# Advanced Calculus

## Tougaloo College

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# Basic Topology

# Finite, Countable, and Uncountable Sets

**Definition 1.**

For any positive integer $n$, let $J_n$ be the set whose elements are the integers $1, 2, 3, \ldots, n$.

Let $J$ be the set consisting of all positive integers.

For any set $A$, we say,

1. $A$ is finite, if $A \sim J_n$.
2. $A$ is infinite if $A$ is not finite. (or $A$ is infinite if $A$ is equivalent to one of its proper subsets.)
3. $A$ is countable if $A \sim J$.
4. $A$ is uncountable if $A$ is neither finite nor countable.
5. $A$ is at most countable if $A$ is finite or countable.

# Finite, Countable, and Uncountable Sets (ii)

**Example:** Let $A$ be the set of all integers (of course not finite). Prove that $A$ is countable.

# Finite, Countable, and Uncountable Sets (iii)

**Definition 2:** Sequence.

A sequence is a function $f$ defined on the set of all positive integers. If $f(n) = x_n$, for $n \in \mathbb{Z}^+$ denote the sequence $f$ by the symbol $\{x_n\}$. The values of $f$ (the elements $x_n$), are called the **terms** of the sequence.

If $A$ is a set and if $x_n \in A$ for all $n \in \mathbb{Z}^+$, then $\{x_n\}$ is said to be a sequence in $A$.

Examples:

$$\{2, 4, 6, 8, \ldots\}$$

$$\{1, 1, 1, \ldots\}$$

# Finite, Countable, and Uncountable Sets (iv)

**Theorem 1.**

Every infinite subset of a countable set $A$ is countable.

**Definition 3.**

Let $A$ and $\Omega$ be sets, and suppose that with each element $\alpha$ of $A$ there is associated a subset $E_\alpha$ of $\Omega$.

The set whose elements are the sets $E_\alpha$ is denoted by $\{E_\alpha\}$.
This is set of sets.

The union of the sets $E_\alpha$ is defined to be the set $S$ such that $x \in S$ if and only if $x \in E_\alpha$ for at least on $\alpha \in A$.

$$S = \cup_{\alpha \in A} E_\alpha$$

If $A$ consists of the integers $1, 2, \ldots, n$

$$S = \cup_{m=1}^{n} E_m = E_1 \cup E_2 \cup E_n$$

# Finite, Countable, and Uncountable Sets (v)

If $A$ is the set of all positive integers,

$$S = \cup_{m=1}^{\infty} E_m$$

The intersection of the sets $E_\alpha$ is the set $P$ such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$.

$$P = \cap_{\alpha \in A} E_\alpha$$

or

$$P = \cap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \ldots \cap E_m$$

or

$$P = \cap_{m=1}^{\infty} E_m$$

If $A \cap B$ is empty, we say that $A$ and $B$ are disjoint.

# Finite, Countable, and Uncountable Sets (vi)

**Example:**

Suppose

$$E_1 = \{1, 2, 3\}$$

and

$$E_2 = \{2, 3, 4\}$$

1. What is $E_1 \cup E_2$?
2. What is $E_1 \cap E_2$?

# Finite, Countable, and Uncountable Sets (vii)

**Example:**

Let $A$ be the set of real numbers $x$ such that $0 < x \leq 1$. For every $x \in A$, let $E_x$ be the set of real numbers $y$ such that $0 < y < x$. Prove that

1. $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$.
2. $\cup_{x \in A} E_x = E_1$
3. $\cap_{x \in A} E_x$ is empty.