MAT426: Advanced Calculus

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2.41 Theorem

Theorem

If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof:

2.42 Theorem

Theorem - Weierstrass

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Perfect Sets

2.43 Theorem

Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Corollary

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Every interval $[a, b]$ is uncountable. In particluar, the set of all real numbers is uncountable.

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Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right]$$

The Cantor set

Continuing in this way, we obtain a sequence of compact sets E_n, such that

1. $E_1 \supset E_2 \supset E_3 \supset \dots$
2. E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \cap_{n=1}^{\infty} E_n$$

is called the **Cantor set**.

Connected Sets

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2.45 Definition

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Example

The segment $[0, 1]$ and the segment $(1, 2)$.

2.47 Theorem

A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property:

$$\text{If } x \in E, y \in E, \text{ and } x < z < y, \text{ then } z \in E.$$