Computational Improvements for the BEM including Viscothermal Effects

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The Boundary Element Method

The basis of the (Acoustical) Boundary Element Method is the integral equation

$$\alpha\zeta(\mathbf{x})p(\mathbf{x}) - \int_\Gamma \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}p(\mathbf{y})\ \mathrm{d}S_{\mathbf{y}} + s(\alpha)k\int_\Gamma G(\mathbf{x},\mathbf{y})v_{\mathbf{n}}(\mathbf{y})\ \mathrm{d}S_{\mathbf{y}} = 0. \tag{1}$$

In order to discrete the integrals we need to approximate the geometry. This can be done using e.g. elements

$$\mathbf{x}^e(\mathbf{u}) = \mathbf{X}^e\mathbf{N}(\mathbf{u}) \in \Gamma^e, \quad \forall \mathbf{u} \in \mathcal{L}, \quad \cup_{e=1}^N \Gamma^e \approx \Gamma \tag{2}$$

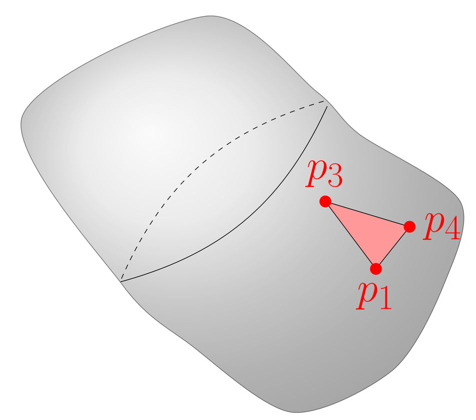


Figure 1. The original domain in shown in gray while a two (linear) elements are shown in red. The red points denote the interpolation nodes of the elements (the columns of $\mathbf{X}^e$).

The pressure (and its normal derivative) on an element can be approximated as

$$p(\mathbf{x}^e(\mathbf{u})) = \mathbf{T}(\mathbf{x}^e(\mathbf{u}))\mathbf{p} = \underbrace{\mathbf{T}(\mathbf{x}(\mathbf{u}))(\mathbf{L}^e)^\top}_{\mathbf{T}^e(\mathbf{u})}\underbrace{\mathbf{L}^e\mathbf{p}}_{\mathbf{p}^e} = \mathbf{T}^e(\mathbf{u})\mathbf{p}^e, \quad \mathbf{u} \in \mathcal{L}, \tag{3}$$

where $\mathbf{L}^e$ is a matrix that extracts the relevant rows of $\mathbf{p}$, i.e. for the element in Figure 1 we have that

$$\mathbf{L}^e = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}, \quad \text{so that} \quad \begin{bmatrix} p_1 \\ p_4 \\ p_3 \end{bmatrix} = \mathbf{L}^e\mathbf{p}. \tag{4}$$

The final ingredient is to approximate the integral on the eth element by using a quadrature scheme

$$\int_{\Gamma^e} f(\mathbf{y})\ \mathrm{d}S_{\mathbf{y}} = \int_{\mathcal{L}} \text{jacobian}(\mathbf{u})f(\mathbf{u})\ \mathrm{d}\mathbf{u} \approx \sum_{i=1}^{Q} \text{jacobian}(\mathbf{u}_i)w_i f(\mathbf{u}_i). \tag{5}$$

Using all of the above it is found that the discrete form of (1) is

$$(\text{diag}(\boldsymbol{\zeta}) - \mathbf{F})\mathbf{p} + s(\alpha)k\mathbf{G}\mathbf{v_n} = \mathbf{H}\mathbf{p} + s(\alpha)k\mathbf{G}\mathbf{v_n} = \mathbf{0}, \tag{6}$$

References

[1] M. Berggren, A. Bernland, and D. Noreland, "Acoustic boundary layers as boundary conditions," *Journal of Computational Physics*, vol. 371, pp. 633–650, 2018.

[2] M. Brunaeu, P. Herzog, J. Kergomard, and J. Polack, "General formulation of the dispersion equation in bounded visco-thermal fluid, and application to some simple geometries," *Wave Motion*, vol. 11, no. 5, pp. 441–451, 1989.

[3] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, "Julia: A fresh approach to numerical computing," *SIAM review*, vol. 59, no. 1, pp. 65–98, 2017.

[4] V. Cutanda Henríquez and P. R. Andersen, "A three-dimensional acoustic boundary element method formulation with viscous and thermal losses based on shape function derivatives," *Journal of Computational Acoustics*, vol. 26, no. 3, 2018.

[5] T. Askham, Z. Gimbutas, L. Greengard, L. Lu, J. Magland, D. Malhotra, M. O'Neil, M. Rachh, and V. Rokhlin, "FMM3D." `https://github.com/flatironinstitute/FMM3D`, 2023.

[6] D. Panagiotopoulos, E. Deckers, and W. Desmet, "Krylov subspaces recycling based model order reduction for acoustic bem systems and an error estimator," *Computer Methods in Applied Mechanics and Engineering*, vol. 359, 2020.

Reduced Order Series Expansion Boundary Element Method (ROSEBEM)

A drawback of the BEM is that the discrete form of (6) depends on the frequency, meaning that for every frequency of interest it needs to be recomputed. A simple Taylor expansion of the Green's function can alleviate this pain as it transforms the discrete form into[6]

$$\left(\text{diag}(\boldsymbol{\zeta}) - \sum_{m=0}^{M-1}\frac{(k-k_0)^m}{m!}\mathbf{F}_m(k_0)\right)\mathbf{p} + s(\alpha)k\left(\sum_{m=0}^{M-1}\frac{(k-k_0)^m}{m!}\mathbf{G}_m(k_0)\right)\mathbf{v}_n \approx \mathbf{0}. \tag{7}$$

This approach, however, increase memory usage due to the storing of M-times the matrices. So solve this a reduced basis can be used, i.e. introduce $\mathbf{U}_\ell$ such that

$$\mathbf{p} \approx \mathbf{U}_\ell\mathbf{p}_\ell, \quad \frac{\partial \mathbf{p}}{\partial \mathbf{n}} \approx \mathbf{U}_\ell\frac{\partial \mathbf{p}_\ell}{\partial \mathbf{n}}. \tag{8}$$

Including the Boundary Layer Impedance condition

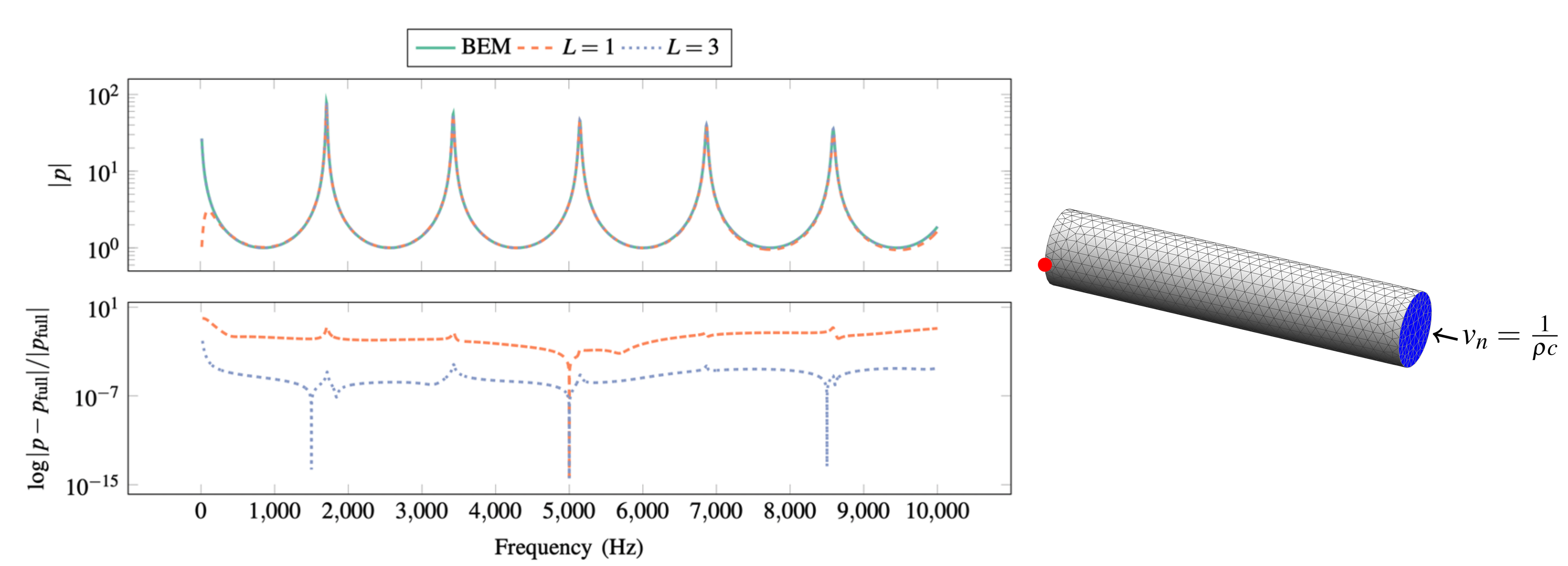
The inclusion of viscothermal losses can sometimes be handed by using the Boundary Layer Impedance condition[1]

$$\frac{\partial p}{\partial \mathbf{n}}(\mathbf{x}) = \left[(\gamma - 1)\frac{\mathrm{i}k^2}{k_h} - \frac{\mathrm{i}\Delta^{\|}}{k_v}\right]p(\mathbf{x}), \tag{9}$$

Using this the ROSEBEM system becomes

$$\sum_{m=0}^{M-1}\frac{(k-k_0)^m}{m!}\left(\left[\mathbf{F}_{\ell m}(k_0) + \frac{(\gamma-1)\mathrm{i}k^2}{k_h}\mathbf{H}_{\ell m}(k_0) + \frac{\mathrm{i}}{k_v}\mathbf{T}_{\ell m}(k_0)\right]\mathbf{p}_\ell + \mathbf{G}_{\ell m}(k_0)\frac{\partial \mathbf{p}_\ell}{\partial \mathbf{n}}\right) \approx \mathbf{0}. \tag{10}$$

Results



Improvements in the full formulation

The basis of the full formulation is the Kirchhoff decomposition which splits the problem into three modes[2,4]

$$\begin{aligned} &\text{Acoustic Mode:} && (\Delta + k_a^2)p_a(\mathbf{x}) = 0, && (11)\\ &\text{Thermal Mode:} && (\Delta + k_h^2)p_h(\mathbf{x}) = 0, && (12)\\ &\text{Viscous Mode:} && (\Delta + k_v^2)\mathbf{v}_v(\mathbf{x}) = \mathbf{0}, \quad \text{with} \quad \nabla\cdot\mathbf{v}_v(\mathbf{x}) = 0. && (13) \end{aligned}$$

The modes are coupled on the boundary as

$$\begin{aligned} &\text{Isothermal:} && p_a(\mathbf{x})\tau_a + p_h(\mathbf{x})\tau_h = 0, \quad \mathbf{x} \in \Gamma, && (14)\\ &\text{No-slip:} && \nabla p_a(\mathbf{x})\phi_a + \nabla p_h(\mathbf{x})\phi_h + \mathbf{v}_v(\mathbf{x}) = \mathbf{v}_{\text{boundary}}(\mathbf{x}). && (15) \end{aligned}$$

The new formulation utilizes that the gradient can be split in two as

$$\nabla p = \nabla^{\|}p + \nabla^{\perp}p. \tag{16}$$

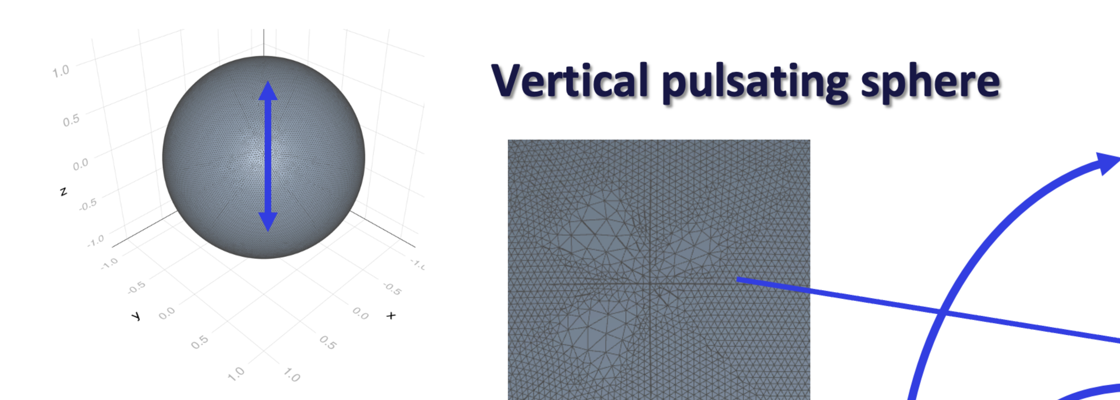
Using this it turns out that the solution to the KD-BEM can be found by solving

$$\left[\mathbf{G}_a\left(\mu_a(\mathbf{RN})^{-1}\mathbf{RD}_c + \mu_h\mathbf{G}_h^{-1}\mathbf{H}_h\right) - \phi_a\mathbf{H}_a\right]\mathbf{p}_a = \mathbf{G}_a(\mathbf{RN})^{-1}\mathbf{R}\mathbf{v}_s. \tag{17}$$

Note that in (17) every matrix except $\mathbf{G}_a$ and $\mathbf{H}_a$ are *sparse*. However, both matrices can be approximated using an acceleration method such as e.g. the Fast Multipole Method or the H-Matrix approach[4]. In the case of the $\mathbf{G}_a$-matrix the acceleration is as follows

$$\begin{aligned} \left(\underbrace{\int_\Gamma G_a(\mathbf{t}_k,\mathbf{y})\mathbf{T}(\mathbf{y})\ \mathrm{d}S_{\mathbf{y}}}_{k\text{th row of } \mathbf{G}}\right)\mathbf{z} &= \left(\sum_{j=1}^{NQ} G_a(\mathbf{t}_k,\mathbf{y}_j)\underbrace{\text{jacobian}(\mathbf{u}_j)w_j\mathbf{T}^{e(j)}(\mathbf{u}_j)\mathbf{L}^{e(j)}}_{j\text{th row of } \mathbf{C}}\right)\mathbf{z} \\ &= \left[G_a(\mathbf{t}_k,\mathbf{y}_1)\ G_a(\mathbf{t}_k,\mathbf{y}_2,)\ \dots\ G_a(\mathbf{t}_k,\mathbf{y}_{NQ})\right]\mathbf{Cz} \end{aligned} \tag{18}$$

Results



Storing the systems (solving not included)

No. DOFs	13k	39k	69k	154k	323k
10-Variable	254 GB	2323 GB	7102 GB	35394 GB	156132 GB
1-variable Dense	**6.1 GB**	**60 GB**	183 GB	909 GB	4000 GB
1-variable FMM	**0.054 GB**	**0.16 GB**	**0.29 GB**	**0.64 GB**	**1.16 GB**