Calculus I

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Calculating Limits Using the Limit Laws

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[cf(x)] = c \lim_{x \to a}[f(x)]$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
6. **Power Law**

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n,$$

Limit Laws (ii)

where n is a positive number.

7. **Root Law**

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)},$$

where n is a positive integer. If n is even we assume that $f(x) > 0$.

8. $\lim_{x \to a} c = c$
9. $\lim_{x \to a} x = a$
10. $\lim_{x \to a} x^n = a^n$, where n is a positive integer.
11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer.

Limit Laws (iii)

2. $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} &= \frac{\lim_{x\to -2} x^3+2x^2-1}{\lim_{x\to -2}(5-3x)} \text{ by law (5)} \\
&= \frac{\lim_{x\to -2} x^3 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - \lim_{x\to -2} 3x} \text{ by laws (1) and (2)} \\
&= \frac{\lim_{x\to -2} x^3 + \lim_{x\to -2} 2x^2 - 1}{5 - \lim_{x\to -2} 3x} \text{ by law (8)} \\
&= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - 1}{5 - 3\lim_{x\to -2} x} \text{ by law (3)} \\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5-3(-2)} \\
&= \frac{-8+8-1}{5+6} = -\frac{1}{11}
\end{aligned}$$

Evaluating Limits by Direct Substitution

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain f, then

$$\lim_{x \to a} f(x) = f(a)$$

Example: Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$.
You cannot just substitute the value $x = 1$.

For $x \neq 1$.

$$\lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1}(x+1) = 2$$

Evaluating Limits by Direct Substitution (ii)

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided that the limits exists.

Example: Find $\lim_{x \to 1} g(x)$, where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

Define a function $f(x) = x + 1$. Then $f(x) = g(x)$ when $x \neq 1$.

Then

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} f(x) = \lim_{x \to 1} x + 1 = 2$$

Evaluating Limits by Direct Substitution (iii)

Example Evaluate $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$.

$$\begin{aligned}
\lim_{h\to 0} \frac{(3+h)^2-9}{h} &= \lim_{h\to 0} \frac{\cancel{9}+6\cancel{h}+h^{\cancel{2}}-\cancel{9}}{\cancel{h}} \\
&= \lim_{h\to 0} 6+h \\
&= 6
\end{aligned}$$

Evaluating Limits by Direct Substitution (iv)

Example: Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

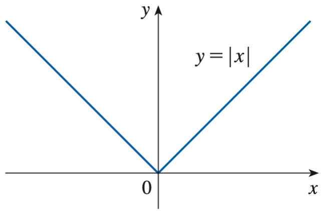
$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} &= \lim_{t\to 0} \frac{t^2+9-9}{t^2\sqrt{t^2+9}+3t^2} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} \\
&= \frac{1}{\sqrt{9}+3} = \frac{1}{6}
\end{aligned}$$

Using one-sided limits

Theorem 1.

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example: Show that $\lim_{x\to 0}|x| = 0$.

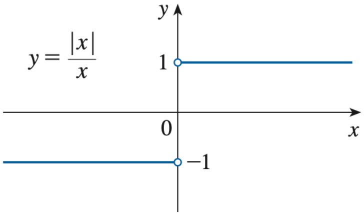


$$|x| = \begin{cases} x \text{ if } & x \geq 0 \\ -x \text{ if } & x < 0 \end{cases}$$

Take $\lim_{x\to 0^-}|x| = \lim_{x\to 0}(-x) = 0$. Then $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$. Then limit is exist and and 0.

Using one-sided limits (ii)

Example: Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.



Using one-sided limits (iii)

Example: If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.