Calculus I

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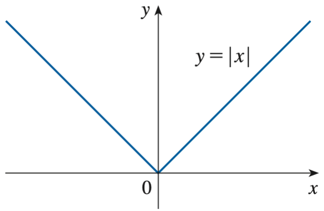
Calculating Limits Using the Limit Laws

Using one-sided limits

Theorem 1.

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example: Show that $\lim_{x\to 0}|x| = 0$.

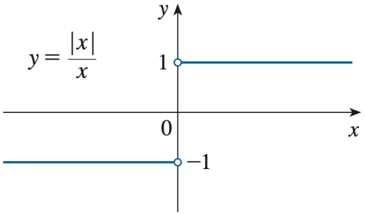


$$|x| = \begin{cases} x \text{ if } & x \geq 0 \\ -x \text{ if } & x < 0 \end{cases}$$

Take $\lim_{x\to 0^-}|x| = \lim_{x\to 0}(-x) = 0$. Then $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$. Then limit is exist and and 0.

Using one-sided limits (ii)

Example: Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.



$$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{if } \ x > 0 \\ -\frac{x}{x} = -1 & \text{if } \ x < 0 \end{cases}$$

Using one-sided limits (iii)

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -1 = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1$$

Limit does not exist, bcause $\lim_{x \to 0^-} \frac{|x|}{x} \neq \lim_{x \to 0^+} \frac{|x|}{x}$.

Using one-sided limits (iv)

Example: If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 8 - 2x = 8 - 2(4) = 8 - 8 = 0$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

We can see that $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = 0$. Therefore $\lim_{x \to 4} f(x)$ is exists.

The Squeeze Theorem

Theorem 2.

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

The Squeeze Theorem (ii)

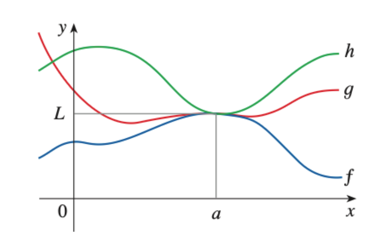
Theorem 3: The Squeeze Theorem.

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$



The Squeeze Theorem (iii)

Example: Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

We know that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist. However

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Note that

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

By the Squeeze theorem,

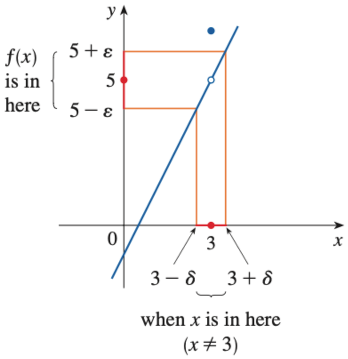
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Precise Definition of Limit

Definition 1.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.



Precise Definition of Limit (ii)

Example: Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Choose $\varepsilon > 0$ such that $|(4x - 5) - 7| < \varepsilon$. Note that

$$\begin{aligned} |(4x - 5) - 7| &= |4x - 12| \\ &= 4|x - 3| < \varepsilon \\ |x - 3| &< \frac{\varepsilon}{4} \end{aligned}$$

Then we can choose $\delta = \frac{\varepsilon}{4}$. For every $\varepsilon > 0$ we have if $0 < |x - 3| < \frac{\varepsilon}{4}$, then $|(4x - 5) - 7| < \varepsilon$.

Precise Definition of Limit (iii)

Example: Prove that $\lim_{x\to 3} x^2 = 9$.