MAT426: Advanced Calculus - Numerical Sequences and Series

Convergent Sequences [1]

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Convergent Sequences

3.1 Definition

A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \in X$ with the following property:

For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$.

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- If $\{p_n\}$ does not converge, it is said to *diverge.*
- Convergent sequence depends not only on $\{p_n\}$ but also on X
  Ex: Sequence $1/n$ in $\mathbb{R}^1$ and in set of all positive real numbers with $d(x, y) = |x - y|$

Convergent Sequences

- The set of all points p_n $(n = 1, 2, 3, \dots)$ is the range of $\{p_n\}$.
- The range of a sequence may be finite set, or may be infinite.
- The sequence $\{p_n\}$ is said to be bounded if its range is bounded.

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- If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.
- If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

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- $\lim_{n\to\infty} cs_n = cs$ for any number c.
- $\lim_{n\to\infty}(c \pm s_n) = c \pm s$ for any number c.
- $\lim_{n\to\infty}(s_n t_n) = st$
- $\lim_{n\to\infty}(1/t_n) = 1/t$ provided $t_n \neq 0$ and $t \neq 0$.

3.4 Theorem

- Suppose $\mathbf{x}_n \in \mathbb{R}^k$ $(n = 1, 2, 3, \dots)$ and

$$\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n}).$$

Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \dots, \alpha_k)$ if and only if $\lim_{n\to\infty} \alpha_{i,n} = \alpha_i$ for $i = 1, 2, \dots, k$.

Subsequences

Definition

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Given a sequnce $\{p_n\}$, consider a sequence n_k of positive integers, such that $n_1 < n_2 < n_3 \ldots$. Then the sequence $\{p_{n_l}\}$ is called a subseqece of $\{p_n\}$. If $\{p_{n_l}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

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It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

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- If $\{p_n\}$ is a sequence in a metric space X, then some subsequence of p_n converges to a point of X.
- Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequences.

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3.7 Theorem

The subsequential limit of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

References

Walter Rudin.
Principles of Mathematical Analysis.
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