Mathcamp 2019 | Mira Bernstein

Homework, Day 1: The case of the MCSP

Probabilistic models and machine learning | Handout #1

How biased is the MCSP?

One day you overhear two staff members in a dark corner of the Main Lounge whispering about the MCSP. Of course, you stop to eavesdrop![1]

It turns out that MCSP stands for the Mysterious Coin of Secret Probability. The MCSP looks like a regular penny, but it comes up heads with probability q and tails with probability $1-q$. No one has any idea what q is. The problem of finding q has been vexing the Mathcamp staff since the year 2000.

That night, you sneak into the inner office, determined to find out the value of q for yourself. You find a drawer containing a single penny, and decide it must be the MCSP. You toss the penny n times and get k heads. Then you run back to your room to do your Bayesian computation.

Since you have absolutely no idea what q is, you decide to assume it is equally likely to be any value between 0 and 1. In other words, your prior density function for q (before you toss the coin) is the uniform density function:

$$p(q) = \begin{cases} 1 & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Even though we are dealing with continuous random variables, you can use Bayes' rule as usual:

$$p(q|D) = \frac{p(D|q)\, p(q)}{p(D)},$$

where D is the data – the particular sequence of k heads and $n-k$ tails that you observed.

In class, we saw that

$$p(q|D) = (n+1)\binom{n}{k} q^k (1-q)^{n-k}.$$

1. Based on the data you've observed, what is the most likely value of q? (Your first guess is probably k/n, and this is correct. But can you prove it? There's a hint on the last page if you need it.)

[1]This is a fictional you. The real you would never eavesdrop on people's conversations, since it violates Rule 1.

2. Based on the data you've observed, what is the probability that if you were to toss the MCSP one more time, it would come up heads?

   Note that this time the answer is *not* k/n. For example, suppose $k = n = 1$, i.e. so far you've tossed the coin only once and it came up heads. That shouldn't lead you to conclude that the probability of it coming up heads every time is 1, right?

   Indeed, you don't know for sure that $q = k/n$: that's just the most likely value, but other values are also possible. What you want is the expected value for q under your posterior distribution for q (i.e. $p(q|D)$). Ponder what that means and see if you can figure it out! (Again, there's a hint on the last page.)

Bayesian hypothesis testing

Suddenly you are struck with doubt: what if the coin you were tossing in the inner office was not actually the MCSP? What if it was some random penny that just happened to fall into an empty drawer? Come to think of it, you actually have no idea which hypothesis is more likely:

- H_1: it was the MCSP;
- H_2: it was just a random coin.

In other words, your *prior odds*[2] for the coin being the MCSP are:

$$\frac{P(H_1)}{P(H_2)} = 1.$$

By Bayes Theorem, after observing the sequence of k heads and $n-k$ tails, your *posterior odds* for the coin being MCSP are:

$$\begin{array}{ccccc} \frac{P(H_1|D)}{P(H_2|D)} & = & \frac{P(D|H_1)}{P(D|H_2)} & \times & \frac{P(H_1)}{P(H_2)} \\ \textit{posterior odds} & = & \textit{Bayes factor} & \times & \textit{prior odds.} \end{array}$$

The ratio

$$\frac{P(D|H_1)}{P(D|H_2)}$$

is called the *Bayes factor*;[3] it tells you how much the data D affects the odds of H_1 being true. You and I may have different prior odds, but the Bayes factor is something we can both agree on; it basically says how much evidence the data D give in support of H_1. Just as there is an arbitrary agreed-upon standard for "statistical significance" in regular statistics ($p < .05$), [4] there is also a standard way of interpreting Bayes factors in Bayesian stistics.

3. Compute the Bayes factor for H_1 (MCSP) vs H_2 (regular coin) after observing a sequence of k heads and $n-k$ tails.

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[2] If an outcome has probability p, then the *odds* for that outcome are $\frac{p}{1-p}$. As you're about to see, it is often more convenient to work with odds rather than probabilities. But of course, you can always figure out one from the other.

[3] Notice that the denominator $P(D)$ from Bayes theorem cancels! That's why we like working with odds.

[4] In regular (frequentist) statistics, in order to decide between the "null hypothesis" (H_0) and the "alternative hypothesis" (H_1), we compute the *p-value*:

$$p = P(\text{getting data more extreme than } D \mid H_0).$$

If $p < .05$, we "reject H_0 in favor of H_1".

This should strike you as really weird: you're computing the probability of data that you didn't actually get, and H_1 itself doesn't even appear in the calculation! That's why more and more scientists these days are switching from p-values to Bayes factors.

4. Here is a statistical statement that appeared in *The Guardian* on Friday January 4, 2002 (soon after the Euro was first introduced):

   > *When spun on edge 250 times, a Belgian one-euro coin came up heads 140 times and tails 110. "It looks very suspicious to me", said Barry Blight, a statistics lecturer at the London School of Economics. "If the coin were unbiased, the chance of getting a result as extreme as that would be less than 7%."*[5]

   In most scientific fields, $p < .07$ is not considered enough for statistical significance. But if the coin had come out heads 141 times out of 250, the p-value would have been less than .05, and Dr. Blight would have confidently declared that the coin was biased.

   Evaluate the Bayes factor for $n = 250$ and $k = 140$. (Easy to do on Wolfram Alpha.) Also try it for $k = 141$. What conclusion would you draw from these data about the probability of H_1 vs H_2?

Hint for Problem 1: Use logarithmic differentiation.

Hint for Problem 2: The quantity you want is

$$\int_0^1 q \cdot p(q|D)dq.$$

Ask me if you don't understand why!

Hint for Problem 3: Convince yourself that $P(D|H_1) = \int_0^1 P(D|q)dq$.

[5]This example comes from the wonderful book *Information Theory, Inference, and Learning Algorithms*, by David MacKay. The book is freely available online and I cannot recommend it highly enough!