

MAT426: Advanced Calculus

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Numerical Sequences and Series

Convergent Sequences

3.1 Definition

A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \in X$ with the following property:

For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$.

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- ▶ If $\{p_n\}$ does not converge, it is said to *diverge*.
- ▶ Convergent sequence depends not only on $\{p_n\}$ but also on X
Ex: Sequence $1/n$ in \mathbb{R}^1 and in set of all positive real numbers with $d(x, y) = |x - y|$

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- ▶ The set of all points p_n ($n = 1, 2, 3, \dots$) is the range of $\{p_n\}$.
- ▶ The range of a sequence may be finite set, or may be infinite.
- ▶ The sequence $\{p_n\}$ is said to be bounded if its range is bounded.

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- ▶ If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- ▶ If $E \subset X$ and if p is a limit point of E , then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n \rightarrow \infty} p_n$.