

MAT414 - Modern Algebra

Isomorphisms [1]

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Motivation

Counting objects.....

Group Isomorphism

Definition

An *Isomorphism* ϕ from a group G to a group \overline{G} is a bijective mapping such that:

$$\phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in G$$

If there is an isomorphism from G to \overline{G} , we say that G and \overline{G} are *isomorphic* and write $G \cong \overline{G}$.

Example

G Operation	\overline{G} Operation	Operation Preservation
\cdot	\cdot	$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$
\cdot	$+$	$\phi(a \cdot b) = \phi(a) + \phi(b)$
$+$	\cdot	$\phi(a + b) = \phi(a) \cdot \phi(b)$
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3. “Surjective.” Show that ϕ is surjective (onto). That is, show that for every $b \in \overline{G}$, there exists an $a \in G$ such that $\phi(a) = b$.
4. “O.P.” Prove ϕ is operation-preserving. That is, show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Example 1

Let G be the real numbers under addition, and let \overline{G} be the positive real numbers under multiplication. Prove that the G and the \overline{G} isomorphic under the mapping $\phi(x) = 2^x$.

Example 2

Any infinite cyclic group is isomorphic to \mathbb{Z} .

Example 3

Prove that the mapping from \mathbb{R} under addition to itself given by $\phi(x) = x^3$ is not an isomorphism.

Example 4

Prove that $U(10) \cong \mathbb{Z}_4$ and $U(5) \cong \mathbb{Z}_4$.

Example 5

Prove that there is no isomorphism from \mathbb{Q} , the group of rational numbers under addition, to \mathbb{Q}^* , the group of non-zero rational numbers under multiplication.

Example 6

Let $G = SL(2, \mathbb{R})$, the group of 2×2 real matrices with determinant 1. Let M be any 2×2 real matrix with determinant 1.

References



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