# MAT102 - College Algebra - Polynomial and Rational Functions

## 3.2 Introduction to Polynomial Functions [1]

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# Determine the End Behavior of a Polynomial Function

## Definition of a Polynomial Function

Let $n$ be a natural number and $a_n, a_{n-1}, \ldots, a_1, a_0$ be real numbers, where $a_n \neq 0$. Then a function defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is called a **Polynomial function of degree** $n$.

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Examples for non-polynomial functions.

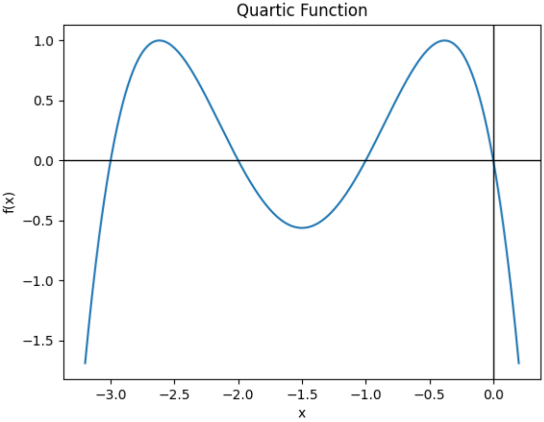
# Several Special Cases of Polynomial Functions

Let $a \neq 0$.

| | | |
|---|---|---|
| $f(x) = c$ | constant function | degree 0 |
| $g(x) = ax + b$ | linear function | degree 1 |
| $h(x) = ax^2 + bx + c$ | quadratic function | degree 2 |
| $j(x) = ax^3 + bx^2 + cx + d$ | cubic function | degree 3 |
| $k(x) = ax^4 + bx^3 + cx^2 + dx + e$ | quartic function | degree 4 |

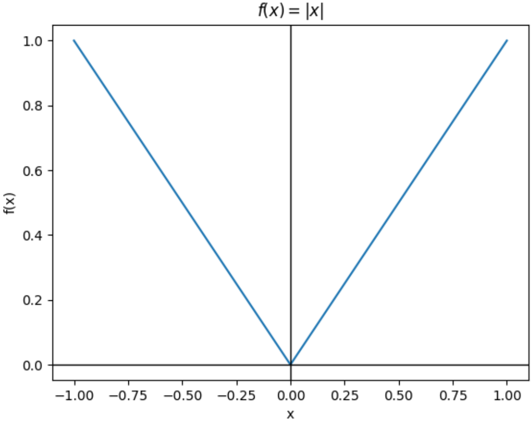
# Smoothness and Continuity

## Smooth and Continuous



# Smoothness and Continuity

Not Smooth



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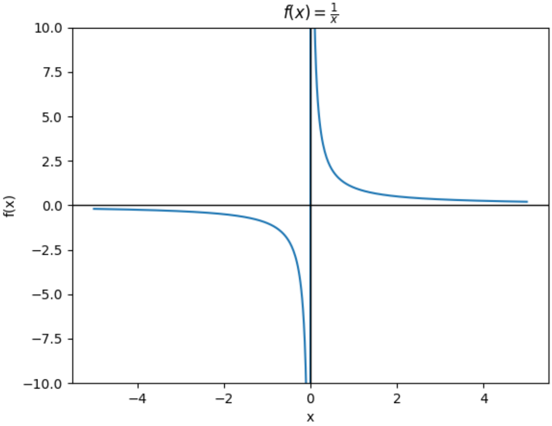

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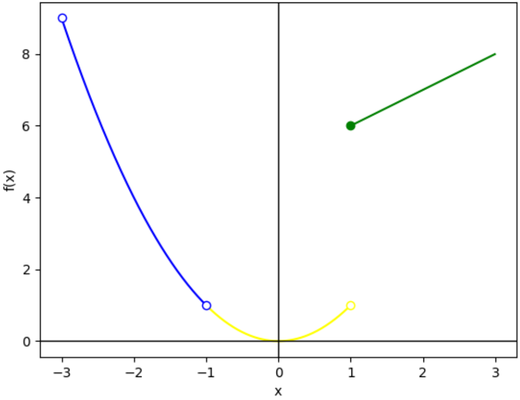

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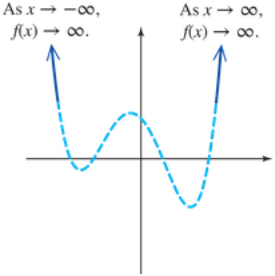
The leading term has the greatest exponent on $x$.

# The Leading Term Test

Consider a polynomial function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

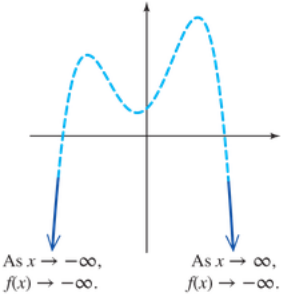
**$n$ is even and $a_n$ positive**




End behavior: up left/up right

# The Leading Term Test

**$n$ is even and $a_n$ negative**




End behavior: down left/down right

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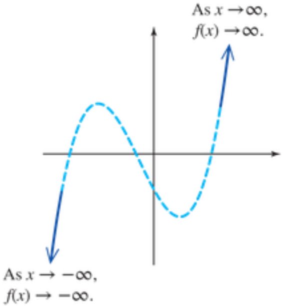
**$n$ is odd and $a_n$ positive**

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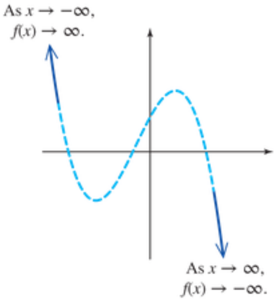


End behavior: up left/down right

# Example - Determining End Behavior

Use the leading term to determine the end behavior of the graph of the function.

1. $f(x) = -4x^5 + 6x^4 + 2x$
   pause
2. $g(x) = \frac{1}{4}x(2x-3)^3(x+4)^2$

# Example - Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by

$$f(x) = x^3 + x^2 - 9x - 9.$$

# Example - Determine the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = -x^3 + 8x^2 - 16x$.

# Touch Points and Cross Points

Let $f$ be a polynomial function and let $c$ be a real zero of $f$. Then the point $(c, 0)$ is an $x-$intercept of the graph of $f$. Furethermore,

- If $c$ is a zero of odd multiplicity, then the graph crosses the $x-$axis at $c$. The point $(c, 0)$ is called a **cross point**.

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- If $c$ is a zero of odd multiplicity, then the graph crosses the $x-$axis at $c$. The point $(c, 0)$ is called a **cross point**.
- If $c$ is a zero of even multiplicity, then the graph touches the $x-$axis at $c$ and turns back around. The point $(c, 0)$ is called a **touch point**.

# Determining Zeros and Multiplicities

Determine the zeros and their multiplicities for the given functions.

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- $m(x) = \dfrac{1}{10}(x-4)^2(2x+5)^3$
- $n(x) = x^4 - 2x^2$

# Intermediate Value Theorem

Let $f$ be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then $f$ has at least one zero on the interval $[a, b]$.

# Example - Applying the Intermediate Value Theorem

Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval $[1, 2]$.

# Number of Turning Points of a Polynomial Function

Let $f$ represent a polynomial function of degree $n$. Then the graph of $f$ has at most $n - 1$ turning points.

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4. Plot the $x-$and $y-$intercepts and sketch the end behavior.

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4. Plot the $x-$and $y-$intercepts and sketch the end behavior.
5. Draw a sketch starting from the left left-end behavior. Connect the $x-$end and $y-$intercepts in the order that they appear from left to right using these rules.
   - 5.1 The curve will cross the $x-$axis at an $x-$ intercept if the corresponding zero has an odd multiplicity.
   - 5.2 The curve will touch but not cross the $x-$axis at an $x-$intercept if the corresponding zero has an even multiplicity.

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6. If a test for symmetry is easy to apply, use symmetry to plot addiotional points.
   6.1 $f$ is an even function if $f(-x) = f(x)$ - symmetric to the $y-$axis.
   6.2 $f$ is an odd function if $f(-x) = -f(x)$ - symmetric to the origin.

# Example - Graphing a Polynomial Function

Graph $f(x) = x^3 - 9x$.

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# References

Julie Miller and Donna Gerken.
*College Algebra*.
McGraw-Hill Education, New York, 2nd edition, 2016.