MAT414 - Modern Algebra - Permutation Groups

Cycle Notation [1]

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Cycle Notation

Write the followings in the cyclic notations:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{bmatrix}$$

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Find $\alpha\beta$.

Properties of Permutations

Theorem 5.1 - Products of Disjoint Cycles

Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

Theorem 5.2

Disjoint Cycles Commute

If the pair of cycles $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

Theorem 5.3

Order of a Permutation

The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

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Find

- $|(132)(45)|$
- $|(1432)(56)|$
- $|(123)(456)(78)|$
- $|(123)(145)|$

Example 5

Determine the orders of the elements of S_7.

Example 6

Determine the number of elements in S_7 of order 12.

Example 7

Determine the number of elements in S_7 of order 3.

Theorem 5.4

Product of 2-Cycles

Every permutation of in S_n, $n > 1$, is a product of $2-$cycles.

Example

$$\begin{aligned}(1\ 2\ 3\ 4\ 5) &= \\ (1\ 6\ 3\ 2)(4\ 5\ 7) &= \end{aligned}$$

Lemma

In S_n, if $\epsilon = \beta_1\beta_2\beta_3\ldots\beta_r$, where the β_i's are 2-cycles, then r is even.

Theorem 5.5

Always Even or Always Odd

If a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2−cycles must have an an even (odd) number of 2−cycles.

In symbols, if

$$\alpha = \beta_1\beta_2 \ldots \beta_r \text{ and } \alpha = \gamma_1\gamma_2 \ldots \gamma_s,$$

where the β's and γ's are 2−cycles, then r and s are both even or both odd.

Even and Odd Permutations

Definition

A permutation that can be expressed as a product of an even number of $2-$cycles is called an **even permutation**. A permutation that can be expressed as a product of an odd number of $2-$cycles is called an **odd permutation**.

Even Permutations Form a Group

Theorem 5.6

The set of all even permutations of S_n is a subgroup of S_n and is denoted by A_n.

Alternating Group of Degree n

Definition

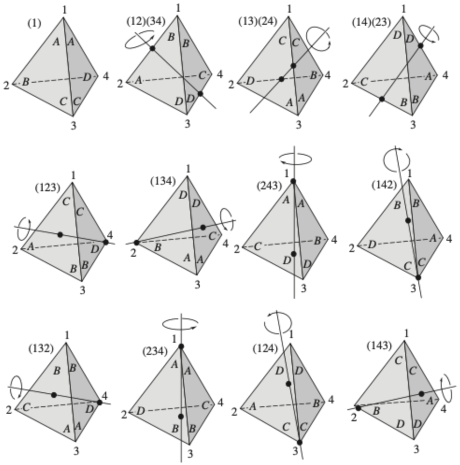
The alternating group of degree n, denoted A_n, is the set of all even permutations of S_n.

Theorem

Theorem 5.7
For $n > 1$, A_n has order $n!/2$.

Example 10 - Rotations of a Tetrahedron

The 12 rotations of a regular tetrahedron can be conveniently described with the elements of A_4.



Example 10 - Rotations of a Tetrahedron

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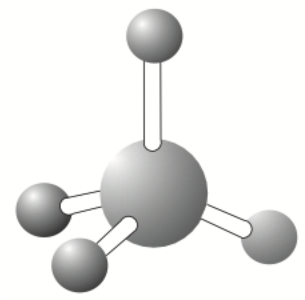
Table 5.1 The Alternating Group A_4 of Even Permutations of {1, 2, 3, 4}

(In this table, the permutations of A_4 are designated as $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ and an entry k inside the table represents α_k. For example, $\alpha_3 \alpha_8 = \alpha_6$.)

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$(1) = \alpha_1$	1	2	3	4	5	6	7	8	9	10	11	12
$(12)(34) = \alpha_2$	2	1	4	3	6	5	8	7	10	9	12	11
$(13)(24) = \alpha_3$	3	4	1	2	7	8	5	6	11	12	9	10
$(14)(23) = \alpha_4$	4	3	2	1	8	7	6	5	12	11	10	9
$(123) = \alpha_5$	5	8	6	7	9	12	10	11	1	4	2	3
$(243) = \alpha_6$	6	7	5	8	10	11	9	12	2	3	1	4
$(142) = \alpha_7$	7	6	8	5	11	10	12	9	3	2	4	1
$(134) = \alpha_8$	8	5	7	6	12	9	11	10	4	1	3	2
$(132) = \alpha_9$	9	11	12	10	1	3	4	2	5	7	8	6
$(143) = \alpha_{10}$	10	12	11	9	2	4	3	1	6	8	7	5
$(234) = \alpha_{11}$	11	9	10	12	3	1	2	4	7	5	6	8
$(124) = \alpha_{12}$	12	10	9	11	4	2	1	3	8	6	5	7

Applications

Many molecules with chemical formulas of the form AB_4, such as methane (CH_4) and carbon tetrachloride (CCl_4), have A_4 as thier symmetry group.



References

Joseph A. Gallian.
Contemporary Abstract Algebra.
Cengage Learning, 9th edition, 2017.