Contents

1 Euclidean Space

Euclidean Space - Cont.

Definition

For eaach positive integer k, let $\mathbb{R}^k$ be the set of all ordered k —tuples, $\boldsymbol{x} = (x_1, x_2, ..., x_k)$, where $x_1, x_2, ..., x_k$ are real numbers, called coordinates of $\boldsymbol{x}$.

- The elements of $\mathbb{R}^k$ is called points, or vectors, especially when $k > 1$.

If $\boldsymbol{y} = (y_1, y_2, ..., y_k)$ and if α is a real number

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, ..., x_k + y_k)$$
$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, ..., x_n)$$

- These two operations satisfy the commutative, associative and distributive laws.
- That makes $\mathbb{R}^k$ into a vector space over real field.
- The zero element of $\mathbb{R}^k$ is the point $\mathbf{0} = (0, 0, ..., 0)$ (origin)

Euclidean Space - Cont. (ii)

Definition (Inner Product)

$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{k} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_k y_k$

Definition (Norm)

$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}$

This structure is called Euclidean k —space.

Theorem

Theorem

Suppose $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$, and α is real. Then

1. $|\boldsymbol{x}| \geq 0$
2. $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = 0$.
3. $|\alpha \boldsymbol{x}| = |\alpha||\boldsymbol{x}|$
4. $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}||\boldsymbol{y}|$
5. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$
6. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$

Theorem - Proof

1. $|\boldsymbol{x}| \geq 0$

$$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}}$$

$$|\boldsymbol{x}|^2 = \sum_{i=1}^{k} x_i^2 \geq 0 \text{ for any } \; \boldsymbol{x}$$

Theorem - Proof (ii)

2. $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = 0$

$$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} = 0$$

$$\sum_{i=0}^{k} x_i^2 = 0$$

$$x_1^2 + x_2^2 + \ldots + x_k^2 = 0$$

$$\text{so} \quad x_i = 0 \text{ for all } \; i \in \{1, \ldots, k\}$$

$$\boldsymbol{x} = 0$$

Theorem - Proof (iii)

3. $|\alpha \boldsymbol{x}| = |\alpha| |\boldsymbol{x}|$

$$
\begin{aligned}
|\alpha| \, |\boldsymbol{x}| &= |\alpha| \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\alpha^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\alpha^2 \sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{k} \alpha^2 x_i^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k} \left(\alpha x_i\right)^2 \right)^{\frac{1}{2}} = |\alpha \boldsymbol{x}|
\end{aligned}
$$

Theorem - Proof (iv)

5. $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$

$$\begin{aligned}
|\boldsymbol{x}+\boldsymbol{y}|^2 &= (\boldsymbol{x}+\boldsymbol{y}) \cdot (\boldsymbol{x}+\boldsymbol{y}) \\
&= \boldsymbol{x} \cdot \boldsymbol{x} + 2\boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{y} \\
&= |\boldsymbol{x}|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + |\boldsymbol{y}|^2
\end{aligned}$$

Consider, $2\boldsymbol{x} \cdot \boldsymbol{y} \leq |2\boldsymbol{x} \cdot \boldsymbol{y}| = 2 \mid \boldsymbol{x} \cdot \boldsymbol{y}| \leq 2|\boldsymbol{x}\|\boldsymbol{y}|$

$$\begin{aligned}
|\boldsymbol{x}+\boldsymbol{y}|^2 &\leq |\boldsymbol{x}|^2 + 2|\boldsymbol{x}\ \|\boldsymbol{y}| + |\boldsymbol{y}|^2 \\
&= (|\boldsymbol{x}| + |\boldsymbol{y}|)^2
\end{aligned}$$

Theorem - Proof (v)

6. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$

Theorem - Proof (vi)

7. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$