Table of Contents

Why Variational Problems are important?

Why Variational Problems are important? Examples

Why Variational Problems are important? Examples

- Architecture

Figure: Frei Otto's Olympic Stadium Roof [1]

Why Variational Problems are important? Examples

- Architecture

Figure: Frei Otto's Olympic Stadium Roof [1]

- Physics

Why Variational Problems are important? Examples

- Architecture

Figure: Frei Otto's Olympic Stadium Roof [1]

- Physics
- Economics

Why Variational Problems are important? Examples

- Architecture

Figure: Frei Otto's Olympic Stadium Roof [1]

- Physics
- Economics
- Computer Science

Why Variational Problems are important? Examples

- Architecture

Figure: Frei Otto's Olympic Stadium Roof [1]

- Physics
- Economics
- Computer Science
- Biology

History

History of the Variational Problem of Curves

- Originated from the pioneering works of the Bernoulli family and L. Euler about the theory of elasticity.

History

History of the Variational Problem of Curves

- Originated from the pioneering works of the Bernoulli family and L. Euler about the theory of elasticity.
- D. Bernoulli proposed to investigate extrema of the functionals

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

where
$\kappa-$ the curvature of the curve γ;
$s-$ arc length;
$p \in \mathbb{R}$, with $p = 2$ being the classical case (bending energy) and general p considered afterwards.

History

History of the Variational Problem of Surfaces

- It is believed that the Plateau problem is the origin of the first variational problem in surfaces.

History

History of the Variational Problem of Surfaces

- It is believed that the Plateau problem is the origin of the first variational problem in surfaces.
- This problem involves finding the surface with the minimal area that spans a given closed curve.

History

History of the Variational Problem of Surfaces

- It is believed that the Plateau problem is the origin of the first variational problem in surfaces.
- This problem involves finding the surface with the minimal area that spans a given closed curve.
- This problem was solved independently by Jesse Douglas and Tibor Radó. Jesse Douglas was one of the mathematicians who won the first Fields Medal for this contribution.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.
- $p = 1/2$; Blaschke and catenaries.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.
- $p = 1/2$; Blaschke and catenaries.
- Some $p \in \mathbb{Q} \cap (0, 1)$; Generating curves of rotational biconservative hyper-surfaces.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.
- $p = 1/2$; Blaschke and catenaries.
- Some $p \in \mathbb{Q} \cap (0, 1)$; Generating curves of rotational biconservative hyper-surfaces.
- $p = 1$; Total curvature and topological invariant.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.
- $p = 1/2$; Blaschke and catenaries.
- Some $p \in \mathbb{Q} \cap (0, 1)$; Generating curves of rotational biconservative hyper-surfaces.
- $p = 1$; Total curvature and topological invariant.
- $p = 2$; Bending/elastic energy.

Closed $p-$Elastic Curves in 2-Space Forms

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

Special Cases of p-elastic curves

- $p = -1$; Cycloids and Brachistochrone problem.
- $p = 0$; Length functional and geodesics.
- $p = 1/3$; Blaschke, parabolas equi-affine length.
- $p = 1/2$; Blaschke and catenaries.
- Some $p \in \mathbb{Q} \cap (0,1)$; Generating curves of rotational biconservative hyper-surfaces.
- $p = 1$; Total curvature and topological invariant.
- $p = 2$; Bending/elastic energy.
- $p > 2$ for $p \in \mathbb{N}$; Generating curves of Willmore-Chen submanifolds.

Problem

There are plenty of papers studying p-elastic curves in some other special cases. Some of them have examined whether closed curves exist or not.

Problem

There are plenty of papers studying p-elastic curves in some other special cases. Some of them have examined whether closed curves exist or not.

Are there closed p-elastic curves for $p \in \mathbb{R}$, and what are the conditions for a p-elastic curve to be closed?

Some Previous Results

- For $p = 2$, Contributors: J. Langer, D.A. Singer, P. Griffiths, R. Bryant
- Other values of p, Contributors: J. Arroyo, M. Barros, O.J. Garay, R. López, J. Mencía, S. Montaldo, E. Musso, C. Oniciuc, A. Pámpano, M. Toda, A. Gruber, H. Tran and others.

Some Previous Results

- For $p = 2$, Contributors: J. Langer, D.A. Singer, P. Griffiths, R. Bryant
- Other values of p, Contributors: J. Arroyo, M. Barros, O.J. Garay, R. López, J. Mencía, S. Montaldo, E. Musso, C. Oniciuc, A. Pámpano, M. Toda, A. Gruber, H. Tran and others.

Theorem [A. Gruber, Á. Pámpano, M. Toda (2023)]

Let n and m be two relatively prime natural numbers satisfying $m < 2n < \sqrt{2}m$. Then, for every $p \in (0, 1)$, there exists a closed $p-$elastic curve with non-constant curvature, winding number n and $m-$fold symmetry.

Some Previous Results

- For $p = 2$, Contributors: J. Langer, D.A. Singer, P. Griffiths, R. Bryant
- Other values of p, Contributors: J. Arroyo, M. Barros, O.J. Garay, R. López, J. Mencía, S. Montaldo, E. Musso, C. Oniciuc, A. Pámpano, M. Toda, A. Gruber, H. Tran and others.

Theorem [A. Gruber, Á. Pámpano, M. Toda (2023)]

Let n and m be two relatively prime natural numbers satisfying $m < 2n < \sqrt{2}m$. Then, for every $p \in (0, 1)$, there exists a closed $p-$elastic curve with non-constant curvature, winding number n and $m-$fold symmetry.

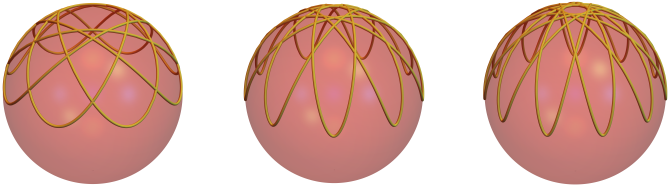


Figure: Closed $p-$elastic curves for $p = 0.3$ in $\mathbb{S}^2$ of type $\gamma_{5,8}$, $\gamma_{5,9}$ and $\gamma_{6,11}$. [6]

Problem

What happens when $p \notin [0, 1]$?

Problem

What happens when $p \notin [0, 1]$?

Are there any closed $p-$elastic curves similar to those we observed in the $\mathbb{S}^2$ case?

Results

Theorem [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

For every $p > 1$ and pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}m$, there exists a closed $p-$elastic curve in the hyperbolic plane with non-constant curvature, winding number n and m-fold symmetry.

Results

Theorem [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

For every $p > 1$ and pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}m$, there exists a closed $p-$elastic curve in the hyperbolic plane with non-constant curvature, winding number n and m-fold symmetry.

Remark

For the pseudo-hyperbolic case the same result follows with $p < 0$.

Results

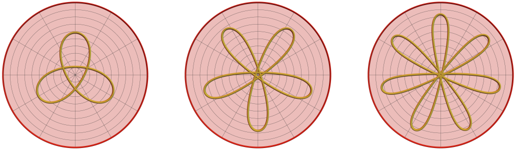


Figure: Closed $p-$elastic curves for $p = 3/2$ in $\mathbb{H}^2_0$ of type $\gamma_{2,3}$, $\gamma_{3,5}$ and $\gamma_{4,7}$, respectively. They are represented in the Poincaré disk model. [8]

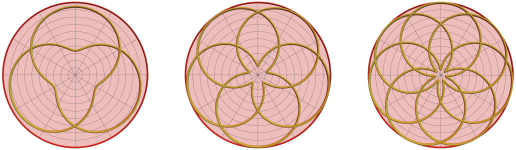


Figure: Closed $p-$elastic curves for $p = -1$ in $\mathbb{H}^2_1$ of type $\gamma_{2,3}$, $\gamma_{3,5}$ and $\gamma_{4,7}$, respectively. They are represented in the once punctured unit disk. [8]

Preliminaries

Let (x, y, z) be standard coordinates of $\mathbb{R}^3$. The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ endowed with the canonical metric of index one $g \equiv \langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$.

Preliminaries

Let (x, y, z) be standard coordinates of $\mathbb{R}^3$. The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ endowed with the canonical metric of index one $g \equiv \langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$.

The Hyperbolic Plane

The hyperbolic plane denoted as $\mathbb{H}_0^2$ is a space like surface of $\mathbb{L}^3$ and is represented by the top part of the hyperboloid of two sheets.

$$\mathbb{H}_0^2 = \{x^2 + y^2 - z^2 = -1, z > 0\}$$

Preliminaries

Let (x, y, z) be standard coordinates of $\mathbb{R}^3$. The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ endowed with the canonical metric of index one $g \equiv \langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$.

The Hyperbolic Plane

The hyperbolic plane denoted as $\mathbb{H}_0^2$ is a space like surface of $\mathbb{L}^3$ and is represented by the top part of the hyperboloid of two sheets.

$$\mathbb{H}_0^2 = \{x^2 + y^2 - z^2 = -1, z > 0\}$$

For visualization purposes, we identify $\mathbb{H}_0^2$ with the Poincaré disk model

$$(x, y, z) \in \mathbb{H}_0^2 \to \frac{1}{1+z}(x, y) \in \mathbb{D}$$

Preliminaries

The de Sitter 2-Space

The de Sitter 2-space, denoted by $\mathbb{H}_1^2$, is a time like surface of $\mathbb{L}^3$ and is represented by the hyperbolic of one sheet

$$\mathbb{H}_1^2 = \{x^2 + y^2 - z^2 = 1\}.$$

Preliminaries

The de Sitter 2-Space

The de Sitter 2-space, denoted by $\mathbb{H}^2_1$, is a time like surface of $\mathbb{L}^3$ and is represented by the hyperbolic of one sheet

$$\mathbb{H}^2_1 = \{x^2 + y^2 - z^2 = 1\}.$$

We can identify the bottom half $\mathbb{H}^2_{10} = \mathbb{H}^2_1 \cap \{z < 0\}$ with the once punctured unit disk via the diffeomorphism

$$(x, y, z) \in \mathbb{H}^2_{10} \to \frac{1}{x^2 + y^2}(x, y) \in \mathring{\mathbb{D}}$$

Variational Problem

Let $C_*^\infty(I, \mathbb{H}_\epsilon^2)$ be the space of smooth non-null immersed convex curves $\gamma : I \subset \mathbb{R} \to \mathbb{H}_\epsilon^2$ parametrized by the arc length $s \in I$.

Variational Problem

Let $C^{\infty}_{*}(I, \mathbb{H}^2_{\epsilon})$ be the space of smooth non-null immersed convex curves $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2_{\epsilon}$ parametrized by the arc length $s \in I$.

For fixed real number $p \in \mathbb{R}$, the $p-$elastic functional is given by

$$\Theta_p(\gamma) := \int_{\gamma} \kappa^p \, ds,$$

and acts on $C^{\infty}_{*}(I, \mathbb{H}^2_{\epsilon})$.

Euler-Lagrange Equation

The critical points for Θ_p must satisfy the Euler-Lagrange equation:

$$p\frac{d^2}{ds^2}\kappa^{p-1} + \epsilon_1\epsilon_2(p-1)\kappa^{p+1} - \epsilon_2 p\kappa^{p-1} = 0,$$

where $\epsilon_1, \epsilon_2 = \pm 1$.

Euler-Lagrange Equation

The critical points for Θ_p must satisfy the Euler-Lagrange equation:

$$p\frac{d^2}{ds^2}\kappa^{p-1} + \epsilon_1\epsilon_2(p-1)\kappa^{p+1} - \epsilon_2 p\kappa^{p-1} = 0,$$

where $\epsilon_1, \epsilon_2 = \pm 1$.
Therefore $\kappa(s)$ is either constant or a solution to the first-order ordinary differential equation:

$$p^2(p-1)^2\kappa^{2(p-2)}(\kappa')^2 + \epsilon_1\epsilon_2(p-1)^2\kappa^{2p} - \epsilon_2 p^2\kappa^{2(p-1)} = a,$$

where $a \in \mathbb{R}$ is a constant of integration.

Constant Curvature Case

Proposition [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

Let γ be a non-geodesic $p-$elastic circle immersed in $\mathbb{H}^2_\epsilon \subset \mathbb{L}^3$. Then, γ is space like and its constant curvature is given by

$$\kappa = \sqrt{\frac{p}{p-1}}.$$

Equivalently, the radius of γ, viewed as a curve in $\mathbb{L}^3$, is $r = \sqrt{(-1)^\epsilon (p-1)}$.

Constant Curvature Case

Proposition [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

Let γ be a non-geodesic $p-$elastic circle immersed in $\mathbb{H}^2_\epsilon \subset \mathbb{L}^3$. Then, γ is space like and its constant curvature is given by

$$\kappa = \sqrt{\frac{p}{p-1}}.$$

Equivalently, the radius of γ, viewed as a curve in $\mathbb{L}^3$, is $r = \sqrt{(-1)^\epsilon (p-1)}$.

Moreover:

- If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, then $p > 1$ holds.
- If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then $p < 0$ holds.

Existence of Periodic Curvatures

Theorem [A. Pámpano, M. Samarakkody, H. Tran (2025)]

Let γ be a p-elastic curve in $\mathbb{H}^2_\epsilon$ with non-constant periodic curvature. Then γ is a space like curve with $0 > a > a_* := -((-1)^\epsilon p)^p((-1)^\epsilon(p-1))^{1-p}$.
Moreover,

- if it is a hyperbolic curve, then $p > 1$.
- if it is a pseudo-hyperbolic curve, then $p < 0$.

Existence of Periodic Curvatures

Theorem [A. Pámpano, M. Samarakkody, H. Tran (2025)]

Let γ be a p-elastic curve in $\mathbb{H}^2_\epsilon$ with non-constant periodic curvature. Then γ is a space like curve with $0 > a > a_* := -((-1)^\epsilon p)^p((-1)^\epsilon(p-1))^{1-p}$.
Moreover,

- if it is a hyperbolic curve, then $p > 1$.
- if it is a pseudo-hyperbolic curve, then $p < 0$.

Conversely, under the above restriction on p and a, there exists a space like convex $p-$elastic curve $\gamma_a : I \subset \mathbb{R} \to \mathbb{H}^2_\epsilon$ with non-constant curvature κ_a. If, in addition, γ_a is defined on its maximal domain, then it is complete ($I = \mathbb{R}$) and its curvature κ_a is a periodic function.

Parameterization

We use an approach by Langer and Singer involving Killing vector fields along curves to find the parameterization. [7]

Parameterization

We use an approach by Langer and Singer involving Killing vector fields along curves to find the parameterization. [7]

Up to rigid motions, the $p-$elastic curve with periodic curvature γ in $\mathbb{H}^2_\epsilon$ can be parameterized in terms of its arc length parameter $s \in \mathbb{R}$, as

$$\gamma = \frac{1}{\sqrt{-a}}(\sqrt{\epsilon_2 a + p^2\kappa^{2(p-1)}}\cos\theta(s), \sqrt{\epsilon_2 a + p^2\kappa^{2(p-1)}}\sin\theta(s), p\kappa^{p-1}),$$

where

$$\theta(s) := \epsilon_2(p-1)\sqrt{-a}\int \frac{k^p}{\epsilon_2 a + p^2\kappa^{2(p-1)}}ds,$$

is the angular progression and $\epsilon_2 = (-1)^\epsilon$.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0, 0, 1)$.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0, 0, 1)$. If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then it is entirely contained in ${\mathbb{H}^2_1}_0 = \mathbb{H}^2_1 \cap \{z < 0\}$.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0, 0, 1)$. If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then it is entirely contained in $\mathbb{H}^2_{10} = \mathbb{H}^2_1 \cap \{z < 0\}$.
2. The curve γ meets the bounding parallels tangentially at the maximum and minimum values of its curvature.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0, 0, 1)$. If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then it is entirely contained in $\mathbb{H}^2_{1}{}_0 = \mathbb{H}^2_1 \cap \{z < 0\}$.
2. The curve γ meets the bounding parallels tangentially at the maximum and minimum values of its curvature.
3. The angular progression is monotonic with respect to the arc length parameter of the curve.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0,0,1)$. If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then it is entirely contained in ${\mathbb{H}^2_1}_0 = \mathbb{H}^2_1 \cap \{z < 0\}$.
2. The curve γ meets the bounding parallels tangentially at the maximum and minimum values of its curvature.
3. The angular progression is monotonic with respect to the arc length parameter of the curve.
4. The $p-$elastic curve γ is closed if and only if the angular progression along a period of the curvature is a rational multiple of 2π.

Results

Theorem [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

For every $p > 1$ and pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}m$, there exists a closed $p-$elastic curve in the hyperbolic plane $\mathbb{H}_0^2$ with non-constant curvature, winding number n and m-fold symmetry.

Results

Theorem [Á. Pámpano, M. Samarakkody, H. Tran (2025)]

For every $p > 1$ and pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}m$, there exists a closed $p-$elastic curve in the hyperbolic plane $\mathbb{H}^2_0$ with non-constant curvature, winding number n and m-fold symmetry.

Remark

For the pseudo-hyperbolic case the same result follows with $p < 0$.

Uniqueness

According to the numerical experiments, the angular progression is monotonic with respect to a. This would imply that for every $\frac{n}{m} \in \mathbb{Q}$ satisfying $m < 2n < \sqrt{2}m$, there exist a unique closed $p-$elastic curve for every $p \in (-\infty, 0) \cup (1, \infty)$ in $\mathbb{H}^2_\epsilon$ with non-constant curvature, up to isometries. In addition, none of them would be embedded.

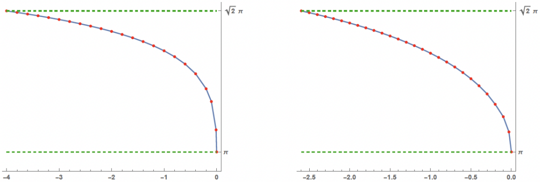


Figure: Angular progression for $p = -1$(Left) and $p = 3/2$(Right). [8]

Uniqueness

The angular progression can be described as a function $\Lambda_p(a) : (a_*, 0) \subset \mathbb{R} \to \mathbb{R}$ given by

$$\Lambda_p(a) := 2p(p-1)^2\sqrt{-a}U,$$

where

$$U = \int_\beta^\alpha \frac{\kappa^{2(p-1)}}{(a + \epsilon_2 p^2 \kappa^{2(p-1)})\sqrt{a - \epsilon_2 (p-1)^2 \kappa^{2p} + \epsilon_2 p^2 \kappa^{2(p-1)}}}\, d\kappa,$$

$\epsilon_2 = (-1)^\epsilon$ and $0 < \beta < \alpha$ are the only positive solutions of

$$f_{p,a}(\kappa) := a - \epsilon_2 (p-1)^2 \kappa^{2p} + \epsilon_2 p^2 \kappa^{2(p-1)} = 0.$$

Uniqueness

Theorem [Á. Pámpano, M. Samarakkody, H. Tran (2025)]
Let $p = 3/2$ and so $\epsilon_2 = 1$. Then, the function $\Lambda_{3/2} : (a_* = -3\sqrt{3}/2, 0) \to \mathbb{R}$ is given by

$$\Lambda_{3/2}(a) = \frac{2\sqrt{\alpha\beta(\alpha+\beta)}}{3\sqrt{2\alpha+\beta}} K(\zeta) + \pi\hat{\Lambda}\left(\arcsin\sqrt{\frac{\chi-\zeta^2}{\chi(1-\zeta^2)}}, \zeta\right),$$

where $\alpha > \beta > 0$ are the only two positive roots of polynomial $\tilde{Q}_{3/2,a}(\kappa) = (-\kappa^3 + 9\kappa + 4a)/4$ and

$$\zeta := \sqrt{\frac{\alpha-\beta}{2\alpha+\beta}}, \quad \chi := \frac{9(\alpha-\beta)}{9\alpha - \alpha\beta(\alpha+\beta)}.$$

Moreover, the function $\Lambda_{3/2}$ decreases monotonically form $\sqrt{2}\pi$ to π.

Results

Closed p-elastic curves $\gamma_{n,m}$ in $\mathbb{H}_0^2$ expand as p varies in $(1, \infty)$. On the other hand, in $\mathbb{H}_1^2$, they shrink as p varies in $(-\infty, 0)$, approaching the unit circle as $p \to 0^-$. In the spherical case, as $p \to 0^+$, they expand to the unit circle and shrink to the point $(0, 0, 1)$ as $p \to 1^-$.

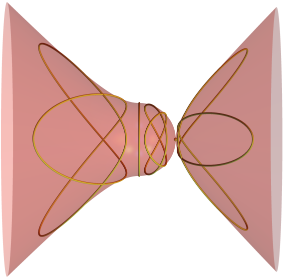
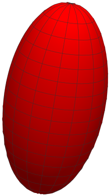


Figure: Evolution of closed $p-$elastic curves of type $\gamma_{2,3}$. [8]

Free Boundary Minimal Surfaces in Spheroids

Free Boundary Minimal Surfaces in Spheroids



The parameterization of Spheroids can be written as

$$\mathbb{E}(t, \theta) = (a\sqrt{1 - t^2}\cos\theta, a\sqrt{1 - t^2}\sin\theta, t),$$

for a is the parameter, $a \in (0, \infty)$, $\theta \in [0, 2\pi)$ and $t \in [-1, 1]$.

Free Boundary Minimal Surfaces

First Variation

Let $\Omega \subset \mathbb{R}^3$ be the domain bounded by a fixed spheroid. For a properly immersed surface $\Sigma \subset \Omega$, let $F : \Sigma \times (-\epsilon, \epsilon) \to \Omega$ be a variation of Σ. Then, the first variation of the area functional can be written as

$$\frac{d}{dt}_{|t=0} \text{Area}(F(\Sigma, t)) = -\int_{\Sigma} \langle F_t, H \rangle \; dA + \int_{\partial\Sigma} F_t^T \cdot \eta \; ds,$$

where F_t is the derivative of F with respect to t, F_t^T is its tangent component, H represents the mean curvature of Σ, and η is the outward-pointing normal vector on $\partial\Sigma$.

Free Boundary Minimal Surfaces

First Variation

Let $\Omega \subset \mathbb{R}^3$ be the domain bounded by a fixed spheroid. For a properly immersed surface $\Sigma \subset \Omega$, let $F : \Sigma \times (-\epsilon, \epsilon) \to \Omega$ be a variation of Σ. Then, the first variation of the area functional can be written as

$$\frac{d}{dt}_{|t=0} \text{Area}(F(\Sigma, t)) = -\int_{\Sigma} \langle F_t, H \rangle \; dA + \int_{\partial\Sigma} F_t^T \cdot \eta \; ds,$$

where F_t is the derivative of F with respect to t, F_t^T is its tangent component, H represents the mean curvature of Σ, and η is the outward-pointing normal vector on $\partial\Sigma$.

Free Boundary Minimal Surfaces

A surface $\Sigma \subset \Omega$ is a **free boundary minimal surface** if and only if its mean curvature vanishes and $\partial\Sigma$ meets $\partial\Omega$ perpendicularly.

Free Boundary Minimal Surfaces

Equatorial disks are a trivial example of free boundary minimal surfaces of a spheroid.

Free Boundary Minimal Surfaces

Equatorial disks are a trivial example of free boundary minimal surfaces of a spheroid.

Are there any other free boundary minimal surfaces in a spheroid?

Existence of the Critical Catenoid inside Spheroids

The Catenoid can be described by the conformal map

$$X : S^1 \times [-T, T] \to \mathbb{R}^3$$

such that

$$X(t, \theta) = (c \cosh(t/c) \cos\theta, c \cosh(t/c) \sin\theta, t),$$

where $\theta \in [0, 2\pi)$, $t \in [-T, T]$, and c is a non-zero constant.

Existence of the Critical Catenoid inside Spheroids

The Catenoid can be described by the conformal map

$$X : S^1 \times [-T, T] \to \mathbb{R}^3$$

such that

$$X(t, \theta) = (c \cosh(t/c) \cos \theta, c \cosh(t/c) \sin \theta, t),$$

where $\theta \in [0, 2\pi)$, $t \in [-T, T]$, and c is a non-zero constant.

Theorem [M. Samarakkody, H. Tran, Á. Pámpano] (preprint)

For fixed $a > 0$, there exists a free boundary catenoid with fixed c and T.

Existence of the Critical Catenoid inside Spheroids

The Catenoid can be described by the conformal map

$$X : S^1 \times [-T, T] \to \mathbb{R}^3$$

such that

$$X(t, \theta) = (c \cosh(t/c) \cos\theta, c \cosh(t/c) \sin\theta, t),$$

where $\theta \in [0, 2\pi)$, $t \in [-T, T]$, and c is a non-zero constant.

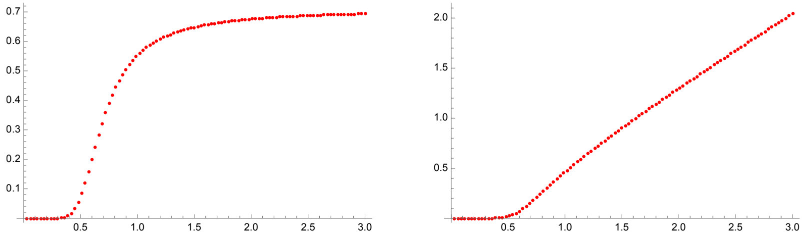
Theorem [M. Samarakkody, H. Tran, Á. Pámpano] (preprint)

For fixed $a > 0$, there exists a free boundary catenoid with fixed c and T.

Remark

From numerical evidence, we can see that this catenoid is unique.

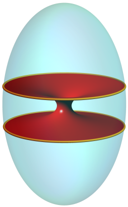
Figures



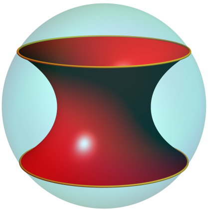
(a) a vs. T (b) a vs. c

Figure: Function $T(a)$ and function $c(a)$

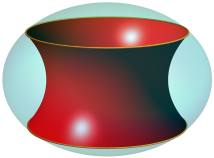
Figures



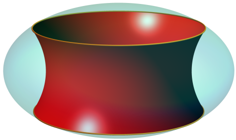
(a) $a = 0.6$



(b) $a = 1$



(c) $a = 1.4$



(d) $a = 1.8$

Second Variation

Second Variation Formula

We can let $F_t = uN$, where N is a unit normal vector field and u is any smooth function. Then the second variation of area is the index bilinear form

$$\begin{aligned} S(u,u) &= \frac{d^2}{dt^2}_{|_{t=0}} \mathrm{Area}(F(\Sigma,t)) \\ &= \int_\Sigma \left(|\nabla^\Sigma u|^2 - |h|^2 u^2 \right) dA + \int_{\partial\Sigma} \langle \nabla_N^\Omega N, \eta \rangle \, dS \end{aligned}$$

where $|h|$ is the norm of the shape operator of Σ and η is the outward conormal vector along $\partial\Sigma$.

Morse Index

Morse Index

The Morse index gives us the number of admissible deformations that decreases area to the second order.

Morse Index

Morse Index

The Morse index gives us the number of admissible deformations that decreases area to the second order.

To calculate the Morse index for our problem, we use the approach by H. Tran. [9].

Morse Index

Morse Index

The Morse index gives us the number of admissible deformations that decreases area to the second order.

To calculate the Morse index for our problem, we use the approach by H. Tran. [9].

The Morse index is equals to:

The number of non-positive eigenvalues of the fixed boundary problem

$+$

The number of eigenvalues, counting multiplicities, that are less than c (a constant) in the Dirichlet-to-Neumann map, where c represents the second fundamental form with respect to the outward unit normal of the spheroid.

Morse Index of the Equatorial Disk

In a spheroid, we have two types of equatorial disks: circular shape and ellipsoidal shape. We will calculate the Morse index for the one with a circular shape.

Morse Index of the Equatorial Disk

In a spheroid, we have two types of equatorial disks: circular shape and ellipsoidal shape. We will calculate the Morse index for the one with a circular shape.

The parameterization of the equatorial disk can be written as

$$\mathcal{D}(s, \theta) = (s\cos\theta, s\sin\theta, 0),$$

where $s \in [0, a]$ and $\theta \in (0, 2\pi]$

Morse Index of the Equatorial Disk

In a spheroid, we have two types of equatorial disks: circular shape and ellipsoidal shape. We will calculate the Morse index for the one with a circular shape.

The parameterization of the equatorial disk can be written as

$$\mathcal{D}(s, \theta) = (s \cos\theta, s \sin\theta, 0),$$

where $s \in [0, a]$ and $\theta \in (0, 2\pi]$

Theorem [M. Samarakkody, H. Tran, Á. Pámpano](preprint)

For a circular-shaped equatorial disk inside a spheroid, the Morse index is given by the number of non-negative integers n such that $n < a^2$.

Future Work

- Proving the uniqueness and existence of the closed curve for general $p-$values in sphere, hyperbolic and pseudo-hyperbolic planes.
- Finding the Morse indices for the ellipse-shaped equatorial disk and the critical catenoid.

References I

Google Image
https://images.app.goo.gl/akRRLWM31yhUrjKg9.
[Accessed 19-01-2025].

J. Arroyo, O. J. Garay, and M. Barros.
Closed free hyperelastic curves in the hyperbolic plane and chen-willmore rotational hypersurfaces.
Israel Journal of Mathematics, 138(1):171–187, Mar 2003.

A. Fraser.
On the free boundary variational problem for minimal disks.
Communications on Pure and Applied Mathematics, 53(8):931–971, 2000.

A. Fraser.
Index estimates for minimal surfaces and k-convexity.
Proceedings of the American Mathematical Society, 135(11):3733–3744, 2007.

References II

A. Fraser and R. Schoen.
The first steklov eigenvalue, conformal geometry, and minimal surfaces.
Advances in Mathematics, 226(5):4011–4030, 2011.

A. Gruber, Á. Pámpano, and M. Toda.
Instability of closed p-elastic curves in $\mathbb{S}^2$.
Analysis and Applications, 21(06):1533–1559, 2023.

J. Langer and D. A. Singer.
The total squared curvature of closed curves.
Journal of Differential Geometry, 20(1):1 – 22, 1984.

Á. Pámpano, M. Samarakkody, and H. Tran.
Closed p-elastic curves in spheres of $\mathbb{L}^3$.
Journal of Mathematical Analysis and Applications, 545(2):129147, 2025.

References III

H. Tran.
Index characterization for free boundary minimal surfaces.
Communications in Analysis and Geometry, 28(1):189–222, 2020.

Thank You!!!