Introduction to Probability Theory

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Random Variables and Their Distributions

Random Variables

Definition 1: Random Variable.

Given an experiment with sample space S, a random variable is a function from the sample space S to the real number $\mathbb{R}$.

A random variable X assigns a numerical value $X(s)$ to each possible outcome s of the experiment.

Example: Coin Tosses

Consider an experiment where we toss a fair coin twice. The sample space consists of four possible outcomes:

$$S = \{HH, TT, TH, TT\}.$$

Here are some random variables on this space.

- Let X be the number of Heads.
- Let Y be the number of Tails.
- Let I be 1 if the first toss lands Heads and 0 otherwise. (This is called and indicator random variable.)

Random Variables (ii)

We can also encode the sample space as $\{(1,1),(1,0),(0,1),(0,0)\}$, where 1 is the code for Heads and 0 is the code for Tails.

Distributions and Probability Mass Functions

There are two main types of random variables:

1. Discrete random variables
2. Continuous random variables.

Definition 2: Discrete Random Variable.

A random variable X is said to be **discrete** is there is a finite list of values $a_1, a_2, \ldots, a_n$ or an infinite list of values $a_1, a_2, \ldots$ such that $P\left(X = a_j \text{ for some } j\right) = 1$.
If X is a discrete random variable, then the finite or countable infinite set of values x such that $P(X = x) > 0$ is called the **support** of X.

Definition 3: Probability Mass Function.

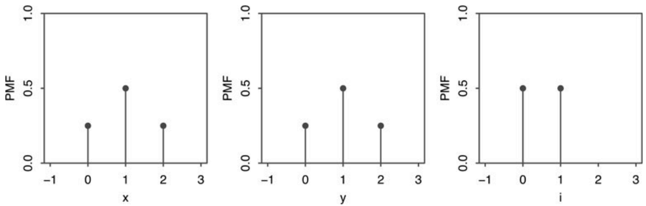
The probability mass function (PMF) of a discrete random variable X is the function $p_X(x) = P(X = x)$. This is positive if x is in the support of X, and 0 otherwise.

Distributions and Probability Mass Functions (ii)

Example

$$\{HH, HT, TH, TT\}$$

- X — the number of Heads
- Y — number of Tails
- I — the indicator of the first toss landing Heads.



Distributions and Probability Mass Functions (iii)

Example: Sum of die rolls

We roll two fair dice. Let $T = X + Y$ be the total of the two tolls, where X and Y are the individual rolls.

$$S = \{(1, 1), (1, 2), ..., (6, 5), (6, 6)\}$$

Since the dice are fair, the PMF of X is

$$P(X = j) = \frac{1}{6}$$

for $j = 1, ..., 6$ and $P(X = j) = 1$ otherwise.

We say that X has a **Discrete Uniform Distribution** on $1, 2, ..., 6$. Similarly, Y is also Discrete Uniform on $1, 2, ..., 6$.

Note that Y also has the same distribution as X but is not the same random variable as x. However,

Distributions and Probability Mass Functions (iv)

$$P(X = Y) = \frac{6}{36} = \frac{1}{6}$$

$$P(X = 7 - X) = \frac{1}{6}$$

$$P(X = 7 - Y) = \frac{1}{6}$$

You can think as, for a standard die, $7 - X$ is the value on the bottom if X is the value on the top.