} \neq 0$. And this is a contradiction. Consequently, there cannot exist an odd map $f : \mathbb{S}^n \longrightarrow \mathbb{S}^m$. □

11.8.30 Note. There is an alternative way of proving the Borsuk–Ulam theorem, in the formulation of Theorem 11.8.29, by using the theory of covering maps as well as cohomology theory. Specifically, the square diagram

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{S}^m \arrow[d, "q"] \\ \mathbb{RP}^n \arrow[r, "\overline{f}"'] \arrow[ur, dashed, "\widetilde{f}"] & \mathbb{RP}^m, \end{tikzcd}$$

which we used in the proof of Theorem 11.8.29, is a diagram of covering maps (see 4.5.3). Now we pose the question of the existence of a lift $\widetilde{f} : \mathbb{RP}^n \longrightarrow \mathbb{S}^m$ of $\overline{f}$, as is indicated in the previous diagram. Such a lift exists if and only if $\overline{f}$ sends the fundamental group $\pi_1(\mathbb{RP}^n)$ into the image under q of the fundamental group $\pi_1(\mathbb{S}^m)$, as we have seen in Exercise 4.5.14. There are two cases. In the first case, when we have $m = 1$, it follows that $\pi_1(\mathbb{RP}^1) = \mathbb{Z}$ and, since $n > 1$, that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$. Therefore, the homomorphism $\overline{f}_* : \pi_1(\mathbb{RP}^n) \longrightarrow \pi_1(\mathbb{RP}^1)$ is zero and the lift exists. In the second case, when we have $m > 1$, we again want to show that $\overline{f}_* = 0$. We just give a sketch of the proof as follows. First we note that $\pi_1(\mathbb{RP}^k) \cong H_1(\mathbb{RP}^k;\mathbb{Z}/2) \cong H^1(\mathbb{RP}^k;\mathbb{Z}/2)$. But for $k = m, n$ there is a correspondence under these isomorphisms of $\overline{f}_*$ with $\overline{f}^*$ in cohomology. But this last map is zero, as we have already seen in the proof of Theorem 11.8.29.

In both cases, then, by Exercise 4.5.14 there exists a lift $\widetilde{f} : \mathbb{RP}^n \longrightarrow \mathbb{S}^m$. In this way, both of the maps

$$\widetilde{f} \circ p,\ f : \mathbb{S}^n \longrightarrow \mathbb{S}^m$$

are lifts of $\overline{f} \circ p : \mathbb{S}^n \longrightarrow \mathbb{RP}^m$. Then for every $x \in \mathbb{S}^n$ we have that $q\widetilde{f}p(x) = qf(x)$, which implies either that $\widetilde{f}p(x) = f(x)$ or that $\widetilde{f}p(x) = -f(x) = f(-x)$. Consequently, the two lifts are equal either in x or in $-x$, where we use the fact that $p(x) = p(-x)$.

But since $\mathbb{S}^n$ is path connected, the two lifts must then be identically equal. However, this is impossible, since one separates antipodal points while the other sends antipodal points to the same point. Therefore, such a map f cannot exist.

The following exercise uses the multiplicative structure of the cohomology to distinguish between two spaces having the same additive structure in their cohomology groups.

11.8.31 Exercise. Let $X = \mathbb{S}^2 \vee \mathbb{S}^4$ and $Y = \mathbb{CP}^2$. Show that X and Y have the same cohomology groups, but multiplicatively their cohomology rings are different. Conclude that X and Y are not of the same homotopy type.

Chapter 12

Cohomology Theories and Brown Representability

In Chapter 7 we presented cohomology theory, and in Chapter 9 we introduced K-theory. Both theories have some properties in common. In this chapter we unify these properties and define the generalized cohomology theories. From this point of view we shall be able to obtain several results that follow from the formal properties rather than from the specific characteristics of the theory in question. Further, we shall prove a theorem that shows that our approach to both theories is quite general. Namely, we prove the Brown representability theorem, which shows that in an adequate category of spaces every generalized cohomology theory is represented by some classifying spaces, such as the Eilenberg–Mac Lane spaces in the case of cohomology and the spaces $\mathrm{BU} \times \mathbb{Z}$ and BU in the case of K-theory. Thus cohomology can always be expressed in homotopical terms. Finally we see that the representability of the cohomology theories implies the existence of certain objects, called spectra, which topologically, or better, homotopically, encode all the information concerning their associated cohomology and homology theories.

12.1 Generalized Cohomology Theories

The cohomology groups in Chapter 7 as well as K-theory in Chapter 9 have some properties in common; namely, they are contravariant functors, they are homotopy invariants, both produce exact sequences for pairs of spaces, and they have some excision property. All these conditions make these theories cohomology theories. In this section we define in general what a cohomology theory is, and then from its properties we derive several results that were obtained in the special cases from the particular definitions of the theories studied earlier.

12.1.1 Definition. Let $\mathcal{T}op_2$ be the category of pairs (X, A) of topological spaces and maps of pairs. Let, moreover, $\mathcal{A}$ be the category of abelian groups and homomorphisms. A *cohomology theory* h^* on $\mathcal{T}op_2$ is a collection of contravariant functors and natural transformations indexed by $q \in \mathbb{Z}$,

$$h^q : \mathcal{T}op_2 \longrightarrow \mathcal{A} \quad \text{and} \quad \delta^q : h^q \circ R \stackrel{\cdot}{\longrightarrow} h^{q+1},$$

these last called *connecting homomorphisms*, where $R : \mathcal{T}op_2 \longrightarrow \mathcal{T}op_2$ is the functor that sends a pair (X, A) to the pair $(A, \emptyset)$ and the map of pairs $f : (X, A) \longrightarrow (Y, B)$ to $f|A$, satisfying the following axioms:

Homotopy. If $f_0 \simeq f_1 : (X, A) \longrightarrow (Y, B)$ (a homotopy of pairs), then

$$f_0^* = f_1^* : h^q(Y, B) \longrightarrow h^q(X, A)$$

for all $q \in \mathbb{Z}$.

Excision. For every pair of spaces (X, A) and a subset $U \subset A$ satisfying $\overline{U} \subset \mathring{A}$, the inclusion $j : (X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism

$$h^q(X, A) \cong h^q(X - U, A - U)$$

for all $q \in \mathbb{Z}$.

Exactness. For every pair of spaces (X, A) we have a long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} h^q(X, A) \xrightarrow{i^*} h^q(X) \xrightarrow{i^*} h^q(A) \xrightarrow{\delta^q} h^{q+1}(X, A) \longrightarrow \cdots,$$

where $i : (X, \emptyset) \hookrightarrow (X, A)$ and $j : (A, \emptyset) \hookrightarrow (X, \emptyset)$ are the inclusions, and we write $h^q(X)$ instead of $h^q(X, \emptyset)$.

12.1.2 Examples.

(a) The functors $(X, A) \mapsto H^q(X, A; G)$ constitute a cohomology theory for every abelian group G in the category $\mathcal{T}op_2$ of all pairs of spaces.

(b) The functors $(X, A) \mapsto K^q(X, A)$ form a cohomology theory in the category of pairs of paracompact spaces and closed subspaces. (See 9.5.9, (9.5.8), and 9.5.10.)

12.1.3 Remark. There is also the dual concept of a *homology theory* h_* on $\mathcal{T}op_2$, which is a collection of covariant functors and natural transformations indexed by $q \in \mathbb{Z}$,

$$h_q : \mathcal{T}op_2 \longrightarrow \mathcal{A} \quad \text{and} \quad \partial_q : h_q \stackrel{\cdot}{\longrightarrow} h_{q-1} \circ R\,,$$

these last called *connecting homomorphisms*, where as before, $R : \mathcal{T}op_2 \longrightarrow \mathcal{T}op_2$ maps a pair of spaces to the second space of the pair, and they satisfy the same axioms as the cohomology with the obvious modifications.

Some examples we have of this are the ordinary homology groups with coefficients in an abelian group G as introduced in Section 5.3, and given by $(X, A) \mapsto H_q(X, A; G)$.

Sometimes it is more convenient to work with the so-called reduced cohomology theories defined on the category $\mathcal{T}op_*$ of pointed spaces and pointed maps.

12.1.4 Definition. Let $\mathcal{T}op_*$ be the category of pointed spaces (X, x_0) and pointed maps. Let, as before, $\mathcal{A}$ be the category of abelian groups and homomorphisms. A *reduced cohomology theory* k^* on $\mathcal{T}op_*$ is a collection of contravariant functors and natural equivalences indexed by $q \in \mathbb{Z}$,

$$k^q : \mathcal{T}op_* \longrightarrow \mathcal{A} \quad \text{and} \quad s^q : k^q \circ S \stackrel{\cdot}{\longrightarrow} k^{q-1}\,,$$

these last called *suspension isomorphisms*, where $S : \mathcal{T}op_* \longrightarrow \mathcal{T}op_*$ is the functor that sends a pointed space (X, x_0) to its reduced suspension $(\Sigma X, *)$ and the pointed map $f : (X, x_0) \longrightarrow (Y, y_0)$ to Σf (see 2.10.1), satisfying the following axioms:

Homotopy. If $f_0 \simeq f_1 : (X, x_0) \longrightarrow (Y, y_0)$ (a homotopy of pointed maps), then

$$f_0^* = f_1^* : k^q(Y, y_0) \longrightarrow k^q(X, x_0)$$

for all $q \in \mathbb{Z}$.

Exactness. For every pointed pair (X, A) we have an exact sequence

$$k^q(X \cup CA, *) \xrightarrow{j^*} k^q(X, x_0) \xrightarrow{i^*} k^q(A, x_0)\,,$$

where $i : (A, x_0) \hookrightarrow (X, x_0)$ is the inclusion and $j : (X, x_0) \hookrightarrow (X \cup CA, *)$ is the canonical inclusion into the cone of i.

12.1.5 Examples.

(a) The functors

$$(X, x_0) \mapsto \widetilde{H}^q(X; G) = H^q(X, \{x_0\}; G)$$

constitute a reduced cohomology theory for every abelian group G in the category of all pointed spaces.

(b) The functors

$$(X, x_0) \mapsto \widetilde{K}^q(X)$$

constitute a reduced cohomology theory in the category $\mathcal{T}op_*$ of pointed paracompact spaces. (See 9.3.3 and 9.5.11.)

12.1.6 Remark. Also in the reduced case one has the dual concept of a *reduced homology theory* k_* on $\mathcal{T}op_*$, which again is a collection of covariant functors and natural equivalences indexed by $q \in \mathbb{Z}$,

$$k_q : \mathcal{T}op_* \longrightarrow \mathcal{A} \quad \text{and} \quad s_q : h_q \stackrel{\cdot}{\longrightarrow} h_{q+1} \circ S,$$

these last called *suspension isomorphisms*, where $S : \mathcal{T}op_* \longrightarrow \mathcal{T}op_*$ maps a pointed space to its suspension as before, and they satisfy the same axioms as the reduced cohomology with the obvious modifications.

There is another property that was included in the list of axioms of Eilenberg and Steenrod for homology or cohomology. It is the **Dimension** axiom, which in the case of cohomology states that $h^q(\{*\}) = 0$ for the one-point space if $q \neq 0$, and $k^q(\mathbb{S}^0, *) = 0$ if $q \neq 0$ in the reduced case. In the case of homology it states that $h_q(\{*\}) = 0$ for the one-point space if $q \neq 0$, and $k_q(\mathbb{S}^0, *) = 0$ if $q \neq 0$ in the reduced case. Cohomology and homology theories that satisfy this axiom are called *ordinary.* Examples of this type are of course the cohomology with coefficients in G, $H^*(-; G)$, and the homology with coefficients in G, $H_*(-; G)$. A cohomology or homology theory that does not satisfy this axiom is called *extraordinary* or *generalized.* An example of this type of cohomology theory is of course the K-theory, $K^*(-)$.

In what follows we restrict ourselves to the case of cohomology theories, although all of the results have a counterpart in homology.

There are several important properties of cohomology theories that are deduced from the axioms. We state them in what follows.

Assume that $i : A \hookrightarrow X$ is a homotopy equivalence. Since then $i^* : h^q(X) \longrightarrow: h^q(A)$ is an isomorphism by the homotopy axiom, then taking the long exact sequence of the pair, we obtain the following result.

12.1.7 **Proposition**. *Let h be a cohomology theory. If $i : A \hookrightarrow X$ is a homotopy equivalence, then $h^q(X, A) = 0$ for all q.* □

12.1.8 **Corollary**. *Let h be a cohomology theory. If X is a (strongly) contractible space, then $h^q(X, \{x_0\}) = 0$ for all q.* □

Assume that $A \subset X$ is a cofibration. Then the quotient map $(X \cup CA, CA) \longrightarrow (X \cup CA/CA, *) \approx (X/A, *)$ is a homotopy equivalence by 4.2.3. We thus have the following.

12.1.9 **Proposition**. *Let h be a cohomology theory. If $A \subset X$ is a cofibration, then the quotient map $p : (X \cup CA, CA) \longrightarrow (X/A, *)$ induces an isomorphism $p^* : h^q(X/A, \{*\}) \longrightarrow h^q(X \cup CA, CA)$.* □

Moreover, one can delete the base point of the unreduced cone CA and then deform the pair $(X \cup CA - *, CA - *)$ to (X, A); that is, the inclusion $(X, A) \hookrightarrow (X \cup CA - *, CA - *)$ is a homotopy equivalence. Thus by the homotopy and the excision axioms we have the following consequence.

12.1.10 **Corollary**. *Let h be a cohomology theory. If $A \subset X$ is a cofibration, then the quotient map $p : (X, A) \longrightarrow (X/A, \{*\})$ induces an isomorphism*

$$p^* : h^q(X/A, *) \longrightarrow h^q(X, A) \text{ for all } q \in \mathbb{Z}\,.$$ □

From the exactness axiom, one has also the following.

12.1.11 **Proposition**. *Suppose that X is a topological space and that $B \subset A \subset X$ are subspaces. If h is a cohomology theory, then there is a long exact sequence*

$$\cdots \longrightarrow h^{q-1}(A, B) \xrightarrow{\overline{\delta}} h^q(X, A) \longrightarrow h^q(X, B) \longrightarrow$$
$$\longrightarrow h^q(A, B) \longrightarrow \cdots,$$

where the homomorphisms are induced by the inclusions, except for $\overline{\delta}$, which is defined as the composite

$$\overline{\delta} : h^{q-1}(A, B) \longrightarrow h^{q-1}(A) \xrightarrow{\delta} h^q(X, A)\,.$$

This is the so-called *exact sequence of the triple* (X, A, B). (See 7.1.33.)

The *proof* uses the exact sequences of (X, A), (X, B), and (A, B). □

There is a way of passing from an unreduced cohomology theory to a reduced one and vice versa. Let h^* be a cohomology theory defined in $\mathcal{T}op_2$ and consider the family $\widetilde{h}^*$ of functors on $\mathcal{T}op_*$ defined by

$$\widetilde{h}^q(X, x_0) = h^q(X, \{x_0\}).$$

Recall that $\Sigma X = CX/X$, where CX is the reduced cone on X, and consider the exact sequence of the triple $\{*\} \subset X \subset CX$ (see 12.1.11),

$$\cdots \longrightarrow h^q(CX, \{*\}) \longrightarrow h^q(X, \{x_0\}) \xrightarrow{\bar{\delta}} h^{q+1}(CX, X) \longrightarrow \\ \longrightarrow h^{q+1}(CX, \{*\}) \longrightarrow \cdots.$$

Since CX is (strongly) contractible, $h^q(CX, \{*\}) = 0$ for all q, and hence $\bar{\delta}$ is always an isomorphism. On the other hand, by Corollary 12.1.10, $p^* : h^q(\Sigma X, \{*\}) \longrightarrow h^q(CX, X)$ is an isomorphism, where $p : (CX, X) \twoheadrightarrow (\Sigma X, \{*\})$ is the quotient map. Therefore, we define the isomorphism s^q as the composite

$$s^q : \widetilde{h}^{q+1}(\Sigma X, *) = h^{q+1}(\Sigma X, \{*\}) \xrightarrow{p^*} h^{q+1}(CX, X) \xrightarrow{\bar{\delta}^{-1}} \\ \longrightarrow h^q(X, \{x_0\}) = \widetilde{h}^q(X, x_0).$$

Using the exactness axiom for the cohomology theory h, it is immediate to check that the reduced cohomology exactness axiom holds for $\widetilde{h}$. We thus have the following.

12.1.12 **Theorem**. *If h^q, δ^q, is a cohomology theory on $\mathcal{T}op_2$, then $\widetilde{h}^q$, s^q as defined above is a reduced cohomology theory on $\mathcal{T}op_*$.* □

Conversely, given a reduced cohomology theory k^* defined on $\mathcal{T}op_*$, we consider the family of contravariant functors $\widehat{k}^*$ defined on $\mathcal{T}op_2$ on objects by setting

$$\widehat{k}^q(X, A) = k^q(X^+ \cup CA^+, *)$$

and on maps $f : (X, A) \longrightarrow (Y, B)$ by letting $f^* : \widehat{k}^q(Y, B) \longrightarrow \widehat{k}^q(X, A)$ be given by the induced pointed map $\widehat{f} : X^+ \cup CA^+ \longrightarrow Y^+ \cup CB^+$, where Z^+ is the space $Z \sqcup \{*\}$ for any space Z with the obvious base point. The natural transformations $\widehat{\delta} : \widehat{k}^q(A) \longrightarrow \widehat{k}^{q+1}(X, A)$ are given by the composite

$$\widehat{k}^q(A) = k^q(A^+, *) \xrightarrow{s_{q+1}^{-1}} k^{q+1}(\Sigma A^+, *) \xrightarrow{(\tau \circ\ p)^*} \\ \longrightarrow k^{q+1}(X^+ \cup CA^+, *) = \widehat{k}^{q+1}(X, A),$$

since $A \cup C\emptyset = A^+$, where $p : X^+ \cup CA^+ \longrightarrow X^+ \cup CA^+/X^+ = \Sigma A^+$ is the collapsing map (see 3.1.3), and $\tau : \Sigma A^+ \longrightarrow \Sigma A^+$ is the homotopy inverse of the H-cogroup ΣA^+ (see 2.10.3).

12.1.13 Note. The inclusion $(X, A) \hookrightarrow (X^+, A^+)$ induces a homeomorphism $X \cup C'A \approx X^+ \cup CA^+$ from the unreduced cone of $A \hookrightarrow X$ onto the reduced cone of $A^+ \hookrightarrow X^+$, which maps the vertex of the unreduced cone $C'A$ to the base point of the reduced cone CA^+. Therefore, we may use either one or the other. Moreover, if the pair (X, A) has a *nondegenerate base point*, that is, if the inclusion of the base point $* \hookrightarrow A$ is a cofibration, then by 4.2.3 the canonical quotient map $X \cup C'A \twoheadrightarrow X \cup CA$ is a homotopy equivalence.

12.1.14 Exercise. Prove that one has a long exact sequence for the pair (X, A) for the functors $\widehat{k}^q$ and natural transformations $\widehat{\delta}$. (Hint: Use the exactness axiom for k and compare with the Barratt–Puppe sequence construction, Section 3.5.)

We have the following result similar to Theorem 12.1.12.

12.1.15 **Theorem**. *If k^q, s^q, is a reduced cohomology theory on $\mathcal{T}op_*$, then $\widehat{k}^q$, $\widehat{\delta}$, as defined above, is a cohomology theory on $\mathcal{T}op_2$.* $\square$

One might think that the two constructions above are inverse to each other, that is, that starting with an unreduced cohomology theory, constructing its associated reduced theory and then passing from the latter to its unreduced theory, we come back to the original theory. This is generally not so. In what follows we establish criteria to see to what extent the given (unreduced) theory and the one obtained after two steps coincide. Similarly, we consider what happens when we start with a reduced theory. The following definition will be useful.

12.1.16 Definition. Let h_1^* and h_2^* be cohomology theories. A *transformation* $T : h_1^* \longrightarrow h_2^*$ of cohomology theories is a family of natural transformations $T_q : h_1^q \longrightarrow h_2^q$ for $q \in \mathbb{Z}$ such that for every pair of spaces (X, A) one has a commutative square

$$\begin{tikzcd} h_1^q(A) \arrow[r, "\delta_1"] \arrow[d, "T_q"'] & h_1^{q+1}(X, A) \arrow[d, "T_{q+1}"] \\ h_2^q(A) \arrow[r, "\delta_2"] & h_2^{q+1}(X, A) . \end{tikzcd}$$

The transformation T is called an *equivalence* if each T_q is a natural equivalence. There are corresponding notions of transformation and equivalence of reduced cohomology theories.

In order to compare the two constructions given above, we produce transformations between $\widehat{\widetilde{h}}^*$ and h^* and between $\widetilde{\widehat{k}}^*$ and k^* and analyze under what circumstances they are equivalences.

Since the inclusion of pairs $(X^+ \cup CA^+, \{*\}) \hookrightarrow (X^+ \cup CA^+, CA^+)$ induces isomorphisms in cohomology (just take the exact sequences of both pairs and observe that CA^+ is contractible), given a cohomology theory h^*, the inclusion $(X, A) \hookrightarrow (X^+ \cup CA^+, CA^+)$ induces a homomorphism

$$\begin{aligned} T_q : \widehat{\widetilde{h}}^q(X, A) &= \widetilde{h}^q(X^+ \cup CA^+, *) \\ &\cong h^q(X^+ \cup CA^+, CA^+) \longrightarrow h^q(X, A) \, . \end{aligned}$$

This is obviously a natural transformation compatible with the connecting homomorphisms.

On the other hand, if the spaces involved have *nondegenerate base points*, that is, if the inclusions of their base points are cofibrations, then the canonical inclusion of pointed spaces $(X, x_0) \hookrightarrow (X^+ \cup CA^+, *)$ is a homotopy equivalence. Hence, given a reduced cohomology theory k^*, there is an isomorphism

$$T'_q : \widetilde{\widehat{k}}^q(X, x_0) = \widehat{k}(X, \{x_0\}) = k(X^+ \cup C\{x_0\}^+, *) \cong k(X, x_0) \, .$$

This is a natural equivalence, and one may prove that it is compatible with the suspension isomorphisms. Therefore, starting from a reduced cohomology theory we come back to the same theory, provided that the spaces we are dealing with have nondegenerate base points. However, if we start with an unreduced theory, this is not the case.

In order to get a one-to-one correspondence between reduced and unreduced theories, we need to introduce another axiom for a cohomology theory h^*.

Weak homotopy equivalence. Given a weak homotopy equivalence of pairs of spaces $f : (X, A) \longrightarrow (Y, B)$, then $f^* : h^q(Y, B) \longrightarrow h^q(X, A)$ is an isomorphism for all $q \in \mathbb{Z}$.

There is the corresponding axiom for a reduced theory k^*.

Weak homotopy equivalence. Given a weak homotopy equivalence $f : X \longrightarrow Y$, then $f^* : k^q(Y, f(x)) \longrightarrow k^q(X, x)$ is an isomorphism for all $x \in X$, $q \in \mathbb{Z}$.

We have the following result (cf. [76, 7.42, 7.44]).

12.1.17 **Theorem**. *Let h^* be a cohomology theory and k^* a reduced cohomology theory, each satisfying the weak homotopy equivalence axiom. Then*

(a) $T : \widehat{\widehat{h}}^*$ *en*h^* *is an equivalence of cohomology theories on the category* $\mathcal{T}op_2$*, and*

(b) $T' : \widetilde{\widehat{k}}^* \longrightarrow k^*$ *is an equivalence of reduced cohomology theories on the category* $\mathcal{T}op_\odot$ *of topological spaces with nondegenerate base points.* □

12.1.18 Remark. If we are working in the category $\mathcal{WT}op_*$ of pointed spaces that have the same homotopy type as CW-complexes or the category $\mathcal{WT}op_2$ of pairs of spaces of the same homotopy type as CW-pairs, then by the Whitehead theorem 5.1.37, any cohomology theory satisfies the weak homotopy equivalence axiom. Therefore, in these categories we have a one-to-one correspondence between unreduced and reduced cohomology theories.

Of course, the corresponding result holds for homology theories.

Milnor introduced a further axiom to study infinite CW-complexes, which allows us to prove a uniqueness theorem for homology and cohomology theories.

Additivity. For every collection $\{(X_\lambda,A_\lambda)\}_{\lambda\in\Lambda}$ of pairs of topological spaces, the inclusions $i_\lambda : (X_\lambda, A_\lambda) \hookrightarrow \coprod_{\mu\in\Lambda}(X_\mu, A_\mu)$ induce an isomorphism

$$(i_\lambda^*) : h^q\left(\coprod_\lambda X_\lambda, \coprod_\lambda A_\lambda\right) \longrightarrow \prod_{\lambda\in\Lambda} h^q(X_\lambda, A_\lambda)\,.$$

And similarly, for a reduced cohomology theory k^* we have the following axiom.

Wedge. For every collection $\{(X_\lambda, x_\lambda)\}_{\lambda\in\Lambda}$ of pointed topological spaces, the inclusions $i_\lambda : X_\lambda \hookrightarrow \bigvee_{\mu\in\Lambda} X_\mu$ induce an isomorphism

$$(i_\lambda^*) : k^q\left(\bigvee_\lambda X_\lambda, *\right) \longrightarrow \prod_{\lambda\in\Lambda} k^q(X_\lambda, x_\lambda)\,.$$

There are the corresponding axioms in the case of homology, where the direct products are exchanged for direct sums and the isomorphisms point in the opposite direction. Theories that satisfy either axiom are called *additive*.

Milnor proved the following uniqueness result for ordinary homology and cohomology theories [56].

12.1.19 **Theorem**. *Let h^* (respectively h_*) be an additive ordinary cohomology (respectively homology) theory on $\mathcal{WT}op_2$ with $h^0(\{*\}) = G$ (respectively $h_0(\{*\}) = G$). Then there is an equivalence of cohomology (respectively homology) theories*

$$h^* \stackrel{\cdot}{\longrightarrow} H^*(-;G)$$

(respectively

$$h_* \stackrel{\cdot}{\longrightarrow} H_*(-;G)\ ).$$

Moreover, if h^* (respectively h_*) satisfies the weak homotopy equivalence axiom, then both theories are equivalent in the category $\mathcal{T}op_2$ of all pairs of topological spaces.

Later on, in Section 12.3, we give an alternative *proof* to Milnor's of this result in the case of cohomology.

Since our homology and cohomology theories, as defined in Sections 5.3 and 7.1, are additive and satisfy the weak homotopy equivalence axiom (see 5.3.31 and 5.3.25 as well as 7.1.13 and 7.1.15), as do singular homology and cohomology (see [67]), we have the following consequence.

12.1.20 **Corollary**. *$H_*(-;G)$ is equivalent to singular homology with coefficients in G, and $H^*(-;G)$ is equivalent to singular cohomology with coefficients in G, both on the category $\mathcal{T}op_2$ of all pairs of topological spaces.* □

One of the important things that can be obtained from the axioms of a cohomology or homology theory is the Mayer–Vietoris exact sequence, which we obtained using the cellular complexes for ordinary cohomology and homology (see 7.4.13).

12.1.21 Definition. A triad of spaces $(X; A, B)$ is called *excisive with respect to a cohomology theory h^** (respectively *a homology theory h_**) if the inclusions $i : (A, A \cap B) \hookrightarrow (X, B)$ and $j : (B, A \cap B) \hookrightarrow (X, A)$ induce isomorphisms

$$i^* : h^q(X, B) \longrightarrow h^q(A, A \cap B), \quad j^* : h^q(X, A) \longrightarrow h^q(B, A \cap B)\,,$$

(respectively

$$i_* : h_q(A, A \cap B) \longrightarrow h_q(X, B), \quad j_* : h_q(B, A \cap B) \longrightarrow h_q(X, A)\,,)$$

for all q. In fact one can prove that if i^* (respectively i_*) is an isomorphism, then j^* (respectively j_*) is also an isomorphism.

Examples of excisive triads for ordinary cohomology and homology are excisive triads $(X; A, B)$, that is, triads such that $\overset{\circ}{A} \cup \overset{\circ}{B} = X$, and also CW-triads.

The following theorem generalizes Theorem 7.4.13 to every homology and cohomology theory.

12.1.22 **Theorem**. *Suppose that $(X; A, B)$ is an excisive triad for a homology theory h_* and take $C \subset A \cap B$. Then there is an exact sequence in homology*

$$\begin{aligned}\cdots \longrightarrow h_q(A \cap B, C) \xrightarrow{\beta} h_q(A, C) \oplus h_q(B, C) \xrightarrow{\alpha} h_q(X, C) \xrightarrow{\overline{\partial}} \\ \longrightarrow h_{q-1}(A \cap B, C) \longrightarrow \cdots ,\end{aligned}$$

where

$$\beta(c) = (i'_*(c), -j'_*(c)) , \quad \alpha(a, b) = i_*(a) + j_*(b) ,$$

and the homomorphism $\overline{\partial}$ is the composite

$$\overline{\partial} : h_q(X, C) \xrightarrow{k'_*} h_q(X, B) \xrightarrow{k_*^{-1}} h_q(A, A \cap B) \xrightarrow{\partial} h_{q-1}(A \cap B, C)$$

and ∂ is the connecting homomorphism in the homology theory h_* for the triple $(A, A \cap B, C)$.

Also, if the triad is excisive with respect to a cohomology theory h^*, then there is an exact sequence in cohomology

$$\begin{aligned}\cdots \longrightarrow h^{q-1}(A \cap B, C) \xrightarrow{\overline{\delta}} \\ \longrightarrow h^q(X, C) \xrightarrow{\alpha'} h^q(A, C) \oplus h^q(B, C) \xrightarrow{\beta'} h^q(A \cap B, C) \longrightarrow \cdots ,\end{aligned}$$

where

$$\alpha'(c) = (i^*(c), j^*(c)) , \quad \beta'(a, b) = i'^*(a) - j'^*(b) ,$$

and $\overline{\delta}$ is given by the composite

$$\overline{\delta} : h^{q-1}(A \cap B, C) \xrightarrow{\delta} h^q(A, A \cap B) \xrightarrow{k^{*-1}} h^q(X, B) \xrightarrow{k'^*} h^q(X, C)$$

and δ is again the connecting homomorphism in the cohomology theory h^*

for the triple $(A, A \cap B, C)$*. Here* i*,* i'*,* j*,* j'*,* k*, and* k' *are the inclusions*

$$\begin{tikzcd}
 & (A,C) \arrow[dr, hook, "i"] & \\
(A\cap B, C) \arrow[ur, hook, "i'"] \arrow[dr, hook, "j'"'] & & (X,C)\,, \\
 & (B,C) \arrow[ur, hook, "j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A, A\cap B) \arrow[dr, hook, "k"] & \\
 & (X,B)\,. \\
(X,C) \arrow[ur, hook, "k'"'] &
\end{tikzcd}$$

The *proof* is obtained by putting together the exact sequences of the triples $(A, A\cap B, C)$ and (X, B, C) and using the fact that $k : (A, A\cap B) \hookrightarrow (X, B)$ induces isomorphisms. □

12.1.23 Exercise. Take $C = \{x_0\}$, where $x_0 \in A \cap B$ is the base point of X, and construct the corresponding Mayer–Vietoris sequences for reduced homology and cohomology.

12.2 Brown Representability Theorem

In this section we present a beautiful result of E.H. Brown [21] that treats a general class of homotopy invariant functors in the category of path-connected pointed spaces. The main theorem characterizes certain functors on the subcategory of CW-complexes. We follow closely the proof given by E.H. Spanier [67]. We start with some categorical considerations.

Let $\mathcal{C}$ be a category. Each object C_0 of $\mathcal{C}$ defines a contravariant functor

$$\mathcal{C}(-, C_0) : \mathcal{C} \longrightarrow \mathcal{S}et\,,$$

given on objects by $C \mapsto \mathcal{C}(C, C_0)$, where $\mathcal{C}(C, C_0)$ denotes the set of morphisms in $\mathcal{C}$ from C to C_0, and on morphisms $f : C \longrightarrow D$ in $\mathcal{C}$ by $f^{\#} = \mathcal{C}(f, C_0) : \mathcal{C}(D, C_0) \longrightarrow \mathcal{C}(C, C_0)$, $f^{\#}(\varphi) = \varphi \circ f$.

12.2.1 Definition. A contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{S}et$ is said to be *representable* if there is an object C_0 in $\mathcal{C}$ and a natural equivalence $e : \mathcal{C}(-, C_0) \stackrel{\cdot}{\longrightarrow} F$. In this case one says that C_0 *represents* F. C_0 will also be called a *classifying object* for F.

The following is known as the Yoneda lemma.

12.2.2 **Lemma**. *Let $F : \mathcal{C} \longrightarrow \mathcal{S}et$ be a contravariant functor. Then there is a one-to-one correspondence between natural transformations $e : \mathcal{C}(-,C_0) \stackrel{\cdot}{\longrightarrow} F$ and elements $u \in F(C_0)$. The correspondence is such that for each object C in $\mathcal{C}$, $e_C : \mathcal{C}(C, C_0) \longrightarrow F(C)$ is given by $e_C(\varphi) = F(\varphi)(u)$ for any $\varphi : C \longrightarrow C_0$.*

Proof: Let $e : \mathcal{C}(-, C_0) \stackrel{\cdot}{\longrightarrow} F$ be a natural transformation. Hence, given any morphism $\varphi : C \longrightarrow C_0$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(C_0, C_0) \arrow[r, "e_{C_0}"] \arrow[d, "\varphi^{\#}"'] & F(C_0) \arrow[d, "F(\varphi)"] \\
\mathcal{C}(C, C_0) \arrow[r, "e_C"'] & F(C) .
\end{tikzcd}$$

If $u = e_{C_0}(1_{C_0}) \in F(C_0)$, then by chasing this element around the diagram, we have

$$\begin{tikzcd}
1_{C_0} \arrow[r, mapsto] \arrow[d, mapsto] & u \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] & F(\varphi)(u) ,
\end{tikzcd}$$

and therefore $e_C(\varphi) = F(\varphi)(u)$.

Conversely, given $u \in F(C_0)$ and any object C in $\mathcal{C}$, define

$$e_C : \mathcal{C}(C, C_0) \longrightarrow F(C)$$

by $e_C(\varphi) = F(\varphi)(u)$. Then e is a natural transformation. □

12.2.3 Definition. If F is a representable functor and

$$e : \mathcal{C}(-, C_0) \stackrel{\cdot}{\longrightarrow} F$$

is a natural equivalence, then the associated element according to the Yoneda lemma, $u_F = e_{C_0}(1_{C_0}) \in F(C_0)$, is called the *universal element* for F.

12.2.4 **Proposition**. *Let $F, G : \mathcal{C} \longrightarrow \mathcal{S}et$ be contravariant functors represented by C_0 and C_0', respectively. Let $\kappa : F \stackrel{\cdot}{\longrightarrow} G$ be a natural transformation. Then there exists a unique morphism $\rho : C_0 \longrightarrow C_0'$ such that for each object C in $\mathcal{C}$ the diagram*

$$\begin{tikzcd}
\mathcal{C}(C, C_0) \arrow[r, dashed, "\rho_{\#}"] \arrow[d, "e_C"', "\cong"] & \mathcal{C}(C, C_0') \arrow[d, "e'_C", "\cong"'] \\
F(C) \arrow[r, "\kappa_C"'] & G(C)
\end{tikzcd}$$

commutes, where $\rho_\#(\varphi) = \rho \circ \varphi$ *and* e_C, e'_C *are the corresponding natural equivalences. Furthermore, if* κ *is a natural equivalence, then* ρ *is an isomorphism in* $\mathcal{C}$*.*

Proof: First, we shall try to make the diagram commute in the special case where $C = C_0$. So take $1_{C_0} \in \mathcal{C}(C_0, C_0)$. Then $u_F = e_{C_0}(1_{C_0}) \in F(C_0)$. Since e'_{C_0} is a bijection, there is a unique element $\rho \in \mathcal{C}(C_0, C'_0)$ such that $e'_{C_0}(\rho) = \kappa_{C_0}(u_F)$.

Now take $\varphi \in \mathcal{C}(C, C_0)$. Then, by Lemma 12.2.2, the naturality of κ and the definition of ρ, we have that

$$\kappa_C e_C(\varphi) = \kappa_C F(\varphi)(u_F) = G(\varphi)\kappa_{C_0}(u_F) = G(\varphi)e'_{C_0}(\rho)\,.$$

On the other hand,

$$e'_C\rho_\#(\varphi) = e'_C(\rho \circ \varphi) = G(\rho \circ \varphi)(u_G) = G(\varphi)G(\rho)(u_G) = G(\varphi)e'_{C_0}(\rho)\,,$$

where $u_G = e'_{C'_0}(1_{C'_0}) \in G(C'_0)$ is the universal element for G. Therefore, $\kappa_C e_C(\varphi) = e'_C\rho_\#(\varphi)$, and so the diagram commutes.

The uniqueness of ρ follows immediately from the first paragraph of this proof, since ρ is the unique morphism making the diagram commute in the special case $C = C_0$.

Finally, assume that κ is a natural equivalence. Since for each object C in $\mathcal{C}$, $\kappa_C : F(C) \longrightarrow G(C)$ is a bijection, we have that the inverse functions $\kappa_C^{-1} : G(C) \longrightarrow F(C)$ determine a natural transformation $\overline{\kappa} : G \stackrel{\cdot}{\longrightarrow} F$ such that $\overline{\kappa}_C = \kappa_C^{-1}$. By the first part, there is a unique morphism $\overline{\rho} : C'_0 \longrightarrow C_0$ corresponding to $\overline{\kappa}$. For each C in $\mathcal{C}$, the composite $\overline{\kappa}_C \circ \kappa_C$ is the identity $F(C) \longrightarrow F(C)$, and the composite $\kappa_C \circ \overline{\kappa}_C$ is the identity $G(C) \longrightarrow G(C)$. But these composites also correspond to $\overline{\rho} \circ \rho$ and $\rho \circ \overline{\rho}$ according to the first part of the proposition. By the uniqueness we have that $\overline{\rho} \circ \rho = 1_{C_0}$ and $\rho \circ \overline{\rho} = 1_{C'_0}$. Hence, if κ is a natural equivalence, then ρ is an isomorphism. □

12.2.5 **Corollary**. *Let* $F : \mathcal{C} \longrightarrow \mathcal{S}et$ *be a representable contravariant functor. If* C_0, C'_0 *are representing objects for* F *with universal elements* u_F, u'_F, *respectively, then there is an isomorphism* $\rho : C_0 \longrightarrow C'_0$ *such that* $F(\rho)(u'_F) = u_F$*.*

Proof: By assumption we have natural equivalences

$$e : \mathcal{C}(-, C_0) \stackrel{\cdot}{\longrightarrow} F\,, \quad e' : \mathcal{C}(-, C'_0) \stackrel{\cdot}{\longrightarrow} F\,,$$

so that $\lambda = e'^{-1} \circ e : \mathcal{C}(-, C_0) \overset{\cdot}{\longrightarrow} \mathcal{C}(-, C_0')$ is a natural equivalence. By the previous proposition, λ determines a unique isomorphism $\rho : C_0 \longrightarrow C_0'$ such that for every object C in $\mathcal{C}$, $\lambda_C : \mathcal{C}(C, C_0) \longrightarrow \mathcal{C}(C, C_0')$ is given by $\lambda_C(f) = \rho \circ f$. So in particular, $\lambda_{C_0}(1_{C_0}) = \rho$.

Recall that the universal elements are given by $u_F = e_{C_0}(1_{C_0})$ and $u_F' = e'_{C_0'}(1_{C_0'})$. Thus by the naturality of e' and the equality for ρ shown above we have that $F(\rho)(u_F') = F(\rho)e'_{C_0'}(1_{C_0'}) = e'_{C_0}(\rho) = e'_{C_0}\lambda_{C_0}(1_{C_0})$. But by the definition of λ we have $e' \circ \lambda = e$ and hence $e'_{C_0}\lambda_{C_0}(1_{C_0}) = e_{C_0}(1_{C_0}) = u_F$. Thus $F(\rho)(u_F') = u_F$, as desired. $\square$

12.2.6 Exercise. State and prove the converse of 12.2.4.

Recall that we defined the cohomology groups of a CW-complex X with coefficients in G by $H^n(X) = [X, K(G,n)]$, where $K(G,n)$ is the Eilenberg–Mac Lane space with a single nonvanishing homotopy group in dimension n, this group being isomorphic to G. Notice that from 4.4.7, since $K(G,n)$ is a path-connected H-space for $n \geq 1$, pointed and unpointed homotopy classes coincide, since (X, x_0) is a well-pointed space (x_0 is a 0-cell). Therefore, for any pointed CW-complex (X, x_0),

$$H^n(X) \cong [X, x_0; K(G,n), *] \quad (n \geq 1)\,.$$

More generally, for any fixed pointed topological space (Y, y_0), we set $\pi^Y(X) = [X, x_0; Y, y_0]$. This is obviously a contravariant functor from the category $\mathcal{T}op_*$ of pointed topological spaces and continuous maps preserving base points to the category $\mathcal{S}et_*$ of pointed sets and pointed functions, since for any pointed map $f : (X', x_0') \longrightarrow (X, x_0)$ we define a pointed function

$$f^* : \pi^Y(X) \longrightarrow \pi^Y(X')$$

by $f^*[\alpha] = [\alpha \circ f]$, which satisfies the required functor axioms. Here the base points of the sets $\pi^Y(X)$, $\pi^Y(X')$ are the homotopy classes of the constant maps.

Strictly speaking, there is another category structure on the objects of $\mathcal{T}op_*$, where the morphisms are homotopy classes of maps between pointed spaces. Specifically, given pointed spaces X, Y, a morphism $[f] : X \longrightarrow Y$ is a pointed homotopy class $[f]$, where $f : X \longrightarrow Y$ is any pointed map. The composition is given by $[g] \circ [f] = [g \circ f]$, and the identity morphism of X is the class $1_X = [\mathrm{id}_X]$. Observe that these morphisms are not functions of the underlying sets anymore. The corresponding category is denoted by $\mathcal{T}op_*^h$ and is called the *pointed homotopy category*. Then, given pointed spaces X,

Y, the morphism set $\mathcal{T}op_*^h(X,Y)$ is precisely $[X,Y]_*$, and thus the functor π^Y defined above is nothing but the functor $\mathcal{T}op_*^h(-,Y)$, which is a special case of the situation considered at the beginning of this section.

What we shall study in the sequel are conditions that characterize the functors π^Y restricted to the category $\mathcal{PT}op_\odot$ of path-connected spaces with nondegenerate base points; that is, we shall study the conditions a functor T must satisfy in order that it become naturally equivalent to one of the form π^Y or, in other words, to be representable.

12.2.7 Definition. Consider a contravariant *homotopy functor*, that is, a functor $T : \mathcal{T}op_*^h \longrightarrow \mathcal{S}et_*$, from the pointed homotopy category to the category of pointed sets and pointed functions. We use the following notation. If $X \subset Y$ and $v \in T(Y)$, then $v|X$ denotes the element $T([i])(v) \in T(X)$, where $i : X \hookrightarrow Y$ denotes the inclusion map. We call T a *Brown functor* if it fulfills the following two axioms.

Wedge. If $\{X_\alpha\}$ is a family of pointed spaces and $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ is the inclusion, then

$$(T(i_\alpha)) : T\left(\bigvee_\alpha X_\alpha\right) \longrightarrow \prod_\alpha T(X_\alpha)$$

is an equivalence of sets.

Mayer–Vietoris. Let $(X; A, B)$ be an excisive triad. Then for any $u \in T(A)$ and $v \in T(B)$ such that $u|A\cap B = v|A\cap B$, there exists $z \in T(X)$ such that $z|A = u$ and $z|B = v$.

12.2.8 Example. Using the axioms of functoriality, homotopy, and exactness (namely, the Eilenberg–Steenrod axioms excluding the dimension axiom) for an ordinary cohomology theory, one has for each q that H^q is a homotopy functor that satisfies the axioms for a Brown functor, except for the fact that the wedge axiom need hold only for finite families. However, by 7.1.13 our ordinary cohomology does satisfy the wedge axiom fully.

12.2.9 Exercise. Show that the Mayer–Vietoris axiom for H^q follows from the reduced Mayer–Vietoris exact sequence for H^*. (Cf. 7.4.13 and 12.1.23.)

The next is an important concept for what follows.

12.2.10 Definition. Given pointed homotopy classes $[f]$, $[g] : C \longrightarrow Y$, a *coequalizer* for them is a pointed homotopy class $[j] : Y \longrightarrow X$, such that:

(i) $[j] \circ [f] = [j] \circ [g]$, or speaking informally, $[f]$ and $[g]$, become equal after composition with $[j]$.

(ii) If $[j'] : Y \longrightarrow X'$ is a pointed homotopy class such that $[j'] \circ [f] = [j'] \circ [g]$, then there exists a unique $[g] : X \longrightarrow X'$ such that $[j'] = [g] \circ [j]$.

In other words, the underlying pointed map $g : X \longrightarrow X'$ is such that in the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "j"] \arrow[dr, "j'"'] & X \arrow[d, dashed, "g"] \\ & & X' \end{tikzcd}$$

the two composites on the top are homotopic, and if the two composites down diagonally to the right are also homotopic, then the vertical map exists uniquely up to homotopy, so that the triangle commutes up to homotopy.

Coequalizers exist. Namely, given pointed maps $f, g : C \longrightarrow Y$, take X to be the *double attaching cylinder* $Y \cup_f^g C \times I = C \times I \sqcup Y/\sim$, where $(c, 0) \sim f(c)$, $(c, 1) \sim g(c)$, $(c_0, t) \sim y_0$ for $c \in C$, $t \in I$ and where c_0, y_0 are the corresponding base points. It is then easy to prove the following result.

12.2.11 **Proposition**. *The homotopy class $[j] : Y \longrightarrow X$ of the map j such that $j(y) = q(y)$, where $q : C \times I \sqcup Y \twoheadrightarrow X$ is the quotient map, is a coequalizer for $[f]$ and $[g]$.* □

12.2.12 **Proposition**. *Assume that the functor T satisfies the Mayer–Vietoris axiom. Then it has the following property. If $f, g : C \longrightarrow Y$ are pointed maps and $w \in T(Y)$ satisfies $T([f])(w) = T([g])(w) \in T(C)$, then there exists $v \in T(X)$ such that $T([j])(v) = w$, where $[j] : Y \longrightarrow X$ is a coequalizer for $[f]$ and $[g]$.*

Proof: Let $X' = Y \cup_f^g C \times I$ be the double attaching cylinder of f and g. Take $A = Y \cup_f C \times [0, 1)$ and $B = Y \cup^g C \times (0, 1] \subset X'$. Then the triple $(X'; A, B)$ is excisive, and $A \cap B = C \times (0, 1)$, which has the homotopy type of C. Let $p : A \longrightarrow Y$, $q : B \longrightarrow Y$ be the canonical projections, which are also homotopy equivalences, and let $u = T([p])(w) \in T(A)$ and $v = T([q])(w) \in T(B)$. Then $T([f])(w) = T([g])(w) \in T(C)$ implies $u|A \cap B = v|A \cap B$. By the Mayer–Vietoris axiom for T, there exists $z \in T(X')$ such that $z|A = u$ and $z|B = v$.

Now, the inclusion $j' : Y \hookrightarrow A = Y \cup_f C \times [0, 1) \twoheadrightarrow X$ is such that $j' \circ f \simeq j' \circ g$. Since $[j] : Y \longrightarrow X$ is a coequalizer, there exists a map

$g : X \longrightarrow X'$ such that $g \circ j \simeq j'$. Then the element $v = T([g])(z) \in T(X)$ is such that $T([j])(v) = w$. □

Let T be a Brown functor. In order to show that it is representable, say by a pointed space Y, by the Yoneda lemma, it is enough to construct a space Y and a universal element $u \in T(Y)$. The space Y will be called a *classifying space* for T

One can produce universal elements, as we shall see below. First we have the following result.

12.2.13 **Proposition**. *If T is a Brown functor and $*$ denotes the one-point space, then $T(*)$ is a set that also consists of a single element.*

Proof: By the wedge axiom, there is an equivalence of sets

$$T(* \vee *) \cong T(*) \times T(*)\,.$$

Since $* \vee * = *$, the equivalence becomes the diagonal function $T(*) \longrightarrow T(*) \times T(*)$, and this equivalence holds only if $T(*)$ has a single element. □

12.2.14 **Proposition**. *If T is a Brown functor and $X = \Sigma X'$ is the suspension of some space, then $T(X)$ can be given a group structure with the distinguished element in the pointed set $T(X)$ as neutral element. It is abelian if $X' = \Sigma X''$.*

Proof: This follows from the fact that if X is a suspension, then it is an H-cospace and has an H-comultiplication $X \longrightarrow X \vee X$ (see 2.10.4), which, using the wedge axiom, induces a multiplication

$$T(X) \times T(X) \cong T(X \vee X) \longrightarrow T(X)\,,$$

making $T(X)$ a group.

If X is a double suspension, namely $X = \Sigma^2 X''$, then $T(X)$ inherits two group structures, which have a common bilateral unit and are mutually distributive. By 2.10.10, these two structures coincide and turn $T(X)$ into an abelian group. □

If T is a (Brown) functor and $u \in T(Y)$, then by the Yoneda lemma 12.2.2 there is a natural transformation $e : \pi^Y \stackrel{\cdot}{\longrightarrow} T$.

12.2.15 Definition. Given a Brown functor T and a space Y, we say that an element $u \in T(Y)$ is an *n-universal element* if the function

$$\varphi_u : \pi^Y(\mathbb{S}^q) = \pi_q(Y) \longrightarrow T(\mathbb{S}^q)$$

given by $\varphi_u([f]) = e_{\mathbb{S}^q}([f]) = T([f])(u)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. An element $u \in T(Y)$ is an *∞-universal element* if it is n-universal for all $n \geq 1$.

We shall construct below n-universal elements for T by induction on n.

12.2.16 **Lemma**. *Given a Brown functor T, a topological space X, and an element $v \in T(X)$, there exists a space $Y_1 \supset X$ together with a 1-universal element $u_1 \in T(Y_1)$ such that $u_1|X = v$.*

Proof: For every element $\alpha \in T(\mathbb{S}^1)$ take a copy $\mathbb{S}^1_\alpha$ of $\mathbb{S}^1$ and construct $Y_1 = X \vee \bigvee_\alpha \mathbb{S}^1_\alpha$. Then by the wedge axiom, there is an equivalence of sets

$$T(Y_1) \cong T(X) \times \prod_\alpha T(\mathbb{S}^1_\alpha)\,.$$

Take $u_1 \in T(Y_1)$ corresponding to the element

$$(v, (\alpha)) \in T(X) \times \prod_\alpha T(\mathbb{S}^1_\alpha)$$

under the equivalence. Then $\varphi_{u_1} : \pi_1(Y_1) \longrightarrow T(\mathbb{S}^1)$ is surjective, since every $\alpha \in T(\mathbb{S}^1)$ satisfies $\varphi_{u_1}([i_\alpha]) = T([i_\alpha])(u) = \alpha$, where $i_\alpha : \mathbb{S}^1 \longrightarrow Y_1$ includes $\mathbb{S}^1$ as $\mathbb{S}^1_\alpha$. Moreover, $X \subset Y_1$ and $u_1|X = v$. □

12.2.17 **Lemma**. *Given a Brown functor T, a space X, and an element $v \in T(X)$, there exists a space Y_n, obtained from X by attaching cells of dimension less than or equal to n, together with an n-universal element $u_n \in T(Y_n)$ such that $u_n|X = v$.*

Proof: We can assume inductively that we have constructed Y_{n-1} such that $X \subset Y_{n-1}$ (obtained from X attaching cells of dimension less than or equal to $n-1$) together with an $(n-1)$-universal element $u_{n-1} \in T(Y_{n-1})$ such that $u_{n-1}|X = v$.

We construct Y_n as follows. For every element $\beta \in T(\mathbb{S}^n)$ take a copy $\mathbb{S}^n_\beta$ of $\mathbb{S}^n$ and set $Y'_n = Y_{n-1} \vee \bigvee_\beta \mathbb{S}^n_\beta$. By the wedge axiom, there is an equivalence of sets

$$T(Y'_n) \cong T(Y_{n-1}) \times \prod_\beta T(\mathbb{S}^n_\beta)\,.$$

Take $u'_n \in T(Y'_n)$ corresponding to the element

$$(u_{n-1}, (\beta)) \in T(Y_{n-1}) \times \prod_\beta T(\mathbb{S}^n_\beta)$$

under the equivalence. Then as before, $\varphi_{u'_n} : \pi_n(Y'_n) \longrightarrow T(\mathbb{S}^n)$ is surjective.

Now, every element $\alpha \in \pi_{n-1}(Y'_n)$ such that $\varphi_{u'_n}(\alpha) = 0 \in T(\mathbb{S}^{n-1})$ is represented by a map $f_\alpha : \mathbb{S}^{n-1}_\alpha = \mathbb{S}^{n-1} \longrightarrow Y'_n$. For each α we shall attach an n cell with f_α as attaching map. In other words, define Y_n as the mapping cone C_f of the map $f : \bigvee_\alpha \mathbb{S}^{n-1}_\alpha \longrightarrow Y'_n$, where $f|\mathbb{S}^{n-1}_\alpha = f_\alpha$.

Since Y_n is obtained from Y'_n, and thus also from Y_{n-1}, by attaching n-cells and since $\pi_q(Y_n)$ depends only on the $(n-1)$-skeleton of Y_n for $q \le n-2$, it follows that the map

$$\pi^{Y_{n-1}}(\mathbb{S}^q) = \pi_q(Y_{n-1}) \longrightarrow \pi_q(Y_n) = \pi^{Y_n}(\mathbb{S}^q)$$

induced by the inclusion is an isomorphism for $q \le n-2$ and an epimorphism for $q = n-1$.

We now construct an n-universal element $u_n \in T(Y_n)$ such that $u_n|Y_{n-1} = u_{n-1}$. It will then follow that $u_n|X = v$.

Consider

$$\begin{tikzcd} \bigvee_\alpha \mathbb{S}^{n-1}_\alpha \arrow[r, shift left, "i"] \arrow[r, shift right, "t"'] & Y'_{n-1} \arrow[r, hook, "j'"] & Y_n , \end{tikzcd}$$

where i is the inclusion and t is the constant map. Then $T([i])(u'_n) = T([t])(u'_n)$. Moreover, $[j] : Y'_{n-1} \hookrightarrow Y_n$ is a coequalizer for $[i]$ and $[t]$. Thus, by the Mayer–Vietoris axiom, there exists $u_n \in T(Y_n)$ such that $u_n|Y_{n-1} = u_{n-1}$. We now show that u_n is n-universal. We have a commutative triangle

$$\begin{tikzcd} \pi_q(Y_{n-1}) \arrow[rr, "j_*"] \arrow[dr, "\varphi_{u_{n-1}}"'] & & \pi_q(Y_n) \arrow[dl, "\varphi_{u_n}"] \\ & T(\mathbb{S}^q) , & \end{tikzcd}$$

where j_* is an isomorphism for $q \le n-2$ and an epimorphism for $q = n-1$. Moreover, $\varphi_{u_{n-1}}$ is an isomorphism for $q \le n-2$ and an epimorphism for $q = n-1$. Thus φ_{u_n} is an isomorphism for $q \le n-2$ and an epimorphism for $q = n-1$. In order to show that it is a monomorphism for $q = n-1$, assume that $\varphi_{u_n}(\gamma) = 0$ for some $\gamma \in \pi_{n-1}(Y_n)$. Since j_* is an epimorphism for $q = n-1$, there exists $\gamma' \in \pi_{n-1}(Y_{n-1})$ with $j_*(\gamma') = \gamma$. But then $\varphi_{u_{n-1}}(\gamma') = 0$ and thus $\gamma' = \alpha \in \ker(\varphi_{u_{n-1}})$ and $j_*(\alpha) = 0$, since we attached a cell for every element $\alpha \in \ker(\varphi_{u_{n-1}})$. Thus $\gamma = 0$, and φ_{u_n} is an isomorphism also for $q = n-1$.

Now it is clear that φ_{u_n} is an epimorphism for $q = n$, since $\varphi_{u'_n}$ is an epimorphism and the triangle

$$\begin{tikzcd}
\pi_n(Y'_n) \arrow[rr] \arrow[dr, "\varphi_{u'_n}"'] & & \pi_n(Y_n) \arrow[dl, "\varphi_{u_n}"] \\
& T(\mathbb{S}^n) &
\end{tikzcd}$$

commutes. Hence, u_n is n-universal. □

12.2.18 **Theorem**. *Let T be a Brown functor, Y_0 a pointed space, and $u_0 \in T(Y_0)$. Then there is a pointed space Y obtained from Y_0 by attaching cells together with an ∞-universal element $u \in T(Y)$ such that $u|Y_0 = u_0$.*

Proof: We construct a space Y and an element $u \in T(Y)$ such that $\varphi_u : \pi_q(Y) \longrightarrow T(\mathbb{S}^q)$ is an isomorphism for all q.

Given a space Y_0 and $u_0 \in T(Y_0)$, by 12.2.17 we have a sequence

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots$$

together with n-universal elements $u_n \in T(Y_n)$, where each Y_n is obtained from Y_{n-1} by attaching cells of dimension less than or equal to n. Let $Y = \bigcup_n Y_n$ with the topology of the union. One has

$$\operatorname*{colim}_n \pi_q(Y_n) \cong \pi_q(Y)\,.$$

Consider the maps

$$f_0,\, f_1 : \bigvee_n Y_n \longrightarrow \bigvee_n Y_n\,,$$

where $f_0|Y_n = i_n : Y_n \hookrightarrow Y_{n+1}$ and $f_1 = \mathrm{id}_{\bigvee_n Y_n}$. Then the homotopy class of $i : \bigvee_n Y_n \longrightarrow Y$ such that $i|Y_n : Y_n \hookrightarrow Y$ is a coequalizer for $[f_0]$ and $[f_1]$. Moreover, the element $(u_n) \in \prod T(Y_n)$ maps to (u_n) under both $T([f_0])$ and $T([f_1])$. Hence, by the Mayer–Vietoris axiom, there exists $u \in T(Y)$ such that $u|Y_n = u_n$. Then

$$\begin{tikzcd}
\operatorname{colim} \pi_q(Y_n) \arrow[rr, "\cong"] \arrow[dr, "\cong", "(\varphi_{u_n})"'] & & \pi_q(Y) \arrow[dl, "\varphi_u"] \\
& T(\mathbb{S}^q) &
\end{tikzcd}$$

commutes, implying that φ_u is an isomorphism for all q. Thus $u \in T(Y)$ is an ∞-universal element. □

12.2.19 **Theorem**. *Let T be a Brown functor. If Y and Y' are pointed* CW-*complexes with ∞-universal elements $u \in T(Y)$ and $u' \in T(Y')$, then there exists a homotopy equivalence $f : Y \longrightarrow Y'$ such that $T([f])(u') = u$.*

Proof: Take $Y_0 = Y \vee Y'$. Let $u_0 \in T(Y_0)$ correspond to $(u, u') \in T(Y) \times T(Y')$ using the wedge axiom. Then by 12.2.18 there exists Y'' containing Y_0 together with an ∞-universal element $u'' \in T(Y'')$ such that $u''|Y_0 = u_0$. The composite $j : Y \hookrightarrow Y_0 = Y \vee Y' \hookrightarrow Y''$ induces

$$\begin{tikzcd} \pi_q(Y) \arrow[rr, "j_*"] \arrow[dr, "\varphi_u"', "\cong"] & & \pi_q(Y'') \arrow[dl, "\varphi_{u''}", "\cong"'] \\ & T(\mathbb{S}^q)\,, & \end{tikzcd}$$

so that j_* is an isomorphism for all q. Hence, $j : Y \hookrightarrow Y''$ is a weak homotopy equivalence, and thus a homotopy equivalence, since Y and Y'' are CW-complexes. Similarly, $j' : Y' \hookrightarrow Y''$ is a homotopy equivalence. If $j'' : Y'' \longrightarrow Y'$ is a homotopy inverse of j', then the composite

$$f : Y \xrightarrow{j} Y'' \xrightarrow{j''} Y'$$

is a homotopy equivalence such that $T([f])(u') = u$. □

12.2.20 **Proposition**. *Let T be a Brown functor, Y a* CW-*complex, and $u \in T(Y)$ an ∞-universal element and (X, A) a* CW-*pair. Given a pointed map $g : A \longrightarrow Y$ and an element $v \in T(X)$ such that $v|A = T([g])(u)$, then there exists an extension $f : X \longrightarrow Y$ of g such that $v = T([f])(u)$.*

Proof: Consider the diagram

$$\begin{tikzcd} & X \arrow[dr, "i_0"] & & \\ A \arrow[ur, "i"] \arrow[dr, "g"'] & & X \vee Y \arrow[r, "j"] & Z\,, \\ & Y \arrow[ur, "i_1"'] & & \end{tikzcd}$$

where i, i_0, i_1 are the inclusions and j is such that $[j]$ is a coequalizer for $[i_0] \circ [i]$ and $[i_1] \circ [g]$. By the construction of coequalizers (see 12.2.11) Z is a CW-complex. By the wedge axiom there is an element $v' \in T(X \vee Y)$ such that $v'|X = v$ and $v'|Y = u$. By the Mayer–Vietoris axiom there exists $w \in T(Z)$ such that $T([j])(w) = v'$.

By 12.2.18 there is a pointed space Y' obtained from Z by attaching cells, hence a CW-complex together with an ∞-universal element $u' \in T(Y')$ such that $u'|Z = v'$. Since we already have a pointed space Y together with a universal element $u \in T(Y)$, 12.2.19 implies that there is a homotopy equivalence $h : Y' \longrightarrow Y$ such that $T([h])(u) = u'$.

Define f' as the composite

$$f' : X \xrightarrow{i_0} X \vee Y \xrightarrow{j} Z \overset{j'}{\hookrightarrow} Y' \xrightarrow{h} Y .$$

Then $g \simeq f' \circ i$. Since $i : A \hookrightarrow X$ is a cofibration, we may extend a homotopy between $f' \circ i$ and g starting with f' and thus obtain $f \simeq f'$ such that $f \circ i = g$. $\square$

12.2.21 **Proposition**. *Let $u \in T(Y)$ be an ∞-universal element. Then u is a universal element in the category of pointed* CW*-complexes, and therefore Y is a classifying space for T. In other words, if X is a pointed* CW*-complex, then $\varphi_u : \pi^Y(X) \longrightarrow T(X)$ is a bijection, and thus it determines a natural equivalence $\pi^Y \overset{\cdot}{\longrightarrow} T$.*

Proof: We shall prove that φ_u is one-to-one and onto. To see that it is onto, take an element $v \in T(X)$. We may apply Proposition 12.2.20 for $A = \{x_0\}$ the base point of X. Therefore, there exists a map $f : X \longrightarrow Y$ extending the inclusion $g : \{x_0\} \hookrightarrow Y$ onto the base point of Y in such a way that $T([f])(u) = v$. Hence $\varphi_u([f]) = T([f])(u) = v$, and so φ_u is surjective.

To see that φ_u is one-to-one, suppose that $\varphi_u([g_0]) = \varphi_u([g_1])$, $[g_0], [g_1] \in \pi^Y(X)$. That is, $T([g_0])(u) = T([g_1])(u)$. The space $X' = X \times I/\{x_0\} \times I$ is a CW-complex with q-skeleton $X'^q = (X^q \times I/\{x_0\} \times I) \cup X^q \times \partial I$. Take now $A = X \times \partial I/\{x_0\} \times \partial I$. Observe that $A \approx X \vee X$. Define $g : A \longrightarrow Y$ by $g\rho(x, 0) = g_0(x)$ and $g\rho(x, 1) = g_1(x)$, where $\rho : X \times \partial I \twoheadrightarrow A$ is the quotient map. On the other hand, the projection $p : X' \longrightarrow X$ is a homotopy equivalence. Take $v' = T([p])T([g_0])(u) \in T(X')$. Then, if $j : A \hookrightarrow X'$ is the inclusion, $T([j])(v')$ corresponds to the element $(T([g_0])(u), T([g_1])(u)) \in T(X) \times T(X) \cong T(A)$ by the wedge axiom. By Proposition 12.2.20 there exists an extension of g to $f : X' \longrightarrow Y$ such that $T([f])(u) = v'$. But then the composite

$$H : X \times I \xrightarrow{\rho} X' \xrightarrow{f} Y ,$$

where $\rho : X \times I \in X'$ is the quotient map, is a homotopy between g_0 and g_1. Thus $[g_0] = [g_1]$, and φ_u is injective. $\square$

Assume that T is a Brown functor. Take the singleton space $*$ and the single element $u_0 \in T(*)$ according to Proposition 12.2.13. From Theorems

12.2.18 and 12.2.19, taking $Y_0 = *$, there is a pointed space Y, unique up to homotopy, and an ∞-universal element $u \in T(Y)$. Finally, by Proposition 12.2.21 there is a natural equivalence $\pi^Y \stackrel{\cdot}{\longrightarrow} T$ in the category of pointed CW-complexes; in other words, for every pointed CW-complex X there is a bijection

$$\Phi_X : [X, Y]_* \longrightarrow T(X)$$

such that $\Phi_X[f] = T([f])(u)$. That is, the functor T is representable. We have then the main result of this section.

12.2.22 **Theorem**. (Brown representability theorem) *Every Brown functor T is representable in the category of path-connected pointed* CW*-complexes. More specifically, there is a pointed* CW*-complex Y, unique up to homotopy, and a natural equivalence*

$$\Phi : [-, Y]_* \stackrel{\cdot}{\longrightarrow} T\,. \qquad \square$$

12.3 Spectra

In this section we show, using the Brown theorem, that any generalized cohomology theory determines a family of topological spaces linked together with a special structure, which constitutes a so-called *spectrum*.

Let k^* be a cohomology theory defined on $\mathcal{WT}op_*$, the category of pointed CW-complexes, and satisfying the wedge axiom. For simplicity in what follows, we omit writing the base point. We thus write $k^n(X)$ instead of $k^n(X, x_0)$. If $(X; A, B)$ is a pointed CW-triad, then it is excisive with respect to k^* and there is a Mayer–Vietoris sequence for this triad (see 12.1.23). The exactness of this sequence at $k^n(A) \oplus k^n(B)$ implies that each homotopy functor k^n satisfies the Mayer–Vietoris axiom for a Brown functor (see 12.2.9). Then by the Brown theorem 12.2.22 there exists a pointed connected CW-complex L_n, unique up to homotopy, and a natural equivalence

$$[Y, L_n]_* \longrightarrow k^n(Y)$$

for each connected pointed CW-complex Y. Define spaces P_n as the loop spaces

$$P_n = \Omega L_{n+1}\,,$$

for each $n \in \mathbb{Z}$. Moreover, if X is any pointed CW-complex, then its reduced suspension ΣX is connected, and so $k^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}]_*$. Now, since k^* is a reduced cohomology theory, there is a natural equivalence

$s_n : k^{n+1}(\Sigma X) \cong k^n(X)$ for each n. On the other hand, by 2.10.5 there is another natural equivalence $[\Sigma X, L_{n+1}]_* \cong [X, \Omega L_{n+1}]_*$. Therefore, putting all these natural equivalences together, we have that

$$k^n(X) \cong k^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}]_* \cong [X, \Omega L_{n+1}]_* = [X, P_n]_*$$

for any pointed CW-complex X.

Since each L_n is unique up to homotopy, P_n is also unique up to homotopy. Thus we can associate to the reduced cohomology theory k^* the family $\{P_n\}_{n\in\mathbb{Z}}$. Milnor proved in [54] that the loop space of a CW-complex has the homotopy type of a CW-complex; therefore, each space P_n has the homotopy type of a CW-complex.

We now establish a relationship between the spaces P_n for different values of n. For this, consider again the suspension isomorphisms $s_n : k^{n+1}(\Sigma X) \cong k^n(X)$. We have the composite of natural equivalences

$$[X, P_n]_* \cong k^n(X) \cong k^{n+1}(\Sigma X) \cong [\Sigma X, P_{n+1}]_* \cong [X, \Omega P_{n+1}]_*$$

for any pointed CW-complex X. Since there are CW-complexes K, L such that $K \simeq P_n$ and $L \simeq \Omega P_{n+1}$, then we have a natural equivalence

$$[X, K]_* \cong [X, L]_*$$

for any CW-complex X. By 12.2.4, for this natural equivalence there is a corresponding homotopy equivalence $K \longrightarrow L$, which in turn corresponds to a homotopy equivalence $\varepsilon_n : P_n \longrightarrow \Omega P_{n+1}$.

Such a family of spaces $\{P_n\}_{n\in\mathbb{Z}}$ together with the homotopy equivalences $\varepsilon_n : P_n \longrightarrow \Omega P_{n+1}$ is an instance of what used to be called an Ω-spectrum. Now it is called an Ω-prespectrum, as defined by May [51]. Observe that from the bijection $[\Sigma P_n, P_{n+1}]_* \cong [P_n, \Omega P_{n+1}]_*$ the maps ε_n have adjoints $\widehat{\varepsilon}_n : \Sigma P_n \longrightarrow P_{n+1}$. We are led to the following definition.

12.3.1 DEFINITION. An *Ω-prespectrum* consists of a collection of pointed spaces $\{P_n\}_{n\in\mathbb{Z}}$ and weak homotopy equivalences $\varepsilon_n : P_n \longrightarrow \Omega P_{n+1}$.

Therefore, we have the following result.

12.3.2 **Theorem**. *Each additive reduced cohomology theory k^* on the category $\mathcal{WT}op_\odot$ of pointed spaces of the homotopy type of a* CW*-complex determines an Ω-prespectrum P such that for any X, $k^n(X) \cong [X, P_n]_*$.* This is called the *associated Ω-prespectrum of k^*.* □

Conversely, let $P = \{P_n\}$ be an Ω-prespectrum. Then we can define an *associated reduced cohomology theory*, usually denoted by the same letter P, such that if X is any pointed CW-complex, then

$$\widetilde{P}^n(X) = [X, P_n]_* \,.$$

The suspension isomorphisms s^n are given by

$$\widetilde{P}^{n+1}(\Sigma X) = [\Sigma X, P_{n+1}]_* \cong [X, \Omega P_{n+1}]_* \xrightarrow{(\varepsilon_n)_*^{-1}} [X, P_n]_* = \widetilde{P}^n(X)\,,$$

where the weak homotopy equivalence ε_n induces a bijection by 5.1.33. In particular, the bijection $\widetilde{P}^n(X) \cong [\Sigma^2 X, P_{n+2}]$ induces the structure of an abelian group on $\widetilde{P}^n(X)$. Proposition 3.3.8 shows that if $A \subset X$, then we have an exact sequence

$$\widetilde{P}^n(X \cup CA) \longrightarrow \widetilde{P}^n(X) \longrightarrow \widetilde{P}^n(A)\,.$$

This shows that the exactness axiom is satisfied. Using CW-approximations, we can extend the theory P^* to the category $\mathcal{T}op_*$ of pointed topological spaces. Hence we have the following.

12.3.3 **Theorem**. *If P is an Ω-prespectrum, then the functors $\widetilde{P}^n : \mathcal{T}op_* \longrightarrow \mathcal{A}$ together with the isomorphisms $s^n : \widetilde{P}^{n+1}(\Sigma X) \longrightarrow \widetilde{P}^n(X)$ for any pointed space X are an additive reduced cohomology theory.* □

Let $S : k^* \longrightarrow k'^*$ be a transformation of additive reduced cohomology theories (see 12.1.16) such that for the 0-sphere $\mathbb{S}^0$ one has an isomorphism

$$S_{\mathbb{S}^0} : k^n(\mathbb{S}^0) \longrightarrow k'^n(\mathbb{S}^0)$$

for all $n \in \mathbb{Z}$. The following is a commutative diagram:

$$\begin{tikzcd}
k^{n-q}(\mathbb{S}^0) \arrow[r, "S_{\mathbb{S}^0}", "\cong"'] \arrow[d, "s^q"'] & k'^{n-q}(\mathbb{S}^0) \arrow[d, "s'^q"] \\
k^n(\mathbb{S}^q) \arrow[r, "S_{\mathbb{S}^q}"'] & k'^n(\mathbb{S}^q)\,,
\end{tikzcd}$$

where s^q and s'^q are the corresponding composites of suspension isomorphisms. Thus $S_{\mathbb{S}^q} : k^n(\mathbb{S}^q) \longrightarrow k'^n(\mathbb{S}^q)$ is an isomorphism for all q.

Assume now that P is the Ω-spectrum associated to k and P' that associated to k'. So one has natural equivalences $\varepsilon : k^n \stackrel{\cdot}{\longrightarrow} [-, P_n]_*$, $\varepsilon' : k'^n \stackrel{\cdot}{\longrightarrow} [-, P'_n]_*$. By 12.2.4, there is a map $\rho_n : P_n \longrightarrow P'_n$ such that $\varepsilon'_X S_X = \rho_{n\#}\varepsilon_X$.

Therefore, for any sphere $X = \mathbb{S}^q$, $\rho_{n\#} : [\mathbb{S}^q, P_n]_* \longrightarrow [\mathbb{S}^q, P'_n]_*$ is an isomorphism. Thus ρ_n is a weak homotopy equivalence, and since P_n, P'_n have the homotopy type of a CW-complex, ρ_n is a homotopy equivalence. Consequently, $\rho_{n\#} : [X, P_n]_* \longrightarrow [X, P'_n]_*$ is also an isomorphism for every pointed space X, and so $S_X : k^n(X) \longrightarrow k'^n(X)$ is an isomorphism.

We have proved the following comparison theorem, which, in a sense, generalizes 12.1.19.

12.3.4 **Theorem**. *Assume that k^*, k'^* are additive reduced cohomology theories on $\mathcal{WT}op_\odot$ and let $S : k^* \longrightarrow k'^*$ be a transformation such that*

$$S_{\mathbb{S}^0} : k^n(\mathbb{S}^0) \longrightarrow k'^n(\mathbb{S}^0)$$

is an isomorphism for all n. Then S is an equivalence of cohomology theories, that is,

$$S_X : k^n(X) \longrightarrow k'^n(X)$$

is an isomorphism for all n and every pointed space X of the homotopy type of a CW-*complex.* □

12.3.5 Remark. If the theories k^* and k'^* above satisfy the weak homotopy equivalence axiom, then they are equivalent in $\mathcal{T}op_*$.

In the case of ordinary cohomology theories, we have the following result.

12.3.6 **Theorem**. *Let k^*, k'^* be ordinary additive reduced cohomology theories on $\mathcal{WT}op_\odot$ such that there is an isomorphism of* coefficients

$$\tau : k^0(\mathbb{S}^0) \longrightarrow k'^0(\mathbb{S}^0)\,.$$

Then τ induces an equivalence of cohomology theories

$$S : k^* \stackrel{\cdot}{\longrightarrow} k'^*\,.$$

Proof: By 12.3.2 there are associated Ω-prespectra P, P' such that P_n, and P'_n have the homotopy type of a CW-complex and

$$k^n(X) \cong [X, P_n]_*\,, \qquad k'^n(X) \cong [X, P'_n]_*$$

for all n and for all pointed spaces X of the homotopy type of a CW-complex. We have

$$\pi_q(P_n) = [\mathbb{S}^q, P_n]_* \cong k^n(\mathbb{S}^q) \cong k^{n-q}(\mathbb{S}^0) \cong \begin{cases} 0 & \text{if } q \neq n, \\ G & \text{if } q = n, \end{cases}$$

where $G = k^0(\mathbb{S}^0)$. In other words, each P_n is an Eilenberg–Mac Lane space of type (G, n). Analogously, P'_n is an Eilenberg–Mac Lane space of type (G', n), where $G' = k'^0(\mathbb{S}^0)$.

By 6.4.9, the isomorphism τ can be realized by a homotopy equivalence $\rho_n : P_n \longrightarrow P'_n$ for each $n \in \mathbb{Z}$. Thus it defines an equivalence

$$S : k^* \stackrel{\cdot}{\longrightarrow} k'^* .$$

□

12.3.7 Remark. If the theories k^* and k'^* in the two previous theorems also satisfy the weak homotopy equivalence axiom, then they are equivalent in $\mathcal{T}op_*$.

12.3.8 Examples.

1. Let G be an abelian group. Then the family of Eilenberg–Mac Lane spaces $\{K(G, n)\}$ constitutes an Ω-prespectrum, where the homotopy equivalences
$$\varepsilon_n : K(G, n) \longrightarrow \Omega K(G, n+1)$$
are given as follows. Since
$$\pi_q(\Omega K(G, n+1)) \cong \pi_{q+1}(K(G, n+1))$$
and
$$\pi_q(K(G, n)) \cong \pi_{q+1}(K(G, n+1)) ,$$
one has
$$\pi_q(K(G, n)) \cong \pi_q(\Omega K(G, n+1)) \quad \text{for all } q \geq 0 .$$
Therefore, by 6.4.6, there is a map
$$\varepsilon_n : K(G, n) \longrightarrow \Omega K(G, n+1)$$
inducing the isomorphism
$$\pi_n(K(G, n)) \cong \pi_n(\Omega K(G, n+1)) .$$
Since all the other homotopy groups are zero, ε_n is a weak homotopy equivalence. Moreover, $\Omega K(G, n+1)$ has the homotopy type of a CW-complex, and so by the Whitehead theorem 5.1.37, ε_n is a homotopy equivalence.

   The Ω-prespectrum HG, where $HG_n = K(G, n)$ for $n \geq 0$ and $HG_n = \{*\}$ for $n < 0$, is called an *Eilenberg–Mac Lane (pre)spectrum.* Hence the cohomology theory defined by HG is precisely the cohomology theory $\widetilde{H}(-;G)$ defined in Chapter 7. Thus for any n,

$$\widetilde{HG}^n(X) = \widetilde{H}^n(X;G)$$

for every pointed space X.

2. The family of spaces $P_{2n} = \mathrm{BU} \times \mathbb{Z}$ and $P_{2n+1} = \Omega\mathrm{BU}$ for $n \in \mathbb{Z}$, has the property that $P_{2n-1} = \Omega\mathrm{BU} = \Omega(\mathrm{BU} \times \mathbb{Z}) = \Omega P_{2n}$ and $P_{2n} = \mathrm{BU} \times \mathbb{Z} \simeq \Omega^2\mathrm{BU} = \Omega P_{2n+1}$ by the Bott periodicity theorem 9.5.1. Hence this family is an Ω-prespectrum called the BU-*spectrum*, usually denoted by $\mathcal{K}$. The associated cohomology theory $\mathcal{K}^*$ is called *complex K-cohomology.* This is the theory defined in Section 9.5. If X is a finite-dimensional CW-complex, then by 9.4.9, $\widetilde{\mathcal{K}}^0(X) \cong \widetilde{K}(X)$. Taking unpointed homotopy classes gives $K(X) \cong [X, \mathrm{BU} \times Z] = \widetilde{\mathcal{K}}^0(X^+)$ for any finite-dimensional CW-complex X.

3. Similarly, the family of spaces $P_{8n-r} = \Omega^r\mathrm{BO} \times \mathbb{Z}$, $0 \leq r < 8$, $n \in \mathbb{Z}$, together with $\varepsilon_{8n} : \mathrm{BO} \times \mathbb{Z} \longrightarrow \Omega^8\mathrm{BO} \times \mathbb{Z}$ given by the real Bott periodicity, and the identity in other dimensions, has the structure of an Ω-prespectrum called the BO-*spectrum.*

12.3.9 Definition. A family of pointed spaces $\{P_n\}$ together with pointed maps $\sigma_n : \Sigma P_n \longrightarrow P_{n+1}$ (where the adjoint maps $\widehat{\sigma}_n : P_n \longrightarrow \Omega P_{n+1}$ are not necessarily weak homotopy equivalences) is called a *prespectrum* P. If P and P' are prespectra, then a *map of prespectra* $f : P \longrightarrow P'$ consists af a family of maps $f_n : P_n \longrightarrow P'_n$ such that the diagram

$$\begin{tikzcd}
\Sigma P_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma P'_n \arrow[d, "\sigma'_n"] \\
P_{n+1} \arrow[r, "f_{n+1}"'] & P'_{n+1}
\end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

A typical example of a prespectrum is the so-called *suspension spectrum* SX associated to any pointed space X, which is defined by

$$SX_n = \begin{cases} \Sigma^n X & \text{if } n \geq 0, \\ * & \text{if } n < 0, \end{cases}$$

with the maps σ_n given by the obvious homeomorphisms $\sigma_n : \Sigma\Sigma^n X \longrightarrow \Sigma^{n+1}X$. A special case of this is the sphere prespectrum $\mathbb{S}$ consisting of all spheres. Other examples are the *Thom spectra*, which appear in cobordism theories (see [76]), as we shall see below.

12.3.10 Definition. Given a prespectrum $P = \{P_n\}$, we define its *homotopy groups* by

$$\pi_n(P) = \operatorname*{colim}_k \pi_{n+k}(P_k)\,,$$

where the colimit is taken over the homomorphisms given by the composite

$$\pi_{n+k}(P_k) \longrightarrow \pi_{n+k+1}(\Sigma P_k) \xrightarrow{\sigma_{k*}} \pi_{n+k+1}(P_{k+1})\,.$$

12.3.11 Example. If X is a pointed space and SX is its suspension prespectrum, then its homotopy groups $\pi_n(SX)$ are the so-called *stable homotopy groups* of X and are usually denoted by $\pi_n^s(X)$. In particular, taking $X = \mathbb{S}^0$, that is, if one takes the sphere prespectrum $\mathbb{S}$, then $\pi_n(\mathbb{S})$ is known as the *n-stem* and is simply denoted by π_n^s.

In order to study invariants that are, so to speak, independent of the dimension, like the stable groups that appear in the Freudenthal suspension theorem, it is necessary to define a good stable homotopy category. This is not an easy matter; the first satisfactory construction was given by Boardman. We now follow May's approach [44].

The first step is to consider prespectra as the objects of this category $\mathcal{P}$ and their maps as the morphisms of $\mathcal{P}$.

The next step is to consider a good family of prespectra; this is the family of CW-*prespectra*. A CW-*prespectrum* W is a collection of CW-complexes W_n and cellular inclusions $\sigma_n : \Sigma W_n \hookrightarrow W_{n+1}$.

12.3.12 Definition. Given a CW-prespectrum $W = \{W_n\}$ and a pointed CW-complex X, one can define groups

$$\widetilde{W}^n(X) = [X, \operatorname*{colim}_k \Omega^k W_{n+k}]_*\,,$$

where the colimit is taken over the maps $\Omega^k\widehat{\sigma}_{n+k} : \Omega^k W_{n+k} \longrightarrow \Omega^{k+1}W_{n+k+1}$ and where $\widehat{\sigma}_{n+k}$ is the adjoint of σ_{n+k}. We also define groups

$$\widetilde{W}_n(X) = \pi_n(W \wedge X)\,,$$

where $W \wedge X$ is the prespectrum given by $(W \wedge X)_n = W_n \wedge X$ with structure maps $\sigma'_n = \sigma_n \wedge \mathrm{id}_X$.

One easily defines isomorphisms $\widetilde{W}^{n+1}(\Sigma X) \longrightarrow \widetilde{W}^n(X)$ and $\widetilde{W}_n(X) \longrightarrow \widetilde{W}_{n+1}(\Sigma X)$, and one has the following theorem.

12.3.13 **Theorem**. *Let* $W = \{W_n\}$ *be a* CW*-prespectrum. Then the groups* $\widetilde{W}^n(X)$ *define an additive reduced cohomology theory and the groups and* $\widetilde{W}_n(X)$ *define an additive reduced homology theory, both on the category* $\mathcal{WTop}_*$ *of pointed* CW*-complexes.* These are the *associated reduced cohomology and homology theories* for W.

12.3.14 REMARK. These theories can be extended to any pointed space X by taking a CW-approximation $\widetilde{X}$.

For the *proof* we refer the reader to [76]. □

12.3.15 EXERCISE. Prove that the associated reduced cohomology and homology theories for the CW-prespectrum HG are ordinary; more precisely, prove that

$$\widetilde{HG}^n(\mathbb{S}^0) \cong \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \qquad \widetilde{HG}_n(\mathbb{S}^0) \cong \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

(Hint: Prove that for any CW-prespectrum W that is an Ω-prespectrum, $\operatorname{colim}_k \Omega^k W_{n+k} \simeq W_n$.) By applying 12.1.19, conclude that $\widetilde{HG}^*$ and $\widetilde{HG}_*$ are equivalent to $\widetilde{H}^*(-;G)$ and $\widetilde{H}_*(-;G)$, respectively, on the category $\mathcal{WTop}_*$.

12.3.16 EXAMPLE. For the sphere prespectrum $\mathbb{S}$ the associated cohomology theory is given by the stable cohomotopy groups $\pi_s^n(X)$ and is the so-called *stable cohomotopy theory.* Its associated homology theory is given by the stable homotopy groups $\pi_n^s(X)$ and is the so-called *stable homotopy theory.* There are also K-homology theories associated to the prespectra BU and BO.

12.3.17 DEFINITION. A *spectrum* is a prespectrum, $\{E_n\}_{n\in\mathbb{Z}}$ together with $\sigma_n : \Sigma E_n \longrightarrow E_{n+1}$, such that the adjoint maps $\widehat{\sigma}_n : E_n \longrightarrow \Omega E_{n+1}$ are homeomorphisms.

If E and E' are spectra, then a *map of spectra* $f : E \longrightarrow E'$ is a map of the underlying prespectra.

Let $\mathcal{S}$ denote the category of spectra and let $\mathcal{P}$ be the category of prespectra. Then the functor $F : \mathcal{S} \longrightarrow \mathcal{P}$ that "forgets" the spectrum structure has a left adjoint $L : \mathcal{P} \longrightarrow \mathcal{S}$ defined as follows. If P is a prespectrum such that each $\widehat{\sigma}_n : P_n \longrightarrow \Omega P_{n+1}$ is an inclusion, then let $L(P)$ be the spectrum such that $L(P)_n = \operatorname{colim}_k \Omega^k P_{n+k}$, where the colimit is taken with respect to the maps $\Omega^k\widetilde{\sigma}_{n+k} : \Omega^k P_{n+k} \longrightarrow \Omega^{k+1}P_{n+k+1}$ for $k \geq 0$. If $f : P \longrightarrow P'$ is a map of prespectra, then $L(f) : L(P) \longrightarrow L(P')$ is given by $L(f)_n = \operatorname{colim}_k \Omega^k f_{n+k}$, for each n. The definition of L for an arbitrary prespectrum is more complicated (see [44]).

Since L is a left adjoint functor of F, there is a bijection between morphism sets

$$\mathcal{S}(L(P), E) \longleftrightarrow \mathcal{P}(P, F(E))$$

for any prespectrum P and any spectrum E.

The category CW-$\mathcal{S}$ of CW-spectra is the image under L of the category CW-$\mathcal{P}$ of CW-prespectra.

To define the stable homotopy category we consider the following. For any spectrum E take the prespectrum whose nth space is $E_n \wedge [0,1]^+$ and apply the functor L to it. The result is denoted by $\mathrm{Cyl}(E)$. We say that the maps $f_0, f_1 : E \longrightarrow E'$ of spectra are *homotopic* if there is a map of spectra $h : \mathrm{Cyl}(E) \longrightarrow E'$ such that $h|E \times \{\nu\} = f_\nu$, $\nu = 0, 1$. The *stable homotopy category* has the same objects as CW-$\mathcal{S}$, and the homotopy classes of maps of spectra as morphisms.

In the category of spectra we have the obvious concept of a weak homotopy equivalence and similar results to the more common ones presented in Chapter 5.

12.3.18 **Theorem**. *In the category $\mathcal{S}$ of spectra we have the following facts:*

(a) *For any spectrum E there is a* CW*-spectrum W and a weak homotopy equivalence $f : W \longrightarrow E$.*

(b) *Let E and E' be spectra and let $f : E \longrightarrow E'$ be a weak homotopy equivalence. Then for any* CW*-spectrum K we have that $f : [K, E] \longrightarrow [K, E']$ is bijective.*

(c) *Every weak homotopy equivalence between* CW*-spectra is a homotopy equivalence.*

Finally, we remark that there is a homology (covariant) version of the Brown representability theorem, expressed in terms of spectra, which is due to Adams [3].

12.3.19 **Theorem**. *Let k be a reduced homology theory defined on the category $\mathcal{WT}op_*$ of pointed* CW*-complexes satisfying*

$$\operatorname*{colim}_i k(X_i) = k(X)\,,$$

where $\{X_i\}$ is the family of all finite subcomplexes of X. Then there is an Ω-prespectrum P such that k is the homology theory corresponding to P. That is, there is an equivalence of homology theories

$$k_n(X) \stackrel{\cdot}{\longrightarrow} \pi_n(P \wedge X)\,,$$

where X is any pointed CW*-complex.*

12.3.20 REMARK. To define products in cohomolgy one needs a good definition of the smash product of spectra to obtain the so-called *ring spectra.* Although it is possible to do this with the conventional spectra (as we did for products in cohomology in Section 7.2 and shall do again below for cobordism), it is more convenient to take spectra indexed not by the integers $\mathbb{Z}$, but rather by finite-dimensional subspaces of the inner product space $\mathbb{R}^\infty$ (see [44] or [29]). These are the so-called *coordinate-free spectra.* Another approach is given in [33]. For the comparison of these approaches and others see [49].

We introduce in what follows a very important family of spectra.

From 8.5.17 (b) we obtain the pullback diagram

$$\begin{tikzcd}
\mathrm{E}_k \oplus \varepsilon^1 \arrow[r] \arrow[d] & \mathrm{E}_{k+1} \arrow[d] \\
\mathrm{BO}_k \arrow[r] & \mathrm{BO}_{k+1}\,,
\end{tikzcd}$$

where BO_k denotes the real Grassmann space $\mathrm{G}_k(\mathbb{R}^\infty)$ and $\mathrm{E}_k \longrightarrow \mathrm{BO}_k$ represents the universal k-vector bundle. Therefore a Riemannian metric on E_{k+1} induces one on $\mathrm{E}_k \oplus \varepsilon^1$. So we have for the Thom spaces an induced embedding $T(\mathrm{E}_k \oplus \varepsilon^1) \hookrightarrow T(\mathrm{E}_{k+1})$. By 11.7.4 (b) we have a homeomorphism $T(\mathrm{E}_k \oplus \varepsilon^1) \approx \Sigma T(\mathrm{E}_k)$. Defining $\mathrm{MO}_k = T(\mathrm{E}_k)$, we have embeddings

$$\Sigma\mathrm{MO}_k \hookrightarrow \mathrm{MO}_{k+1}$$

for all $k \geq 0$. Since each BO_k is a CW-complex (see [58]), MO_k is also a CW-complex. Hence these spaces constitute a CW-prespectrum MO, where $\mathrm{MO}_k = *$ when $k < 0$.

The cohomology theory $\widetilde{\mathrm{MO}}^*$ associated to MO is called *unoriented cobordism* and the homology theory $\widetilde{\mathrm{MO}}_*$ is called *unoriented bordism.* These theories were introduced by Atiyah [11]. There is another pullback diagram

$$\begin{tikzcd}
\mathrm{E}_k \times \mathrm{E}_l \arrow[r] \arrow[d] & \mathrm{E}_{k+l} \arrow[d] \\
\mathrm{BO}_k \times \mathrm{BO}_l \arrow[r] & \mathrm{BO}_{k+l}\,,
\end{tikzcd}$$

which by 11.7.4 (d) induces maps $\mathrm{MO}_k \wedge \mathrm{MO}_l \longrightarrow \mathrm{MO}_{k+l}$. This makes MO into a *ring spectrum*. The *coefficients* of this theory are the graded ring $\widetilde{\mathrm{MO}}_*(\mathbb{S}^0) = \pi_*(\mathrm{MO})$. This ring has the following geometric interpretation.

Consider two smooth closed (i.e., compact with empty boundary) n-manifolds M, N. We say that they are *cobordant* if there is a compact smooth $(n+1)$-manifold W such that its boundary ∂W is diffeomorphic to the topological sum $M \sqcup N$. One can show that this is an equivalence relation. Clearly, if two manifolds are diffeomorphic, then they are cobordant. So cobordism is a weaker equivalence relation than diffeomorphism, but one that allows us to study the topology of smooth manifolds. We denote by $\mathcal{N}_n$ the set of cobordism classes of n-manifolds. Taking the topological sum of manifolds turns $\mathcal{N}_n$ into a group. Taking the Cartesian product of manifolds we can define a graded product

$$\mathcal{N}_m \times \mathcal{N}_n \longrightarrow \mathcal{N}_{m+n} \,,$$

so that $\mathcal{N}_*$ is a graded ring. Thom [78] proved that $\mathcal{N}_*$ and $\pi_*(\mathrm{MO})$ are isomorphic as graded rings. This is the fundamental result in cobordism theory, and it translates a classification problem of manifolds into a problem in homotopy theory. Then, using the tools of algebraic topology, Thom calculated the ring $\pi_*(\mathrm{MO})$, obtaining the following remarkable result [78].

12.3.21 **Theorem**. *$\mathcal{N}_*$ is a polynomial ring over $\mathbb{Z}_2$ with one generator $x_n \in \mathcal{N}_n$ for each $n \neq 2^k - 1$ $(k \geq 0)$.*

Furthermore, using Stiefel–Whitney classes, Thom defined algebraic invariants that characterize the cobordism class of a manifold.

Atiyah [11] gave a geometric interpretation of the groups $\widetilde{\mathrm{MO}}_n(X)$ in terms of cobordism classes of pairs (M, φ), where M is a closed smooth n-manifold that is the boundary of a compact smooth $(n+1)$-manifold W and $\varphi : M \longrightarrow X$ is continuous. A similar interpretation for the cobordism groups $\widetilde{\mathrm{MO}}^n(X)$ was given by Quillen [61], who also gave another proof of Thom's result using *formal groups*.

Using the complex universal bundles $\mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow \mathrm{BU}_k$ one can construct a spectrum MU, where $\mathrm{MU}_{2n} = T(E_n(\mathbb{C}^\infty))$ and $\mathrm{MU}_{2n+1} = \Sigma \mathrm{MU}_{2n}$ $(n \geq 0)$, and whose coefficients are isomorphic to the cobordism ring of stably almost complex smooth manifolds. This theory was studied by Milnor [55] and independently by S.P. Novikov. Complex cobordism can be used to study the stable homotopy groups of spheres [63]. There are cobordism theories associated to other families of Lie groups. For example, certain bordism groups of spin manifolds are used to study the Gromov–Lawson–Rosenberg conjecture about the existence of a positive scalar curvature metric on a spin manifold [72].

Algebraic K-theory yields an important family of spectra.

Let R be a ring (associative with unit). Consider the category of finitely generated left projective R-modules. Let $S(R)$ be the semigroup (under the product) of isomorphism classes of such modules. We define $K_0(R)$ to be the Grothendieck group associated to $S(R)$ (see 9.1.1). Let $C(X;F)$ be the ring of continuous functions from X to $F = \mathbb{R}$ or $\mathbb{C}$. We can assign to a vector bundle $p : E \longrightarrow X$ the $C(X;F)$-module $\Gamma(E)$ (see 8.3.10). If X is a finite-dimensional paracompact space with a finite number of components, then by the Serre–Swan theorem [75] there is an isomorphism $K(X) \cong K_0(C(X;\mathbb{C}))$ (similarly, $KO(X) \cong K_0(C(X;\mathbb{R}))$). Quillen ([62]) defined groups $K_i(R)$ for all $i > 0$, called the *algebraic K-theory of R*. He considered the group $\mathrm{GL}(R) = \operatorname{colim} \mathrm{GL}_n(R)$, its classifying space $B\mathrm{GL}(R)$ (cf. 4.6.17), and then he applied his plus construction to obtain a space $B\mathrm{GL}(R)^+$ (not the disjoint union with a point!). He set $K_i(R) = \pi_i(B\mathrm{GL}(R)^+)$. These groups have applications in topology. For example, for a space X dominated by a finite CW-complex (see 6.3.22), C.T.C. Wall defined an obstruction in $\widetilde{K}_0(\mathbb{Z}\pi_1(X))$, where $\mathbb{Z}\pi_1(X)$ denotes the group ring of $\pi_1(X)$, for X to have the homotopy type of a finite CW-complex. There are also applications in number theory, algebraic geometry, operator theory, etc. (see [64]).

Consider now the space $KR = K_0(R) \times B\mathrm{GL}(R)^+$. This is (like $\mathrm{BU}\times\mathbb{Z}$ and $\mathrm{BO}\times\mathbb{Z}$) a remarkable space, namely an *infinite loop space*, i.e., it has the homotopy type of the 0th space of an Ω-spectrum. Therefore the algebraic K-theory groups of R are the homotopy groups of this spectrum.

Spectra allow us to classify the cohomology operations of its associated cohomology theory.

12.3.22 DEFINITION. Let P be a CW-prespectrum that is also an Ω-prespectrum. A *cohomology operation of type* $(n, n+i)$ of the cohomology theory P^* is a natural transformation $\theta_n^i : P^n \stackrel{\cdot}{\longrightarrow} P^{n+i}$ of contravariant functors from the category of pointed CW-complexes to $\mathcal{S}et$. We denote by $\mathcal{A}_n^i(P)$ the set of cohomology operations of type $(n, n+i)$.

Since $P^n(X) = [X, P_n]_*$ for any CW-complex X, by the Yoneda lemma 12.2.2 there is a bijection

$$\Phi_n^i : \mathcal{A}_n^i(P) \longrightarrow P^{n+i}(P_n)\,.$$

Obviously, $\mathcal{A}_n^i(P)$ has a natural group structure, and Φ_n^i is an isomorphism with respect to it and the group structure of $P^{n+i}(P_n)$.

12.3.23 Definition. A *cohomology operation of degree* i is a family of cohomology operations of type $(n, n+i)$, $\theta^i = \{\theta_n^i\}$, for all $n \in \mathbb{Z}$. Such an operation is called *stable* if the diagram

$$\begin{tikzcd} P^n(X) \arrow[r, "\theta_n^i(X)"] & P^{n+i}(X) \\ P^{n+1}(\Sigma X) \arrow[u, "s^n"', "\cong"] \arrow[r, "\theta_{n+1}^i(\Sigma X)"'] & P^{n+i+1}(\Sigma X) \arrow[u, "\cong", "s^{n+i}"'] \end{tikzcd}$$

commutes for all X and all n. We denote by $\mathcal{A}^i(P)$ the group of stable operations of degree i in P^*.

The proof of the next result is an *exercise.*

12.3.24 **Theorem**. *The isomorphisms* Φ_n^i *induce an isomorphism*

$$\Phi^i : \mathcal{A}^i(P) \longrightarrow \lim_n P^{i+n}(P_n)\,,$$

where the homomorphisms of the limit are given by the composites

$$P^{i+n+1}(P_{n+1}) \xrightarrow{\sigma_n^*} P^{i+n+1}(\Sigma P_n) \xrightarrow{s^{i+n}} P^{i+n}(P_n)\,. \qquad \square$$

12.3.25 Examples.

1. The Adams operations defined in 10.1.7 are cohomology operations in K-theory of type $(0,0)$. Although these operations were defined using vector bundles only for compact spaces X, it is possible to extend them as maps BU$\longrightarrow$BU. See [2].

2. Let $H\mathbb{Z}_2$ be the Eilenberg–Mac Lane spectrum with coefficients in $\mathbb{Z}_2$. In this case, $\mathcal{A}_2^* = \mathcal{A}^*(H\mathbb{Z}_2)$ is called the mod 2 *Steenrod algebra.* By 12.3.24, $\mathcal{A}_2^i \cong \lim_n H^{i+n}(K(\mathbb{Z}_2,n);\mathbb{Z}_2) = \lim_n [K(\mathbb{Z}_2,n), K(\mathbb{Z}_2,n+i)]_*$. $K(\mathbb{Z}_2,n)$ is $(n-1)$-connected. Hence by 7.3.18, $H^k(K(\mathbb{Z}_2,n)) = 0$ for $k < n$. Therefore, $\mathcal{A}_2^i = 0$ for $i < 0$, i.e., there are no operations that lower the degree. One can show that there are stable operations Sq^i of degree i for each $i \geq 0$, called *Steenrod squares*, which are characterized by the following properties:

   (i) $\mathrm{Sq}_i^i(x) = x^2$, (ii) $\mathrm{Sq}_r^i(x) = 0$ for $r < i$.

The Steenrod algebra $\mathcal{A}_2^*$ is indeed an algebra if one takes the composition as a multiplication. Serre in [66] showed that the Steenrod squares generate $\mathcal{A}_2^*$ as an algebra. They do not generate it freely; there are relations among the squares known as *Adem relations* [6].

For an n-dimensional real vector bundle $p : E \longrightarrow B$, Thom discovered that $w_i(E) = \varphi^{-1}\mathrm{Sq}_n^i(t_E)$ (see 11.7.20). See [58], where this formula is used to define the Stiefel–Whitney classes.

There are also cohomology operations in $H\mathbb{Z}_p^*$, called *Steenrod pth powers*, for all other prime numbers p. These generalize the Steenrod squares.

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Appendix A

Proof of the Dold–Thom Theorem

In this appendix we shall give a version of the results presented in [26], and this will lead to a proof of Theorem 5.2.17. As far as we know, the original proof in German is the only one available in the literature, besides the one in the Spanish version of the present text.

A.1 Criteria for Quasifibrations

In this section we study some conditions that guarantee that a given map is a quasifibration.

A.1.1 Definition. Let $p : E \longrightarrow B$ be a continuous map. A subset $U \subset B$ is called *distinguished* (with respect to p) if $U \subset p(E)$ and if the restriction of p, $p_U : p^{-1}(U) \longrightarrow U$, is a quasifibration (see 4.3.39).

We have the following criterion.

A.1.2 **Theorem**. *Let $p : E \longrightarrow B$ be a continuous map. Let $\mathcal{U} = \{U_i\}$ be an open cover of B such that each element U_i is distinguished with respect to p. If for each $b \in U_i \cap U_j$ there exists $U_k \in \mathcal{U}$ such that $b \in U_k \subset U_i \cap U_j$, then B is distinguished, that is, $p : E \longrightarrow B$ is a quasifibration.*

We shall give the proof later, after making some comments and proving some lemmas. The following is an immediate consequence of A.1.2.

A.1.3 **Corollary**. *If $p : E \longrightarrow B$ is continuous and U, V, and $U \cap V$ are distinguished, then so also is $U \cup V$.* $\square$

A.1.4 REMARK. The second hypothesis of Theorem A.1.2 cannot be eliminated; that is, it is not sufficient that the distinguished sets cover B, as the following counterexample shows.

A.1.5 EXAMPLE. Suppose that $B = \mathbb{R}^2$ and E is the plane with a cut along the interval $0 < x < 1$, $y = 0$ without the lower boundary, that is, without the boundary of the region $y < 0$. In other words, E is the result of taking the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and the part of the lower half-plane $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}$ from which one takes away the said interval, and identifying the half-lines $\{(x, 0) \mid x \leq 0\} \cup \{(x, 0) \mid x \geq 1\}$ of both via the identity. Let $p : E \longrightarrow B$ be the natural projection (see Figure A.1).

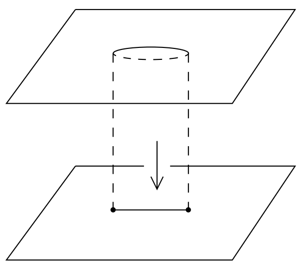

Figure A.1

The open half-planes $U = \{(x, y) \mid x > 0\}$ and $V = \{(x, y) \mid x < 1\}$ are distinguished, since the groups $\pi_n(p^{-1}(U), p^{-1}(b))$, $\pi_n(U)$, $\pi_n(p^{-1}(V), p^{-1}(b))$, and $\pi_n(V)$ are all trivial. Moreover, they cover B. If p were a quasifibration, then we would have an isomorphism $p_* : \pi_n(E) \cong \pi_n(B)$, since all of the fibers are points. However, the group $\pi_n(B)$ is trivial, while $\pi_n(E)$ is infinite cyclic because E has the homotopy type of the circle $\mathbb{S}^1$ (see 4.5.13).

The previous example shows also that a subset of a distinguished set is not necessarily distinguished. The half-plane U is distinguished, but the strip $0 < x < 1$ is not (otherwise, the whole plane would be distinguished by Theorem A.1.2). In particular, this proves that a map $B' \longrightarrow B$ into the base space of a quasifibration $E \longrightarrow B$ does not in general induce a quasifibration $E' \longrightarrow B'$.

In the following we shall prepare ourselves for the proof of A.1.2.

A.1.6 **Lemma**. *Let $p : E \longrightarrow B$ be a continuous map and $U \subset B$ a distinguished subset. Then the following statements are equivalent:*

(a) $p_* : \pi_n(E, p^{-1}(b), e) \cong \pi_n(B, b)$ *for any* $b \in U$, $e \in p^{-1}(b)$, *and* $n \geq 0$.

(b) $p_* : \pi_n(E, p^{-1}(U), e) \cong \pi_n(B, U, b)$ *for any* $b \in U$, $e \in p^{-1}(b)$, *and* $n \geq 0$.

Proof: For every $e \in p^{-1}(b)$ the map p induces a homomorphism between the long exact homotopy sequences of the triples $(E, p^{-1}(U), p^{-1}(b))$, and (B, U, b), as follows (see 3.5.10):

$$\begin{array}{ccccccccc} \cdots \to & \pi_n(E, p^{-1}(b)) & \to & \pi_n(E, p^{-1}(U)) & \to & \pi_{n-1}(p^{-1}(U), p^{-1}(b)) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \pi_n(B) & \to & \pi_n(B, U) & \to & \pi_{n-1}(U) & \to \cdots . \end{array}$$

In the diagram above, under either hypothesis (a) or (b), all of the vertical homomorphisms, with the possible exception of one out of each four (the first or the second in the part shown), are isomorphisms. The assertion is obtained by applying the five lemma (see [47, I.3.3]). □

A.1.7 Remark. For $n = 0, 1$ in the previous diagram, the sets with distinguished element are not necessarilly groups. Nonetheless, the five lemma remains true. It is an *exercise* to verify that the proof of the lemma (by chasing elements) is still valid. Note that, in this case, the kernel of a function is simply the inverse image under the function of the distinguished element.

A.1.8 **Lemma**. *Assume that* $p : F \longrightarrow U$ *is a continuous map,* $V \subset U$, $G = p^{-1}(V)$, *and* $r \geq 0$. *For every* $b \in V$ *and* $e \in p^{-1}(b)$ *assume that* $p_* : \pi_n(F, G) \longrightarrow \pi_n(U, V)$ *(which are groups based on* e *and* b*, respectively) is a monomorphism for* $n = r$ *and an epimorphism for* $n = r + 1$. *Suppose that we are given maps*

(i) $\overline{H} : (\mathbb{D}^r \times I, \mathbb{D}^r \times 1) \longrightarrow (U, V)$,

(ii) $h : (\mathbb{D}^r \times 0 \cup \mathbb{S}^{r-1} \times I, \mathbb{S}^{r-1} \times 1) \longrightarrow (F, G) = (p^{-1}(U), p^{-1}(V))$,

(iii) $d : ((\mathbb{D}^r \times 0 \cup \mathbb{S}^{r-1} \times I) \times I, (\mathbb{S}^{r-1} \times 1) \times I) \longrightarrow (U, V)$,

such that $d(z, t, 0) = \overline{H}(z, t)$ *and* $d(z, t, 1) = p \circ h(z, t)$.

Then there exist extensions of h *and* d*, that is, continuous maps*

(a) $H : (\mathbb{D}^r \times I, \mathbb{D}^r \times 1) \longrightarrow (F, G)$, *such that* $H|(\mathbb{D}^r \times 0 \cup \mathbb{S}^{r-1} \times I) = h$,

(b) $D : (\mathbb{D}^r \times I \times I, \mathbb{D}^r \times 1 \times I) \longrightarrow (U, V)$, *such that* $D|(\mathbb{D}^r \times 0 \cup \mathbb{S}^{r-1} \times I) \times I = d$ *and* $D(z, t, 0) = \overline{H}(z, t)$, $D(z, t, 1) = p \circ H(z, t)$.

Proof: Since $(\mathbb{D}^r \times 0 \cup \mathbb{S}^{r-1} \times I, \mathbb{S}^{r-1} \times 1) \approx (\mathbb{D}^r, \mathbb{S}^{r-1}) \approx \Sigma^{r-1}(I, \partial I)$, the map h defines an element $\alpha \in \pi_r(F, G)$, whose projection in $\pi_r(U, V)$ is zero. Namely, by (iii), $p \circ h$ is homotopic to $\overline{H}$ by means of d. But since $\overline{H}$ is defined on all of $\mathbb{D}^r \times I$, which is contractible, it is nullhomotopic. Therefore, since $\alpha = 0$ and by assumption, $p_* : \pi_r(F, G) \longrightarrow \pi_r(U, V)$ is a monomorphism, h can be extended to a map $H' : (\mathbb{D}^r \times I, \mathbb{D}^r \times 1) \longrightarrow (F, G)$.

On the other hand, we have two nullhomotopies of $p \circ h$; namely, the first is $p \circ H'$ and the second is given by d and $\overline{H}$. Both nullhomotopies determine an element $\beta \in \pi_{r+1}(U, V)$. We can modify β by an arbitrary element of $p_*(\pi_{r+1}(F, G))$, modifying H' appropriately at the same time. Since p_* is an epimorphism in this dimension, in particular we can choose $H' = H$ so that $\beta = 0$ holds. Then D is the corresponding nullhomotopy.

We can assume that H' maps a small $(r+1)$-disk of the form $K \times [s, 1]$ constantly to a point, say to $y \in p^{-1}(V)$. Then K is a homothetic reduction of $\mathbb{D}^r$ and $0 < s < 1$ (see Figure A.2).

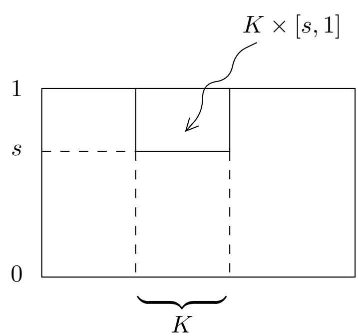

Figure A.2

We now consider the $(r+1)$-disk

$$\begin{aligned} \overline{K} &= \partial(\mathbb{D}^r \times I \times I) - \mathbb{D}^r \times 1 \times I \\ &= \mathbb{D}^r \times I \times 0 \cup \mathbb{D}^r \times I \times 1 \cup \partial\mathbb{D}^r \times I \times I\,, \end{aligned}$$

and we define a map D' from this disk to U that for $z \in K$ and $t \in I$ maps the boundary to V as follows:

$$D'(z, t, 0) = \overline{H}(z, t)\,;\ D'(z, t, 1) = p \circ H'(z, t)\,;$$
$$D'|(\mathbb{D}^r \times 0 \cup \partial\mathbb{D}^r \times I) \times I = d\,.$$

Then D' maps $K \times [s, 1] \times 1$ to the point $x = p(y)$ and represents a certain element $\beta \in \pi_{r+1}(U, V)$. We now choose a map $H'' : (\overline{K}, \partial\overline{K}) \longrightarrow (F, G)$

whose projection $p \circ k''$ represents the element $-\beta$ and that maps the complement of $K' \times [s,1] \times 1$ constantly to the point y. Then we define $H : (\mathbb{D}^r \times I, \mathbb{D}^r \times 1) \longrightarrow (F, G)$ by

$$H(z,t) = \begin{cases} H''(z,t,1) & \text{if } (z,t) \in K \times [s,1], \\ H'(z,t) & \text{if } (z,t) \notin K \times [s,1]. \end{cases}$$

We also define $D : (\overline{K}, \partial\overline{K}) \longrightarrow (U, V)$ by

$$\begin{aligned} D(z,t,1) &= p \circ H''(z,t,1) = p \circ H(z,t) & \text{if } (z,t) \in K \times [s,1]\,, \\ D(z,t,u) &= D'(z,t,u) & \text{if } (z,t) \notin K \times [s,1]\,. \end{aligned}$$

Then D represents the element $(-\beta) + \beta = 0 \in \pi_{r+1}(U,V)$ and so can be extended to a map $D : (\mathbb{D}^r \times I \times I, \mathbb{D}^r \times 1 \times I) \longrightarrow (U,V)$. The maps H and D so constructed satisfy conditions (a) and (b). $\square$

As Example A.1.5 shows, in a quasifibration it is not possible in general to lift an arbitrary homotopy of a finite polyhedron (that is, one with a finite number of simplices). A weak form of the homotopy covering theorem is, however, true. In fact, we have the following result.

A.1.9 **Theorem**. *Let $p : E \longrightarrow B$ be continuous and let $\mathcal{U} = \{U_\lambda\}$ be an open cover of B by distinguished sets that satisfy the hypotheses of Theorem* A.1.2. *(Then according to Theorem* A.1.2, *p is a quasifibration.) Suppose that P is a finite polyhedron and that $h : P \longrightarrow E$ and $\overline{H} : P \times I \longrightarrow B$ are continuous maps such that $\overline{H}(z,0) = p \circ h(z)$ for $z \in P$. Moreover, assume that $K_\lambda \subset P \times I$ are a finite number of compact sets such that $\overline{H}(K_\lambda) \subset U_\lambda \in \mathcal{U}$. Then there exist maps $H : P \times I \longrightarrow E$ and $D : P \times I \times I \longrightarrow B$ that satisfy*

(a) $H(z,0) = h(z)$,

(b) $D(z,t,0) = \overline{H}(z,t)$, $D(z,t,1) = p \circ H(z,t)$, $D(z,0,u) = \overline{H}(z,0)$ *for* $u, t \in I$,

(c) $D(K_\lambda \times I) \subset U_\lambda$.

Obviously, given $\overline{H}$, we can pick the compact sets K_λ so that they cover $P \times I$. Then we can reformulate Theorem A.1.9 in an abbreviated form as follows: *The homotopy $\overline{H}$ can be lifted to E up to a suitable deformation relative to $P \times 0$. This deformation can be picked sufficiently small so that the images of all the points vary inside one element of the cover $\mathcal{U}$.*

Let $\{\sigma_\mu\}$ and $\{I_\nu\}$ be cellular decompositions of P and I, respectively. (Here $I_\nu = [t_\nu, t_{\nu+1}]$ for $0 = t_1 < t_2 < \cdots < t_m = 1$.) For the proof of Theorem A.1.9 we now need a lemma.

A.1.10 **Lemma**. *By picking $\{\sigma_\mu\}$ and $\{I_\nu\}$ suitably, we can associate a set $U^\rho \in \mathcal{U}$ to every cell ρ of the product cellular decomposition $\{\sigma_\mu\} \times \{I_\nu\}$ of $P \times I$, so that we have*

(a) $\overline{H}(\rho) \subset U^\rho$;

(b) *if ρ is a face of ρ', then $U^\rho \subset U^{\rho'}$;*

(c) *if $\rho \cap K_\lambda \neq \emptyset$, then $U^\rho \subset U_\lambda$.*

Proof: We shall show this by inductive descent on the dimension of the cells of $P \times I$. So we suppose that there are decompositions $\{\sigma_\mu\}$ of P and $\{I_\nu\}$ of K and a mapping $\rho \mapsto U^\rho$ such that (a), (b), and (c) hold for all the cells of dimension bigger than k. Under this induction hypothesis, let τ be a k-cell of $P \times I$. According to the assumption of Theorem A.1.2, for every $y \in \tau$ there is a neighborhood u_y in $P \times I$ and an open set $U^y \in \mathcal{U}$ such that

(i) $\overline{H}(u_y) \subset U^y$,

(ii) $U^y \subset U^\rho$ for each ρ that has τ as a face, and

(iii) $U^y \subset U_\lambda$ for all K_λ that intersect u_y nontrivially.

If we make a sufficiently fine subdivision of τ, then every cell $\overline{\tau}$ of this subdivision lies in one of the sets u_y, and so $\overline{H}(\overline{\tau}) \subset U^y = U^{\overline{\tau}}$. We can obtain such subdivisions simultaneously for all τ if we subdivide sufficiently finely the decompositions $\{\sigma_\mu\}$ and $\{I_\nu\}$. Thereby the cells ρ of dimension bigger than k are subdivided further. To the cells $\overline{\rho}$ that we obtain from ρ we associate the set $U^{\overline{\rho}} = U^\rho$. □

Proof of A.1.9: Using Lemma A.1.10 we associate a subdivision to the cells $\sigma_\mu \times I_\nu$ in the following way. First we take the cells $\sigma_\mu \times I_1$, starting with those of the lowest dimension. Then we take the cells $\sigma_\mu \times I_2$, again in order of increasing dimension, and so on. The maps H and D are constructed successively on the cells $\sigma_\mu \times I_\nu$ and $\sigma_\mu \times I_\nu \times I$, respectively, in such a way that $D(\rho \times I) \subset U^\rho$ for all the cells ρ in the subdivision of $P \times I$. Using (c) of Lemma A.1.10, we automatically satisfy (c) of A.1.9. In each stage of the construction we have the following problem: Given $\overline{H} : (\sigma \times I, \sigma \times 1) \longrightarrow (u^{\sigma \times I}, U^{\sigma \times 1})$ and given H defined on $\sigma \times 0 \cup \partial\sigma \times I$ and D defined on $(\sigma \times 0 \cup \partial\sigma \times I) \times I$, we have to find extensions of H and D. These extensions exist according to Lemma A.1.8. (One takes $V = U^{\sigma \times 1}$ and $U = U^{\sigma \times I}$, so that V and U satisfy the hypotheses of A.1.8 by A.1.10(b) and by Lemma A.1.6.) □

Proof of A.1.2: Take $U \in \mathcal{U}$, $b \in U$ and $e \in p^{-1}(x)$. We shall show that $p_* : \pi_n(E, p^{-1}(U), e) \longrightarrow \pi_n(B, U, b)$ is an isomorphism. Since the sets U cover B, the assertion is obtained from A.1.6.

(a) p_* *is an epimorphism.* First note that the case $n = 0$ is trivial, since p is onto. For $n > 0$, an element $\alpha \in \pi_n(B, U, b)$ is represented by a map $\overline{H} : (I^{n-1} \times I, I^{n-1} \times 0) \longrightarrow (B, U)$ from an n-cube that maps $I^{n-1} \times 0 \cup \partial I^{n-1} \times I$ constantly to the point b. We rewrite $I^{n-1} \times I$ in the form $P \times I$ with $P = I^{n-1} \times 0 \cup \partial I^{n-1} \times I$, as Figure A.3 illustrates.

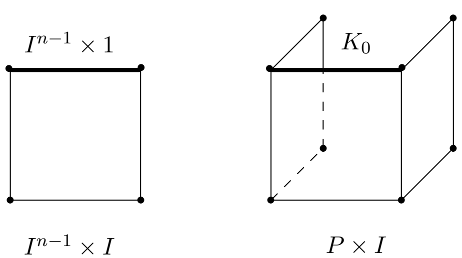

Figure A.3

Now we apply Theorem A.1.9 with $h(P) = e$, $K_0 = I^{n-1} \times 1$, and $U_0 = U$. So we obtain an extension

$$H : (I^{n-1} \times I, I^{n-1} \times 1) \approx (P \times I, K_0) \longrightarrow (E, p^{-1}(U))$$

of h whose projection $p \circ H$ is homotopic to $\overline{H}$ by means of the homotopy

$$D : (I^{n-1} \times I \times I, I^{n-1} \times 1 \times I) \approx (P \times I \times I, K_0 \times I) \longrightarrow (B, U)\,,$$

under which the image of P remains fixed. Then $p \circ H$ turns out to be a representative of α.

(b) p_* *is a monomorphism.* Let $\alpha \in \pi_n(E, p^{-1}(U), e)$ be such that $p_*(\alpha) = 0$. Suppose that $h : (I^n, \partial I^n) \longrightarrow (E, p^{-1}(U))$ is a representative of α and that $\overline{H} : (I^n \times I, \partial I^n \times I) \longrightarrow (B, U)$ is a homotopy of $p \circ H$ to the constant map $\overline{H}(I^n \times 1) = b$. We apply Theorem A.1.9 with $P = I^n$, $K_0 = \partial I^n \times I \cup I^n \times 1$ and $U_0 = U$, and so we get a map $H : (I^n \times I, \partial I^n \times I) \longrightarrow (E, p^{-1}(U))$ such that $H(z, 0) = h(z)$ and $H(z, 1) \in p^{-1}(U)$; that is, $\alpha = 0$. (Note that in the construction of a nullhomotopy it is not necessary to hold the base point fixed.) $\square$

To finish this section we shall give two more criteria for determining when a map is a quasifibration as well as a useful application for the second appendix.

A.1.11 **Lemma**. *Let $q : E \longrightarrow B$ be continuous and surjective. Let $B' \subset B$ be a distinguished subset with respect to q and put $E' = q^{-1}(B')$. Assume that we have homotopies $D_t : E \longrightarrow E$ and $d_t : B \longrightarrow B$ such that*

$$\begin{aligned} D_0 &= \mathrm{id}\,, & D_t(E') &\subset E'\,, & D_1(E) &\subset E'\,,\\ d_0 &= \mathrm{id}\,, & d_t(B') &\subset B'\,, & d_1(B) &\subset B'\,, \end{aligned}$$

and

(A.1.12) $$q \circ D_1 = d_1 \circ q\,.$$

For every $b \in B$ and $n \geq 0$ suppose that we have

(A.1.13) $$D_{1*} : \pi_n(q^{-1}(b)) \cong \pi_n(q^{-1}(d_1(b))).$$

Then B is also a distinguished set with respect to q, that is, q is a quasifibration.

Proof: Since d_t and D_t are homotopies, we have for all n that

(A.1.14) $$d_{1*} : \pi_n(B, b) \cong \pi_n(B', b')\,, \qquad b' = d_1(b),$$

(A.1.15) $$D_{1*} : \pi_n(E, e) \cong \pi_n(E', e')\,, \qquad e' = D_1(e).$$

Then D_1 maps $q^{-1}(b)$ to $q^{-1}(b')$, and so it induces a homomorphism from the homotopy sequence of the pair $(E, q^{-1}(b))$ to the homotopy sequence of the pair $(E', q^{-1}(B'))$. By (A.1.13) and (A.1.15) the absolute homotopy groups are mapped isomorphically, and then by the five lemma so also are the relative groups, namely,

(A.1.16) $$D_{1*} : \pi_n(E, q^{-1}(b)) \cong \pi_n(E', q^{-1}(b')), \qquad e' = D_1(e).$$

Now let us consider the diagram

$$\begin{tikzcd} \pi_n(E, q^{-1}(b)) \arrow[r, "D_{1*}"] \arrow[d, "q_*"'] & \pi_n(E', q^{-1}(b')) \arrow[d, "(q|E')_*"] \\ \pi_n(B, b) \arrow[r, "d_{1*}"'] & \pi_n(B', b')\,. \end{tikzcd}$$

According to (A.1.12) the diagram is commutative. Also, d_{1*} and D_{1*} are isomorphisms by (A.1.14) and (A.1.16), and likewise so is $(q|E')_*$, since by hypothesis B' is a distinguished subset. Thus q^* is also an isomorphism. $\square$

The following theorem is important for CW-complexes, since it implies that every map $p : E \longrightarrow B$ with B a CW-complex is itself a quasifibration, provided that it is a quasifibration when restricted to every skeleton of B.

A.1.17 **Theorem**. *Assume that $p : E \longrightarrow B$ is continuous and that $B = \bigcup B_i$ is Hausdorff with the union topology. If each B_i is distinguished with respect to p, then so is B itself; that is, p is a quasifibration.*

Proof: We have to prove that $p_* : \pi_k(E, p^{-1}(b)) \longrightarrow \pi_k(B, b)$ is an isomorphism. It is enough to notice that the elements of both groups are homotopy classes of maps defined on compact sets, and so their images lie in one of the spaces of the union (see 5.1.10). So, we have to consider elements, whether in the first group or in the second, that also represent elements in the corresponding groups of each space in the union, for which the corresponding assertions are found to be true because each B_n is a distinguished subset. □

We conclude this section by proving a result that will be used in Appendix B to prove the Bott periodicity theorem.

Let us consider a map $p : E \longrightarrow B$, where B is Hausdorff. Also assume that $B = \bigcup_{i \geq 0} B_i$, where $B_i \subset B_{i+1}$ for $i \geq 0$ and where each B_i is closed in B. Suppose, moreover, that p is trivial over each difference $B_{i+1} - B_i$, that is, we have a commutative triangle

$$\begin{tikzcd}
E_{i+1} - E_i \arrow[rr, "\approx"] \arrow[dr, "p|"'] & & (B_{i+1} - B_i) \times F \arrow[dl, "\mathrm{proj}"] \\
& B_{i+1} - B_i \,, &
\end{tikzcd}$$

where $E_i = p^{-1}(B_i)$. In particular, taking B_{-1} to be the empty set, $p_0 = p|E_0 : E_0 \longrightarrow B_0$ also is trivial. So for every $x \in B$ we have the fiber $p^{-1}(x) \approx F$.

Suppose, moreover, that for each i there exists an open neighborhood U_i of B_i in B_{i+1} and a deformation retraction (that is, a homotopy equivalence) $r_i : U_i \longrightarrow B_i$ that lifts to a deformation retraction $\widetilde{r}_i : p^{-1}U_i \longrightarrow E_i$. This means that we have the commutative diagram

(A.1.18)
$$\begin{tikzcd}
p^{-1}U_i \arrow[r, "\widetilde{r}_i"] \arrow[d, "p|"'] & E_i \arrow[d, "p_i"] \\
U_i \arrow[r, "r_i"'] & B_i .
\end{tikzcd}$$

Then, by restricting the maps $\widetilde{r}_i$ to each fiber, we obtain maps $\widetilde{r}_i^x : p^{-1}(x) \longrightarrow p^{-1}(r_i(x))$.

Under the above hypotheses, we have the next result.

A.1.19 **Theorem**. *If $\widetilde{r}_i^x : p^{-1}(x) \longrightarrow p^{-1}(r_i(x))$ is a homotopy equivalence for every i and every $x \in U_i$, then $p : E \longrightarrow B$ is a quasifibration.*

Proof: We are going to apply A.1.17, for which it is enough to check that each space B_i is distinguished with respect to p. We shall verify this by induction on i. Since by hypothesis p_0 is trivial, it follows that B_0 is distinguished with respect to p. So let us assume that B_i is distinguished with respect to p for some $i \geq 0$ and let us prove that B_{i+1} also is distinguished. To do this, we shall apply Theorem A.1.2 to the cover of B_{i+1} formed by the open sets U_i, $V_i = B_{i+1} - B_i$, and $W_i = U_i - B_i$, and so it is sufficient to show that each of these open sets is distinguished.

Because $p|(E_{i+1} - E_i)$ is trivial, V_i is evidently distinguished. Since $W_i \subset V_i$, we also have that $p|p^{-1}(W_i)$ is trivial, and so W_i is distinguished.

To prove that U_i is distinguished it is enough to observe that by the commutativity in (A.1.18) and the naturality of the long exact homotopy sequence of a pair (3.4.6), we have commutative squares for all $x \in U_i$ and $k > 0$ in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(p^{-1}U_i) \arrow[r] \arrow[d, "\widetilde{r}_{i*}"', "\cong"] & \pi_k(p^{-1}U_i, p^{-1}(x)) \arrow[r, "\partial"] \arrow[d, "\widetilde{r}_{i*}"'] & \pi_{k-1}(p^{-1}(x)) \arrow[r] \arrow[d, "\cong"', "\widetilde{r}^x_{i*}"] & \cdots \\
\cdots \arrow[r] & \pi_k(E_i) \arrow[r] & \pi_k(E_i, p^{-1}(r_i(x))) \arrow[r, "\partial"] & \pi_{k-1}(p^{-1}(r_i(x))) \arrow[r] & \cdots ,
\end{tikzcd}$$

and so, by the five lemma, the vertical homomorphism in the middle is an isomorphism.

Let us consider the commutative square

$$\begin{tikzcd}
\pi_k(p^{-1}U_i, p^{-1}(x)) \arrow[r] \arrow[d, "\widetilde{r}_{i*}"', "\cong"] & \pi_k(U_i, x) \arrow[d, "\cong"', "r_{i*}"] \\
\pi_k(E_i, p^{-1}(r_i(x))) \arrow[r, "\cong"'] & \pi_k(B_i, r_i(x)) .
\end{tikzcd}$$

We have just proved that the left vertical arrow is an isomorphism. The right vertical arrow is an isomorphism because r_i is a homotopy equivalence. Finally, the lower horizontal arrow is an isomorphism by the induction hypothesis. Consequently, the upper horizontal arrow is an isomorphism, which proves that U_i is distinguished. □

A.2 Symmetric Products

In this section we shall make use of the definition of symmetric product that we presented in Section 5.2, and we shall study in more detail its properties.

A.2.1 Definition. Let X be a Hausdorff space with base point x_0. In the infinite symmetric product $\mathrm{SP}\ X$ we introduce a *sum* in a natural way, $+ : \mathrm{SP}\ X \times \mathrm{SP}\ X \longrightarrow \mathrm{SP}\ X$, which consists in putting together q-tuples and r-tuples as follows:

$$[x_1, x_2, \ldots, x_q] + [y_1, y_2, \ldots, y_r] = [x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_r]\,.$$

In particular, if we simply write x_i for $[x_i]$, then $[x_1, x_2, \ldots, x_q] = x_1 + x_2 + \cdots + x_q$.

A.2.2 Exercise. (a) Prove that the operation $+$ is well defined and converts $\mathrm{SP}\ X$ into a free abelian semigroup over X with $0 = [x_0]$.

(b) Prove that if $f : X \longrightarrow Y$ is continuous, then the induced map $\widehat{f} : \mathrm{SP}\ X \longrightarrow \mathrm{SP}\ Y$ is a homomorphism of semigroups.

The problem of continuity of $+$ is not trivial. We clearly have that the restriction $+ : \mathrm{SP}^{\,q}X \times \mathrm{SP}^{\,r}X \longrightarrow \mathrm{SP}\ X$ is continuous, since it factors through the continuous map $\mathrm{SP}^{\,q}X \times \mathrm{SP}^{\,r}X \longrightarrow \mathrm{SP}^{\,q+r}X$, which is obtained, by passing to the quotient, starting from the map $\overline{X}^r \times \overline{X}^s \longrightarrow \overline{X}^{r+s} \longrightarrow \mathrm{SP}^{\,q+r}X$ given by

$$((x_1, x_2, \ldots, x_q), (y_1, y_2, \ldots, y_r)) \mapsto [x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_r]\,.$$

As the diagram

$$\begin{tikzcd}
\overline{X}^q \times \overline{X}^r \arrow[r] \arrow[d] & \overline{X}^{q+r} \arrow[d] & \\
\mathrm{SP}^{\,q}X \times \mathrm{SP}^{\,r}X \arrow[r, "+"] & \mathrm{SP}^{\,q+r}X \arrow[r, hook] & \mathrm{SP}\ X
\end{tikzcd}$$

illustrates, by taking the quotient map and then $+$ we get the same thing as by first taking the product and then the quotient map. The next statement is immediate.

A.2.3 **Proposition**. *If* $\mathrm{SP}\ X \times \mathrm{SP}\ X$ *has the union topology with respect to the spaces* $Z^n = \bigcup_{i=1}^n (\mathrm{SP}^{\,i}X \times \mathrm{SP}^{\,n-i}X)$, *then the sum is continuous.* □

Nonetheless, it is not true that $\mathrm{SP}\, X \times \mathrm{SP}\, X$ is always equipped with the union topology. There are some results that tell us when this condition does hold. In the first place, Steenrod proves in [70] that *if* $X = \bigcup X^n$ *and* $Y = \bigcup Y^n$ *have the union topology, then* $X \times Y = \bigcup Z^n$ *also has the union topology, where we define* $Z^n = \bigcup_{i=1}^n (X^i \times Y^{n-i})$ *and where* $\times$ *represents the product in the category of compactly generated spaces* (see 4.3.22). Therefore, we have the following result.

A.2.4 **Proposition**. *If* X *is compactly generated and the product* $\mathrm{SP}\, X \times_k \mathrm{SP}\, X = k(\mathrm{SP}\, X \times \mathrm{SP}\, X)$ *is the product in the category of compactly generated spaces, then* $+ : \mathrm{SP}\, X \times_k \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, X$ *is continuous.* □

The case of CW-complexes is particularly important for us. Let us recall that a CW-space has the union topology with respect to its skeletons (or its closed cells). In general it is not true that the product of CW-complexes is a CW-complex; however, it is indeed true if we take the compactly generated product $\times_k$ (see [82, II(1.6)]). On the other hand, this product coincides with the usual one in some cases, namely, as we saw in Chapter 5, we have that *if* X *and* Y *are* CW*-complexes such that either* X *or* Y *is finite (i.e., it has finitely many cells) or such that both* X *and* Y *are countable (i.e., they have countably many cells), then* $X \times_k Y = X \times Y$ (see 5.1.46). So we have the following important particular case of A.2.4.

A.2.5 **Theorem**. *If* X *is a countable* CW*-complex, then the sum* $+ : \mathrm{SP}\, X \times \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, X$ *is continuous.* □

By what we have said before, the following result is always true.

A.2.6 **Theorem**. *The sum* $+ : \mathrm{SP}\, X \times \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, X$ *is continuous on each* $\mathrm{SP}^q X \times \mathrm{SP}^r X$ *as well as on every compact subset of* $\mathrm{SP}\, X \times \mathrm{SP}\, X$. □

A.2.7 **Corollary**. *For any compact space or, more generally, for any compactly generated space, say* W*, we have that* $+ : \mathrm{SP}\, X \times \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, X$ *induces an additive structure on* $[W, \mathrm{SP}\, X]$. □

A.2.8 Exercise. Analyze what corresponds to the additive structure on $\mathrm{SP}\, \mathbb{S}^2$ after identifying the elements of this space with the complex polynomials (see 5.2.4).

The equation $a + x = b$ in SP X either does not have a solution or has a unique solution. In other words, the "difference" $x = b - a$ is unique if it is defined. Then we have the following.

A.2.9 **Lemma**. *The difference function $(a, b) \mapsto a - b$ is continuous in the intersection of its domain of definition with* $\mathrm{SP}^r X \times \mathrm{SP}^s X$ *for all r and s. It also is continuous on every compact subset of its domain of definition.*

Proof: We can assume that $r \geq s$ and so define $q = r - s$ (≥ 0). Let us consider the set $X^{q,s}$ of points $((a_1, a_2, \ldots, a_{q+s}), (b_1, b_2, \ldots b_s))$ of $X^{q+s} \times X^s$ that satisfy $b_i = a_{q+i}$ for all $i \geq 0$. The image of $X^{q,s}$ under the identification map $\sigma : X^{q+s} \times X^s \longrightarrow \mathrm{SP}^{q+s} X \times \mathrm{SP}^s X$ is precisely the domain of definition of the difference $a - b$. The map $X^{q,s} \longrightarrow \mathrm{SP}\ X$ given by

$$((a_1, a_2, \ldots, a_{q+s}), (b_1, b_2, \ldots, b_s)) \mapsto [a_1, a_2, \ldots, a_q]$$

is compatible with the identification map $\sigma | X^{q,s}$. Passing to the quotient $\sigma(X^{q,s})$ we therefore obtain a continuous map $\sigma(X^{q,s}) \longrightarrow \mathrm{SP}\ X$, namely, the difference function. □

A.2.10 **Corollary**. *Let a be a given point in* SP X. *The maps $x \mapsto a + x$ ("left translation") and $x \mapsto a - x$, wherever they are defined, where $x \in$* SP X, *are continuous.*

Proof: By A.2.6, left translation $x \mapsto a + x$ is continuous on each $\mathrm{SP}^q X$ and so is continuous. The map $x \mapsto a - x$ is continuous on the intersection of its domain of definition with $\mathrm{SP}^q X$. This intersection is closed, as we see from the proof of A.2.9, since $X^{q,s}$ is closed and σ is a closed map. Thus the entire domain of definition is closed in SP X, and the assertion is obtained from the fact that therefore this domain has the union topology given by its intersections with each $\mathrm{SP}^q X$ (see A.2.11). □

A.2.11 Exercise. Prove that if $Y = \bigcup Y_n$ is Hausdorff and has the union topology and if $C \subset Y$ is closed, then $C = \bigcup (Y_n \cap C)$ has the union topology.

A.2.12 Exercise. Analyze the relationship between the operation $\Omega(+) : \Omega\mathrm{SP}\ X \times \Omega\mathrm{SP}\ X \longrightarrow \Omega\mathrm{SP}\ X$ and the operation on $\Omega\mathrm{SP}\ X$ as a loop space.

Before ending this section it is worthwhile to present a result about the symmetric product of the wedge $X \vee Y$ of two pointed spaces X and Y. We define a map $\rho : \mathrm{SP}\ X \times \mathrm{SP}\ Y \longrightarrow \mathrm{SP}\ (X \vee Y)$ by

$$\rho([x_1, x_2, \ldots, x_q], [y_1, y_2, \ldots, y_r]) = [x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_r]\,.$$

Here we are considering X and Y as subspaces of $X \vee Y$. Obviously, ρ establishes a bijection between its domain and its codomain. We shall analyze the possiblity that ρ is continuous. To do this, we factor it into the maps

$$\mathrm{SP}\ X \times \mathrm{SP}\ Y \xrightarrow{\widehat{\imath}\times\widehat{\jmath}} \mathrm{SP}\ (X \vee Y) \times \mathrm{SP}\ (X \vee Y) \xrightarrow{\alpha} \mathrm{SP}\ (X \vee Y),$$

where $\widehat{\imath}$ and $\widehat{\jmath}$ are induced by the canonical inclusions $i : X \hookrightarrow X \vee Y$ and $j : Y \hookrightarrow X \vee Y$, and α is the addition. As before, the restriction of ρ to $\mathrm{SP}^q X \times \mathrm{SP}^r Y$ is continuous for every q and r. So just as in A.2.5, we obtain from this the continuity of ρ itself in the case that X and Y are countable CW-complexes.

Note that ρ^{-1} always is continuous. To show this it is enough to prove that the composite

$$\mathrm{SP}\ (X \vee Y) \xrightarrow{\rho^{-1}} \mathrm{SP}\ X \times \mathrm{SP}\ Y \xrightarrow{p_1} \mathrm{SP}\ X$$

is continuous, where p_1 is the projection onto the first factor (and analogously for the projection p_2 onto the second factor). This composite $p_1 \circ \rho^{-1}$ is nothing other than $\widehat{r}_1$, where $r_1 : X \vee Y \longrightarrow X$ is the canonical retraction. Therefore, $p_1 \circ \rho^{-1}$ is continuous.

A.2.13 **Theorem**. *The map* $\rho^{-1} : \mathrm{SP}\ (X \vee Y) \longrightarrow \mathrm{SP}\ X \times \mathrm{SP}\ Y$ *is well defined and is continuous. Its inverse is continuous on each* $\mathrm{SP}^q X \times \mathrm{SP}^r Y$ *as well as on each compact subset of* $\mathrm{SP}\ X \times \mathrm{SP}\ Y$. *Consequently,* ρ^{-1} *is a weak homotopy equivalence. In the case that X and Y are countable* CW-*complexes,* ρ *is a homeomorphism.*

Proof: It remains only to note that ρ^{-1} induces isomorphisms of homotopy groups (that is, it is a weak homotopy equivalence), since both ρ^{-1} and ρ determine bijections between the set of continuous maps of any compact space W into $\mathrm{SP}\ (X \vee Y)$ and the set of continuous maps of W into $\mathrm{SP}\ X \times \mathrm{SP}\ Y$. □

A.3 PROOF OF THE DOLD–THOM THEOREM

In this section we shall give a proof of Theorem 5.2.17. Before doing that, we present the reformulation as it appears in [26].

Suppose that X is a Hausdorff space with base point x_0 and that $A \subset X$ is a closed subset that contains x_0. Let X/A be the quotient space that results by identifying the set A to a single point, which will serve as the base point of the quotient space. Let $p : X \longrightarrow X/A$ be the identification (or quotient) map, which turns out to be a pointed map. We also shall suppose that X/A is Hausdorff, which is always true if X is a regular space. The map p induces a map $\widehat{p} : \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, (X/A)$ between the symmetric products. Under certain conditions this map is a quasifibration.

A.3.1 **Theorem**. *If A is path connected and has a neighborhood W that is deformable to A in X, then the map $\widehat{p} : \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, (X/A)$ defined above is a quasifibration with fiber $\widehat{f}^{-1}(0) = \mathrm{SP}\, A$.*

Proof: According to Theorem A.1.17, it is enough to show that the restriction of $\widehat{p}$ to $\mathrm{SP}_q X = \widehat{p}^{-1}(\mathrm{SP}^q(X/A))$, denoted by $p_q : \mathrm{SP}_q X \longrightarrow \mathrm{SP}^q(X/A)$, is a quasifibration for each q. We shall do this by induction on q.

If we define $\mathrm{SP}^0(X/A) = 0$ (the singular space), then using 5.2.8 we have that $\mathrm{SP}_0 X = \mathrm{SP}\, A$, and so the statement for $q = 0$ is trivial. Let us assume that $q > 0$ and that the statement is true for $q - 1$. We shall construct a system of distinguished sets in $\mathrm{SP}^q(X/A)$ that satisfy the hypotheses of Theorem A.1.2. First we take the set $V = \mathrm{SP}^q(X/A) - \mathrm{SP}^{q-1}(X/A)$. A point $P \in p_q^{-1}(V)$ has exactly q elements $x_1, x_2, \dots, x_q$ in $X - A$. Any other elements $y_1, y_2, \dots, y_r$ in V, viewed as a subset of $\mathrm{SP}\, (X/A)$, lie in A. The map $\sigma : p_q^{-1}(V) \longrightarrow V \times \mathrm{SP}\, A$, defined by $P \mapsto (p_q[x_1, x_2, \dots, x_q], [y_1, y_2, \dots, y_r])$, is a bijection. We shall prove that σ and σ^{-1} are continuous on compact sets. Then σ will behave like a homeomorphism with respect to compact subsets, and so V will be a distinguished subset with respect to p_q.

First we shall consider the following maps:

$$X \supset X - A \xrightarrow{\ p\ } X/A - \overline{x}_0 \subset X/A\,.$$

These induce maps, some of which are homeomorphisms (see 5.2.8), namely,

$$\begin{aligned} \mathrm{SP}^q X \supset \mathrm{SP}^q(X - A) &\approx \mathrm{SP}^q(X/A - \overline{x}_0) \\ &\approx \mathrm{SP}^q(X/A) - \mathrm{SP}^{q-1}(X/A) = V\,. \end{aligned}$$

Therefore, we can identify V with a subset of $\mathrm{SP}^q X$ by means of the map p_q. In order to prove continuity of σ, as desired, we have to prove that the maps $\sigma_1 : P \mapsto p_q[x_1, x_2, \dots, x_q]$ and $\sigma_2 : P \mapsto [y_1, y_2, \dots, y_r]$ are continuous on compact sets. But $\sigma_1 = p_q$ and $\sigma_2(P) = P - p_q(P)$ (where we are considering $p_q(P)$ as a point of $\mathrm{SP}^q X$), and the statement is obtained from A.2.9.

The inverse σ^{-1} is obtained by taking the sum $\mathrm{SP}\, X \times \mathrm{SP}\, X \longrightarrow \mathrm{SP}\, X$ and restricting it to V in the first factor and to $\mathrm{SP}\, (X/A)$ in the second factor. Thus it also is continuous on compact sets by A.2.6.

Second, we shall find an open subset $U \subset \mathrm{SP}^q(X/A)$ that contains $\mathrm{SP}^{q-1}(X/A)$. And with this we shall have finished the proof, since U, V, and $U \cap V$ constitute a system of distinguished sets, as we wished to construct.

Since there exists a neighborhood W of A in X that can be deformed to A (see 5.2.15), we can take the set U to consist of those points in $\mathrm{SP}^q(X/A)$ that have at least one element in the open set $\widetilde{W} = p(W) \subset X/A$. Then U can be deformed to $\mathrm{SP}^{q-1}(X/A)$; namely, if d_t is a deformation of W in A that maps the set A to itself, then $\widetilde{d_t} = p \circ d_t \circ p^{-1}$ is a deformation of $\widetilde{W}$ to $\overline{x}_0$ that leaves fixed $\overline{x}_0$ and so contracts $\widetilde{W}$ to a point. The restriction of $\widehat{\widetilde{d_t}}$ to U contracts U to $\mathrm{SP}^{q-1}(X/A)$. Analogously, the deformation $\widehat{d_t}$ contracts the subset $(p_q)^{-1}(U)$ to $(p_q)^{-1}(\mathrm{SP}^{q-1}(X/A)) = \mathrm{SP}_{q-1}X$, and we have the equality $p_q \circ \widehat{d_t} = \widehat{\widetilde{d}}_t \circ p_q$. According to Lemma A.1.11, U is distinguished with respect to p_q if $d^x : p_q^{-1}(x) \longrightarrow p_q^{-1}(x')$ (where $x' = \widehat{\widetilde{d}}_1(x)$), the restriction of $\widehat{d_1}$ to $p_q^{-1}(x)$, is a (weak) homotopy equivalence. To show this, let $x^0 \in p_q^{-1}(x)$ be the point that does not have any element different from x_0 (the base point) in A. We define $'x^0 \in p_q^{-1}(x')$ in an analogous way. The maps $y \mapsto x^0 + y$ and $y \mapsto {'x^0} + y$ are homeomorphisms of $\mathrm{SP}\ A$ to $p_q^{-1}(x)$ and $p_q^{-1}(x')$, respectively (see A.2.10). Through these homeomorphisms we turn d^x into a map of $\mathrm{SP}\ A$ to itself, namely, into the map that sends $y \mapsto y^0 + \widehat{d_1}(y)$, where $y^0 = \widehat{d_1}(x^0) - {'x^0}$. (Note that this difference is defined, since $\widehat{d_1}(x^0) \in p_q(x)$.) But this map can be deformed into the identity of $\mathrm{SP}\ A$, namely, since A is path connected, we can connect y^0 with 0 by a path y^t in $\mathrm{SP}\ A$ and so obtain the desired deformation by defining $y \mapsto y^t + d_{1-t}(y)$. $\square$

Appendix B

Proof of the Bott Periodicity Theorem

In this appendix we shall present a topological proof of the Bott periodicity theorem in the complex case 9.5.1, as we announced in Chapter 9. The proof essentially follows the lines indicated by D. McDuff in [53]. We shall make use of one of the results of Dold and Thom that we presented in Appendix A. This appendix is based on the article [7] by M.A. Aguilar and C. Prieto.

B.1 A Convenient Description of $\mathrm{BU} \times \mathbb{Z}$

In this section we shall slightly modify the definitions of U and BU given before, with the idea of giving a description of $\mathrm{BU} \times \mathbb{Z}$.

Let us recall that the unitary group U_n consists of unitary matrices in $\mathrm{GL}_n(\mathbb{C})$, that is, of those matrices whose column vectors form an orthonormal basis of $\mathbb{C}^n$ with respect to the canonical Hermitian inner product in that vector space. In other words, a matrix A belongs to U_n if and only if $AA^* = \mathrm{I}$, where A^* represents the transposed conjugate matrix of A and I is the identity matrix.

B.1.1 Definition. We define the *unitary group of infinite dimension* as

$$\mathrm{U} = \operatorname*{colim}_n \mathrm{U}_n,$$

with respect to the closed inclusions $\mathrm{U}_n \hookrightarrow \mathrm{U}_{n+1}$ given by sending the matrix $M \in \mathrm{U}_n$ to

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & \mathrm{I} \end{array}\right) \in \mathrm{U}_{n+1}\,.$$

Let us observe that the inclusion of U_n in U_{n+1} is that of a subgroup, as well as that of a closed subspace, so that the colimit is the same, whether as group or as space, and has the structure of a topological group (of infinite dimension).

Let us now recall the definition of BU, which, even though it is equivalent to that given in Definition 9.2.8, we shall express in a more convenient form for what we have in mind here. To do this we shall introduce some more notation and definitions.

Suppose that $-\infty \leq p \leq q \leq \infty$ (with at least two of the inequalities strict) and define

$$\mathbb{C}_p^q = \{z : \mathbb{Z} \longrightarrow \mathbb{C} \mid z_i = 0 \text{ for almost all } i \text{ and if } i \leq p \text{ or } i > q\}$$

with the usual topology in the finite-dimensional case, and the topology of the union in the infinite-dimensional case. Clearly, we then have $\mathbb{C}_0^q = \mathbb{C}^q$, $\mathbb{C}_0^\infty = \mathbb{C}^\infty$, $\mathbb{C}_0^1 = \mathbb{C}$, $\mathbb{C}_0^0 = \{0\}$, and so forth. All of the spaces $\mathbb{C}_p^q$ are thus subspaces of $\mathbb{C}_{-\infty}^\infty$. With these definitions we have that if $-\infty < p \leq q < \infty$, then $\dim \mathbb{C}_p^q = q - p$. Moreover, if $p \leq q \leq r$, then $\mathbb{C}_p^q \oplus \mathbb{C}_q^r = \mathbb{C}_p^r$.

We then have the Grassmann manifold

$$\mathrm{G}_n(\mathbb{C}_0^q) = \{W \mid W \text{ is a subspace of } \mathbb{C}_0^q \text{ of dimension } n\}$$

as well as

$$\mathrm{BU}_n = \mathrm{G}_n(\mathbb{C}_0^\infty) = \operatorname*{colim}_q \mathrm{G}_n(\mathbb{C}_0^q)\,,$$

where the colimit is taken with respect to the maps

$$\mathrm{G}_n(\mathbb{C}_0^q) \longrightarrow \mathrm{G}_n(\mathbb{C}_0^{q+1})\,,$$

which send $W \subset \mathbb{C}_0^q$ to $W = W \oplus 0 \subset \mathbb{C}_0^q \oplus \mathbb{C}_q^{q+1} = \mathbb{C}_0^{q+1}$. Then BU_n can be seen as the set

$$\{W \mid W \text{ is a subspace of } \mathbb{C}^\infty \text{ of dimension } n\}\,.$$

B.1.2 DEFINITION. For every $k \in \mathbb{Z}$ we define the *shift operator* by k coordinates

$$t_k : \mathbb{C}_{-\infty}^\infty \longrightarrow \mathbb{C}_{-\infty}^\infty$$

to be $t_k(z)_i = z_{i-k}$. These shift operators are continuous linear isomorphisms such that $t_0 = \mathrm{I}$ and $t_k \circ t_l = t_l \circ t_k = t_{k+l}$ hold.

The shift operator t_k has the property of shifting the coordinates k spaces to the right.

B.1.3 Definition. For each n we have a map $j_n^{n+1} : \mathrm{BU}_n \longrightarrow \mathrm{BU}_{n+1}$ that sends $W \subset \mathbb{C}^\infty$ to $\mathbb{C} \oplus t_1(W) \subset \mathbb{C}^\infty$. Then we define BU as

$$\mathrm{BU} = \operatorname{colim} \mathrm{BU}_k.$$

In order to compare this definition with an alternative way of stabilizing, we shall prove a lemma. But first we introduce the next definition.

B.1.4 Definition. Take $W \subset \mathbb{C}_k^l$ and let m be such that $\mathbb{C}_k^m \subset W$. Then $W/\mathbb{C}_k^m$ denotes the orthogonal complement of $\mathbb{C}_k^m$ in W; that is, if $\{e_{k+1}, \ldots, e_m\}$ is the canonical basis for $\mathbb{C}_k^m$, we complete it to an orthonormal basis $\{e_{k+1}, \ldots, e_m, w_1, \ldots, w_q\}$ of W; then $W/\mathbb{C}_k^m$ is spanned by $\{w_1, \ldots, w_q\}$, and we have $\mathbb{C}_k^m \oplus (W/\mathbb{C}_k^m) = W$.

B.1.5 **Lemma**. *There exists a homeomorphism*

$$\Phi : \mathrm{BU} \longrightarrow \overline{\mathrm{BU}}^0,$$

where $\overline{\mathrm{BU}}^0 = \{W \subset \mathbb{C}_0^\infty \mid \dim W < \infty$ *and* $\mathbb{C}_0^k \subset W \Leftrightarrow k = 0\}$.

Proof: Take $W \in \mathrm{BU}_n$ and let k be maximal with respect to the property $\mathbb{C}_0^k \subset W$. We define $\Phi_n(W) = t_{-k}(W/\mathbb{C}_0^k) \in \overline{\mathrm{BU}}^0$. Clearly, the map $\Phi_n : \mathrm{BU}_n \longrightarrow \overline{\mathrm{BU}}^0$ determines in the colimit the map Φ that we seek.

The map Φ is surjective, since if $W \in \overline{\mathrm{BU}}^0$ and $\dim W = n$, then $W \in \mathrm{BU}_n$ and $\Phi_n(W) = W$, because in this case $k = 0$. (In fact, the map $\Psi : \overline{\mathrm{BU}}^0 \longrightarrow \mathrm{BU}$ such that $W \mapsto W$ is the inverse.)

It also is injective, since if $V \in \mathrm{BU}_m$ and $W \in \mathrm{BU}_n$ satisfy $\Phi_m(V) = \Phi_n(W)$, then, provided that p and q are maximal for the properties $\mathbb{C}_0^p \subset V$ and $\mathbb{C}_0^q \subset W$, respectively, we have that

$$t_{-p}(V/\mathbb{C}_0^p) = t_{-q}(W/\mathbb{C}_0^q). \tag{B.1.6}$$

So the dimensions $m - p$ and $n - q$ are equal. Without loss of generality we may assume that $p \leq q$, so that in particular, we have $q - p = n - m \geq 0$. If we now apply t_q and sum on the left with $\mathbb{C}_0^q$ on both sides of (B.1.6), we obtain on the left side

$$\begin{aligned}\mathbb{C}_0^q \oplus t_{q-p}(V/\mathbb{C}_0^p) &= \mathbb{C}_0^{q-p} \oplus \mathbb{C}_{q-p}^q \oplus t_{q-p}(V/\mathbb{C}_0^p) \\ &= \mathbb{C}_0^{q-p} \oplus t_{q-p}(\mathbb{C}_0^p \oplus V/\mathbb{C}_0^p) = \mathbb{C}_0^{q-p} \oplus t_{q-p}(V),\end{aligned}$$

which is the image of V in $\mathrm{BU}_{m+q-p} = \mathrm{BU}_n$. And on the right side we get

$$\mathbb{C}_0^q \oplus t_0(W/\mathbb{C}_0^q) = W,$$

so that $j_m^n(V) = W$, where $j_m^n = j_{n-1}^n \circ \cdots \circ j_m^{m+1}$, and therefore V and W represent the same element in BU. $\square$

B.1.7 Definition. We define $\widetilde{\mathrm{BU}} = \{W \mid \mathbb{C}^p_{-\infty} \subset W \subset \mathbb{C}^q_{-\infty} \mid -\infty < p \leq q < \infty\}$, which is covered by the subspaces $\widetilde{\mathrm{BU}}^p = \{W \in \widetilde{\mathrm{BU}} \mid \mathbb{C}^p_{-\infty} \subset W$ and p is maximal$\}$ for $p \in \mathbb{Z}$.

Clearly, the map $W \mapsto \mathbb{C}^0_{-\infty} \oplus W$ determines a homeomorphism $\overline{\mathrm{BU}}^0 \longrightarrow \widetilde{\mathrm{BU}}^0$. Likewise, $W \mapsto t_{-k}(W)$ determines a homeomorphism $\widetilde{\mathrm{BU}}^k \longrightarrow \widetilde{\mathrm{BU}}^0$, so that we have a canonical homeomorphism

$$\widetilde{\mathrm{BU}}^0 \times \mathbb{Z} \longrightarrow \widetilde{\mathrm{BU}}$$

given by the composite $(W, k) \mapsto t_k(W) \in \widetilde{\mathrm{BU}}^k \hookrightarrow \widetilde{\mathrm{BU}}$. By Lemma B.1.5 we have proved the following.

B.1.8 **Theorem**. *There exists a homeomorphism*

$$\mathrm{BU} \times \mathbb{Z} \longrightarrow \widetilde{\mathrm{BU}}.$$

□

B.2 Proof of the Bott Periodicity Theorem

In this section we shall prove the periodicity theorem in the complex case. To do this we shall construct a quasifibration $p : \mathrm{E} \longrightarrow \mathrm{U}$ over the unitary group of infinite dimension, such that the total space E turns out to be contractible and the fiber is $\mathrm{BU} \times \mathbb{Z}$ (see 9.2.8). In this way we shall have a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \pi_i(\mathrm{BU} \times \mathbb{Z}) \longrightarrow \pi_i(\mathrm{E}) \longrightarrow \pi_i(\mathrm{U}) \longrightarrow \\ \longrightarrow \pi_{i-1}(\mathrm{BU} \times \mathbb{Z}) \longrightarrow \pi_{i-1}(\mathrm{E}) \longrightarrow \cdots, \end{aligned} \tag{B.2.1}$$

in which $\pi_i(\mathrm{E}) = 0 = \pi_{i-1}(\mathrm{E})$, and so we shall obtain for $i > 1$ that

$$\pi_i(\mathrm{U}) \cong \pi_{i-1}(\mathrm{BU} \times \mathbb{Z}) \cong \pi_{i-1}(\mathrm{BU}), \tag{B.2.2}$$

and for $i = 1$ we shall get

$$\pi_1(\mathrm{U}) \cong \mathbb{Z}\,. \tag{B.2.3}$$

As of now, as we proved in Chapter 9, we have (locally trivial) fibrations $\mathrm{E}_k(\mathbb{C}^\infty) \longrightarrow \mathrm{BU}_k$ with fiber U_k, where the base spaces are the classifying spaces of the unitary groups given by the colimits of Grassmann manifolds, and the total spaces are the corresponding colimits of Stiefel manifolds, such that, on passing again to the colimit, they determine a (locally trivial) fibration $\mathrm{EU} \longrightarrow \mathrm{BU}$ with fiber U and contractible total space EU and U as fiber (see [76]).

On the other hand, let us consider $\mathrm{PBU} = \{\omega : I \longrightarrow \mathrm{BU} \mid \omega(0) = x_0\}$, the *path space* of BU, where $x_0 \in \mathrm{BU}$ is the base point.

From 4.3.16 we obtain the following particular case.

B.2.4 **Proposition**. *The path space* PBU *is contractible and the map* $q : \mathrm{PBU} \longrightarrow \mathrm{BU}$ *given by* $q(\omega) = \omega(1)$ *is a Hurewicz fibration with fiber* $\Omega\mathrm{BU}$.□

The following is a proposition of a general character, which we include in this appendix for its particular interest here.

B.2.5 **Proposition**. *Let* $p : E \longrightarrow B$ *be a quasifibration with fiber* F *and* $p' : E' \longrightarrow B$ *a Hurewicz fibration with fiber* F', *such that the total spaces* E *and* E' *are contractible. Then there is a weak homotopy equivalence* $F \longrightarrow F'$, *and the homotopy groups (or sets, as the case may be) satisfy* $\pi_{i-1}(F) \cong \pi_i(B) \cong \pi_{i-1}(F')$ *for* $i \geq 1$.

Proof: Let $x_0 \in B$, $e_0 \in F \subset E$, and $e'_0 \in F' \subset E'$ be the base points. Since E is contractible, there exists a homotopy $H : E \times I \longrightarrow E$ such that $H(e,0) = e_0$ and $H(e,1) = e$ for all $e \in E$. Because $p' : E' \longrightarrow B$ is a Hurewicz fibration, we can complete the diagram

$$\begin{tikzcd} E \arrow[r, "\overline{e'_0}"] \arrow[d, "j_0"'] & E' \arrow[d, "p'"] \\ E \times I \arrow[r, "p\circ H"'] \arrow[ur, dashed, "\widetilde{H}"] & B, \end{tikzcd}$$

where $\overline{e'_0}$ is the constant map with value e'_0, in order to obtain the homotopy $\widetilde{H}$. Defining $\varphi(e) = \widetilde{H}(e,1)$, we therefore obtain a map $\varphi : E \longrightarrow E'$ that makes the triangle

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

commutative. In this way φ determines by restriction a map $\varphi_0 : F \longrightarrow F'$ that we shall see is a weak homotopy equivalence.

Since $p : E \longrightarrow B$ is a quasifibration, it has a long exact homotopy sequence, and because both E as well as E' are contractible, from the long exact sequences of each one of p and p', we get isomorphisms

$$\pi_i(B) \cong \pi_{i-1}(F), \qquad \pi_i(B) \cong \pi_{i-1}(F'),$$

which by the naturality of these sequences, namely,

$$\begin{tikzcd}
& 0 \arrow[d, equal] & & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & \pi_i(E) \arrow[r] \arrow[d, "\varphi_*"', "\cong"] & \pi_i(B) \arrow[r] \arrow[d, equal] & \pi_{i-1}(F) \arrow[r] \arrow[d, "\varphi_{0*}"'] & \pi_{i-1}(E) \arrow[r] \arrow[d, "\varphi_*"', "\cong"] & \cdots \\
\cdots \arrow[r] & \pi_i(E') \arrow[r] \arrow[d, equal] & \pi_i(B) \arrow[r] & \pi_{i-1}(F') \arrow[r] & \pi_{i-1}(E') \arrow[r] \arrow[d, equal] & \cdots , \\
& 0 & & & 0 &
\end{tikzcd}$$

determine the commutative triangle

$$\begin{tikzcd}
& \pi_i(B) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\pi_{i-1}(F) \arrow[rr, "\varphi_{0*}", "\cong"'] & & \pi_{i-1}(F').
\end{tikzcd}$$

Consequently, F and F' have the same weak homotopy type. □

B.2.6 **Corollary**. *There exists a homotopy equivalence* $\Omega\mathrm{BU} \simeq \mathrm{U}$ *and therefore isomorphisms* $\pi_{i-2}(\mathrm{U}) \cong \pi_{i-2}(\Omega\mathrm{BU}) \cong \pi_{i-1}(\mathrm{BU})$ *for* $i \geq 2$.

Proof: This is obtained from Proposition B.2.5 and from the fact that both $\Omega\mathrm{BU}$ and U have the homotopy type of CW complexes [54]. □

Then from (B.2.2) and B.2.6 we obtain the desired theorem.

B.2.7 **Theorem**. (Bott periodicity) *There is a homotopy equivalence* $\mathrm{BU} \times \mathbb{Z} \simeq \Omega\mathrm{U}$*; hence, for every* $i \geq 2$*, there exists an isomorphism*

$$\pi_i(\mathrm{U}) \cong \pi_{i-2}(\mathrm{U})$$

or, equivalently,

$$\pi_{i+1}(\mathrm{BU}) \cong \pi_{i-1}(\mathrm{BU}).$$

□

Or put in other terms, again by (B.2.2) and B.2.6 we have that $\pi_i(\mathrm{BU} \times \mathbb{Z}) \cong \pi_{i+1}(\mathrm{U}) \cong \pi_{i+1}(\Omega\mathrm{BU}) \cong \pi_i(\Omega^2\mathrm{BU})$; that is, we get an isomorphism

$$\pi_i(\mathrm{BU} \times \mathbb{Z}) \cong \pi_i(\Omega^2\mathrm{BU})\,,$$

which implies the earlier version of the periodicity theorem 9.5.1.

Having said this, in order to arrive at the proof of the existence of the desired quasifibration, we recall that an $n \times n$ matrix C with complex entries is *Hermitian* if $C = C^*$, where C^* denotes, as before, the transposed conjugate matrix of C. If $\langle -, - \rangle$ denotes the canonical Hermitian product on $\mathbb{C}^n$, then C satisfies the identity $\langle Cv, w\rangle = \langle v, Cw\rangle$ for arbitrary $v, w \in \mathbb{C}^n$. This implies in particular that the eigenvalues of the matrix C are real.

The set $\mathrm{H}_n(\mathbb{C})$ of all the $n \times n$ Hermitian matrices has the structure of a real vector space. Let E_n be the topological subspace of $\mathrm{H}_n(\mathbb{C})$ consisting of those matrices whose eigenvalues lie in the interval I. The space E_n is contractible by means of the homotopy $h : \mathrm{E}_n \times I \longrightarrow \mathrm{E}_n$ given by $h(C, \tau) = (1-\tau)C$, $0 \le \tau \le 1$, which begins with the identity map and ends with the constant map whose value is the zero matrix.

Let $M_{n\times n}(\mathbb{C})$ be the complex vector space of complex $n \times n$ matrices and let $\mathrm{GL}_n(\mathbb{C})$ (general linear group) be the group of the invertible matrices in $M_{n\times n}(\mathbb{C})$. We have a (differentiable) map

$$\exp : M_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

defined by

$$\exp(B) = \mathrm{e}^B = \sum_{i=0}^{\infty} \frac{B^i}{i!} = \mathrm{I}_n + B + \frac{B^2}{2!} + \cdots\,,$$

which satisfies the usual exponential laws precisely when the matrices involved in the exponents commute among themselves. After observing that $(TBT^{-1})^n = TB^nT^{-1}$, one can easily check the property

$$\mathrm{e}^{TBT^{-1}} = T\mathrm{e}^BT^{-1}$$

for any invertible operator T; moreover, for a diagonal matrix one has the property

$$\mathrm{e}^D = \begin{pmatrix} \mathrm{e}^{\lambda_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n} \end{pmatrix} \quad \text{if} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Let $M^a_{n\times n}(\mathbb{C}) \subset M_{n\times n}(\mathbb{C})$ be the real subspace of *skew-Hermitian* matrices, that is, of those matrices A such that $A^* = -A$. If A is skew-Hermitian, then $(\mathrm{e}^A)^* = \mathrm{e}^{A^*} = \mathrm{e}^{-A}$ and therefore

$$(\mathrm{e}^A)^*\mathrm{e}^A = \mathrm{e}^{-A}\mathrm{e}^A = \mathrm{e}^0 = \mathrm{I}_n\,.$$

Consequently, the map exp defined above can be restricted to

$$\exp : M^a_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{U}_n.$$

We have an isomorphism $\mathrm{H}_n(\mathbb{C}) \longrightarrow M^a_{n\times n}(\mathbb{C})$ given by $C \mapsto 2\pi\mathrm{i}C$. We define a map $p_n : \mathrm{E}_n \longrightarrow \mathrm{U}_n$ by $p_n(C) = \exp(2\pi\mathrm{i}C)$, so that the following diagram commutes:

$$\begin{tikzcd}
M^a_{n\times n}(\mathbb{C}) \arrow[r, "\exp"] & \mathrm{U}_n \\
\mathrm{H}_n(\mathbb{C}) \arrow[u, "\cong"'] & \\
\mathrm{E}_n. \arrow[u, hook] \arrow[uur, "p_n"'] &
\end{tikzcd}$$

B.2.8 **Proposition**. *The map p_n is surjective.*

Proof: Suppose that $U \in \mathrm{U}_n$ is arbitrary. We can diagonalize this matrix by taking another matrix $T \in \mathrm{U}_n$ and forming the product $T^{-1}UT$. Since the eigenvalues of a unitary matrix have norm 1, we have that

$$T^{-1}UT = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\lambda_1} & & & 0 \\ & \mathrm{e}^{2\pi\mathrm{i}\lambda_2} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_n} \end{pmatrix},$$

where $\lambda_i \in I$ for $i = 1, 2, \ldots, n$. Put

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

and consider the matrix TDT^{-1}. Because $T \in \mathrm{U}_n$ we have that $T^{-1} = T^*$, and so $(TDT^{-1})^* = (TDT^*)^* = TD^*T^* = TDT^{-1}$. This means that TDT^{-1} is Hermitian, and so $TDT^{-1} \in \mathrm{E}_n$. Thus we have that

$$p_n(TDT^{-1}) = \mathrm{e}^{2\pi\mathrm{i}(TDT^{-1})} = \mathrm{e}^{T(2\pi\mathrm{i}D)T^{-1}} = T\mathrm{e}^{2\pi\mathrm{i}D}T^{-1}$$

$$= T\begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\lambda_1} & & & 0 \\ & \mathrm{e}^{2\pi\mathrm{i}\lambda_2} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_n} \end{pmatrix}T^{-1} = U\,. \tag{B.2.9}$$

The third equality here is obtained from the fact that $\mathrm{e}^{TAT^{-1}} = T\mathrm{e}^{A}T^{-1}$, as shown above. $\square$

Let us now analyze the fibers of p_n. To do this suppose that we are given a matrix $C \in \mathrm{E}_n$ and let us consider the subspaces $\ker(C - \mathrm{I})$ and $\ker(p_n(C) - \mathrm{I})$.

If $v \in \ker(C - \mathrm{I})$, then $Cv = v$, and we have that

$$\begin{aligned} p_n(C)v &= \left(\mathrm{e}^{2\pi\mathrm{i}C}\right) v = \left(\mathrm{I} + 2\pi\mathrm{i}C + \frac{(2\pi\mathrm{i})^2}{2!}C^2 + \cdots\right) v \\ &= \mathrm{I}v + 2\pi\mathrm{i}Cv + \frac{(2\pi\mathrm{i})^2}{2!}C^2 v + \cdots \\ &= v + 2\pi\mathrm{i}v + \frac{(2\pi\mathrm{i})^2}{2!}v + \cdots \\ &= \left(1 + 2\pi\mathrm{i} + \frac{(2\pi\mathrm{i})^2}{2!} + \cdots\right) v = \mathrm{e}^{2\pi\mathrm{i}}v = v. \end{aligned}$$

Consequently, we have $\ker(C - \mathrm{I}) \subset \ker(p_n(C) - \mathrm{I})$. In this way for each $U \in \mathrm{U}_n$ we can define a map $g : p_n^{-1}(U) \longrightarrow \mathrm{G}(\ker(U - \mathrm{I}))$, the *Grassmann space* of all finite-dimensional vector subspaces of $\ker(U - \mathrm{I})$, by sending $C \in p_n^{-1}(U)$ to the subspace $\ker(C - \mathrm{I})$ of $\ker(U - \mathrm{I})$.

B.2.10 **Lemma**. *The map* $g : p_n^{-1}(U) \longrightarrow \mathrm{G}(\ker(U - \mathrm{I}))$ *is bijective.*

Proof: To show that g is surjective we take an arbitrary subspace $V \subset \ker(U - \mathrm{I})$. We then wish to construct a matrix $C_V \in p_n^{-1}(U) \subset \mathrm{E}_n$ such that $\ker(C_V - \mathrm{I}) = V$. To do this we shall construct a matrix T that suitably diagonalizes U. Note that $\ker(U - \mathrm{I})$ is the subspace of eigenvectors of U with eigenvalue 1, which we denote by $\mathcal{E}_1(U)$. Analogously, we have $\ker(C_V - \mathrm{I}) = \mathcal{E}_1(C_V)$. So we have $V \subset \mathcal{E}_1(U)$.

Let $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+s}\}$ be an orthonormal basis of $\mathcal{E}_1(U)$ such that $\{v_1, \ldots, v_r\}$ is a basis of V. Since U is a unitary matrix, the orthogonal complement of $\mathcal{E}_1(U)$ in $\mathbb{C}^n$, namely $\mathcal{E}_1(U)^\perp$, is a subspace invariant under U. This is so because if $w \in \mathcal{E}_1(U)^\perp$ and $v \in \mathcal{E}_1(U)$, then $\langle Uw, v\rangle = \langle w, U^*v\rangle = \langle w, Uv\rangle = \langle w, v\rangle = 0$. In other words, $U(\mathcal{E}_1(U)^\perp) \subset \mathcal{E}_1(U)^\perp$, and so we can find an orthonormal basis $\{v_{r+s+1}, \ldots, v_n\}$ of $\mathcal{E}_1(U)^\perp$ made out of eigenvectors of U whose eigenvalues are different from 1.

Let $T \in \mathrm{U}_n$ be such that $Te^i = v_i$ for $i = 1, \ldots, n$, where the e^i denote the vectors in the canonical basis of $\mathbb{C}^n$. Then $T^{-1}UT = \widehat{D}$ is the diagonal matrix

$$\widehat{D} = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & e^{2\pi i \lambda_{r+s+1}} & & \\ & & & & \ddots & \\ 0 & & & & & e^{2\pi i \lambda_n} \end{pmatrix}$$

with $r+s$ ones on the diagonal and the remaining eigenvalues different from 1.

Now put

$$D = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & \lambda_{r+s+1} & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \lambda_n \end{pmatrix},$$

with r ones on the diagonal followed by zeros in locations $r+1$ to $r+s$.

We are going to see that $C_V = TDT^{-1}$ is the desired matrix. Clearly, $e^{2\pi i D} = \widehat{D}$, so that we have $p_n(C_V) = e^{2\pi i C_V} = e^{T(2\pi i D)T^{-1}} = Te^{2\pi i D}T^{-1} = T\widehat{D}T^{-1} = U$. On the other hand, we can immediately verify that $\mathcal{E}_1(C_V) = T(\mathcal{E}_1(D))$. But $\mathcal{E}_1(D) = \{z \in \mathbb{C}^n \mid z_j = 0 \text{ for } r < j \leq n\}$, and since $Te_i = v_i$ for $i = 1, \dots, n$ and V is the subspace generated by $v_1, \dots, v_r$, we have $T(\mathcal{E}_1(D)) = V$.

In order to verify that the assignment $C \mapsto \mathcal{E}_1(C)$ is bijective, we have only to show that $C = C_{\mathcal{E}_1(C)}$. For this we observe that if C is Hermitian, then λ is an eigenvalue of C if and only if $e^{2\pi i \lambda}$ is an eigenvalue of $e^{2\pi i C}$. Indeed, let $R \in \mathrm{U}_n$ be such that $D = R^{-1}CR$ is a diagonal matrix. Then we have $R^{-1}e^{2\pi i C}R = e^{R^{-1}2\pi i CR} = e^{2\pi i R^{-1}CR} = e^{2\pi i D}$, which is a diagonal matrix with a diagonal entry $e^{2\pi i \lambda}$ for each diagonal entry λ of D. If now $C_1, C_2 \in \mathrm{E}_n$ satisfy $e^{2\pi i C_1} = e^{2\pi i C_2}$ and $\mathcal{E}_1(C_1) = \mathcal{E}_1(C_2)$, then $C_1 = C_2$. Indeed, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of C_1 and $\mu_1, \dots, \mu_n$ the eigenvalues of C_2. Since $\mathcal{E}_1(C_1) = \mathcal{E}_1(C_2)$, we can suppose that $\lambda_k = \mu_k = 1$ for $1 \leq k \leq r = \dim \mathcal{E}_1(C_1)$ and moreover, $\lambda_k \neq 1 \neq \mu_k$ when $r < k \leq n$. As we showed before, $e^{2\pi i \lambda_k}$ and $e^{2\pi i \mu_k}$ for $1 \leq k \leq n$ are the eigenvalues of $e^{2\pi i C_1}$ and $e^{2\pi i C_2}$, respectively. Consequently, $e^{2\pi i \lambda_k} = e^{2\pi i \mu_k}$ for all k, and by taking $r < k \leq n$

this implies that $\lambda_k = \mu_k$. Thus we have proved that $\lambda_k = \mu_k$ for all k, so that $C_1 = C_2$ follows.

In particular, if we apply what we have done to $C_1 = C$ and $C_2 = C_{\mathcal{E}_1(C)}$, then we have that $C = C_{\mathcal{E}_1(C)}$. $\square$

We can summarize all the above in the following theorem.

B.2.11 **Theorem**. *Let* E_n *be the space of Hermitian* $n \times n$ *matrices whose eigenvalues lie in the unit interval and let* $p_n : \mathrm{E}_n \longrightarrow \mathrm{U}_n$ *be given by* $p_n(C) = \mathrm{e}^{2\pi \mathrm{i} C}$. *Then* E_n *is contractible,* p_n *is surjective, and the fiber over each matrix* $U \in \mathrm{U}_n$ *is homeomorphic to the Grassmann space* $\mathrm{G}(\mathcal{E}_1(U))$. $\square$

Let us now see two ways of stabilizing this result. The usual way is by taking the canonical embeddings $\rho^n_{n+1} : \mathrm{E}_n \longrightarrow \mathrm{E}_{n+1}$ and $\tau^n_{n+1} : \mathrm{U}_n \longrightarrow \mathrm{U}_{n+1}$ given by

$$\rho_n(C) = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathrm{E}_{n+1}$$

and by

$$\tau_n(U) = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & \mathrm{I} \end{array}\right) \in \mathrm{U}_{n+1}.$$

We immediately verify that we have a commutative diagram:

$$\begin{tikzcd} \mathrm{E}_n \arrow[r, "\rho^n_{n+1}"] \arrow[d, "p_n"'] & \mathrm{E}_{n+1} \arrow[d, "p_{n+1}"] \\ \mathrm{U}_n \arrow[r, "\tau^n_{n+1}"'] & \mathrm{U}_{n+1}. \end{tikzcd}$$

In this way we obtain a map $p' : \operatorname{colim}_n \mathrm{E}_n \longrightarrow \operatorname{colim}_n \mathrm{U}_n$ such that $p' \circ \rho_n = p'|\mathrm{E}_n = p_n$.

Let us now analyze the fibers of p'. It is clear that if $U \in \mathrm{U}_n$, then we have $\mathcal{E}_1(\tau_n(U)) = \mathcal{E}_1(U) \oplus \mathbb{C}$ and $\mathcal{E}_1(\rho_n(C)) = \mathcal{E}_1(C) \oplus 0$. So we have the following commutative diagram:

$$\begin{tikzcd} p_n^{-1}(U) \arrow[r, "\rho^n_{n+1}"] \arrow[d, "\cong"'] & p_{n+1}^{-1}(\tau(U)) \arrow[d, "\cong"] \\ \mathrm{G}(\mathcal{E}_1(U)) \arrow[r, "\delta"'] & \mathrm{G}(\mathcal{E}_1(U) \oplus \mathbb{C}), \end{tikzcd}$$

where $\delta(V) = V \oplus 0$ defines the lower arrow. For example, if we take $U = \mathrm{I}$, then we have $\delta : \mathrm{G}(\mathbb{C}_0^n) \longrightarrow \mathrm{G}(\mathbb{C}_0^{n+1})$. In this way the fibers of f are homeomorphic to $\coprod_{r\geq 0} \mathrm{G}_r(\mathbb{C}_0^\infty) = \coprod_{r\geq 0} \mathrm{BU}_r$.

The map p_n defined above has BU_n as fiber. Now let us construct a new map $\widehat{p} : \widehat{\mathrm{E}} \longrightarrow \widehat{\mathrm{U}}$ with fiber $\mathrm{BU} \times \mathbb{Z}$. This new map $\widehat{p}$ will be in some sense a completion of p'. We define an operator C in $\mathbb{C}_{-\infty}^\infty$ to be *Hermitian* if $\langle Cz, z'\rangle = \langle z, Cz'\rangle$ or, equivalently, if $C = C^*$. Put

$$\widehat{\mathrm{E}} = \{C \mid C \text{ is Hermitian, of finite type, and with eigenvalues in } I\}\,,$$

where we understand by a Hermitian operator of *finite type* one for which there exist $r < s$ such that $Ce^i = 0$ when $i \leq r$ or $i > s$. In other words, a Hermitian operator of finite type is represented by an infinite matrix of the form

$$\begin{pmatrix}
\ddots & & & & & & & & 0 \\
& 0 & & & & & & & \\
& & \ddots & & & & & & \\
& & & 0 & & & & & \\
& & & & \widetilde{C} & & & & \\
& & & & & 0 & & & \\
& & & & & & \ddots & & \\
& & & & & & & 0 & \\
0 & & & & & & & & \ddots
\end{pmatrix},$$

where $\widetilde{C}$ is an $(s-r) \times (s-r)$ Hermitian matrix that acts on $\mathbb{C}_r^s$. Notice that $\widehat{\mathrm{E}}$ is contractible, just as E_n is.

Analogously, we define an operator U in $\mathbb{C}_{-\infty}^\infty$ to be *unitary* if $\langle Uz, Uz'\rangle = \langle z, z'\rangle$ or, equivalently, if $UU^* = \mathrm{I}$. Put

$$\widehat{\mathrm{U}} = \{U \mid U \text{ is unitary and of finite type }\}\,,$$

where we understand by a unitary operator of *finite type* one for which there exist $r < s$ such that $Ue^i = e^i$ when $i \leq r$ or $i > s$. In other words, a unitary operator of finite type is represented by an infinite matrix of the form

$$\begin{pmatrix} \ddots & & & & & & & & 0 \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & \widetilde{U} & & & & \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ 0 & & & & & & & & \ddots \end{pmatrix},$$

where $\widetilde{U}$ is an $(s-r)\times(s-r)$ unitary matrix that acts on $\mathbb{C}_r^s$.

In order to simplify notation, we shall write these two matrices as

$$\begin{pmatrix} 0_{-\infty}^r & & 0 \\ & \widetilde{C} & \\ 0 & & 0_s^\infty \end{pmatrix}, \qquad \begin{pmatrix} \mathrm{I}_{-\infty}^r & & 0 \\ & \widetilde{U} & \\ 0 & & \mathrm{I}_s^\infty \end{pmatrix},$$

where 0_m^n is the zero matrix and I_m^n is the identity matrix, each of these acting on $\mathbb{C}_m^n$ for $-\infty \leq m < \infty$ and $-\infty < n \leq \infty$. For simplicity, we shall write 0 or I when this does not cause confusion.

We can define a map $\widehat{p} : \widehat{\mathrm{E}} \longrightarrow \widehat{\mathrm{U}}$ by $\widehat{p}(C) = \exp(2\pi \mathrm{i} C)$. We then have the matrix identity

$$\widehat{p}(C) = \begin{pmatrix} \mathrm{I}_{-\infty}^r & & 0 \\ & \mathrm{e}^{2\pi\mathrm{i}\widetilde{C}} & \\ 0 & & \mathrm{I}_s^\infty \end{pmatrix}.$$

We shall simply denote the identity matrix that acts on $\mathbb{C}_{-\infty}^\infty$ by I. Suppose that $U \in \widehat{\mathrm{U}}$. The space of eigenvectors of U with eigenvalue equal to 1, namely $\ker(U - \mathrm{I})$, is evidently given by $\mathbb{C}_{-\infty}^r \oplus \ker(\widetilde{U} - \mathrm{I}_r^s) \oplus \mathbb{C}_s^\infty$ and therefore is isomorphic to $\mathbb{C}_{-\infty}^\infty$. Let us consider the *grassmannian* $\mathrm{G}_\infty(\ker(U - \mathrm{I})) = \{W \subset \ker(U-\mathrm{I}) \mid \mathbb{C}_{-\infty}^{r'} \subset W \text{ and } \dim(W/\mathbb{C}_{-\infty}^{r'}) < \infty\}$. We then have the following lemma.

B.2.12 **Lemma**. *For each $U \in \widehat{\mathrm{U}}$ there exists a homeomorphism*

$$\varphi_U : \widetilde{\mathrm{BU}} \approx \mathrm{G}_\infty(\ker(U - \mathrm{I})). \qquad \square$$

Analogous to Lemma B.2.10 we have the following result.

B.2.13 **Proposition**. *If $U \in \widehat{\mathrm{U}}$, then $\widehat{p}^{-1}(U) \approx \widetilde{\mathrm{BU}} = \mathrm{BU} \times \mathbb{Z}$.*

Proof: It is enough to prove that we have a homeomorphism

$$g_U : \widehat{p}^{-1}(U) \longrightarrow \mathrm{G}_\infty(\ker(U - \mathrm{I})).$$

Let us first observe that if $C \in \widehat{\mathrm{E}}$, then $V_C = \ker(C - \mathrm{I}) = \ker(\widetilde{C} - \mathrm{I}_r^s) \subset \mathbb{C}_r^s$. So for $C \in \widehat{p}^{-1}(U)$ we then define $g_U(C) = \mathbb{C}_{-\infty}^r \oplus V_C \in \mathrm{G}_\infty(\ker(U - \mathrm{I}))$.

To show that g_U is surjective we start with an arbitrary $W \in \mathrm{G}_\infty(\ker(U - \mathrm{I}))$ in the codomain. So we have that $W = \mathbb{C}_{-\infty}^{r'} \oplus \widetilde{W}$ with $\dim \widetilde{W} < \infty$.

Without loss of generality we can suppose that $r' = r$. Since $W \subset \ker(U - \mathrm{I}) = \mathbb{C}_{-\infty}^r \oplus \ker(\widetilde{U} - \mathrm{I}_r^s) \oplus \mathbb{C}_s^\infty$, by taking s sufficiently large we have that $\widetilde{W} \subset \ker(\widetilde{U} - \mathrm{I}_r^s) = \mathcal{E}_1(\widetilde{U}) \subset \mathbb{C}_r^s$.

As in Lemma B.2.10, let $\{v_1, \ldots, v_m\}$ be an orthonormal basis of $\widetilde{W}$, $\{v_{m+1}, \ldots, v_{m+n}\}$ an orthonormal basis of the orthogonal complement of $\widetilde{W}$ in $\mathcal{E}_1(\widetilde{U})$, and $\{v_{m+n+1}, \ldots, v_{s-r}\}$ an orthonormal basis of the orthogonal complement of $\mathcal{E}_1(\widetilde{U})$ in $\mathbb{C}_r^s$ (which is invariant under $\widetilde{U}$), this last basis being made up of eigenvectors with eigenvalues different from 1.

If we define $T \in \widehat{\mathrm{U}}$ by

$$Te^i = \begin{cases} e^i & \text{if } i \leq r, \\ v_{i-r} & \text{if } r < i \leq s, \\ e^i & \text{if } i > s, \end{cases}$$

then we have that $T^{-1}UT = \widehat{D}$ is diagonal of the form

$$\begin{pmatrix} \mathrm{I}_{-\infty}^{r+m+n} & & & & 0 \\ & \mathrm{e}^{2\pi \mathrm{i} \lambda_{m+n+1}} & & & \\ & & \ddots & & \\ & & & \mathrm{e}^{2\pi \mathrm{i} \lambda_{s-r}} & \\ 0 & & & & \mathrm{I}_s^\infty \end{pmatrix}.$$

We now take

$$D = \begin{pmatrix} 0_{-\infty}^r & & & & & & 0 \\ & \mathrm{I}_r^{r+m} & & & & & \\ & & 0_{r+m}^{r+m+n} & & & & \\ & & & \lambda_{m+n+1} & & & \\ & & & & \ddots & & \\ & & & & & \lambda_{s-r} & \\ 0 & & & & & & 0_{s-r}^\infty \end{pmatrix}$$

and we define $C_W = TDT^{-1}$ so that we have $\widehat{p}(C_W) = \mathrm{e}^{2\pi \mathrm{i} C_W} = T\mathrm{e}^{2\pi \mathrm{i} D}T^{-1} = T\widehat{D}T^{-1} = U$. Moreover, we have $g_U(C_W) = \mathbb{C}_{-\infty}^r \oplus \ker(C_W - \mathrm{I})$. Using the same argument as in B.2.10, we show that $\ker(C_W - \mathrm{I}) = \widetilde{W}$, and so $g_U(C_W) = W$.

Finally, the map g_U is injective, since if C_1 and C_2 are matrices such that $\exp(2\pi\mathrm{i}C_1) = \exp(2\pi\mathrm{i}C_2) = U$ and $\mathcal{E}_1(C_1) = \ker(C_1 - \mathrm{I}) = \ker(C_2 - \mathrm{I}) = \mathcal{E}_1(C_2)$, then we can argue in the same way as in the corresponding part of the proof of B.2.10 in order to prove that $C_1 = C_2$. □

To prove that $\widehat{p} : \widehat{E} \longrightarrow \widehat{U}$ is a quasifibration, we shall apply the criterion given by Theorem A.1.19, for which we shall need two results.

B.2.14 **Proposition**. *The map* $\widehat{p}|_{\widehat{\mathrm{U}}_n - \widehat{\mathrm{U}}_{n-1}}$ *is trivial; that is, there exists a homeomorphism*

$$h : \widehat{p}^{-1}(\widehat{\mathrm{U}}_n - \widehat{\mathrm{U}}_{n-1}) \longrightarrow (\widehat{\mathrm{U}}_n - \widehat{\mathrm{U}}_{n-1}) \times \widetilde{\mathrm{BU}}$$

such that $\mathrm{proj}_1 \circ h = \widehat{p}$.

Proof: We shall analyze the case where n is even; the case where n is odd is analogous. Take $C \in \widehat{p}^{-1}(\widehat{\mathrm{U}}_n - \widehat{\mathrm{U}}_{n-1})$ and put $U = \widehat{p}(C) \in \widehat{\mathrm{U}}_n - \widehat{\mathrm{U}}_{n-1}$. Therefore, we have

$$U = \begin{pmatrix} \mathrm{I}_{-\infty}^{-n/2} & 0 \\ 0 & \widetilde{U} \end{pmatrix},$$

and $-n/2$ is maximal for this matrix. So

$$U - \mathrm{I} = \begin{pmatrix} 0_{-\infty}^{-n/2} & 0 \\ 0 & U' \end{pmatrix},$$

where U' is not of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & U'' \end{pmatrix},$$

so that $\ker(U - \mathrm{I}) = \mathbb{C}_{-\infty}^{-n/2} \oplus \ker(U')$ with $-n/2$ maximal. Therefore, $\ker(U - \mathrm{I})$ depends continuously on U. Consequently, the homeomorphism $\varphi_U : \mathrm{G}_\infty(\ker(U - \mathrm{I})) \longrightarrow \widetilde{\mathrm{BU}}$ of Lemma B.2.12 also depends continuously on U.

Suppose that $h(C) = (\widehat{p}(C), \varphi(C))$, where $\varphi(C) = \varphi_U(g_U(C))$ and g_U is as in the proof of B.2.13. Since both φ_U and g_U are homeomorphisms that depend continuously on U, h also is a homeomorphism. □

In the complex space $\mathbb{C}_{-\infty}^{\infty}$ let us take the canonical Hermitian inner product given by $\langle z, z' \rangle = \sum_{r\in\mathbb{Z}} z_r \overline{z'}_r$. The canonical basis in this space, namely, the vectors e^r such that $e^r_s = \delta^r_s$, is orthonormal, and the assignment $e^r \mapsto e^{2r}$ for $r > 0$ and $e^r \mapsto e^{2|r|+1}$ for $r \leq 0$ gives an isomorphism

(B.2.15) $$S : \mathbb{C}^\infty_{-\infty} \cong \mathbb{C}^\infty_0 .$$

Through the isomorphism S we have an isomorphism $\mathrm{U} \longrightarrow \widehat{\mathrm{U}}$ given by $U \mapsto SUS^{-1}$ in such a way that if $\widehat{\mathrm{U}}_n$ is the image of U_n under this isomorphism, then $\widehat{\mathrm{U}} = \mathrm{colim}_n\, \widehat{\mathrm{U}}_n$.

In order to verify the second condition of Theorem A.1.19, we have the following result. Its proof needs some elementary facts of differential topology. A general reference for these facts is [20].

B.2.16 **Proposition**. *There is a neighborhood V_n of U_{n-1} in U_n and a strong deformation retraction of V_n onto U_{n-1} that lifts to a strong deformation retraction of $p^{-1}(V_n)$ onto $p^{-1}(\mathrm{U}_{n-1})$ in $p^{-1}(\mathrm{U}_n)$.*

Proof: Since U_{n-1} is a submanifold of U_n, we shall construct a tubular neighborhood V_n of the first in the second as follows.

Recall $\mathrm{H}_n(\mathbb{C})$, the space of Hermitian $n \times n$ matrices, and define $f : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{H}_n(\mathbb{C})$ by $f(A) = A^*A$. One can easily verify that f is smooth and has I as a regular value; therefore, $\mathrm{U}_n = f^{-1}(\mathrm{I})$ is a smooth manifold, and if $W \in \mathrm{U}_n$, then the tangent space of U_n at W, $\mathrm{T}_W(\mathrm{U}_n)$, is the kernel of the differential of f at W; that is,

$$\mathrm{T}_W(\mathrm{U}_n) = \{A \in M_{n\times n}(\mathbb{C}) \mid A^*W = -W^*A\}\,.$$

Now recall that there is a Hermitian product in $M_{n\times n}(\mathbb{C})$, given by $\langle A, B\rangle = \mathrm{trace}(AB^*)$; thus, taking the real part of this product, we get an inner product $M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{R}$. The restriction of this inner product to each tangent space $\mathrm{T}_W(\mathrm{U}_n) \subset M_{n\times n}(\mathbb{C})$ defines a Riemannian metric on U_n. Let $i : \mathrm{U}_{n-1} \hookrightarrow \mathrm{U}_n$ be the inclusion, such that $i(U) = U \oplus I$; then the differential $di : \mathrm{T}_U(\mathrm{U}_{n-1}) \longrightarrow \mathrm{T}_{i(U)}(\mathrm{U}_n)$ is an inclusion mapping a matrix R to $R \oplus 0$; that is,

$$di(R) = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}.$$

One can easily check that the orthogonal complement of the space $\mathrm{T}_U(\mathrm{U}_{n-1})$ in $\mathrm{T}_{i(U)}(\mathrm{U}_n)$ is given by

$$\mathrm{T}_U(\mathrm{U}_{n-1})^\perp = \left\{ \begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix} \in M_{n\times n}(\mathbb{C}) \;\middle|\; b = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1} \text{ and } t \in \mathbb{R} \right\},$$

which is a real $(2n-1)$-dimensional vector space. We denote by $N = \bigcup_{U \in \mathrm{U}_{n-1}} \mathrm{T}_U(\mathrm{U}_{n-1})^\perp$ the normal bundle of U_{n-1} in U_n.

Any vector space basis of $\mathrm{T}_\mathrm{I}(\mathrm{U}_n)$ provides a parallelization of U_n that defines a connection on it. This connection does not depend on the chosen basis and determines a spray on U_n. By [20], there exists $\varepsilon > 0$ such that $N_\varepsilon = \{v \in N \mid \|v\| < \varepsilon\}$ is an open neighborhood of the 0-section, and the exponential map associated to the spray, $\mathrm{Exp} : N_\varepsilon \longrightarrow \mathrm{U}_n$, is an embedding onto a neighborhood of U_{n-1} in U_n. Now, since the geodesics of this spray are the integral curves of the left-invariant vector fields, then $\mathrm{Exp}(A) = L_U \exp((dL_U)^{-1}(A))$, where $A \in \mathrm{T}_U(\mathrm{U}_{n-1})^\perp$, $L_U : \mathrm{U}_n \longrightarrow \mathrm{U}_n$ is given by $L_U(W) = UW$, and exp is the usual exponential map defined above. Evaluating the differential of L_U, we obtain $\mathrm{Exp}(A) = U\exp(U^*A)$.

Therefore, we have the following description of a tubular neighborhood $V_n = \mathrm{Exp}(N_\varepsilon)$ of U_{n-1} in U_n as

$$\left\{ U \exp\left(U^* \begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix} \right) \;\middle|\; U \in \mathrm{U}_{n-1},\ (b,t) \in \mathbb{C}^{n-1} \times \mathbb{R} \text{ and } \|(b,t)\| < \varepsilon \right\}.$$

In order to compute $U \exp\left(U^* \left(\begin{smallmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{smallmatrix}\right)\right)$, first note that

$$U^* \begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix} = \begin{pmatrix} 0 & U^*b \\ -b^*U & \mathrm{i}t \end{pmatrix}.$$

Set $A(b,t) = \left(\begin{smallmatrix} 0 & b \\ -b^* & \mathrm{i}t \end{smallmatrix}\right)$. Assume $b \neq 0$. To diagonalize this matrix one takes an orthonormal basis of eigenvectors and uses it to form a matrix. The $n \times n$ matrix $A(b,t)$ has $n-2$ eigenvalues equal to 0 and two eigenvalues λ_1, λ_2, such that

$$\lambda_\nu = \frac{t + (-1)^\nu \sqrt{4|b|^2 + t^2}}{2}\mathrm{i},$$

so that the matrix

$$W(b,t) = \begin{pmatrix} v_1 & \cdots & v_{n-2} & \mu_1 b & \mu_2 b \\ 0 & \cdots & 0 & \mu_1\lambda_1 & \mu_2\lambda_2 \end{pmatrix},$$

where $\{v_1, \ldots, v_{n-2}\} \subset \mathbb{C}^{n-1}$ is an orthonormal basis of the space $b^\perp = \{v \mid v \perp b\} \subset \mathbb{C}^{n-1}$ and $\mu_\nu = (|b|^2 + |\lambda_\nu|^2)^{-1/2}$, is a unitary $n \times n$ matrix that satisfies

$$D(b,t) = W(b,t)^* A(b,t) W(b,t) = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & \lambda_1 & \\ 0 & & & & \lambda_2 \end{pmatrix}.$$

Since we can write

$$D(b,t) = W(b,t)^*UU^*A(b,t)UU^*W(b,t) \text{ and } A(U^*b,t) = U^*A(b,t)U\,,$$

then

$$A(U^*b,t) = U^*W(b,t)D(b,t)(U^*W(b,t))^*\,.$$

Therefore, the points in the tubular neighborhood are of the form

$$U\exp(A(U^*b,t)) == UU^*W(b,t)\exp(D(b,t))W(b,t)^*U = \exp(A(b,t))U\,.$$

Hence, every element in V_n coming from the fiber over U in N_ε is right translation by U of an element coming from the fiber over I. It is thus enough to study the situation over the identity matrix.

Since we may linearly deform the neighborhood N_ε to the zero section, simply by $v \mapsto (1-\tau)v$, $1 \leq \tau \leq 1$, we obtain a strong deformation retraction $r_n^\tau : V_n \longrightarrow V_n$ such that

$$r_n^\tau(\exp(A(b,t))U) = \exp(A((1-\tau)b,(1-\tau)t))U\,.$$

Observe that for $\tau = 1$, $r_n^1(\exp(A(b,t))U) = \exp(0)U = \mathrm{I}U = U$, so that it is a retraction of V_n onto U_{n-1}.

In what follows, we define the lifting $\widetilde{r}_n^\tau : p^{-1}(V_n) \longrightarrow p^{-1}(V_n)$. Since fiberwise, $p^{-1}(V_n)$ consists of spaces homeomorphic to the grassmannians $G_\infty(\mathcal{E}_1(U'))$, $U' \in V_n$, we shall show how $\widetilde{r}_n^\tau$ acts on these spaces. It is clearly enough to study the case $\tau = 0$.

Take $U' = \exp(A(b,t))U \in V_n$ and let $G_{U,b,t} = G_\infty(\mathcal{E}_1(U'))$; we also have to show that the restriction of the lifting $\widetilde{r}_n^1$, $\widetilde{r}_n^1| : G_{U,b,t} \longrightarrow G_{U,0,0} = G_\infty(\mathcal{E}(U))$ is a homotopy equivalence.

Since $\mathcal{E}_1(\exp(A(b,t))U) = U\mathcal{E}_1(\exp(A(U^*b,t))$ and for $b \neq 0$, $t \neq 0$, $\mathcal{E}_1(\exp(A(U^*b,t))) = \mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^\infty$, because $\mathrm{e}^{\lambda_1} \neq 1 \neq \mathrm{e}^{\lambda_2}$, we have that the grassmannians $G_{U,b,t}$ and $G_{\mathrm{I},b,t}$ differ only by left multiplication by U. It is thus enough to study the case $U = \mathrm{I}$, namely the map $\widetilde{r}$: $G_\infty(\mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^\infty) \longrightarrow G_\infty(\mathbb{C}_{-\infty}^\infty)$. If $V \subset \mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^\infty$ is a subspace, then we define $\widetilde{r}(V) = V \subset \mathbb{C}_{-\infty}^\infty$, i.e., the map induced by the inclusion $\mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^\infty \hookrightarrow \mathbb{C}_{-\infty}^\infty$. The result now follows from the next proposition.□

B.2.17 **Proposition**. *The inclusion $\mathbb{C}_{-\infty}^r \oplus \mathbb{C}_s^\infty \hookrightarrow \mathbb{C}_{-\infty}^\infty$, $r \leq s \in \mathbb{Z}$, induces a homotopy equivalence between the grassmannians*

$$\alpha : G_\infty(\mathbb{C}_{-\infty}^r \oplus \mathbb{C}_s^\infty) \longrightarrow G_\infty(\mathbb{C}_{-\infty}^\infty)\,.$$

Proof: Take $V \in G_\infty(\mathbb{C}^\infty_{-\infty})$ and decompose it as $V = V_1 \oplus V_2$, where $V_1 \subset \mathbb{C}^r_{-\infty}$ and $V_2 \subset \mathbb{C}^\infty_r$, and define $\beta : G_\infty(\mathbb{C}^\infty_{-\infty}) \longrightarrow G_\infty(\mathbb{C}^r_{-\infty} \oplus \mathbb{C}^\infty_s)$ such that $\beta(V) = V_1 \oplus t_{s-r}V_2$, where t_{s-r} is the shift by $s-r$ coordinates (see B.1.2). Then $\alpha\beta(V) = V_1 \oplus t_{s-r}V_2 \subset \mathbb{C}^\infty_{-\infty}$ and $\beta\alpha(W) = W_1 \oplus t_{s-r}W_2$ if $W = W_1 \oplus W_2 \subset \mathbb{C}^r_{-\infty} \oplus \mathbb{C}^\infty_s$. The proposition now follows immediately from the next lemma. □

B.2.18 **Lemma**. *The map $\gamma : G_\infty(\mathbb{C}^r_{-\infty} \oplus \mathbb{C}^\infty_s) \longrightarrow G_\infty(\mathbb{C}^r_{-\infty} \oplus \mathbb{C}^\infty_s)$, $r \leq s \in \mathbb{Z}$, given by $\gamma(V) = V_1 \oplus t_k(V_2)$, $k \geq 0$, where $V = V_1 \oplus V_2$, $V_1 \subset \mathbb{C}^r_{-\infty}$, and $V_2 \subset \mathbb{C}^\infty_s$, is homotopic to the identity.*

Proof: The homotopy $h^1_\tau = \sin(\frac{\pi}{2}\tau)\mathrm{I} + \cos(\frac{\pi}{2}\tau)t_1 : \mathbb{C}^\infty_s \longrightarrow \mathbb{C}^\infty_s$, $0 \leq \tau \leq 1$, starts with t_1 and ends with the identity through monomorphisms, and $h^k_\tau = h^1_\tau \circ \cdots \circ h^1_\tau$ (k times) is such that $h^k_0 = t_k$ and $h^k_1 = \mathrm{I}$. Then $\widehat{h}_\tau(V) = V_1 \oplus h^k_\tau(V_2)$ is a homotopy, as desired. □

We have thus shown that $\widetilde{r}^1_n : p^{-1}(V_n) \longrightarrow p^{-1}(\mathrm{U}_{n-1})$ is fiberwise a homotopy equivalence. This finishes the proof of B.2.16. It should be remarked that for $s < 1$, the deformation $\widetilde{r}^s_n : p^{-1}(V_n) \longrightarrow p^{-1}(V_n)$ is fiberwise a homeomorphism, since after identifying the fibers with the associated grassmannians, it is the identity. This behavior is congruent with the first fact B.2.14 needed for the verification of the criterion A.1.19.

Thus we have our main theorem, which, as already seen at the beginning of this section, implies Bott periodicity in the complex case.

B.2.19 **Theorem**. *Let* E *be the space of Hermitian operators of finite type on $\mathbb{C}^\infty$ and let $p : \mathrm{E} \longrightarrow \mathrm{U}$ be given by $p(C) = \exp(2\pi\mathrm{i}C)$. Then p is a quasifibration with fiber* $\mathrm{BU} \times \mathbb{Z}$ *and contractible total space* E. □

B.2.20 Remark. A proof along the same lines of the real periodicity theorem was given very recently by Behrens (see [15]).

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