

# MAT426: Advanced Calculus

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04/01/2025

## 2.38 Theorem

### Theorem

If  $\{I_n\}$  is a sequence of intervals in  $\mathbb{R}^1$ , such that  $I_n \supset I_{n+1}$  ( $n = 1, 2, 3, \dots$ ), then  $\bigcap_{n=1}^{\infty} I_n$  is not empty.

Proof:

## 2.39 Theorem

### Theorem

Let  $k$  be a positive integer. If  $\{I_n\}$  is a sequence of  $k$ -cells such that  $I_n \supset I_{n+1}$  ( $n = 1, 2, 3, \dots$ ), then  $\bigcap_{i=1}^{\infty} I_n$  is not empty.

Proof:

# Theorem

## Theorem

Every  $k$ -cell is compact.

Proof:

## 2.41 Theorem

### Theorem

If a set  $E$  in  $\mathbb{R}^k$  has one of the following three properties, then it has the other two:

- (a)  $E$  is closed and bounded.
- (b)  $E$  is compact.
- (c) Every infinite subset of  $E$  has a limit point in  $E$ .

Proof:

## 2.42 Theorem

### Theorem - Weierstrass

Every bounded infinite subset of  $\mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .

# Perfect Sets

## 2.43 Theorem

Let  $P$  be a non-empty perfect set in  $\mathbb{R}^k$ . Then  $P$  is uncountable.



# Corollary

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Every interval  $[a, b]$  is uncountable. In particular, the set of all real numbers is uncountable.

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Remove the middle thirds of these intervals, and let  $E_2$  be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right]$$

# The Cantor set

Continuing in this way, we obtain a sequence of compact sets  $E_n$ , such that

1.  $E_1 \supset E_2 \supset E_3 \supset \dots$
2.  $E_n$  is the union of  $2^n$  intervals, each of length  $3^{-n}$ .

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the **Cantor set**.