#2 Even-odd Sudoku - Formulation

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This is the MIP formulation of the puzzle. Statement and solution implementation of all puzzles are available from the main page of the Mip Puzzles page, which is maintained by Mip Wise.

The goal is to fill a 9x9 grid with digits from 1 to 9 in a way that the following conditions are satisfied:

- Usual rules of sudoku, i.e., in each row, column and bold region, all the digits from 1 to 9 must appear, or equivalently, there must be no repetition of any digit in each row, column and bold region;

- The given initial digits on the cells must be preserved at the final solution;

- Some specific cells must have an even/odd digit (follow this link for the puzzle statement).

Input Data

We start by defining the set of indices, which corresponds simultaneously to the available digits, and to the indices of rows and columns:

- `I = {1, 2, 3, 4, 5, 6, 7, 8, 9}`

We also need to describe the cells where we are given: an even digit; an odd digit; an initial digit:

- Even cells:
  `EC = {(2, 1), (3, 2), (3, 5), (2, 6), (2, 7), (2, 8), (3, 8), (4, 8), (5, 7), (8, 7), (8, 9)}`

- Odd cells:
  `OC = {(1, 2), (2, 3), (4, 4), (5, 3), (6, 2), (6, 6), (7, 2), (8, 2), (8, 3), (8, 4), (7, 5), (7, 8), (9, 8)}`

- Given digits:
  `GD = {(1, 6): 4, (1, 7): 6, (1, 9): 9, (2, 5): 5, (3, 4): 1, (3, 9): 7, (4, 3): 4, (4, 9): 8, (5, 2): 2, (5, 8): 9, (6, 1): 1, (6, 7): 3, (7, 1): 9, (7, 6): 8, (8, 5): 6, (9, 1): 8, (9, 3): 5, (9, 4): 7}`

Finally, we describe the bold regions, as a python dictionary:

- Bold regions:
  `BR = {`
  `1: [(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)],`
  `2: [(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5),`

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       (3, 6)],
    3: [(1, 7), (1, 8), (1, 9), (2, 7), (2, 8), (2, 9), (3, 7), (3, 8),
       (3, 9)],
    4: [(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2),
       (6, 3)],
    5: [(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5),
       (6, 6)],
    6: [(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8),
       (6, 9)],
    7: [(7, 1), (7, 2), (7, 3), (8, 1), (8, 2), (8, 3), (9, 1), (9, 2),
       (9, 3)],
    8: [(7, 4), (7, 5), (7, 6), (8, 4), (8, 5), (8, 6), (9, 4), (9, 5),
       (9, 6)],
    9: [(7, 7), (7, 8), (7, 9), (8, 7), (8, 8), (8, 9), (9, 7), (9, 8),
       (9, 9)]}
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Decision Variables

We now define the decision variables. The first attempt could be to define a set of variables $x_{ij} \in \{1, \ldots, 9\}$ for $i, j \in I$ which correspond to the digit that must enter cell (i, j). Although this formulation is possible, it gets harder to establish the constraints later (why? Try yourself).

Alternatively, we propose the following decision variables:

- x_{ijk} for $i, j, k \in I$, where x_{ijk} equals 1 when digit k enters cell (i, j), and 0 otherwise.

At a first glance, this may seems an over-complicated formulation, but it will save us a lot of effort while setting the constraints. Note that in this way, we have a bigger number of variables, but they turn out to be binaries!

Constraints

With the rules of the puzzle in mind and with the decision variables defined, we now establish the constraints precisely:

- *Each cell must have exactly one digit*:

$$\sum_k x_{ijk} = 1 \quad \forall i, j \in I.$$

Here, a pair (i, j) corresponds to one cell in the 9x9 grid, and since the variables are binaries, the sum over k above ensure that one, and exactly one, of the x_{ijk} is set to 1. Therefore, the unique k for which $x_{ijk} = 1$ represents the digit that enters cell (i, j), which guarantees the desired constraint. There would not be such constraint if we had defined the decision variables in the other way.

- *Digits can't repeat in each row*:

$$\sum_j x_{ijk} = 1 \quad \forall i, k \in I.$$

The index i corresponds to each row on the grid, while k stands for each available digit. Then, given a row i and a digit k, the sum over j guarantees the existence of one, and

exactly one, j such that $x_{ijk} = 1$, i.e., only one j for which the value k enters (i, j).

- *Digits can't repeat in each column*:

$$\sum_i x_{ijk} = 1 \quad \forall j, k \in I.$$

The index j represents each column, and k each available digit. It is similar to the previous constraint.

- *Digits can't repeat in each bold region*:

$$\sum_{(i,j) \in b} x_{ijk} = 1 \quad \forall b \in BR, \ \forall k \in I.$$

Now, for a given bold region $b \in BR$ and a given digit k, the sum over $(i, j) \in b$ above ensures that for exactly one cell (i, j) in the region b we will have $x_{ijk} = 1$, i.e., digit k in cell (i, j).

- *Some cells must have the given digits*:

$$x_{ijk} = 1, \quad \forall (i, j) \in GD \text{ and } k = GD[i, j].$$

Each $(i, j) \in GD$ is precisely one cell for which an initial digit is given by the puzzle statement. $k = GD[i, j]$ represents the given value in cell (i, j) (see Input Data section). Therefore, we enforce cell (i, j) to have the value k, i.e., set $x_{ijk} = 1$.

- *Some cells must have even digits*:

$$\sum_{k \text{ even}} x_{ijk} = 1 \quad \forall (i, j) \in EC.$$

Every $(i, j) \in EC$ represents a cell that must have an even digit, and the equality above ensures that there is exactly one even value for k such that $x_{ijk} = 1$, i.e., such that k enters cell (i, j).

- *Some cells must have odd digits*:

$$\sum_{k \text{ odd}} x_{ijk} = 1 \quad \forall (i, j) \in OC.$$

Every $(i, j) \in OC$ represents a cell that must have an odd digit, and the equality above ensures that there is exactly one odd value for k such that $x_{ijk} = 1$, i.e., such that k enters cell (i, j).

- x_{ijk} *must be binary*:

$$x_{ijk} \in \{0, 1\} \quad \forall i, j, k \in I.$$

Objective Function

There is no objective to maximize or minimize in this problem. We only need to find one feasible solution (which turns out to be unique in this case). But there is no problem if we define an objective function, and it can be anything. For instance, set

$$\max \quad x_{111}.$$

Final Formulation

$$
\begin{aligned}
\max \quad & x_{111} \\
\text{s.t.} \quad & \sum_k x_{ijk} = 1, && \forall i, j \in \mathrm{I} \\
& \sum_j x_{ijk} = 1, && \forall i, k \in \mathrm{I} \\
& \sum_i x_{ijk} = 1, && \forall j, k \in \mathrm{I} \\
& \sum_{(i,j) \in b} x_{ijk} = 1, && \forall b \in BR, \ \ \forall k \in \mathrm{I} \\
& x_{ijk} = 1, && \forall (i,j) \in GD, \ k = GD[i,j] \\
& \sum_{k \text{ even}} x_{ijk} = 1, && \forall (i,j) \in EC \\
& \sum_{k \text{ odd}} x_{ijk} = 1, && \forall (i,j) \in OC \\
& x_{ijk} \in \{0,1\}, && \forall i, j, k \in \mathrm{I}.
\end{aligned}
$$