Introduction to Probability Theory

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Random Variables and Their Distributions

Random Variables

Definition 1: Random Variable.

Given an experiment with sample space S, a random variable is a function from the sample space S to the real number $\mathbb{R}$.

A random variable X assigns a numerical value $X(s)$ to each possible outcome s of the experiment.

Example: Coin Tosses

Consider an experiment where we toss a fair coin twice. The sample space consists of four possible outcomes:

$$S = \{HH, HT, TH, TT\}.$$

Here are some random variables on this space.

- Let X be the number of Heads.
  $X(HH) = 2 \quad X(HT) = 1 \quad X(TH) = 1 \quad X(TT) = 0$

Random Variables (ii)

- Let Y be the number of Tails.
  In terms of Y, we have $Y = 2 - X$.
  $Y(s) = 2 - X(s)$.
  $Y(HH) = 0 = 2 - 2 = 2 - X(HH)$

  $Y(HT) = 1 = 2 - 1 = 2 - X(HT)$
  $Y(TH) = 1 = 2 - 1 = 2 - X(TH)$
  $Y(TT) = 2 = 2 - 0 = 2 - X(TT)$
- Let I be 1 if the first toss lands Heads and 0 otherwise. (This is called and indicator random variable.)

$$I(HH) = 1$$

$$I(HT) = 1$$

$$I(TH) = 0$$

$$I(TT) = 0$$

Random Variables (iii)

We can also encode the sample space as $\{(1,1),(1,0),(0,1),(0,0)\}$, where 1 is the code for Heads and 0 is the code for Tails.

$$X((s_1,s_2)) = s_1 + s_2$$
$$Y((s_1,s_2)) = 2 - s_1 - s_2$$
$$I((s_1,s_2)) = s_1$$

Distributions and Probability Mass Functions

There are two main types of random variables:

1. Discrete random variables
2. Continuous random variables.

Definition 2: Discrete Random Variable.

A random variable X is said to be **discrete** if there is a finite list of values $a_1, a_2, \ldots, a_n$ or an infinite list of values $a_1, a_2, \ldots$ such that $P\left(X = a_j \text{ for some } j\right) = 1$.
If X is a discrete random variable, then the finite or countable infinite set of values x such that $P(X = x) > 0$ is called the **support** of X.

Definition 3: Probability Mass Function.

The probability mass function (PMF) of a discrete random variable X is the function $p_X(x) = P(X = x)$. This is positive if x is in the support of X, and 0 otherwise.

Distributions and Probability Mass Functions (ii)

Example

$$\{HH, HT, TH, TT\}$$

- X – the number of Heads

$$p_X(0) = P(X = 0) = \frac{1}{4} \to \{TT\}$$
$$p_X(1) = P(X = 1) = \frac{1}{2} \to \{HT, TH\}$$
$$p_X(2) = P(X = 2) = \frac{1}{4} \to \{HH\}$$

$$p_X(0) + p_X(1) + p_X(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

- Y – number of Tails

Distributions and Probability Mass Functions (iii)

$$Y = 2 - X$$

$$p_Y(y) = P(Y = y) = P(2 - X = y) = P(X = 2 - y) = p_X(2 - y)$$

$$p_Y(0) = P(Y = 0) = p_X(2 - 0) = p_X(2) = \frac{1}{4}$$

$$p_Y(1) = p_X(2 - 1) = \frac{1}{2}$$

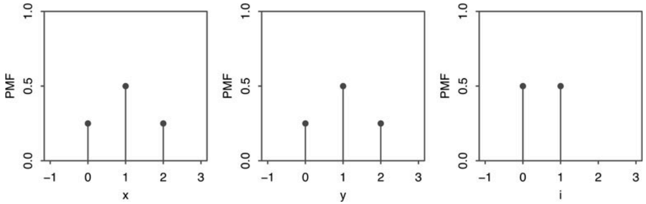
$$p_Y(2) = \frac{1}{4}$$

- I – the indicator of the first toss landing Heads.

$$p_I(0) = P(I = 0) = \frac{1}{2} \to \{TH, TT\}$$

$$p_I(1) = P(I = 1) = \frac{1}{2} \to \{HH, HT\}$$

Distributions and Probability Mass Functions (iv)



Distributions and Probability Mass Functions (v)

Example: Sum of die rolls

We roll two fair dice. Let $T = X + Y$ be the total of the two rolls, where X and Y are the individual rolls.

$$S = \{(1,1), (1,2), ..., (6,5), (6,6)\}$$

Since the dice are fair, the PMF of X is

$$P(X = j) = \frac{1}{6}$$

for $j = 1, ..., 6$ and $P(X = j) = 0$ otherwise.

We say that X has a **Discrete Uniform Distribution** on $1, 2, ..., 6$. Similarly, Y is also Discrete Uniform on $1, 2, ..., 6$.

Note that Y also has the same distribution as X but is not the same random variable as x. However,

Distributions and Probability Mass Functions (vi)

$$P(X = Y) = \frac{6}{36} = \frac{1}{6}$$

$$P(X = 7 - X) = \frac{1}{6}$$

$$P(X = 7 - Y) = \frac{1}{6}$$

You can think as, for a standard die, $7 - X$ is the value on the bottom if X is the value on the top.