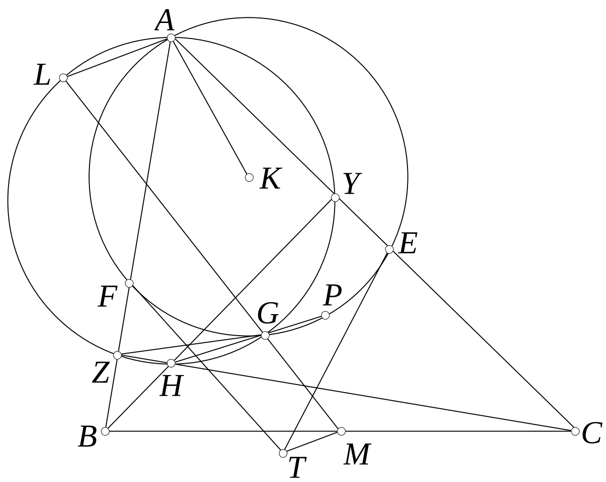
Proposed problems for TST

Tran Quang Hung

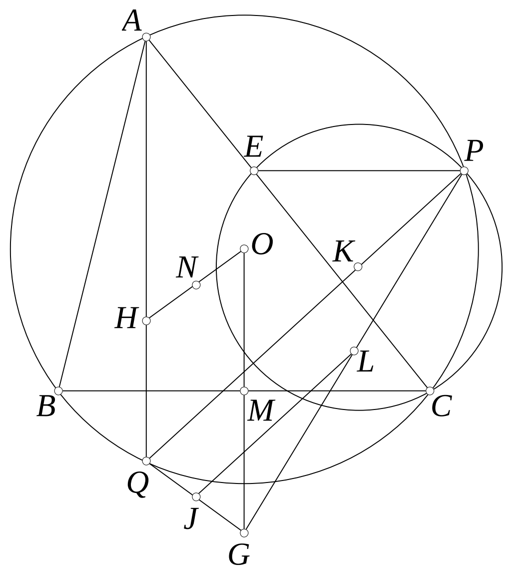
Problem 1 (Prepared)**.** Let ABC be a triangle with orthocenter H. P is a point. (K) is the circle with diameter AP. (K) cuts CA, AB again at E, F. PH cuts (K) again at G. Tangent line at E, F of (K) intersect at T. M is midpoint of BC. L is the point on MG such that $AL \parallel MT$. Prove that $LA \perp LH$.



Hnh 1.

Solution. Let BY, CZ be altitudes of ABC. HP cuts (K) again at G, thus $AG \perp PH$ so A, Z, H, G, Y lie on circle diameter AH. Easily seen MY, MZ are tangent to circle diameter AH. We have the tangent line at E, F of (K) intersect at T. From $\triangle GEF \sim \triangle GYZ$ we have $\triangle GZT \sim \triangle GMF$ deduce $\triangle GZF \sim \triangle GMT$ so $\angle GMT = \angle GZA$ but $\angle GMT = \angle GLA$ from $AL \parallel GT$, therefore $\angle GLA = \angle GZA$ so G, Z, L, A are cyclic. Thus, L lies on circle diameter AH, so $AL \perp AH$. $\square$

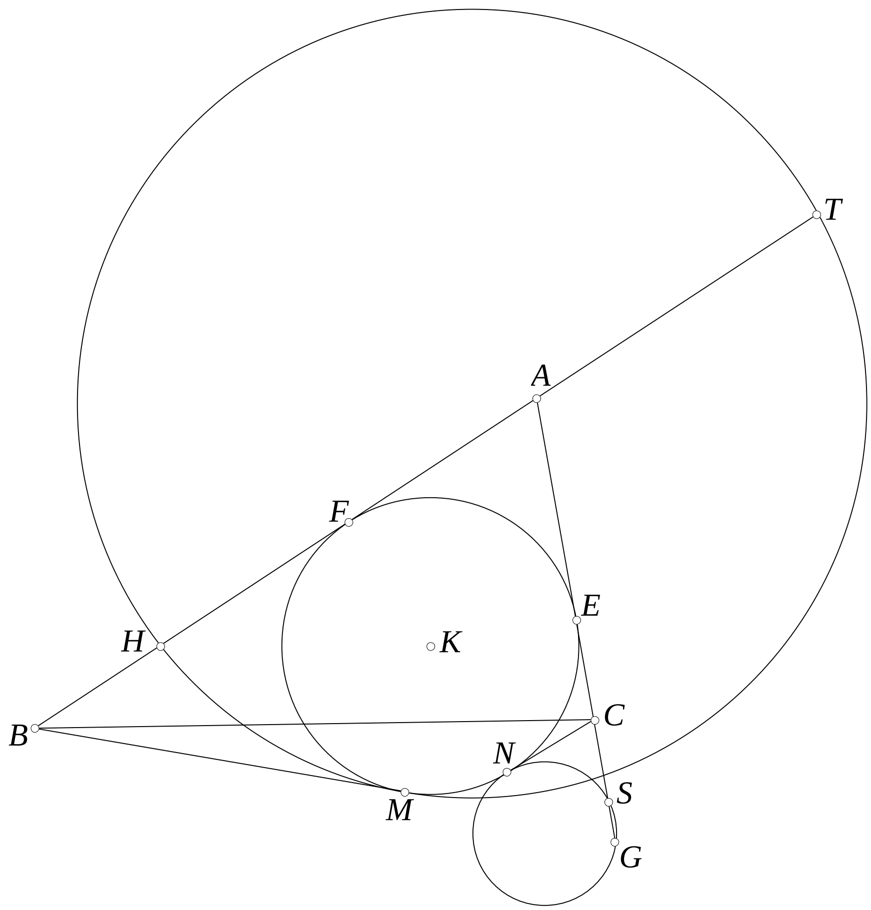
Problem 2 (Hard)**.** Let ABC be a triangle inscribed circle (O). P lies on (O). The line passes through P and parallel to BC cuts CA at E. K is circumcenter of triangle PCE and L is nine point center of triangle PBC. Prove that the line passes through L and parrallel to PK, always passes through a fixed point when P moves.



Hnh 2.

Solution. Let H, N be orthocenter and nine point center of triangle ABC. Let M be midpoint of BC, and G is symmetric of P through L. We are well known that G is reflection of O through BC. AH cuts (O) again at Q. Because of $PE \parallel BC$ we have $\angle CEP = \angle ACB$, so $\angle CPK = 90° - \angle CEP = 90° - \angle ACB = \angle CAQ = \angle CPQ$ thus P, K, Q are collinear. The line passes through L parallel to PK cuts GQ at J. Thus, J is midpoint of LQ. Quadrilateral $OHQG$ is isoceles trapezoid and NJ is its median line. Therefore, J is relfeciton of N through BC. Therefore, the line passes through L and parrallel to PK, always passes through a fixed point, it is the reflection of N through BC. $\square$

Problem 3 (Medium)**.** Let ABC be a triangle and (K) is a circle that touches segments CA, AB at E, F, reps. M, N lie on (K) such that BM, CN are tangent to (K). G, H are symmetric of A through E, F. The circle passes through G and touches to (K) at N that cuts CA again at S. The circle passes through H and touches (K) at M that cuts AB again at T. Prove that the line passes through K and perpendicular to ST always passes through a fixed point when (K) changes.

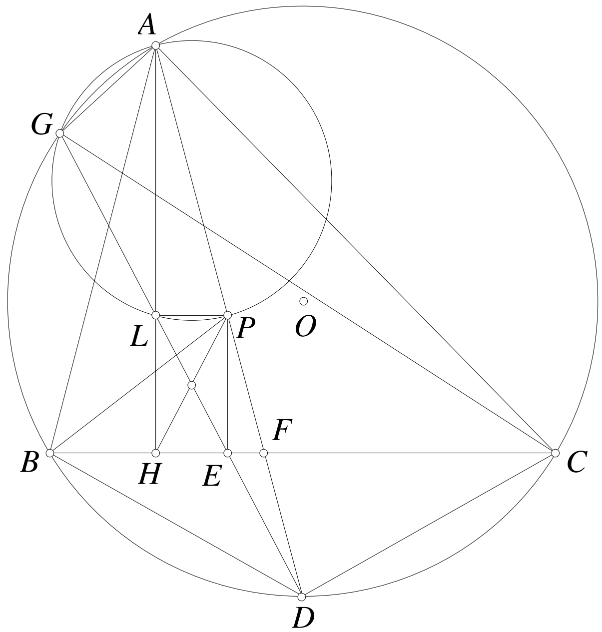


Hnh 3.

Solution. We have $CE^2 = CM^2 = CG.CS$ so $\dfrac{CE}{CS} = \dfrac{CG}{CE} = \dfrac{CE+CG}{CS+CE} = \dfrac{EG}{ES} = \dfrac{EA}{ES}$ deduce $\dfrac{EA}{CE} = \dfrac{ES}{CS} = \dfrac{EA+ES}{CE+CS} = \dfrac{AS}{ES}$ thus $SE^2 = SA.SC$. From this S lies on radical axis of (K) and circumcircle (O) of triangle ABC. Simiarly, with T. Thus ST is radical axis of (K) and (O). Therefore the line passes through K and perpendicular to ST always passes through fixed point O. We are done.

$\square$

Problem 4 (Easy)**.** Let ABC be acute triangle inscribed circle (O), altitude AH, H lies on BC. P is a point that lies on bisector $\angle BAC$ and P is inside triangle ABC. Circle diameter AP cuts (O) again at G. L is projection of P on AH. Assume that GL bisects HP. Prove that P is incenter of ABC.



Hnh 4.

Solution. Let AP cuts BC at F and cuts (O) again at D. Note that $LP \parallel BC$ so we have $\angle AGL = \angle LPF = \angle BFD = \angle FCD + \angle FDC = \angle FBD + \angle AGC = \angle DGC + \angle AGC = \angle AGD$. From this G, L, D are collinear. GL cuts BC at E. Because GL bisects PH so $PLHE$ is rectangle. Follow Thales's theorem $\dfrac{DP}{DA} = \dfrac{PE}{AL} = \dfrac{LH}{AL} = \dfrac{PF}{AP}$ deduce $\dfrac{DA}{AP} = \dfrac{DP}{PF}$ or $\dfrac{DA}{DA - AP} = \dfrac{DP}{DP - PF}$ deduce $\dfrac{DA}{DP} = \dfrac{DP}{DF}$. Thus, $DP^2 = DF.DA$. We have $\angle DBF = \angle DAC = \angle DAB$ so triangles DBF and DAB are similar. Deduce $BD^2 = DF.DA = DP^2$. From this, $\angle PBC = \angle PBD - \angle CBD = \angle BPD - \angle CAD = \angle BPD - \angle DAB = \angle PBA$. Thus, P is incenter of triangle ABC. We are done. $\square$

Proposed problems for Junior Bankan

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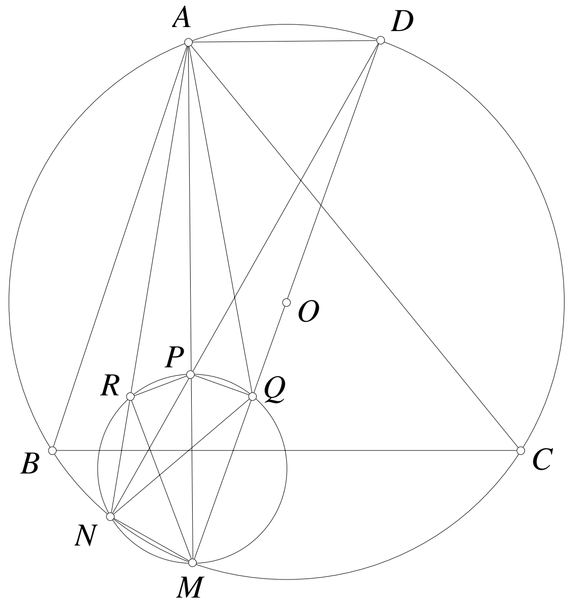
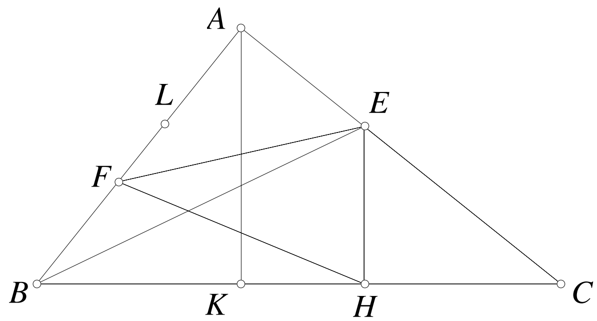
Problem 5. Let ABC be an acute triangle inscried circle (O). M lies on small arc $\overset{\frown}{BC}$. P lies on AM. Circle diameter MP cuts (O) again at N. MO cuts circle diameter MP again at Q. AN cuts circle diameter MP again at R. Prove that $\angle PRA = \angle PQA$.



Hnh 5.

Solution. Let MD be diameter of (O). We have $DN \perp MN \perp NP$. From this N, P, D are collinear. MD is diameter of (O) then $\angle PAD = 90°$. Q lies on circle diameter MP then $\angle PQD = 90°$. Thus $APQD$ is cyclic deduce $\angle NAP = \angle NDM = \angle PAQ$ and $\angle PQN = \angle PMN = \angle ADN = \angle AQP$. Therefore AP is bisector of $\angle NAQ$ and QP is bisector $\angle NQA$. And $\angle PMQ = \angle AMD = \angle AND = \angle RNP = \angle RMP$. From this $\triangle ARM = \triangle AQM(a.s.a)$. We deduce $\angle ARM = \angle AQM$ but $\angle PRM = \angle PQM = 90°$ deduce $\angle PRA = \angle PQA$. We are done. $\square$

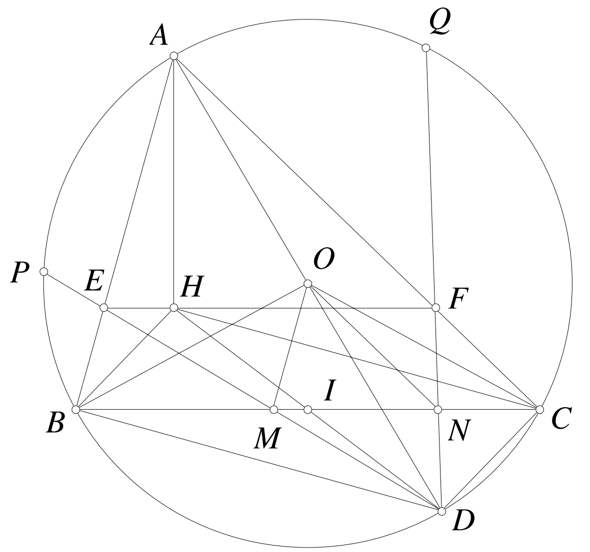
Problem 6. Let ABC be right triangle with hypotenus BC, bisector BE, E lies on CA. Assume that circumcircle of triangle BCE cuts segment AB again at F. K is projection of A on BC. L lies on segment AB such that $BL = BK$. Prove that $\frac{AL}{AF} = \sqrt{\frac{BK}{BC}}$.



Hnh 6.

Solution. Let H be projection of E on BC. We easily seen $EA = EH, BH = BA$ and $BCEF$ is cyclic so $\angle AFE = \angle ECH$ deduce AFE and HCE are congruent. From this, $BC = HB + HC = BA + AF$. We have $BA^2 = BK.BC = BK(BA + AF)$ deduce $BK.AF = BA(BA - BK) = BA(BA - BL) = BA.AL$. From this $\frac{AL}{AF} = \frac{BK}{BA} = \frac{BA}{BC}$ thus $\frac{AL}{AF} = \sqrt{\frac{BK}{BA}.\frac{BA}{BC}} = \sqrt{\frac{BK}{BC}}$. We are done. $\square$

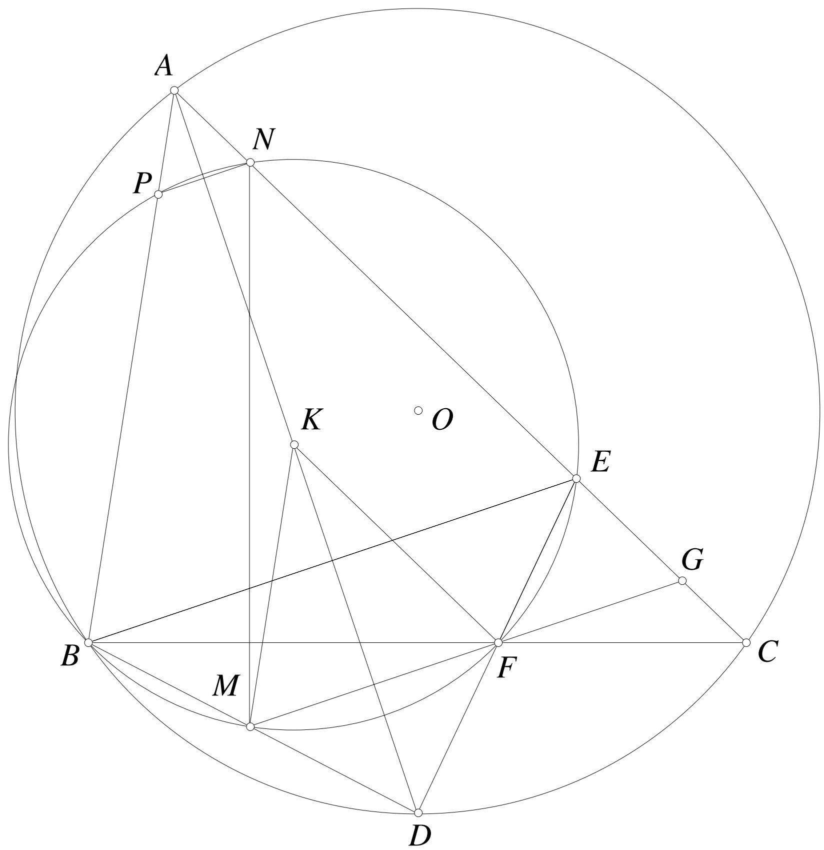
Problem 7. Let ABC be acute triangle inscribed circle (O). AD is diamater of (O). M, N lie on BC such that $OM \parallel AB, ON \parallel AC$. DM, DN cut (O) again at P, Q. Prove that $BC = DP = DQ$.



Hnh 7.

Solution. We easily seen $HCDB$ is parallelogram so H and D are symmetric through midpoint I of BC. Let DM, DN cut CA, AB at E, F, reps. Because O is midpoint of AD and $OM \parallel AE$ so M is midpoint of DE. Similarly N is midpoint of DF. Triangle FDB right at B and M is midpoint of FD so $MB = MD$. But $OB = OD$ thus OM is perpendicular bisector of BD therefore MO is bisector of $\angle FMN$. Similarly, NO is bisector of $\angle ENM$ deduce O is D-excenter of DMN. So O is equidistance to MN, DM and DN so $DP = DQ = BC$. We are done. $\square$

Problem 8. Let ABC be acute triangle with $AB < AC$ inscribed cirle (O). Bisector of $\angle BAC$ cuts (O) again at D. E is reflection of B through AD. DE cuts BC at F. Let (K) be circumcircle of triangle BEF. BD, EA cut (K) again at M, N, reps. Prove that $\angle BMN = \angle KFM$.



Hnh 8.

Solution. We easily seen E lies on AC. AD is perpedicular bisector of BE so K lies on AD or AD is reflection axis of (K). Hence if AB cuts (K) again at P then $AN = AP$. Thus $\angle BMN = \angle ANP = 90^\circ - \angle NAD = 90^\circ - \angle CBD$. We deduce $MN \perp BF$, therefore $\angle BMN = \angle KMF = \angle KFM$. $\square$