Darts - Formulation

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This is a mathematical formulation of the puzzle. The statement of the puzzle is on the mipwise.com/puzzles webpage. Implementations of this formulation using multiple solvers are available in the scripts directory.

Let's start by reviewing the statement of the puzzle:
There are three people darts: Andrea, Antonio, and Luiz. They threw 6 darts each (red marks in the figure), and each scored 71 points. We also know that Andrea's first 2 darts scored 22 points and Antonio's first dart scored 3 points. We need to find out which player hit the bullseye.

Puzzle Image

Input Data

We start by defining three sets of indices.

- The first set corresponds to the **player** (Andrea, Antonio, and Luiz, respectively):
  `I = {1, 2, 3}`
- Next, we define the set of **shots** (each player threw 6 darts):
  `J = {1, 2, 3, 4, 5, 6}`
- Next, we define the set of **scores** (scoring regions of the dartboard):
  `K = {1, 2, 3, 5, 10, 20, 25, 50}`
- Finally, we define a dictionary which indicates the numbers of **marks** in each **scoring region** (as shown in the figure):
  `R = {1: 3, 2: 2, 3: 2, 5: 2, 10: 3, 20: 3, 25: 2, 50: 1}`

Decision variables

With the sets of indices defined above we can now define the decision variables. Can you think how we do this?

We want to answer is this: What is the score of each player in each shot?

So we define decision variables with three indices, and they are all binary variables:

- x_{ijk} equals 1 if player i scores k in shot j, 0 otherwise.

For example, if the variable $x_{1,2,5}$ equals one, it means that player 1 (Andrea in this case) scores 5 in her second shot. We assume that in every shot the player hits the dartboard, so they always score at least one point.

Constraints

With the definition of the decision variables above, we can now model the rules of the game with constraints.

- *Every shot hits one, and only one, scoring region*.

$$\sum_{k} x_{ijk} = 1, \ \forall i, j.$$

For example, for $i = 1$ and $j = 2$, this constraint becomes

$$x_{1,2,1} + x_{1,2,2} + x_{1,2,3} + x_{1,2,5} + x_{1,2,10} + x_{1,2,20} + x_{1,2,50} = 1,$$

which means that, out of the seven possible scores, the first player in her second shot can only hit one score. In fact, the only way for a sum of binary variables to add up to one is that one of those variables equal one and all the other variables equal zero.

- *Every player scored 71 points*.

$$\sum_{j,k} k \cdot x_{ijk} = 71, \ \forall i.$$

This constraint is saying that the total score of player i, across all shots and all possible scores in each shot, must add up to 71. Notice that we multiply x_{ijk} by k so that, if $x_{ijk} = 1$, we add k to the sum, and if $x_{ijk} = 0$, we add zero.

- *Number of marks in each scoring region*.

$$\sum_{i,j} x_{ijk} = R[k], \ \forall k.$$

This is saying that for $k = 25$, for example, there are $R[25] = 2$ darts, across all players and all shots, that hit the scoring region of 25 points. This constraint, in combination with the next two constraints, makes the solution to this puzzle to be unique.

- *Andrea's first two shots scored 22 points*.

$$\sum_{k} x_{11k} + x_{12k} = 22.$$

This constraint is identical to the constraint "*Every player scored 71 points*", except that this one is restricted to player 1 and its first two shots.
Alternatively, given that 22 points can only be achieved in two shots if one dart hits 20 and the other dart hits 2, we could simply enforce $x_{1,1,20} = 1$ and $x_{1,2,2} = 1$ (or $x_{1,1,2} = 1$ and $x_{1,2,20} = 1$).

- *Antonio's first shot scored 3 points*.

$$x_{2,1,3} = 1.$$

Objective Function

This is just a feasibility problem, since there is nothing to optimize. Hence, we can simply define a dummy objective function such as the sum of all variables. The sum of all variables may sounds like a sophisticated choice for a dummy objective function, but this can help the solver when the problem is hard to solve (although this is not the case with this baby problem).

$$\min \sum_{i,j,k} x_{ijk}.$$

Final formulation

$$\begin{aligned}
\min \quad & \textstyle\sum_{i,j,k} x_{ijk} & \\
\text{s.t.} \quad & \textstyle\sum_{k} x_{ijk} = 1, & \forall i, j \\
& \textstyle\sum_{j,k} k \cdot x_{ijk} = 71, & \forall i \\
& \textstyle\sum_{i,j} x_{ijk} = R[k], & \forall k \\
& \textstyle\sum_{k} x_{11k} + x_{12k} = 22 & \\
& x_{2,1,3} = 1 & \\
& x_{ijk} \in \{0,1\} & \forall i, j, k.
\end{aligned}$$