MAT434: Theory of Mathematical Statistics

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03/24/2025

The Normal Distribution

The density function of the normal distribution can be written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

2.3 Functions of a Random Variable

Function of a Random Variable

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- Assume that we need to find the density function of $Y = g(X)$.
- To illustrate techniques for solving such a problem, we first develop some useful facts about the normal distribution.

Function of a Random variable

Suppose $X \sim N(\mu, \sigma^2)$ and that $Y = aX + b$, where $a > 0$. The commulative distribution function of Y is

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

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Upto this point, we haven't used the assumption of normality. So this result holds for a general continuous random varaible, provided that F_X is appropriately differentiable.

Function of a Random Varaible

If f_X is a normal density function with parameters μ and σ

$$f_Y(y) = \frac{1}{a\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2\right]$$

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The case $a < 0$ can be analyzed similarly.

Proposition A

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If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $Y \sim N(a\mu + b, a^2\sigma^2)$

Applications of Proposition A

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- We can see that Z follows a standard normal distribution.
- Also we can see that probabilities for general normal random variables can be evaluated in terms of probabilities fro standard normal random variables.

Example A

Scores on a certain standardized test, IQ scorse, are approximately normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Here we are referring to the distribution of scores over a very large population, and we approximate that discrete cumulative distribution function by a normal continuous commulative distribution function. An individual is selected at random. What is the probability that his score X satisfies $120 < X < 130$?

Example B

Let $X \sim N(\mu, \sigma^2)$, and find the probability that X is less than σ away from μ; that is, find $P(|X - \mu| < \sigma)$.

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Thus, a normal random variable is within 1 standard deviation of its mean with probability 0.68

Example C

Find the density of $X = Z^2$, where $Z \sim N(0, 1)$.