MAT414 - Modern Algebra

Isomorphisms [1]

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Motivation

Counting objects.....

Group Isomorphism

Definition

An *Isomorphism* ϕ from a group G to a group $\overline{G}$ is a bijective mapping such that:

$$\phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in G$$

If there is an isomorphism from G to $\overline{G}$, we say that G and $\overline{G}$ are *isomorphic* and write $G \cong \overline{G}$.

Example

G Operation	$\overline{G}$ Operation	Operation Preservation
$\cdot$	$\cdot$	$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$
$\cdot$	$+$	$\phi(a \cdot b) = \phi(a) + \phi(b)$
$+$	$\cdot$	$\phi(a + b) = \phi(a) \cdot \phi(b)$
$+$	$+$	$\phi(a + b) = \phi(a) + \phi(b)$

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3. "Surjective." Show that ϕ is surjective (onto). That is, show that for every $b \in \overline{G}$, there exists an $a \in G$ such that $\phi(a) = b$.
4. "O.P." Prove ϕ is operation-preserving. That is, show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Example 1

Let G be the real numbers under addition, and let $\overline{G}$ be the positive real numbers under multiplication. Prove that the G and the $\overline{G}$ isomorphic under the mapping $\phi(x) = 2^x$.

Example 2

Any infinte cyclic group is isomorphic to $\mathbb{Z}$.

Example 3

Prove that the mapping from $\mathbb{R}$ under addition to itself given by $\phi(x) = x^3$ is not an isomorphism.

Example 4

Prove that $U(10) \cong \mathbb{Z}_4$ and $U(5) \cong \mathbb{Z}_4$.

Example 5

Prove that there is no isomorphism from $\mathbb{Q}$, the group of rational numbers under addition, to $\mathbb{Q}^*$, the group of non-zero rational numbers under multiplication.

Example 6

Let $G = SL(2, \mathbb{R})$, the group of 2×2 real matrices with deteminant 1. Let M be any 2×2 real matrix with determinant 1. Define a mapping from G to G itself by $\phi_M(A) = MAM^{-1}$ for all $A \in G$. Prove that ϕ_M is an isomorphism.

References

Joseph A. Gallian.
Contemporary Abstract Algebra.
Cengage Learning, 9th edition, 2017.