MAT434: Theory of Mathematical Statistics

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2.3 Functions of a Random Variable

Proposition A

Proposition A

If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $Y \sim N(a\mu + b, a^2\sigma^2)$

Applications of Proposition A

Consider the random variable

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- We can see that Z follows a standard normal distribution.
- Also we can see that probabilities for general normal random variables can be evaluated in terms of probabilities fro standard normal random variables.

Example B

Let $X \sim N(\mu, \sigma^2)$, and find the probability that X is less than σ away from μ; that is, find $P(|X - \mu| < \sigma)$.

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Thus, a normal random variable is within 1 standard deviation of its mean with probability 0.68

Example C

Find the density of $X = Z^2$, where $Z \sim N(0, 1)$.

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The density function is same as the gamma density with $\alpha = \lambda = \dfrac{1}{2}$. ($\Gamma(1/2) = \sqrt{\pi}$).
This density is also called the **chi-square density** with 1 degree of freedom.

Example D

Let U be a uniform random variable on $[0, 1]$ and let $V = 1/U$. Find the density of V.

Proposition B

Proposition

Let X be a continous random variable with density $f(x)$ and let $Y = g(X)$ where g is a differentiable, strictly monotonic function on some interval I. Suppose that $f(x) = 0$ if x is not in I. Then Y has the density function

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for y such that $y = g(x)$ for some x, and $f_Y(y) = 0$ if $y \neq g(x)$ for any x in I. Here g^{-1} is the inverse function of g; that is, $g^{-1}(y) = x$ if $y = g(x)$.

Proposition C

Proposition

Let $Z = F(X)$; then Z has a uniform distribution of $[0, 1]$.

Proof:

Proposition D

Proposition

Let U be uniform on $[0, 1]$, and let $X = F^{-1}(U)$. Then the cdf of X is F.

Proof: