Advanced Calculus

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Euclidean Space

Euclidean Space - Cont.

Definition 1.

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered $k-$tuples, $\boldsymbol{x} = (x_1, x_2, ..., x_k)$, where $x_1, x_2, ..., x_k$ are real numbers, called coordinates of $\boldsymbol{x}$.

- The elements of $\mathbb{R}^k$ is called points, or vectors, especially when $k > 1$.

If $\boldsymbol{y} = (y_1, y_2, ..., y_k)$ and if α is a real number

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, ..., x_k + y_k)$$
$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, ..., x_n)$$

- These two operations satisfy the commutative, associative and distributive laws.
- That makes $\mathbb{R}^k$ into a vector space over real field.
- The zero element of $\mathbb{R}^k$ is the point $\mathbf{0} = (0, 0, ..., 0)$ (origin)

Definition 2: Inner Product.

$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{k} x_i y_i = x_1 y_1 + x_2 y_2 + ... + x_k y_k$

Euclidean Space - Cont. (ii)

Definition 3: Norm.

$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}$

This structure is called Euclidean k —space.

Theorem

Theorem 1.

Suppose $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$, and α is real. Then

1. $|\boldsymbol{x}| \geq 0$
2. $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = 0$.
3. $|\alpha \boldsymbol{x}| = |\alpha\|\boldsymbol{x}|$
4. $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}\|\boldsymbol{y}|$
5. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$
6. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$

Theorem - Proof

1. $|\boldsymbol{x}| \geq 0$

$$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}}$$

$$|\boldsymbol{x}|^2 = \sum_{i=1}^{k} x_i^2 \geq 0 \text{ for any } \boldsymbol{x}$$

Theorem - Proof (ii)

2. $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = 0$

$$|\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} = 0$$

$$\sum_{i=0}^{k} x_i^2 = 0$$

$$x_1^2 + x_2^2 + \dots + x_k^2 = 0$$

$$\text{so} \quad x_i = 0 \text{ for all } \quad i \in \{1, \dots, k\}$$

$$\boldsymbol{x} = 0$$

Theorem - Proof (iii)

3. $|\alpha \boldsymbol{x}| = |\alpha\|\boldsymbol{x}|$

$$
\begin{aligned}
|\alpha\| \ \boldsymbol{x}| &= |\alpha| \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\alpha^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\alpha^2 \sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{k} \alpha^2 x_i^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k} \left(\alpha x_i\right)^2 \right)^{\frac{1}{2}} = |\alpha \boldsymbol{x}|
\end{aligned}
$$

Theorem - Proof (iv)

4. $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}\|\boldsymbol{y}|$

Use Schwarz inequality,

$$
\begin{aligned}
|\boldsymbol{x} \cdot \boldsymbol{y}| = |\sum_{i=1}^{k} x_i y_i|^2 &\leq \sum_{i=1}^{k} |x_i|^2 \sum_{i=1}^{k} |y_i|^2 \\
&= \sum_{i=1}^{k} x_i^2 \sum_{i=1}^{k} y_i^2 \text{ because } \ x_i, y_i \in \mathbb{R} \\
&= |x|^2 \ |y|^2
\end{aligned}
$$

Theorem - Proof (v)

5. $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$

$$\begin{aligned} |\boldsymbol{x}+\boldsymbol{y}|^2 &= (\boldsymbol{x}+\boldsymbol{y}) \cdot (\boldsymbol{x}+\boldsymbol{y}) \\ &= \boldsymbol{x} \cdot \boldsymbol{x} + 2\boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{y} \\ &= |\boldsymbol{x}|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + |\boldsymbol{y}|^2 \end{aligned}$$

Consider, $2\boldsymbol{x} \cdot \boldsymbol{y} \leq |2\boldsymbol{x} \cdot \boldsymbol{y}| = 2 \mid \boldsymbol{x} \cdot \boldsymbol{y}| \leq 2|\boldsymbol{x}\|\boldsymbol{y}|$

$$\begin{aligned} |\boldsymbol{x}+\boldsymbol{y}|^2 &\leq |\boldsymbol{x}|^2 + 2|\boldsymbol{x}\ \|\boldsymbol{y}| + |\boldsymbol{y}|^2 \\ &= (|\boldsymbol{x}|+|\boldsymbol{y}|)^2 \end{aligned}$$

Theorem - Proof (vi)

6. $|\boldsymbol{x} - \boldsymbol{z}| \leq |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$

Use (5.) $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$. Replace x by $x - y$ and y by $y - z$

$$\begin{aligned} |x - \cancel{y} + \cancel{y} - z| &\leq |x - y| + |y - z| \\ |x - z| &\leq |x - y| + |y - z| \end{aligned}$$

Basic Topology

Finite, Countable, and Uncountable Sets

Definition 4.

Consider two sets A and B and suppose that with each element x of A there is associated an element of B, which we denoted by $f(x)$. Then f is said to be a function from A to B.

- The set A is called the **domain of $\boldsymbol{f}$**.
- The elements $f(x)$ are called the values of f.
- The set of all values is called the **range of $\boldsymbol{f}$**

Finite, Countable, and Uncountable Sets (ii)

Definition 5.

Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$.
We call $f(E)$ the **image of $\boldsymbol{E}$ under $\boldsymbol{f}$**.
As previous definition, $f(A)$ is the range of f and clear that $f(A) \subset B$.
If $f(A) = B$, we say that f is maps A **onto** B.

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the **inverse image of $\boldsymbol{E}$ under $\boldsymbol{f}$**.
If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$.
If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be 1-1 (one to one) mapping of A into B.

A 1-1 mapping of A to B also expressed as follows:

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for $x_, x_2 \in A$.