Introduction to Probability Theory

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Conditional Probabilities are Probabilities

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Conditional probability satisfies all the properties of probability.

1. Conditional Probabilities are between 0 and 1.
2. $P(S|E) = 1$, $P(\emptyset|E) = 0$
3. If $A_1, A_2, \ldots$ are disjoint, then $P\left(\cup_{j=1}^{\infty} A_j|E\right) = \sum_{j=1}^{\infty} P(A_j|E)$
4. $P(A^c|E) = 1 - P(A|E)$
5. Inclusion-exclusion: $P(A \cup B|E) = P(A|E) + P(B|E) - P(A \cap B)$

Proof:

1. Conditional Probabilities are between 0 and 1.

Let

$$\tilde{P}(A) = P(A|E) = \frac{P(A \cap E)}{P(E)}$$

It is clear that $\frac{P(A \cap E)}{P(E)} \geq 0$ for $P(E) > 0$. Also, $P(A \cap E) \leq P(E)$, $\frac{P(A \cap E)}{P(E)} \leq 1$

Conditional Probabilities are Probabilities (ii)

2. $P(S|E) = 1$, $P(\emptyset|E) = 0$

$$\tilde{P}(S) = P(S|E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1$$

$$\tilde{P}(\emptyset) = P(\emptyset|E) = \frac{P(\emptyset \cap E)}{P(E)} = \frac{P(\emptyset)}{P(E)} = 0$$

Conditional Probabilities are Probabilities (iii)

3. If $A_1, A_2, \ldots$ are disjoint, then $P\left(\cup_{j=1}^{\infty} A_j | E\right) = \sum_{j=1}^{\infty} P\left(A_j | E\right)$

$$\tilde{P}\left(\cup_{j=1}^{\infty} A_j\right) = P\left(\cup_{j=1}^{\infty} A_j \;| E\right) = \frac{P\left(\left(\cup_{j=1}^{\infty} A_j\right) \cap E\right)}{P(E)} = \frac{P((A_1 \cap E) \cup (A_2 \cap E) \cup \ldots)}{P(E)}$$

Numerator is a union of disjoint sets.

$$\tilde{P}\left(\cup_{j=1}^{\infty} A_j\right) = \frac{\sum_{j=1}^{\infty} P\left(A_j \cap E\right)}{P(E)} = \sum_{j=1}^{\infty} \frac{P\left(A_j \cap E\right)}{P(E)} = \sum_{j=1}^{\infty} P\left(A_j \mid E\right)$$

Conditional Probabilities are Probabilities (iv)

4. $P(A^c|E) = 1 - P(A|E)$

We know that $P(S|E) = 1$.

$$P(S|E) = P(A \cup A^c|E) = P(A|E) + P(A^c|E) \text{ by 3.}$$

$$1 = P(A|E) + P(A^c|E)$$

$$P(A^c|E) = 1 - P(A|E)$$

Conditional Probabilities are Probabilities (v)

5. Inclusion-exclusion: $P(A \cup B|E) = P(A|E) + P(B|E) - P(A \cap B|E)$

H.W.

Independence of Events

Independence of Events

The situation where events provide no information about each other is called **Independence**.

Definition 1: Independence of two events.

Events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

If $P(A) > 0$ and $P(B)$, then this is equivalent to

$$P(A|B) = P(A)$$

and also equivalent to

$$P(B|A) = P(B)$$

- A and B are independent if learning that B occurred gives us no information that would change our probabilities for A occurring.
- Independence is a symmetric relation: if A is independent of B, then B is independent of A.
- Independence is completely different from disjointness. If A and B are disjoint, then $P(A \cap B) = 0$, so disjoint events can be independent if $P(A) = 0$ or $P(B) = 0$.

Independence of Events (ii)

Proposition 1.

If A and B are independent, then A and B^c are independent, A^c and B are independent, and A^c and B^c are independent.

Proof: Let A and B are independent. If $P(A) = 0$, then $P(A \cap B) = 0$ and $P(B) = 0$. $P(A \cap B^c) = 0$. That is trivial.

Assume that $P(A) \neq 0$.

$$P(B^c|A) = 1 - P(B|A) = 1 - P(B) = P(B^c)$$

A and B^c are independent.

In the similar manner, you can prove the others.