MAT222: Calculus II - Technique of Integration

Integration of Rational Functions by Partial Fractions

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Integration of Rational Functions by Partial Fractions

CASE I: The denominator $Q(x)$ is a product of distinct linear factors.

We can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \dots (a_k x + b_k),$$

where no factor is repeated.

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In this case the partial fraction theorem states that there exist constants $A_1, A_2, A_3, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}.$$

Example 2

Write the partial fraction decomposition of

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}$$

Example 2

Use and alternative method to write the partial fraction decomposition of

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}$$

Example 2

Find

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

Example 3

Find

$$\int \frac{dx}{x^2 - a^2},$$

where $a \neq 0$.

CASE II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\dfrac{A_1}{(a_1x + b)}$, we would write

$$\frac{A_1}{(a_1x + b)} + \frac{A_2}{(a_1x + b)^2} + \cdots + \frac{A_r}{(a_1x + b)^r}$$

Example 4

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

Example 5

Evaluate

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} \, dx$$

Example 6

Evaluate

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx$$

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$.

Example 7

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example 8

Evaluate

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$$

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Evaluate

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$$

$$= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

Rationalizing Substitutions

Some non-rational functions can be changed into rational functions by means of appropriate substitutions.

Example 9

Evaluate

$$\int \frac{\sqrt{x+4}}{x}\,dx.$$