Closed $\mathrm{p}-$Elastic Curves in $2-$Space Forms

Miraj Samarakkody

Department of Mathematics and Statistics, Texas Tech University, 79409 Lubbock, Texas, USA

`Miraj.Samarakkody@ttu.edu`

Historical Background

Variational problems involving curves with curvature-dependent energy densities play a crucial role in differential geometry, geometric analysis, and mathematical physics, with significant implications across various disciplines, including biology, physics, architecture, and art. These problems aim to understand the shapes and configurations of curves that minimize or optimize certain energy functionals dependent on curvature. The historical roots of this field trace back to the work of the Bernoulli family and Leonhard Euler, who laid the groundwork for the study of elasticity, pivotal for the development of the calculus of variations. Notably, the concept of p-elastic curves, which generalize classical elastic curves by introducing a power p in the curvature term of the energy density, emerged from a 1738 correspondence between Daniel Bernoulli and Euler, marking a foundational moment in the mathematical exploration of generalized elastic curves.

Variational Problem

Let $C_*^\infty(\mathcal{M}_\epsilon^2(\rho))$ be the space of smooth, closed, non-null immersed convex curves $\gamma:\mathbb{R}\to\mathcal{M}_\epsilon^2(\rho)$, parametrized by the arc length $s\in\mathbb{R}$, where ρ represents the sectional curvature.
For fixed real number $p\in\mathbb{R}$, the $p-$elastic functional is given by

$$\Theta_p(\gamma) := \int_\gamma \kappa^p \, ds \tag{1}$$

and acts on $C_*^\infty(\mathcal{M}_\epsilon^2(\rho))$. It is associated with the Euler-Lagrange equation:

$$p\frac{d^2}{ds^2}\kappa^{p-1}+\epsilon_1\epsilon_2(p-1)\kappa^{p+1}+\epsilon_1\rho p\kappa^{p-1}=0, \tag{2}$$

where $\langle T,T\rangle=\epsilon_1=\pm1$ and $\langle N,N\rangle=\epsilon_2=\pm1$ are casual characters of the Frenet frame.

Therefore $\kappa(s)$ is either constant or a solution to the first-order ordinary differential equation:

$$p^2(p-1)^2\kappa^{2(p-2)}(\kappa')^2+\epsilon_1\epsilon_2(p-1)^2\kappa^{2p}+\epsilon_1 p^2\rho\kappa^{2(p-1)}=a, \tag{3}$$

where $a\in\mathbb{R}$ is a constant of integration.

The Round 2-Sphere

The 2-sphere case for a general p value has been partially addressed in the works of J. Arroyo, M. Barros, O.J. Garay, R. López, J. Mencía, S. Montaldo, E. Musso, C. Oniciuc, J. Langer, D.A. Singer, A. Gruber, Á. Pámpano, and M. Toda.
As a result it is proved that if $\gamma\in\mathcal{M}_0^1(1)=\mathbb{S}^2$ is a closed $p-$elastic curve with non-constant curvature in the round $2-$sphere, then either $p=2$ or else $p\in(0,1)$.

Case $\mathrm{p}=2$

Theorem ([4])

For every pair (n,m) of relatively prime natural numbers satisfying $n<m$, there exists a unique closed $2-$elastic curve $\gamma_{n,m}$ in $\mathbb{S}^2$ with non-constant curvature. (These curves are not convex.)

The number n represents the number of times the curve winds around the pole, i.e., it is the winding number, while m is the number of periods of the curvature needed to close the curve, i.e., the number of lobes.

Case $\mathrm{p}\in(0,1)$

Theorem ([3])

Let n and m be two relatively prime natural numbers satisfying $m<2n<\sqrt{2}m$. Then, for every $p\in(0,1)$, there exists a convex closed $p-$elastic curve in $\mathbb{S}^2$ with non-constant curvature.

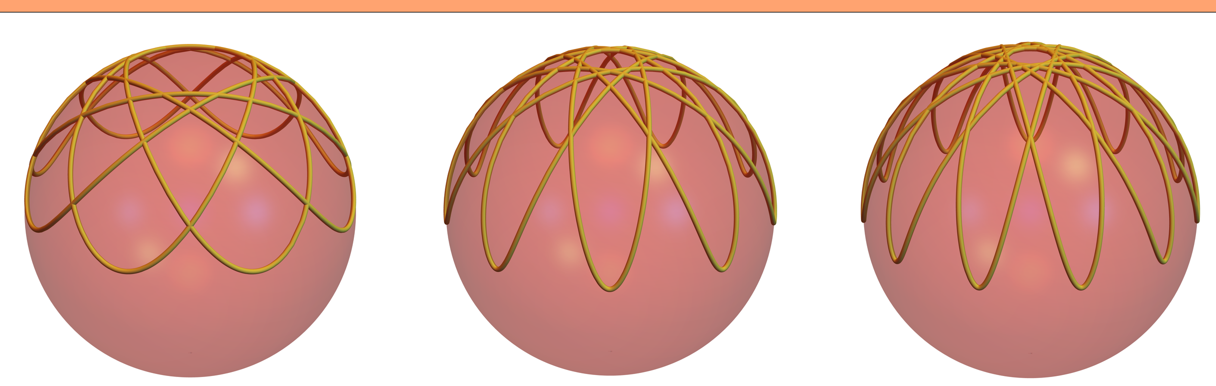

Figure. *Closed $p-$elastic curves for $p=0.3$ in $\mathbb{S}^2$ of type $\gamma_{5,8}$, $\gamma_{5,9}$ and $\gamma_{6,11}$. [3]*

The Hyperbolic Plane and The de-Sitter 2-Space

We specifically focus on the closed $p-$elastic curves in the hyperbolic plane $\mathcal{M}_0^2(-1)=\mathbb{H}_0^2$ and the de-Sitter 2-space $\mathcal{M}_1^2(1)=\mathbb{H}_1^2$.

Proposition ([5])

Let γ be a p-elastic curve in $\mathbb{H}_\epsilon^2$ with non-constant periodic curvature. Then γ is a space-like curve with $0>a>a_*:=-((-1)^\epsilon p)^p((-1)^\epsilon(p-1))^{1-p}$. Moreover,

- *if it is a hyperbolic curve, then $p>1$.*
- *if it is a pseudo-hyperbolic curve, then $p<0$.*

Conversely, under these conditions on p and a, there exists a space-like convex p-elastic curve, denoted $\gamma_a=\gamma:\mathbb{R}\to\mathbb{H}_\epsilon^2$, which exhibits non-constant periodic curvature $\kappa_a=\kappa$.

Parametrization

The $p-$elastic curve γ with periodic curvature in $\mathbb{H}_\epsilon^2$ can be parametrized up to rigid motion in terms of its arc length parameter $s\in\mathbb{R}$, as

$$\gamma=\frac{1}{\sqrt{-a}}\left(\sqrt{\epsilon_2 a+p^2\kappa^{2(p-1)}}\cos\theta(s),\sqrt{\epsilon_2 a+p^2\kappa^{2(p-1)}}\sin\theta(s),p\kappa^{p-1}\right),$$

where

$$\theta(s):=\epsilon_2(p-1)\sqrt{-a}\int\frac{k^p}{\epsilon_2 a+p^2\kappa^{2(p-1)}}ds,$$

is the angular progression and $\epsilon_2=(-1)^\epsilon$.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}_\epsilon^2$. If $\gamma\subset\mathbb{H}_0^2$ is a hyperbolic curve, it never meets the pole $(0,0,1)$. If $\gamma\subset\mathbb{H}_1^2$ is a pseudo-hyperbolic curve, then it is entirely contained in $\mathbb{H}_{10}^2=\mathbb{H}_1^2\cap\{z<0\}$.
2. The curve γ meets the bounding parallels tangentially at the maximum and minimum values of its curvature.
3. The $p-$elastic curve γ is closed if and only if the angular progression along a period of the curvature is a rational multiple of 2π.

Theorem ([5])

For every pair (n,m) of relatively prime natural numbers satisfying $m<2n<\sqrt{2}m$ and $p\in(-\infty,0)\cup(1,\infty)$, there exists a closed $p-$elastic curve $\gamma_{n,m}$ with non constant curvature.

- *If $p>1$, $\gamma_{n,m}$ is immersed in the hyperbolic plane $\mathbb{H}_0^2$.*
- *If $p<0$, $\gamma_{n,m}$ is immersed in the de-Sitter space $\mathbb{H}_1^2$.*

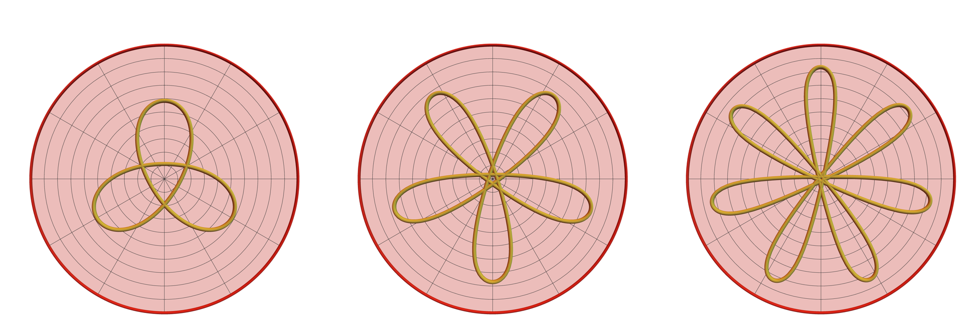

Figure. *Closed $p-$elastic curves for $p=3/2$ in $\mathbb{H}_0^2$ of type $\gamma_{2,3}$, $\gamma_{3,5}$ and $\gamma_{4,7}$, respectively. They are represented in the Poincaré disk model. [5]*

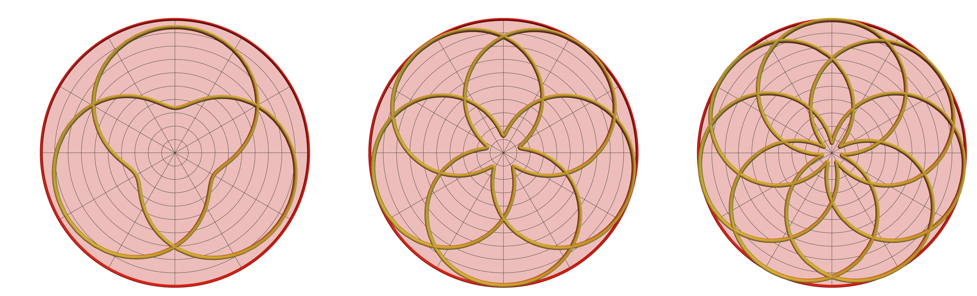

Figure. *Closed $p-$elastic curves for $p=-1$ in $\mathbb{H}_1^2$ of type $\gamma_{3,2}$, $\gamma_{3,5}$ and $\gamma_{4,7}$, respectively. They are represented in the once punctured unit disk. [5]*

Evolution with Energy Parameter

Closed p-elastic curves $\gamma_{n,m}$ in $\mathbb{H}_0^2$ expand as p varies in $(1,\infty)$. On the other hand, in $\mathbb{H}_1^2$, they shrink as p varies in $(-\infty,0)$, approaching the unit circle as $p\to0^-$. In the spherical case, as $p\to0^+$, they expand to the unit circle and shrink to the point $(0,0,1)$ as $p\to1^-$.

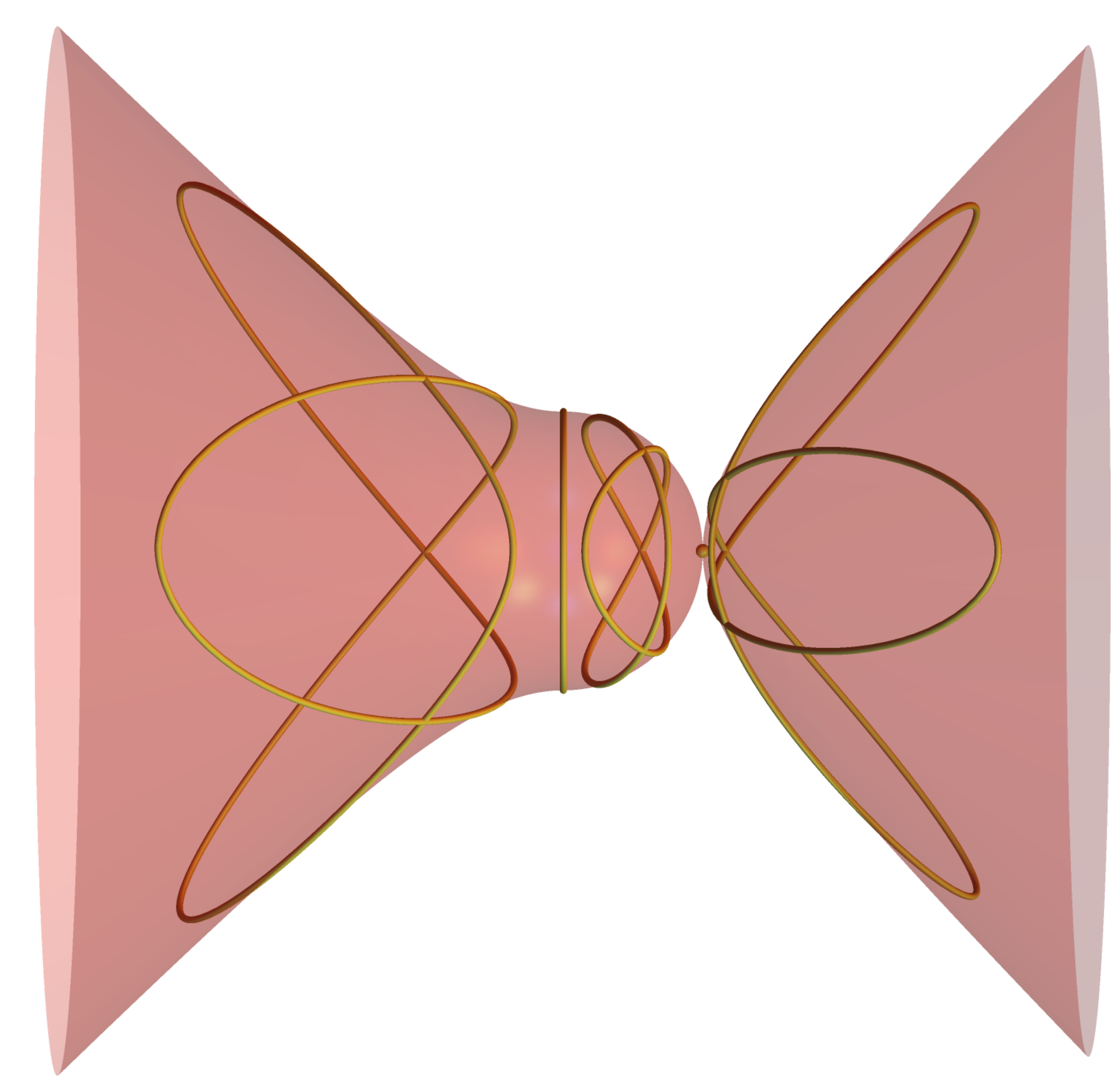

Figure. *Evolution of closed $p-$elastic curves of type $\gamma_{2,3}$. [5]*

References

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[4] J. Langer, D. A. Singer, ***The Total Squared Curvature of Closed Curves***, J. Diff. Geom., 20(1984) 1-22.

[5] A. Pámpano, M. Samarakkody, H. Tran, ***Closed $p-$Elastic Curves in Spheres of $\mathbb{L}^3$***, Submitted (2024). https://arxiv.org/abs/2404.08593