MAT426: Advanced Calculus - Numerical Sequences and Series

Convergent Sequences

Miraj Samarakkody

Tougaloo College

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Convergent Sequences

3.1 Definition

A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \in X$ with the following property:

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- Convergent sequence depends not only on $\{p_n\}$ but also on X
  Ex: Sequence $1/n$ in $\mathbb{R}^1$ and in set of all positive real numbers with $d(x, y) = |x - y|$

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- The set of all points p_n $(n = 1, 2, 3, \dots)$ is the range of $\{p_n\}$.
- The range of a sequence may be finite set, or may be infinite.
- The sequence $\{p_n\}$ is said to be bounded if its range is bounded.

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- If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.