MAT434: Theory of Mathematical Statistics

Miraj Samarakkody

Tougaloo College

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Discrete Random Variables

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Their **joint frequency function** $p(x, y)$ is

$$p(x_i, y_j) = P(X = x_i, Y = y_j)$$

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The p_Y is called the **marginal frequency function** of Y.

Marginal Frequency Function

Let X and Y are discrete random variables defined on the same sample space. The marginal function of X can be written as

$$p_X(x) = \sum_i p(x, y_i)$$

In the similary way, we can write the marginal frequency function for Y.

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- Let N_i be the total number of outcomes of type i in the n trials, $i = 1, \ldots, r$.
- We observe that any particular sequence of trials giving rise to $N_1 = n_1, N_2 = n_2, \ldots, N_r = n_r$ occurs with probability $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$

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- There are $\dfrac{n!}{n_1! n_2! \ldots n_r!}$ such sequences.
- Thus the joint frequency function is

$$p(n_1, n_2, \ldots, n_r) = \binom{n}{n_1 n_2 \ldots n_r} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$$

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- Note that N_i can be interpreted as the number of successes in n trials, each of which has probability p_i of success and $1 - p_i$ of failure.
- Therefore, N_i is a binomial random variable, and

$$p_{N_i}(n_i) = \binom{n}{n_i} p_i^{n_i}(1 - p_i)^{n-n_i}$$

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- The density function $f(x, y)$ is non-negative and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dy\ dx = 1$.
- For two dimensional set A

$$P((X, Y) \in A) = \iint_A f(x, y)\ dy\ dx$$

Continuous Random Variable

- In particular, if $A = \{(X, Y) | X \leq x \text{ and } Y \leq y\}$,

$$F(x, y) \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\ dv\ du$$

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- From the fundamental theorem of multivariable calculus,

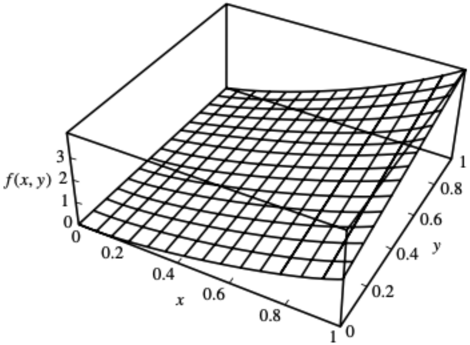
$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

Example A

Consider the bivariate density function

$$f(x,y) = \frac{12}{7}(x^2 + xy),\ 0 \le x \le 1,\ 0 \le y \le 1.$$

Calculate $P(X > Y)$.



Continuous Random Variables

The **marginal cdf** of X, of F_X, is

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \lim_{y \to \infty} F(x, y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u, y)\ dy\ du \end{aligned}$$

The marginal density of X is

$$f_X(x) = F_X'(x) = \int_{-\infty}^{\infty} f(x, y)\ dy$$

Example B

Consider the bivariate density function

$$f(x, y) = \frac{12}{7}(x^2 + xy),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1.$$

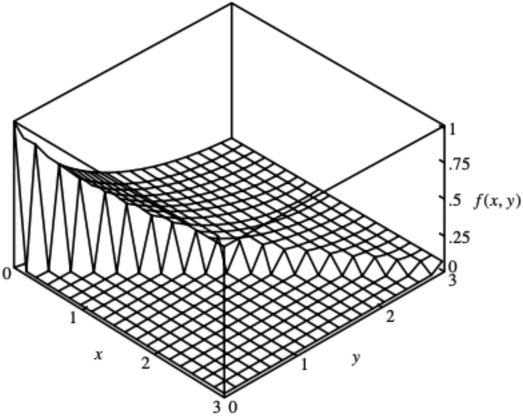
Find the marginal density of X.

Example D

Consider the following joint density:

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y, \ \lambda > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the marginal densities.



Continuous Random Variables

- In some applications, it is useful to analyze distributions that are uniform over some region space.
- For example, in the plane, the random point (X, Y) is uniform over a region R, if for any $A \subset R$,

$$P((X, Y) \in A) = \frac{|A|}{|R|}$$

Example E

A point is chosen randomly in a disk of radius 1. Since the area of the disk is π,

$$f(x, y) = \begin{cases} \dfrac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $F_R(r)$, $f_R(r)$ $f_X(x)$ and $f_Y(y)$.