Partial correlation

2. $$r_{y.12\cdots k} = \frac{Cov(e_1, e_2)}{\sigma_1 \cdot \sigma_2}$$
$$= \frac{E(e_1 e_2)}{\sigma_1 \cdot \sigma_2}$$
$$= \frac{E[(B_1 e_2 + \varepsilon) e_2]}{\sigma_1 \cdot \sigma_2}$$
$$= \frac{B_1 E(e_2^2) + E(\varepsilon e_2)}{\sigma_1 \cdot \sigma_2}$$

Since $E(e_2^2) = \sigma_2^2 \neq 0$, so B_1 must be 0.

Mathematical exercises

Prove

1. $$\Sigma \hat{y}_i \hat{e}_i = \Sigma(A + B\hat{x}_i)\hat{e}_i$$
$$= A\Sigma\hat{e}_i + B\Sigma\hat{x}_i\hat{e}_i$$
$$= A\Sigma\hat{e}_i + B\Sigma\hat{x}_i(\hat{y}_i - A - B\hat{x}_i)$$
$$= A\Sigma\hat{e}_i + B(\Sigma\hat{x}_i\hat{y}_i - A\Sigma\hat{x}_i - B^2\Sigma\hat{x}_i^2)$$

use normal equation $A\Sigma x_i + B\Sigma x_i^2 = \Sigma x_i y_i$ and $\Sigma \hat{e}_i = 0$

$$\Rightarrow \Sigma \hat{y}_i \hat{e}_i = A \cdot 0 + B \cdot 0 = 0$$

2. $$\Sigma(y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) = \Sigma\hat{e}_i(\hat{y}_i - \bar{y})$$
$$= \Sigma\hat{e}_i\hat{y}_i - \bar{y}\Sigma\hat{e}_i$$

Since $\Sigma\hat{y}_i\hat{e}_i = 0$ (proved above) and $\Sigma\hat{e}_i = 0$

$$\Rightarrow \Sigma \hat{e}_i(\hat{y}_i - \bar{y}) = 0 - 0 = 0$$

1. Based on normal equations for simple regression,

$$An + B\Sigma x_i = \Sigma y_i$$

$$A\Sigma x_i + B\Sigma x_i^2 = \Sigma x_i y_i$$

we get $\beta_{y|x} = B = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$, $\alpha_{y|x} = A = \bar{y} - \beta_{y|x}\bar{x}$

Similarly, $\beta_{x|y} = \frac{n\Sigma x_i y_i - \Sigma y_i \Sigma x_i}{n\Sigma y_i^2 - (\Sigma y_i)^2}$, $\alpha_{x|y} = \bar{x} - \beta_{x|y} \cdot \bar{y}$

given $sd(y) = sd(x)$,

we get $\frac{\Sigma(y_i - \bar{y})^2}{n-1} = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$

$$\Sigma x_i^2 = \Sigma y_i^2, \Sigma x_i = \Sigma y_i$$

so, $\beta_{y|x} = \beta_{x|y}$.

since $\bar{y} = \bar{x}$, we get $\alpha_{y|x} = \alpha_{x|y}$.

$$r_{xy} = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{\frac{1}{n-1}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n-1}\sqrt{\Sigma(x_i - \bar{x})^2} \cdot \sqrt{\Sigma(y_i - \bar{y})^2}}$$

$$= \frac{\Sigma x_i y_i - \bar{y}\Sigma x_i - \bar{x}\Sigma y_i + n\bar{x}\cdot\bar{y}}{\Sigma x_i^2 - 2\bar{x}\cdot\Sigma x_i + n\bar{x}^2}$$

let $\bar{x} = \bar{y} = a$, $\Sigma x_i = \Sigma y_i = na$

$$r_{xy} = \frac{\Sigma x_i y_i - 2na^2 + na^2}{\Sigma x_i^2 - 2na^2 + na^2} = \frac{\Sigma x_i y_i - na^2}{\Sigma x_i^2 - na^2}$$

$$\beta_{y|x} = \beta_{x|y} = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} = \frac{n\Sigma x_i y_i - n^2a^2}{n\Sigma x_i^2 - n^2a^2} = \frac{\Sigma x_i y_i - na^2}{\Sigma x_i^2 - na^2}$$

$$\Rightarrow \beta_{y|x} = \beta_{x|y} = r_{xy}$$

2. $Y = \alpha_{y|x} + \beta_{y|x} \cdot X = r_{xy} + r_{xy} \cdot X$

$$X = \alpha_{x|y} + \beta_{x|y} \cdot Y$$

$$\Rightarrow Y = \frac{1}{\beta_{x|y}}(X - \alpha_{x|y})$$

$$= \frac{1}{r_{xy}}X - \frac{1}{r_{xy}}\alpha_{x|y}$$

since $r_{xy} < 1$, so $r_{xy} \neq \frac{1}{r_{xy}}$

so the regression of y on x different from the line for the regression of x on y.