

Advanced Calculus

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Euclidean Space

Euclidean Space - Cont.

Definition 1.

For each positive integer k , let \mathbb{R}^k be the set of all ordered k -tuples, $\mathbf{x} = (x_1, x_2, \dots, x_k)$, where x_1, x_2, \dots, x_k are real numbers, called coordinates of \mathbf{x} .

- The elements of \mathbb{R}^k are called points, or vectors, especially when $k > 1$.

If $\mathbf{y} = (y_1, y_2, \dots, y_k)$ and if α is a real number

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_k + y_k)$$

$$\alpha \mathbf{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

- These two operations satisfy the commutative, associative and distributive laws.
- That makes \mathbb{R}^k into a vector space over real field.
- The zero element of \mathbb{R}^k is the point $\mathbf{0} = (0, 0, \dots, 0)$ (origin)

Definition 2: Inner Product.

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_k y_k$$

Euclidean Space - Cont. (ii)

Definition 3: Norm.

$$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}}$$

This structure is called Euclidean k –space.

Theorem

Theorem 1.

Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then

1. $|\mathbf{x}| \geq 0$
2. $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
3. $|\alpha \mathbf{x}| = |\alpha| |\mathbf{x}|$
4. $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$
5. $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
6. $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

Theorem - Proof

1. $|\mathbf{x}| \geq 0$

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}}$$

$$|\mathbf{x}|^2 = \sum_{i=1}^k x_i^2 \geq 0 \text{ for any } \mathbf{x}$$

Theorem - Proof (ii)

2. $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0$

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}} = 0$$

$$\sum_{i=1}^k x_i^2 = 0$$

$$x_1^2 + x_2^2 + \dots + x_k^2 = 0$$

so $x_i = 0$ for all $i \in \{1, \dots, k\}$

$$\mathbf{x} = 0$$

Theorem - Proof (iii)

3. $|\alpha \mathbf{x}| = |\alpha| \|\mathbf{x}\|$

$$\begin{aligned} |\alpha \mathbf{x}| &= |\alpha| \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}} \\ &= (\alpha^2)^{\frac{1}{2}} \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}} \\ &= \left(\alpha^2 \sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^k \alpha^2 x_i^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^k (\alpha x_i)^2 \right)^{\frac{1}{2}} = |\alpha \mathbf{x}| \end{aligned}$$

Theorem - Proof (iv)

4. $|x \cdot y| \leq |x||y|$

Theorem - Proof (v)

5. $|x + y| \leq |x| + |y|$

$$\begin{aligned}|x + y|^2 &= (x + y) \cdot (x + y) \\&= x \cdot x + 2x \cdot y + y \cdot y \\&= |x|^2 + 2x \cdot y + |y|^2\end{aligned}$$

Consider, $2x \cdot y \leq |2x \cdot y| = 2 |x \cdot y| \leq 2|x||y|$

$$\begin{aligned}|x + y|^2 &\leq |x|^2 + 2|x||y| + |y|^2 \\&= (|x| + |y|)^2\end{aligned}$$

Theorem - Proof (vi)

7. $|x + y| \leq |x - y| + |y - z|$

Basic Topology

Finite, Countable, and Uncountable Sets

Definition 4.

Consider two sets A and B and suppose that with each element x of A there is associated an element of B , which we denoted by $f(x)$. Then f is said to be a function from A to B .

- The set A is called the **domain of f** .
- The elements $f(x)$ are called the values of f .
- The set of all values is called the **range of f**

Finite, Countable, and Uncountable Sets (ii)

Definition 5.

Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$.

We call $f(E)$ the **image of E under f** .

As previous definition, $f(A)$ is the range of f and clear that $f(A) \subset B$.

If $f(A) = B$, we say that f maps A **onto** B .

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the **inverse image of E under f** .

If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$.

If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A , then f is said to be 1-1 (one to one) mapping of A into B .

A 1-1 mapping of A to B also expressed as follows:

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for $x, x_2 \in A$.