Advanced Calculus

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Basic Topology

Finite, Countable, and Uncountable Sets

Definition 1.

For any positive integer n, let J_n be the set whose elements are the integers $1, 2, 3, \ldots, n$.

Let J be the set consisting of all positive integers.

For any set A, we say,

1. A is finite, if $A \sim J_n$.
2. A is infinite if A is not finite. (or A is infinite if A is equivalent to one of its proper subsets.)
3. A is countable if $A \sim J$.
4. A is uncountable if A is neither finite nor countable.
5. A is at most countable if A is finite or countable.

Finite, Countable, and Uncountable Sets (ii)

Example: Let A be the set of all integers (of course not finite). Prove that A is countable.

Goal: $J \sim A$
We can define a piecewise function $f : J \to A$.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ even} \\ -\frac{n-1}{2} & \text{if} \quad n \text{ odd} \end{cases}$$

f is 1-1.
for any $n_1 \neq n_2$, $f(n_1) \neq f(n_2)$.

f is onto.
This is trivial because for any $y \in A$, you have a $n \in J$.

Therefore we have 1-1 correspondence. By the definition, A is countable.

Finite, Countable, and Uncountable Sets (iii)

Definition 2: Sequence.

A sequence is a function f defined on the set of all positive integers. If $f(n) = x_n$, for $n \in \mathbb{Z}^+$ denote the sequence f by the symbol $\{x_n\}$. The values of f (the elements x_n), are called the **terms** of the sequence.

If A is a set and if $x_n \in A$ for all $n \in \mathbb{Z}^+$, then $\{x_n\}$ is said to be a sequence in A.

Examples:

$$\{2, 4, 6, 8, ...\}$$

$$\{1, 1, 1, ...\}$$

Finite, Countable, and Uncountable Sets (iv)

Theorem 1.

Every infinite subset of a countable set A is countable.

Proof:

Suppose $E \subset A$, and E is infinite.
Arranging the elements x of A in a sequence $\{x_n\}$ of distinct elements.
Construct a sequence $\{n_k\}$ as follows:
Let n_1 be the smallest integer such that $x_{n_1} \in E$.
Having chosen $n_1, \ldots, n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Define $f : J \to E$ such that $f(k) = x_{n_k}$
f is one to one and onto.
Therefore we obtain a 1-1 correspondence between E and J.
E is countable.

Finite, Countable, and Uncountable Sets (v)

Definition 3.

Let A and Ω be sets, and suppose that with each element α of A there is associated a subset E_α of Ω.

The set whose elements are the sets E_α is denoted by $\{E_\alpha\}$.
This is set of sets.

The union of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least on $\alpha \in A$.

$$S = \cup_{\alpha \in A} E_\alpha$$

If A consists of the integers $1, 2, \ldots, n$

$$S = \cup_{m=1}^{n} E_m = E_1 \cup E_2 \cup E_n$$

If A is the set of all positive integers,

$$S = \cup_{m=1}^{\infty} E_m$$

Finite, Countable, and Uncountable Sets (vi)

The intersection of the sets E_α is the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$.

$$P = \cap_{\alpha \in A} E_\alpha$$

or

$$P = \cap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \ldots \cap E_m$$

or

$$P = \cap_{m=1}^{\infty} E_m$$

If $A \cap B$ is empty, we say that A and B are disjoint.

Finite, Countable, and Uncountable Sets (vii)

Example:

Suppose

$$E_1 = \{1, 2, 3\}$$

and

$$E_2 = \{2, 3, 4\}$$

1. What is $E_1 \cup E_2$?
2. What is $E_1 \cap E_2$?

Finite, Countable, and Uncountable Sets (viii)

Example:

Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Prove that

1. $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$.
2. $\cup_{x \in A} E_x = E_1$
3. $\cap_{x \in A} E_x$ is empty.