MT222: Calculus II

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7.3 - Trigonometric Substitution

Table of Trigonometric Substitiution

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Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 5

Evaluate

$$\int \frac{dx}{\sqrt{x^2 - a^2}},$$

where $a > 0$. (Use Hyperbolic functions)

Example 6

Find

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

Example 7

Evaluate

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

Integration of Rational Functions by Partial Fractions

Motivation

In this section, we show how to integrate any rational function by expressing it as a sum of simpler fractions, called *partial fraction*, that we already know how to integrate.

Example 1

Find

$$\int \frac{x^3 + x}{x - 1} \, dx$$

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Next we study some different cases.

CASE I: The denominator $Q(x)$ is a product of distinct linear factors.

We can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k),$$

where no factor is repeated.

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In this case the partial fraction theorem states that there exist constants $A_1, A_2, A_3, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}.$$

Example 2

Write the partial fraction decomposition of

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}$$

Example 2

Use and alternative method to write the partial fraction decomposition of

$$\frac{x^2+2x-1}{2x^3+3x^2-2x}$$

Example 2

Find

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

Example 3

Find

$$\int \frac{x^2 - a^2}{dx},$$

where $a \neq 0$.

CASE II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1 x + b_1)$ is repeated r times; that is, $(a_1 x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\dfrac{A_1}{(a_1 x + b)}$, we would write

$$\frac{A_1}{(a_1 x + b)} + \frac{A_2}{(a_1 x + b)^2} + \cdots + \frac{A_r}{(a_1 x + b)^r}$$

Example 4

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\ dx.$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

Example 5

Evaluate

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\ dx$$

Example 6

Evaluate

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \, dx$$

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$.

Example 7

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example 8

Evaluate

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\, dx$$

Rationalizing Substitutions

Some non-rational functions can be changed into rational functions by means of appropriate substitutions.

Example 9

Evaluate

$$\int \frac{\sqrt{x+4}}{x}\,dx.$$