Advanced Calculus

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Basic Topology

Finite, Countable, and Uncountable Sets

Definition 1.

Consider two sets A and B and suppose that with each element x of A there is associated an element of B, which we denoted by $f(x)$. Then f is said to be a function from A to B.

- The set A is called the **domain of $\boldsymbol{f}$**.
- The elements $f(x)$ are called the values of f.
- The set of all values is called the **range of $\boldsymbol{f}$**

Finite, Countable, and Uncountable Sets (ii)

Definition 2.

Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$.
We call $f(E)$ the **image of $\boldsymbol{E}$ under $\boldsymbol{f}$**.
As previous definition, $f(A)$ is the range of f and clear that $f(A) \subset B$.
If $f(A) = B$, we say that f is maps A **onto** B.

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the **inverse image of $\boldsymbol{E}$ under $\boldsymbol{f}$**.
If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$.
If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be 1-1 (one to one) mapping of A into B.

A 1-1 mapping of A to B also expressed as follows:

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for $x_1, x_2 \in A$.

Finite, Countable, and Uncountable Sets (iii)

Definition 3.

If there exists a 1-1 mapping of A onto B, we say that A and B can be put in 1-1 correspondence, or that A and B have the same cardinal number, or that A and B are equivalent, and we write $A \sim B$.

Definition 4: Equivalent Relation.

Equivalent relation has the following properties:

1. Reflexive: $A \sim A$
2. Symmetric: If $A \sim B$, then $B \sim A$.
3. Transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these properties is called an **equivalence relation**.

Finite, Countable, and Uncountable Sets (iv)

Definition 5.

For any positive integer n, let J_n be the set whose elements are the integers $1, 2, 3, \ldots, n$.

Let J be the set consisting of all positive integers.

For any set A, we say,

1. A is finite, if $A \sim J_n$.
2. A is infinite if A is not finite. (or A is infinite if A is equivalent to one of its proper subsets.)
3. A is countable if $A \sim J$.
4. A is uncountable if A is neither finite nor countable.
5. A is at most countable if A is finite or countable.

Finite, Countable, and Uncountable Sets (v)

Example: Let A be the set of all integers (of course not finite). Prove that A is countable.

Finite, Countable, and Uncountable Sets (vi)

Definition 6: Sequence.

A sequence is a function f defined on the set of all positive integers. If $f(n) = x_n$, for $n \in \mathbb{Z}^+$ denote the sequence f by the symbol $\{x_n\}$. The values of f (the elements x_n), are called the **terms** of the sequence.

If A is a set and if $x_n \in A$ for all $n \in \mathbb{Z}^+$, then $\{x_n\}$ is said to be a sequence in A.

Examples:

$$\{2, 4, 6, 8, \ldots\}$$

$$\{1, 1, 1, \ldots\}$$

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Theorem 1.

Every infinite subset of a countable set A is countable.

Definition 7.

Let A and Ω be sets, and suppose that with each element α of A there is associated a subset E_α of Ω.

The set whose elements are the sets E_α is denoted by $\{E_\alpha\}$.
This is set of sets.

The union of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least on $\alpha \in A$.

$$S = \cup_{\alpha \in A} E_\alpha$$

If A consists of the integers $1, 2, \ldots, n$

$$S = \cup_{m=1}^{n} E_m = E_1 \cup E_2 \cup E_n$$

Finite, Countable, and Uncountable Sets (viii)

If A is the set of all positive integers,

$$S = \cup_{m=1}^{\infty} E_m$$

The intersection of the sets E_α is the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$.

$$P = \cap_{\alpha \in A} E_\alpha$$

or

$$P = \cap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \dots \cap E_m$$

or

$$P = \cap_{m=1}^{\infty} E_m$$

If $A \cap B$ is empty, we say that A and B are disjoint.

Finite, Countable, and Uncountable Sets (ix)

Example:

Suppose

$$E_1 = \{1, 2, 3\}$$

and

$$E_2 = \{2, 3, 4\}$$

1. What is $E_1 \cup E_2$?
2. What is $E_1 \cap E_2$?

Finite, Countable, and Uncountable Sets (x)

Example:

Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Prove that

1. $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$.
2. $\cup_{x \in A} E_x = E_1$
3. $\cap_{x \in A} E_x$ is empty.