MAT222: Calculus II - Technique of Integration

7.8 Improper Integrals

Miraj Samarakkody

Tougaloo College

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Introduction

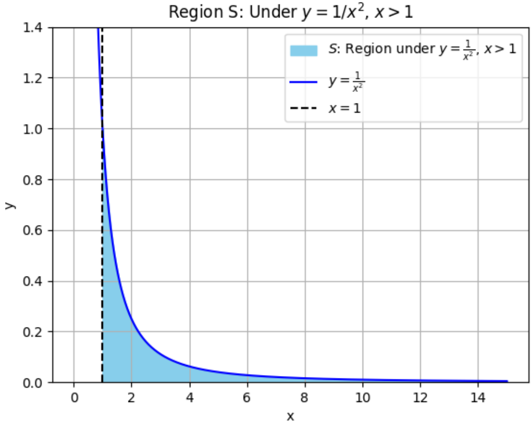
Definition

A **improper integral** is an integral that has one of the following properties:

- The interval of integration is infinite.
- The integrand is infinite at one or more points in the interval of integration.

Type I - Infinite Intervals

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the $x-$axis, and the right of the line $x = 1$.



Example

Evaluate the integral

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

Definition of an Improper Integral of Type I

- If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

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- If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent, the we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

Example

Determine whether the integral $\int_1^\infty (1/x)dx$ is convergent or divergent.

Example 2

Evaluate
$$\int_{-\infty}^{0} xe^{x}\ dx.$$

Example 3

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Example 4

For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Remark

$\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.