Đề bài

1 Đề bài

1. (Czech-Polish-Slovak Junior Match 2016, team p6 CPSJ) Let k be a given positive integer. Find all triples of positive integers a,b,c, such that $a+b+c=3k+1$, $ab+bc+ca=3k^2+2k$.

   Suppose that (a,b,c) is a solution and $a \geq b \geq c$.

   $2=2(3k+1)^2-6(3k^2+2k)$
   $=2(a+b+c)^2-6(ab+bc+ca)$
   $=(a-b)^2+(b-c)^2+(a-c)^2$
   We consider two cases (1),(2) :

   (1) $a-c=b-c=1$ Since $a+b+c=3k+1$, we get $(a,b,c)=(k+1,k,k)$

   (2) $b-c=1,\ a=b$ Since $a+b+c=3k+1$, we have $b+b+(b-1)=3k+1$. However, in this case, we get $-1 \equiv 1 \pmod 3$, which is impossible.

   Conversely, we can verify that $(a,b,c)=(k+1,k,k)$ is a solution, of course.

   Conclusion : $\boxed{(a,b,c)=(k+1,k,k)}$ is the only solution.

2. Let x and y be real numbers such that $x^2+y^2-1<xy$. Prove that $x+y-|x-y|<2$.

   $x^2+y^2-1<xy \iff xy<1-(x-y)^2 \leq 1 \implies \min(x,y)<1 \implies x+y-|x-y|=2\min(x,y)<2$

3. Prove that for all real numbers x,y holds $(x^2+1)(y^2+1) \geq 2(xy-1)(x+y)$. For which integers x,y does equality occur?

   $(x^2+1)(y^2+1)=(xy-1)^2+(x+y)^2 \geq 2(xy-1)(x+y)$

   Equality, if $xy-1=x+y$ and as @above...

4. Decide if there are primes p,q,r such that $(p^2+p)(q^2+q)(r^2+r)$ is a square of an integer.

   Assume that there are primes p,q,r and an integer n, such that

   $$(p^2+p)(q^2+q)(r^2+r)=n^2.$$

   WLOG $p \geq q,r$. Since $p|n^2 \Rightarrow p^2|n^2$ we have

   $$p^2|(q^2+q)(r^2+r).$$

   If $p|q$, then $p=q$ and r^2+r is a perfect square. Contradiction. Hence, $p|(q+1)(r+1)$. WLOG $p|q+1$. As $p \geq q$, this implies $p=q+1$, so $p=3, q=2$. Hence, $r=2$ or $r=3$. In both cases we get a contradiction, since 2^2+2 and 3^2+3 aren't perfect squares. Therefore, there are no solutions.

5. On the board are written 100 mutually different positive real numbers, such that for any three different numbers a,b,c is a^2+bc is an integer. Prove that for any two numbers x,y from the board , number $\frac{x}{y}$ is rational.

   Let S be a set of all numbers written on the board. Let (a,b) be an arbitrary pair of distinct elements of S. We want show that b/a is a rational number.

   Let c be an arbitrary element of S with $c \notin \{a,b\}$. Let d be an arbitrary element of S with $d \notin \{a,b,c\}$.

   We can take $x \in \mathbb{N}$ such that $a^2+bc=x\ \cdots(1)$. We can take $y \in \mathbb{N}$ such that $b^2+ac=y \cdots(2)$. We can take $x \in \mathbb{N}$ such that $a^2+bd=z \cdots(3)$. We can take $w \in \mathbb{N}$ such that $b^2+ad=w \cdots(4)$.

From (1) and (2), we have $x - by/a = a^2 - b^3/a \ \cdots(5)$. From (3) and (4), we have $z - bw/a = a^2 - b^3/a \cdots (6)$.

From (5) and (6), we have $(y-w)\frac{b}{a} = x - z \ \cdots(7)$. Since $c \neq d$, we have $y \neq w \ \cdots(8)$.

From (7) and (8), we have $\frac{b}{a} = \frac{x-z}{y-w}$, which is a rational number.

6. For natural numbers a, bc it holds that $(a+b+c)^2 | ab(a+b) + bc(b+c) + ca(c+a) + 3abc$. Prove that $(a+b+c)|(a-b)^2 + (b-c)^2 + (c-a)^2$

   So, we have $a+b+c | ab+bc+ca$ Let $ab+bc+ca = k(a+b+c)$ $(a-b)^2+(b-c)^2+(c-a)^2 = 2(a+b+c)^2 - 6(ab+bc+ca) = (a+b+c)(2(a+b+c)-6k)$ Q.E.D

7. Given a set $S \subset R^+$, $S \neq \emptyset$ such that for all $a, b, c \in S$ (not necessarily distinct) then $a^3+b^3+c^3-3abc$ is rational number. Prove that for all $a, b \in S$ then $\frac{a-b}{a+b}$ is also rational.

   Let $a, b \in S$ and assume wlog that $a \neq b$. We know that

   $$u = 2a^3 + b^3 - 3a^2 b \in \mathbb{Q}, \quad v = a^3 + 2b^3 - 3ab^2 \in \mathbb{Q}.$$

   Hence,

   $$u + v = 3(a^3 + b^3 - a^2 b - ab^2) = 3(a+b)(a-b)^2 \in \mathbb{Q}$$

   and

   $$u - v = a^3 - 3a^2 b + 3ab^2 - b^3 = (a-b)^3 \in \mathbb{Q}.$$

   This implies $\frac{a-b}{a+b} = \frac{3(u-v)}{u+v} \in \mathbb{Q}$.

8. Determine all triples (x, y, z) of positive rational numbers with $x \leq y \leq z$ such that $x+y+z, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, and xyz are natural numbers.

   Let $x = \frac{a}{b}$, $y = \frac{c}{d}$, and $z = \frac{m}{n}$, where $a, b, c, d, m, n \in \mathbb{Z}$, and $\gcd(a,b) = \gcd(c,d) = \gcd(m,n) = 1$.

   The numbers $n(x+y+z) = \frac{n}{bd}(ad+bc) + m$ and $m(x+y+z) = \frac{m}{ac}(ad+bc) + n$ are integers.

   Then, the numbers $\frac{n}{bd}(ad+bc) = \frac{n^2}{bdn}(ad+bc)$, $\frac{m}{ac}(ad+bc) = \frac{m^2}{acm}(ad+bc)$, and $xyz = \frac{acm}{bdn}$ are integers, so $bdn | n^2(ad+bc)$ and $bd | bdn | acm | m^2(ad+bc)$.

   Thus, $bdn | \gcd(n^2(ad+bc), m^2(ad+bc)) = (ad+bc)\gcd(n^2, m^2) = ad+bc$, so $\frac{ad+bc}{bd} = x+y$ is an integer.

   Therefore, $x+y+z-(x+y) = z$ is an integer, and so are x and y by symmetry. Without loss of generality, let $x \leq y \leq z$. We use cases.

   Suppose $x = 1$. Then $\frac{1}{y} + \frac{1}{z} \leq 2$, so $\frac{1}{y} + \frac{1}{z} = 2$ or $\frac{1}{y} + \frac{1}{z} = 1$.

   In the former case, $y = z = 1$, and in the second case, $y = z = 2$. Now suppose $x = 2$. Then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \frac{3}{2}$, so $\frac{1}{y}+\frac{1}{z} = \frac{1}{2}$. Then either $y = z = 4$, or $y = 3$ and $z = 6$, because if $z \geq 7$, we have $2 < y < 3$, which is impossible. Now suppose $x = 3$. Then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq 1$, so $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, and therefore $\frac{1}{y}+\frac{1}{z} = \frac{2}{3}$. Then $y = z = 3$. Now suppose $x \geq 4$. Then $0 < \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \frac{3}{4} < 1$, which is impossible because $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is an integer. Thus, the only solutions up to order are $(x,y,z) = (1,1,1), (1,2,2), (2,4,4), (2,3,6), (3,3,3)$.

9. (Czech-Polish-Slovak Match Junior 2019, team p1 CPSJ)Rational numbers a, b are such that $a+b$ and a^2+b^2 are integers. Prove that a, b are integers.

   $(a+b)^2 - a^2 - b^2 = 2ab$, that should be an integer. Let $a = \dfrac{x}{y}$, $b = \dfrac{w}{z}$, with $gcd(x,y) = gcd(w,z) = 1$ (1).

   We'll prove that $y = z$:

For that, note that if $a+b=n$, then $xyn = xz+yw$, and that tells us that z divides y. But if $a^2+b^2=m$, then $(yz)^2.m = (xz)^2+(yw)^2$, and that tells us that y divides z.

Thus $y=z=L$. Since $2ab = 2xw/yz = 2xw/(L^2)$, but since (1), L has to be 1, because $L^2=2$ don't give us solutions on the integers.

So we finish, because $a=x$ and $b=w$.

Q.E.D

10. Find all integers a such that $\sqrt{\frac{9a+4}{a-6}}$ is rational number.

$\sqrt{\dfrac{9a+4}{a-6}}$ is a rational number $\rightleftarrows$ $(9a+4)(a-6)$ is a perfect square

$(9a+4)(a-6) = s^2 \quad (s>0, s\in\mathbb{Z})$

$\rightleftarrows\ (9a+3s-25)(9a-3s-25) = 29\cdot 29$

$\rightleftarrows\ 9a+3s-25 = -1 \ \wedge\ 9a-3s-25 = -841$

$\rightleftarrows\ (a,s) \ = \ (-44,\ 140)$

Conclusion: $a=-44$ is the only solution.

11. Let S be a set of rational numbers with the following properties: (a) $\frac{1}{2}$ is an element in S, (b) if x is in S, then both $\frac{1}{x+1}$ and $\frac{x}{x+1}$ are in S. Prove that S contains all rational numbers in the interval $(0,1)$.

Note that $1/2\in S$, and $1/2/(1/2+1) = 1/3\in S$ as well.

Furthermore, note that $1/(1/2+1) = 2/3 \in S$.

We now prove by (strongly) inducting on ℓ that for every $\ell\geqslant 2$, $k/\ell\in S$, where $1\leqslant k\leqslant \ell-1$, and $(k,\ell)=1$.

Note that the base cases $\ell = 2,3$ are already established above.

Suppose the assertion holds for every positive integer up to $\ell-1$. Let $k\in[1,\ell-1]\cap\mathbb{Z}$ with $(k,\ell)=1$. Suppose $\ell-k<k$. Then, $\frac{1}{\frac{\ell-k}{k}+1} = \frac{k}{\ell}\in S$, as requested.

Suppose now $\ell-k>k$. Then $\frac{\frac{k}{\ell-k}}{1+\frac{k}{\ell-k}} = \frac{k}{\ell}\in S$ (note that in both cases above, I've used the facts that $(k,\ell)=(\ell-k,k)=1$, and inductive hypotheses applied on k and $\ell-k$, respectively). This finishes the conclusion

12. Find all pairs (a,b) of positive rational numbers such that $\sqrt[b]{a}=ab$

First, notice that for any positive rational number a, $(a,1)$ is a solution. Now, suppose $b\neq 1$. We prove first the following: Lemma: If $0<p<q$ are positive integers such that $\gcd(p,q)=1$ and $0<x<1$ is a rational number such that $x^{q-p}=\frac{p}{q}$, then $q=p+1$. Proof: Write $x=\frac{u}{v}$ where $\gcd(u,v)=1$ and $0<u<v$. Let $d=q-p$, then $v^d-u^d=d$, but $v^d-u^d\geq(u+1)^d-u^d=\sum_{i=0}^{d-1}\binom{d}{i}u^i\geq d$ with equality if and only if $d=1$.

Now, write $a=\prod_i p_i^{\alpha_i}$ and $b=\prod_i p_i^{\beta_i}$, where $\alpha_i,\beta_i\in\mathbb{Z}$. Write $b=\frac{p}{q}$ where p,q are positive integers s.t. $\gcd(p,q)=1$. The equation implies $q\alpha_i = p(\alpha_i+\beta_i)$, therefore $\alpha_i = p\gamma_i$ for some $\gamma_i\in\mathbb{Z}$, hence $\beta_i=(q-p)\gamma_i$. Let $x=\prod_i p_i^{\gamma_i}$, then $a=x^p$ and $b=x^{q-p}=\frac{p}{q}$.

Case 1: $q>p$ Using the lemma, we get $b=\frac{p}{p+1}$ and $a=\left(\frac{p}{p+1}\right)^p$, which is a solution whenever $p>0$ is an integer.

Case 1: $q < p$ Interchange p and q and apply the lemma to get $b = \frac{q+1}{q}$ and $a = \left(\frac{q}{q+1}\right)^{q+1}$, which is a solution whenever $q > 0$ is an integer.

Conclusion: The solution are: $(a,1)$ for any rational number $a > 0$. $\left(\left(\frac{p}{p+1}\right)^{p}, \frac{p}{p+1}\right)$ for any positive integer p. $\left(\left(\frac{p}{p+1}\right)^{p+1}, \frac{p+1}{p}\right)$ for any positive integer p.

13. (1989 ITAMO p1) Determine whether the equation $x^2 + xy + y^2 = 2$ has a solution (x, y) in rational numbers.

    Equivalent to $p^2 + pq + q^2 = 2c^2$ where $p, q, c \in \mathbb{Z}$, $c \neq 0$

    Which is $(2p+q)^2 + 3q^2 = 8c^2$ Looking at this equation $\pmod 3$, we get $2p + q \equiv c \equiv 0 \pmod 3$ and so $q \equiv 0 \pmod 3$ and so $p \equiv 0 \pmod 3$ Dividing p, q, c by 3 we get a new similar equation and infinite descent implies $p = q = c = 0$

    And so $\boxed{\text{No rational roots}}$ for the original equation (since we'd need $c \neq 0$)

14. $$A = \left\{a + b\sqrt{2}, a, b \in \mathbb{Q}\right\}, B = \left\{a + b\sqrt[3]{2}, a, b \in \mathbb{Q}\right\}$$

    .Find

    $$A \cap B$$

    Write $a + b\sqrt{2} = c + d\sqrt[3]{2}$, write it $(a-c) + b\sqrt{2} = d\sqrt[3]{2}$ and cube this equality

    You get $b = a - c = d = 0$

    And so $A \cap B = \mathbb{Q}$

15. Prove that $x^2 + y^2 + z^2 = x + y + z + 1$ does not have solutions in $\mathbb{Q}$

    Assume that there is one. Then, multiplying both sides by 4, and rearranging, we get $(2x-1)^2 + (2y-1)^2 + (2z-1)^2 = 7$.

    In particular, $u^2 + v^2 + t^2 = 7$ has a solution in rational numbers.

    Now, as an implication, it also follows that $m^2 + n^2 + k^2 = 7\ell^2$ has a solution in integers, with $\ell \neq 0$ (here, ℓ being the common denominator of u, v, t).

    Now, we prove that this equation has no solutions besides $(m, n, k, \ell) = (0, 0, 0, 0)$.

    Suppose there is one. By clearing out (m, n, k, ℓ), we may suppose $(m, n, k, \ell) = 1$. Now, $m^2 + n^2 + k^2 + \ell^2 \equiv 0 \pmod 8$. However, this implies, m, n, k, ℓ are all even; which contradicts with $(m, n, k, \ell) = 1$. Thus, we are done.

16. (Greece JBMO TST 2009 p3) Given are the non zero natural numbers a, b, c such that the number $\frac{a\sqrt{2}+b\sqrt{3}}{b\sqrt{2}+c\sqrt{3}}$ is rational. Prove that the number $\frac{a^2+b^2+c^2}{a+b+c}$ is an integer .

    Rationalizing the denominator, we have

    $$\frac{a\sqrt{2}+b\sqrt{3}}{b\sqrt{2}+c\sqrt{3}} = \frac{(a\sqrt{2}+b\sqrt{3})(b\sqrt{2}-c\sqrt{3})}{(b\sqrt{2}+c\sqrt{3})(b\sqrt{2}-c\sqrt{3})} = \frac{(2ab-3bc)+(b^2\sqrt{6}-ac\sqrt{6})}{2b^2-3c^2}.$$

    Since the number is rational and a, b, c are non-zero natural numbers, we have

    $$(b^2 - ac)\sqrt{6} = 0 \Rightarrow b^2 = ac.$$

    Recall the expansion

    $$(a+b+c)^2 = a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

we plug in $ac = b^2$ and get

$$(a+b+c)^2 = a^2+b^2+c^2-2b(a+b+c) \Rightarrow (a+3b+c)(a+b+c) = a^2+b^2+c^2,$$

so

$$\frac{a^2+b^2+c^2}{a+b+c} = a+3b+c,$$

which is an integer, as desired.

17. Prove that

$$\sqrt[100]{\sqrt{3}+\sqrt{2}}+\sqrt[100]{\sqrt{3}-\sqrt{2}}$$

is irrational. Easy to show that $a+\frac{1}{a}\in\mathbb{Q}$ implies $a^n+\frac{1}{a^n}\in\mathbb{Q}$

Choosing $a = \sqrt[100]{\sqrt{3}-\sqrt{2}}$ and $n = 100$, this shows that if required expression is rational, then $2\sqrt{3}$ is rational.

Hence the conclusion.

18. Let a and b be different positive real numbers, so that $a+\sqrt{ab}$ and $b+\sqrt{ab}$ are both rational. Prove that a and b are also rational.

Let $a+\sqrt{ab} = t, t\in\mathbb{Q}$ and $b+\sqrt{ab} = u, u\in\mathbb{Q}$. Since $a,b\in\mathbb{R}^+$, $t,u>0$. Consider that

$$\frac{t}{u} = \frac{a+\sqrt{ab}}{b+\sqrt{ab}} = \frac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{b}+\sqrt{a})} = \frac{\sqrt{a}}{\sqrt{b}}$$

Since $t,u\in\mathbb{Q}$, then $\frac{t}{u} = \frac{\sqrt{a}}{\sqrt{b}}$ is also rational. Thus, let $\frac{\sqrt{a}}{\sqrt{b}} = j, j\in\mathbb{Q}$. From here, we get $a = j^2b \Rightarrow j^2b+\sqrt{(j^2b)b} = j^2b+jb = bj(j+1) = t$ Since $t\in\mathbb{Q}$ and both $j, j+1\in\mathbb{Q}$, we may conclude that b is rational, which at the end also applies to a as rational number. So, it is proven that a,b are both rational.

19. Do there exist pairwise distinct rational numbers x,y and z such that

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2} = 2014?$$

The answer is no, because

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2} = \left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2$$

and 2014 is clearly not a square of a rational.

20. Positive real numbers a,b are such that $a^3+b^3 = 2$. Show that that $\frac{1}{a}+\frac{1}{b}\geq 2(a^2-a+1)(b^2-b+1)$.
Solution of Zhangyanzong: By AM-GM,

$$\frac{(a+b)(a+1)(b+1)}{ab}\geq 8 = \frac{(a^3+b^3+2)^2}{2}\geq 2(a^3+1)(b^3+1)$$

$$\frac{1}{a}+\frac{1}{b}\geq\frac{8}{(a+1)(b+1)}\geq 2(a^2-a+1)(b^2-b+1)$$

21. (XIII Polish Junior MO 2018 Finals - Problem 1) Positive odd integers a,b are such that $a^b b^a$ is a perfect square. Show that ab is a perfect square.

    WLOG $a \geq b$, We know that $a^b b^a$ is perfect square. Then notice that $a^b b^a = (ab)^b b^{a-b}$. Because a and b are odds, then $a-b$ is even. Therefore, b^{a-b} is perfect square. Because $a^b b^a$ is perfect square, then we get $(ab)^b$ is perfect square. Because b is odd, then ab is perfect square.

22. (XIII Polish Junior MO 2018 Finals - Problem 4) Real numbers a,b,c are not equal 0 and are solution of the system: $\begin{cases} a^2+a=b^2 \\ b^2+b=c^2 \\ c^2+c=a^2 \end{cases}$ Prove that $(a-b)(b-c)(c-a)=1$.

    Adding all of the equation, we'll get $a+b+c=0$. Note that

$$(-c)(a-b)=(a+b)(a-b)=a^2-b^2=-a$$
$$(-a)(b-c)=(b+c)(b-c)=b^2-c^2=-b$$
$$(-b)(c-a)=(a+c)(c-a)=c^2-a^2=-c$$

    Multiply three of these equations, and we'll get the desired result: $(a-b)(b-c)(c-a)=1$.

23. (XIII Polish Junior MO 2018 Second Round - Problem 3) Determine all trios of integers (x,y,z) which are solution of system of equations $\begin{cases} x-yz=1 \\ xz+y=2 \end{cases}$

    First equation gives $x=yz+1$ and second equation becomes $y=\frac{2-z}{z^2+1}$

    This implies : Either $z>2$ and $z^2+1\leq z-2$, impossible Either $z=2$ and so $y=0$ and $x=1$ Either $z<2$ and so $z^2+1\leq 2-z$ and so $z\in\{-1,0\}$, from which only $z=0$ fits, giving $y=2$ and $x=1$

    Hence the result $\boxed{(x,y,z)\in\{(1,0,2),(1,2,0)\}}$

24. (XIII Polish Junior MO 2018 First Round - Problem 6) Positive integers k,m,n satisfy the equation $m^2+n=k^2+k$. Show that $m\leq n$.

    $4m^2+4n+1=(2k+1)^2$. If $m>n$ then $(2m+1)^2>(2k+1)^2>(2m)^2$, contradiction.

25. (Korea junior MO 2018) Find all integer pair (m,n) such that $7^m=5^n+24$.

    We have given the Diophantine equation

    $(1)\ 7^m=5^n+24$.

    According to equation (1)

    $(-1)^m\equiv 7^m=5^n+24\equiv 1^n=1 \pmod 4$,

    implying m is even. Futhermore

    $1\equiv 7^m=5^n+24\equiv(-1)^n \pmod 3$,

    yielding n is even. Hence $(m,n)=(2s,2t)$ $(s,t\in\mathbb{N})$, which inserted in equation (1) result in

    $(2)\ (7^s-5^t)(7^s+5^t)=24$.

    The fact that $2\mid 7^s\pm 5^t$ combined with equation (2) give us $(7^s-5^t,7^s+5^t)=(2,12)$ or $(7^s-5^t,7^s+5^t)=(4,6)$. Consequently $(7^s,5^t)=(7,5)$ or $(7^s,5^t)=(5,1)$, which means $s=t=1$.

    Conclusion The only solution of equation (1) is $m=n=2$.

    Solution 2:

    Note that: $\text{ord}_4(7)=2$ and $\text{ord}_6(5)=2 \implies m=2m_1$ and $n=2n_1$

$$7^m=5^n+\ 24\implies(7^{m_1}+\ 5^{n_1})(7^{m_1}-\ 5^{n_1})=24\implies(m_1,n_1)\equiv(1,1)\implies\boxed{(m,n)\equiv(2,2)}$$

26. Let $a \geq b \geq c \geq d > 0$. Show that
$$\frac{b^3}{a}+\frac{c^3}{b}+\frac{d^3}{c}+\frac{a^3}{d}+3(ab+bc+cd+da) \geq 4\left(a^2+b^2+c^2+d^2\right).$$
$LHS - RHS = \sum_{cyc}(\frac{b^3}{a} - 3b^2 + 3ba - a^2) = \sum_{cyc} \frac{(b-a)^3}{a}$
but $(a-d)^3 \geq (a-b)^3 + (b-c)^3 + (c-a)^3$
So $\sum_{cyc} \frac{(b-a)^3}{a} \geq (a-b)^3(\frac{a-d}{ad}) + (b-c)^3(\frac{b-d}{bd}) + (c-d)^3(\frac{c-d}{cd}) \geq 0$

27. (2015 Korean Junior MO P4) Reals a,b,c,x,y satisfy $a^2+b^2+c^2 = x^2+y^2 = 1$. Find the maximum value of
$$(ax+by)^2+(bx+cy)^2$$
$3(a^2+b^2+c^2)(x^2+y^2) - 2\left((ax+by)^2+(bx+cy)^2\right) = (ax-\frac{1}{2}by)^2 + (cy-\frac{1}{2}bx)^2 + 3(ay-\frac{1}{2}bx)^2 + 3(cx-\frac{1}{2}by)^2 \geq 0$, $(ax+by)^2+(bx+cy)^2 \leq \frac{3}{2}$. Equality holds when $a = c = \frac{1}{2}b, x = y$ and $a^2+b^2+c^2 = x^2+y^2 = 1$.

28. Let $a,b,c > 0$ and x,y,z be real numbers such that $a^2+x^2 = b^2+y^2 = c^2+z^2 = 1$. Prove that
$$(a+b+c)^2+(x+y+z)^2 \geq 1$$
If all of x, y and z are positive or negative, the inequality is trivial. Since replace x, y and z by $-x$, $-y$ and $-z$ respectively, the condition doesn't change. We only consider the case: x, $y \geq 0$, $z \leq 0$.
$$\begin{aligned}(a+b+c)^2+(x+y+z)^2-1 &= 3+2ab+2bc+2ca+2xy+2yz+2zx-1\\ &= 2ab+2c(a+b)+2(1+xy-x-y)+2z(x+y)+2(x+y)\\ &= 2ab+2c(a+b)+2(1-x)(1-y)+2(x+y)(z+1)\\ &\geq 0\end{aligned}$$
The last ineqality is true, since $-1 \leq x,y,z \leq 1$.

29. (KJMO 2014 p5) For positive integers x,y, find all pairs (x,y) such that x^2y+x is a multiple of xy^2+7.
First notice that if $7 \nmid x$, then $\gcd(x,xy^2+7) = 1$ and $xy^2+7|x(xy+1) \implies xy^2+7|xy+1$, impossible, since $xy^2+7 > xy+1$. Hence, $7|x$. Let $x = 7k$. Now, we have $7ky^2+7|49ky+7k \implies ky^2+1|7ky+k$. Now, $\gcd(k,ky^2+1) = 1$, hence, $ky^2+1|7y+1 \implies y \leq 7$. Now we try $y = 1$. So, $7k+7|49k^2+7k \implies k+1|7k^2+k \implies k+1|7k+1 \implies k+1|7k+1-7k-7 = -6$. Hence $k+1|6 \implies k \in \{1,2,5\}$. Notice that all of these work, so we have 3 solutions $(x,y) = (7,1),(14,1),(35,1)$.
Now, for $y = 2$, we have
$$28k+7|98k^2+7k \implies 4k+1|14k^2+k \implies 4k+1|14k+1 \implies 4k+1|2k-2 \implies k = 1$$
Hence we get another working solution $(x,y) = (7,2)$. Now for $y = 3,4,5,6$ easy to check that no solution. For $y = 7$, we get after simplification that $49k+1|k(49k+1)$ which is true for all k. Hence, we get the pairs of form $(x,y) = (7k,7)$. Therefore, all solutions are
$$\boxed{(x,y) = (7,1),(14,1),(35,1),(7,2),(7k,7)}$$

30. Reals a,b,c,x,y satisfies $a^2+b^2+c^2+x^2+y^2=1$. Find the maximum value of
$$(ax+by)^2+(bx+cy)^2$$
$$(ax+by)^2+(bx+cy)^2 \le (ax+by)^2+(bx+cy)^2+(cx-ay)^2$$
$$=(x^2+y^2)(a^2+b^2+c^2)+(2ab+2bc-2ab)xy \le \frac{3}{2}(x^2+y^2)(a^2+b^2+c^2) \le \frac{3}{8}$$
The second to last inequality follows as $2ab+2bc-2ab \le a^2+b^2+c^2$ which is equivalent to $(a+c-b)^2 \ge 0$.

31. (kjmo 2012 pr 1) Prove the following inequality where positive reals a, b, c satisfies $ab+bc+ca=1$.
$$\frac{a+b}{\sqrt{ab(1-ab)}}+\frac{b+c}{\sqrt{bc(1-bc)}}+\frac{c+a}{\sqrt{ca(1-ca)}} \le \frac{\sqrt{2}}{abc}$$
we have $1-ab=bc+ac=c(a+b)$ hence we need to prove
$\sqrt{a+b}+\sqrt{b+c}+\sqrt{a+c} \le \sqrt{\frac{2}{abc}}$
C-S$\sqrt{a+b}+\sqrt{b+c}+\sqrt{a+c} \le \sqrt{6(a+b+c))} \le \sqrt{\frac{2}{abc}} \iff 3abc(a+b+c) \le 1=(ab+bc+ac)^2$
Done!

32. (KJMO 2012 Problem 3) Find all $l,m,n \in \mathbb{N}$ that satisfies the equation $5^l 43^m+1=n^3$
Rearranging, we get $5^l 43^m=n^3-1=(n-1)(n^2+n+1)$. Because $gcd(n-1,n^2+n+1)=gcd(3,n-1)$ by the Euclidean Algorithm, the factors of 5 and factors of 43 must be in separate terms. If $n^2+n+1 \equiv 0 \pmod 5$, then multiplying by 4, we get
$$4n^2+4n+4=(2n+1)^2+3 \equiv 0 \pmod 5$$
or $(2n+1)^2 \equiv 2 \pmod 5$, which is impossible. It follows that $n-1=5^l$ and $n^2+n+1=43^m$.
We now plug in $n=5^l+1$ into the second equation to get
$$(5^l+1)^2+(5^l+1)+1=5^{2l}+3\cdot 5^l+3=43^m$$
If $l \ge 2$, then $43^m \equiv 3 \pmod{25}$, However, because $43 \equiv -7 \pmod{25}$, multiplying by itself gives the repeating cycle $\{-7,-1,7,1\}$, none of which are the term 3.
It follows that $l=1$ is forced, producing $n=6$. We can then conclude that the only solution to the equation is $(l,m,n)=\boxed{(1,1,6)}$.

33. (KJMO 2011 pr 1) Real numbers a, b, c which are differ from 1 satisfies the following conditions; (1) $abc=1$ (2) $a^2+b^2+c^2-\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=8(a+b+c)-8(ab+bc+ca)$ Find all possible values of expression $\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}$
Let $\sum a=p$, $\sum ab=q$ so that the second condition yields $(p^2-2q)-(q^2-2p)=8(p-q) \iff (p-q)(p+q-6)=0$.
Now $\sum_{sym}\frac{1}{a-1}=\frac{3-2p+q}{p-q}$ is defined iff $p \ne q$, whereupon we must have $p+q=6$ so that $\sum_{sym}\frac{1}{a-1}=-\frac{3}{2}$.

34. (KJMO 2011 pr 7) For those real numbers $x_1, x_2, \ldots, x_{2011}$ where each of which satisfies $0 \le x_1 \le 1$ $(i = 1, 2, \ldots, 2011)$, find the maximum of

$$x_1^3 + x_2^3 + \cdots + x_{2011}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2)$$

Note that if $a, b, c \in \{0, 1\}$, then $a^3 + b^3 + c^3 - 3abc \le 2$. Therefore, (for convenience, $x_{i+2011} = x_i$.)

$$\text{(given function)} = \frac{1}{3}\sum_{i=1}^{2011}(x_i^3 + x_{i+1}^3 + x_{i+2}^3 - 3x_ix_{i+1}x_{i+2}) \le \frac{1}{3} \cdot 2011 \cdot 2 = 1340 + \frac{2}{3}$$

Since the maximum value should be an integer, we get an upper bound 1340. The value 1340 is obtained when

$$x_i = \begin{cases} 0 & i \equiv 0 (\text{mod} 3) \\ 1 & i \equiv 1, 2 (\text{mod} 3) \end{cases}$$

35. (KJMO 2009 p1) For primes a, b, c that satisfies the following, calculate abc. $b + 8$ is a multiple of a, and $b^2 - 1$ is a multiple of a and c. Also, $b + c = a^2 - 1$.

$a \mid b + 8$, $a \mid b^2 - 1$, $c \mid b^2 - 1$, $b + c = a^2 - 1$. Let $k_1a = b + 8$, $k_2a = b^2 - 1$. Then, $k_1^2a^2 - 16k_1a + 63 = k_2a \implies a \mid 63$. a can't be 3 due to last given, so $a = 7$. From here, $b + c = 48$, we can guess $b = 7k + 1$ for which both b, c are prime. The only solution is $b = 41$, $c = 7$. It is easy to verify the conditions. Thus, $abc = 2009$.

36. For two arbitrary reals x, y which are larger than 0 and less than 1. Prove that

$$\frac{x^2}{x+y} + \frac{y^2}{1-x} + \frac{(1-x-y)^2}{1-y} \ge \frac{1}{2}.$$

37. (2009 Korean Junior Math Olympiad no. 8) Let a, b, c, d, and e be positive integers. Are there any solutions to $a^2 + b^3 + c^5 + d^7 = e^{11}$?

$a = 2^{105(11n-2)}, b = 2^{70(11n-2)}, c = 2^{42(11n-2)}, d = 2^{30(11n-2)}$ $a^2 + b^3 + c^5 + d^7 = 4 * 2^{210(11n-2)} = 2^{210*11n-420+2}$ $2^{210*11n-418} = 2^{11(210n-38)}$ $e = 2^{210n-38}$

38. (KJMO 2006 p2) Find all positive integers that can be written in the following way $\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c}$. Also, a, b, c are positive integers that are pairwise relatively prime.

Basicly $\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} = (a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) - 3$ so $(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = \frac{(a+b+c)(ab+bc+ca)}{abc}$ must be an integer.

a, b, c are both relatively prime so $a | (a+b+c)(ab+bc+ca) \Rightarrow a | (a+b+c)bc \Rightarrow a | a+b+c \Rightarrow abc | (a+b+c)$

and also we know that $a + b + c \le 3\max(a, b, c) \le 3abc$

$i)$ $a + b + c = abc$

$ii)$ $a + b + c = 2abc$

$iii)$ $a + b + c = 3abc$

only solutions for these are; $\{(1,1,1), (2,1,1), (3,2,1)\}$ and these yields $6, 7, 8$ respectively.

39. For each positive integer n, determine the least possible value of a real number K_n such that the following inequality holds for all real numbers $a_1, a_2, \ldots, a_n$:

$$\frac{a_1 + a_2 + \ldots + a_n}{(1+a_1^2)(1+a_2^2)\ldots(1+a_n^2)} \le K_n$$

$K_n = \frac{n}{\sqrt{2n-1}}\left(1-\frac{1}{2n}\right)^n$, where equality holds if and only if $a_1 = a_2 = \ldots = a_n = \frac{1}{\sqrt{2n-1}}$

Quite obviously, it is sufficient to only consider the case where all the a_i's are all non-negative. Without loss of generality, assume $a_n = \max\{a_1, a_2, \ldots, a_n\}$. If $a_n > 1$,

$$\frac{a_1+a_2+\ldots+a_n}{(1+a_1^2)(1+a_2^2)\ldots(1+a_n^2)} = \frac{\frac{1}{a_n^2}(a_1+a_2+\ldots+a_{n-1})+\frac{1}{a_n}}{(1+a_1^2)(1+a_2^2)\ldots(1+\left(\frac{1}{a_n}\right)^2)} \leq \frac{a_1+a_2+\ldots+a_{n-1}+\frac{1}{a_n}}{(1+a_1^2)(1+a_2^2)\ldots(1+\left(\frac{1}{a_n}\right)^2)}$$

where equality holds if and only if $a_1 = a_2 = \ldots = a_{n-1} = 0$ or $n = 1$, but in either case we would just have

$$LHS = \frac{a_n}{1+a_n^2} < \frac{1}{1+1}$$

i.e. $a_n = 1$ is a better choice anyway, so the left hand side cannot be maximum if $a_n > 1$. Hence, we can now assume $a_n \leq 1$. In particular, $a_i \in [0,1]$ for all $i = 1, 2, \ldots, n$. Now, we observe that for any $x, y \in [0,1]$,

$$(1+x^2)(1+y^2) = (1-xy)^2 + (x+y)^2 \geq \left(1-\left(\frac{x+y}{2}\right)^2\right)^2 + (x+y)^2 \ldots (1)$$

since $xy \leq \left(\frac{x+y}{2}\right)^2 < 1$. This allows us to use the mixing variables method, obtaining us:

$$\frac{a_1+a_2+\ldots+a_n}{(1+a_1^2)(1+a_2^2)\ldots(1+a_n^2)} \leq \frac{nt}{(1+t^2)^n} \ldots (2)$$

where $t = \frac{a_1+a_2+\ldots+a_n}{n}$. Finally, by AM-GM:

$$\frac{nt}{(1+t^2)^n} = \frac{nt}{((2n-1)\frac{1}{2n-1}+t^2)^n} \leq \frac{nt}{(2n)^n\sqrt{(2n-1)^{1-2n}t^2}} = \frac{n}{\sqrt{2n-1}}\left(1-\frac{1}{2n}\right)^n \ldots (3)$$

Conversely, when $a_1 = a_2 = \ldots = a_n = \frac{1}{\sqrt{2n-1}}$ we have

$$\frac{a_1+a_2+\ldots+a_n}{(1+a_1^2)(1+a_2^2)\ldots(1+a_n^2)} = \frac{n}{\sqrt{2n-1}}\left(1-\frac{1}{2n}\right)^n$$

Since equality holds in (1) iff $x = y$, those in (2) holds if and only if $a_1 = a_2 = \ldots = a_n = t$, and by AM-GM as well, equality in (3) holds if and only if $t = \frac{1}{\sqrt{2n-1}}$. To sum up, equality holds if and only if $a_1 = a_2 = \ldots = a_n = \frac{1}{\sqrt{2n-1}}$.

40. (KJMO 2006 p5) Find all positive integers that can be written in the following way $\frac{m^2+20mn+n^2}{m^3+n^3}$ Also, m, n are relatively prime positive integers.

    We can write this fraction:

$$\left(\frac{(m+n)^2+18mn}{(m+n)(m^2-mn+n^2)}\right)$$

So it's clear that $(m+n) \mid (18mn)$ and because of $gcd(m+n, mn) = 1$ so $(m+n) \mid (18)$ So that means $(m+n) = 1, 2, 3, 6, 9, 18$ and m and n are relatively prime numbers. So we can easily see that there is no solution for 1 and 2. Also $(m^3+n^3) \leq (m^2+20mn+n^2)$ If we try possible solutions ; Case 1: If $m+n = 3$

$$(m,n) = (2,1), (1,2)$$

Case 2: If $m+n=6$

$$(m,n)=(5,1)(1,5)$$

Case 3: If $m+n=9$

$$(m,n)=thereisnosolution$$

Case 4: If $m+n=18$

$$(m,n)=thereisnosolution$$

So all integer solutions are

$$(m,n)=(2,1),(1,2),(5,1),(1,5)$$

41. (Greece JBMO TST 2019 p2) Find all pairs of positive integers (x,n) that are solutions of the equation $3\cdot 2^x+4=n^2$.

    I claim that the only solutions are $(x,n)=(6,14),(2,4),(5,10)$. We can easily verify that these are indeed solutions.

    Clearly, $x=1$ leads to no solution. Now assume $x\geq 2$ and let $m=\frac{n}{2}$. After dividing the equation by 4, we get $3\cdot 2^{x-2}+1=m^2$, or $3\cdot 2^{x-2}=(m+1)(m-1)$. This leads to two cases.

    Case 1: $m+1=2^a$ and $m-1=3\cdot 2^b$ for nonnegative a,b

    Then $2^a-3\cdot 2^b=2$. We see that $a,b\geq 1$. Since $v_2(2)=1$, we need at least one of a,b to be 1. The case of $a=1$ leads to no solutions, but $b=1$ leads to the solution $(a,b)=(3,1)$, which corresponds to $m=7$, $n=14$, and $x=6$.

    Case 2: $m+1=3\cdot 2^a$ and $m-1=2^b$ for nonnegative a,b

    Then $3\cdot 2^a-2^b=2$. $a=b=0$ leads to $m=2$, $n=4$, and $x=2$. Now assuming that $a,b>0$, we require $a=1$ or $b=1$. The case of $a=1$ leads to the solution $(a,b)=(1,2)$, which corresponds to $m=5$, $n=10$, and $x=5$. The case of $b=1$ leads to no solutions.

    We have exhausted all cases, so we are done.

42. Let a,b,c be positive real numbers . Prove that

$$\frac{1}{ab(b+1)(c+1)}+\frac{1}{bc(c+1)(a+1)}+\frac{1}{ca(a+1)(b+1)}\geq\frac{3}{(1+abc)^2}.$$

43. Let a, b, c be positive real numbers. Prove the inequality

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}\geq\frac{3}{1+abc}.$$

    Let P be $\sum_{cyc}\frac{1}{a(1+b)}$ Then, apply the well-known inequality: $(x+y+z)^2\geq 3(xy+yz+zx)$ we obtain: $P^2\geq 3(\sum_{cyc}\frac{1}{ab(1+b)(1+c)})=\frac{3}{abc}-\frac{3}{(1+a)(1+b)(1+c)}-\frac{1}{abc((1+a)(1+b)(1+c)}$ Denote $t=\sqrt[3]{abc}$ and apply AM-GM, we have: $(1+a)(1+b)(1+c)\geq(t+1)^3\Rightarrow P^2\geq\frac{3}{t^3}-\frac{3}{(t+1)^3}-\frac{3}{t^3(t+1)^3}=\frac{9}{t^2(t+1)^2}$ Thus $\sum_{cyc}\frac{1}{a(1+b)}\geq\frac{3}{\sqrt[3]{abc}(\sqrt[3]{abc}+1)}$

    $\frac{1+abc}{a+ab}=\frac{1+a+ab+abc}{a+ab}-1=\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}-1$. Hence, rewrite the inequality in the form:

    $\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}+\frac{1+b}{b(1+c)}+\frac{c(1+a)}{1+c}+\frac{1+c}{c(1+a)}+\frac{a(1+b)}{1+a}\geq 6$. This one follows immediately from AM-GM.

44. (Greece JBMO TST 2018 p1) Let a,b,c,d be positive real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that exist two of a,b,c,d with sum less or equal to 2.

    By Cauchy-Swarz, we have

    $$2(a^2+b^2)\geq(a+b)^2$$
    $$2(c^2+d^2)\geq(c+d)^2$$

    Adding the two inequalities, we have

    $$2(a^2+b^2+c^2+d^2)=2\cdot4=8\geq(a+b)^2+(c+d)^2$$

    If $a+b,c+d>2$, then $(a+b)^2+(c+d)^2>2^2+2^2=8$, contradiction. Hence, one of $a+b,c+d$ is less or equal to 2.

45. (Greece JBMO TST Problem 4)Find all positive integers x,y,z with z odd, which satisfy the equation:

    $$2018^x=100^y+1918^z$$

    First assume $x>1$. Taking $\mod 3$ gives that x is odd.

    If $x>z$, $100^y=2018^x-1918^z=2^z(1009^x.2^{x-z}-959^z)$ and looking at the exponent of 2, we conclude $2y=z$, contradiction.

    If $z>x$, analogous argument gives $2y=x$, contradiction. Therefore $z=x$.

    Now $25^{y-1}.2^{2y-2}=2018^{x-1}+\cdots+1918^{x-1}=2^{x-1}.(1009^{x-1}+\cdots+959^{x-1})$ and the second factor is a sum of odd number of odd numbers and therefore is odd. Looking at the exponent of $2, x-1=2y-2$. Now $25^{y-1}=1009^{x-1}+\cdots+959^{x-1}>1009^{x-1}=(1009^2)^{y-1}>25^{y-1}$, contradiction. Therefore $x=1$ and the RHS is bounded and we easily find the only solution $(x,y,z)=(1,1,1)$,$\blacksquare$

46. (Greece JBMO TST 2017, Problem 1) Positive real numbers a,b,c satisfy $a+b+c=1$. Prove that

    $$(a+1)\sqrt{2a(1-a)}+(b+1)\sqrt{2b(1-b)}+(c+1)\sqrt{2c(1-c)}\geq 8(ab+bc+ca).$$

    Also, find the values of a,b,c for which the equality happens.

    $(a+1)\sqrt{2a(1-a)}=(2a+b+c).\sqrt{2a(b+c)}\geq 4.(ab+ac)$

47. (Greece JBMO TST 2017, Problem 3) Prove that for every positive integer n, the number $A_n=7^{2n}-48n-1$ is a multiple of 9.

48. (Bosnia and Herzegovina Junior Balkan Mathematical Olympiad TST 2016) Prove that it is not possible that numbers $(n+1)\cdot 2^n$ and $(n+3)\cdot 2^{n+2}$ are perfect squares, where n is positive integer.

    Case I. n is even.

    Then, we need to show that $n+1$ and $n+3$ cannot be perfect squares.

    Suppose that $n+1=a^2$ and $n+3=b^2$ where a and b are positive integers.

    We have $a^2+2=b^2 \implies (b+a)(b-a)=2$, which has no solutions in positive integers.

    Case II. n is odd.

    Then, we need to show that $2(n+1)$ and $2(n+3)$ cannot be perfect squares.

    Suppose that $2(n+1)=a^2$ and $2(n+3)=b^2$ where a and b are positive integers.

    We have $a^2+4=b^2 \implies (b-a)(b+a)=4$, which also has no solutions in positive integers.

49. Givan the set $S = \{1, 2, 3, ..., n\}$. We want to partition the set S into three subsets A, B, C disjoint (to each other) with $A \cup B \cup C = S$, such that the sums of their elements $S_A S_B S_C$ to be equal .Examine if this is possible when:

   a) $n = 2014$

   b) $n = 2015$

   c) $n = 2018$

   We need $\frac{n(n+1)}{2} \equiv 0(3) \iff n \equiv 0, 2(3)$ which shows that it is impossible for $n = 2014$.

   If it's possible for n then also for $n+9$ since we can do

$$A \to A \cup \{n+9, n+4, n+2\}, B \to B \cup \{n+8, n+6, n+1\}, C \to C \cup \{n+7, n+5, n+3\}$$

   For $n = 8$ we have $A = \{8, 4\}, B = \{7, 5\}, C = \{6, 1, 3, 2\}$ hence we have a solution for $n \equiv 8(9)$ which includes 2015.

   For $n = 11$ we have $A = \{11, 10, 1\}, B = \{9, 8, 5\}, C = \{7, 6, 4, 3, 2\}$ hence soluble for $n \equiv 11(9)$ which includes 2018.

50. If p is a prime positive integer and x, y are positive integers, find , in terms of p, all pairs (x, y) that are solutions of the equation: $p(x-2) = x(y-1)$. (1) If it is also given that $x + y = 21$, find all triplets (x, y, p) that are solutions to equation (1).

   The equation is equivalent to:

$$y - 1 = \frac{px - 2p}{x}$$

   Since $y - 1$ is an integer, $\frac{px-2p}{x}$ must be an integer as well.

$$\frac{px - 2p}{x} = p - \frac{2p}{x}$$

   This shows that x must be a divisor of $2p$ in order for this to be an integer. This gives us $x = 1, 2, p, 2p$.

   Each of these values of x gives $(x, y) = (1, 1-p), (2, 1), (p, p-1), (2p, p)$ respectively.

   Therefore, all positive integer solutions are $(x, y) = \boxed{(2, 1), (p, p-1), (2p, p)}$.

   For the second part, we have $(x, y) = (p, p-1) \implies p + (p-1) = 21 \implies p = 11$, or $(x, y) = (2p, p) \implies 2p + p = 21 \implies p = 7$.

   The solutions are $(x, y) = \boxed{(11, 10, 11), (14, 7, 7)}$

51. Find all triplets of real (a, b, c) that solve the equation $a(a-b-c) + (b^2 + c^2 - bc) = 4c^2 \left(abc - \frac{a^2}{4} - b^2c^2\right)$

52. Find integer solutions of the equation $8x^3 - 4 = y(6x - y^2)$

53. Find all real x, y, z such that $\frac{x-2y}{y} + \frac{2y-4}{x} + \frac{4}{xy} = 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$.

54. Prove the inequality

$$\frac{y^2 - x^2}{2x^2 + 1} + \frac{z^2 - y^2}{2y^2 + 1} + \frac{x^2 - z^2}{2z^2 + 1} \geq 0$$

   where x, y and z are real numbers

55. Positive real numbers a and b verify $a^5 + b^5 = a^3 + b^3$. Find the greatest possible value of the expression $E = a^2 - ab + b^2$

   $a^2 - ab + b^2 = \frac{a^3+b^3}{a+b} = \frac{a^3+b^3}{a+b} \cdot \frac{a^3+b^3}{a^5+b^5}$ But Cauchy Schwarz gives $(a^5 + b^5)(a + b) \geq (a^3 + b^3)^2$ so desired maximum is 1 attainable for $a = 1 = b$

56. $a_1, a_2, ...a_{2018}$ are positive numbers,and $a_{2018}^2 + a_{2017}^2 = a_{2016}^2 - a_{2015}^2 + a_{2014}^2 - ... + a_2^2 - a_1^2$. Prove that $A = a_1 a_2 ... a_{2018} + 2025$ is a difference of two squares

    If $a_1, a_2, ...a_{2018}$ are all odd then $a_{2018}^2 + a_{2017}^2 \equiv 2(\mod 4)$ while $a_{2016}^2 - a_{2015}^2 + a_{2014}^2 - ... + a_2^2 - a_1^2 \equiv 0(\mod 4)$ (contraction) Hence, $A = a_1 a_2 ... a_{2018} + 2025 = 2k+1 = (k+1)^2 - k^2$, where k is a positive integer.

57. Let $a, b, c \in \mathbb{R}_+^*$.Prove the inequality $\frac{a^2+4}{b+c} + \frac{b^2+9}{c+a} + \frac{c^2+16}{a+b} \geq 9$

    Have $\frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b} \geq \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}$ also $\frac{4}{b+c} + \frac{9}{a+c} + \frac{16}{a+b} \geq \frac{81}{2(a+b+c)}$ and $\frac{a+b+c}{2} + \frac{81}{2(a+b+c)} \geq 2\sqrt{\frac{a+b+c}{2}\frac{81}{2(a+b+c)}} = 9$ just by AM-GM. Done.

58. Find all pairs of positive integers (x,y) such that $y^3 = x^3 + 7x^2 + 4x + 15$.

    We see that $x = 1$ and $y = 3$ it is solution. Other solution is $x = 7$, and $y = 9$. We will prove that these are all the solution.

    We have $(x+2)^3 = x^3 + 6x^2 + 12x + 8 < x^3 + 7x^2 + 4x + 15 < (x+3)^3$ if $8x < x^2 + 7$, i.e. for $x \in \{0, 1\} \cup (7, +\infty)$, so we must verify these values: $x \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

    For these value we get:

    $x = 1$, so $y = 3$

    $x = 7$, so $y = 9$

59. Let x,y be real numbers such that $\frac{1}{1+x+x^2} + \frac{1}{1+y+y^2} + \frac{1}{1+x+y} = 1$.Prove that $xy = 1$.

    The relation it is equivalent which: $\frac{x^2+x+1+y^2+y+1}{(x^2+x+1)(y^2+y+1)} = \frac{x+y}{x+y+1}$. Denote $x+y = S$, and $xy = P$. After some calculs we get: $(P-1)(S^2 + (P+2)S + 2) = 0$. One of the situations indeed is $P = 1$, but the other is $S^2 + (P+2)S + 2 = 0$

60. Let $a, b, c \in \mathbb{R}$ and $|a+b| + |b+c| + |c+a| = 8$. Find MIN and MAX: $F = a^2 + b^2 + c^2$.

    Note that $8 = \sum |a+b| \leq \sqrt{3\left[(a+b)^2 + (b+c)^2 + (c+a)^2\right]} \leq \sqrt{12(a^2+b^2+c^2)}$ so $F \geq \dfrac{16}{3}$, holds iff $a = b = c = \pm\dfrac{4}{3}$ And $8 = \sum |a+b| = |a+b| + |-b-c| + |c+a| \geq 2|a| \Rightarrow |a| \leq 4 \Rightarrow a^2 \leq 16$ We also have $b^2 \leq 16, c^2 \leq 16$ too, so $F \leq 48$, equality holds iff $a = b = -c = \pm 4$ and other permutations.

61. Prove that there are not intgers a and b with conditions:

    i) $16a - 9b$ is a prime number.

    ii) ab is a perfect square.

    iii) $a + b$ is also perfect square.

    First $ab \geq 0$ and $a + b \geq 0$ leads to a and b are nonnegative integers. Let $d = \gcd(a, b)$ so $a = dx, b = dy$ and $16a - 9b = d(16x - 9y)$ is prime so $d = 1$ or $16x - 9y = 1$

    -If $16x - 9y = 1$ then $x = 9n + 4$ and $y = 16n + 7$ for some integer n thus $ab = d^2(9n+4)(16n+7)$ is a perfect square so $(9n+4)(16n+7) = \frac{(127+288n)^2-1}{576} = m^2$. This gives us $(127+288n)^2 - 1 = (24m)^2$ which it is a contradiction.

    -If $d = 1$, since ab is a perfect square then both a and b are perfect squares. Let $a = m^2, b = n^2$ for non negative integers m, n. Hence $16a - 9b = (4m - 3n)(4m + 3n)$ is a prime thus $4m - 3n = 1$ so $m = 3k + 1$ and $n = 4k + 1$ for some integer k and in this case $a + b = (3k+1)^2 + (4k+1)^2 = \frac{(7+25k)^2+1}{25}$ cannot be a perfect square.

By the same way if $4m-3n=-1$ then $m=3k+2, n=4k+3$ and in this case $a+b=\frac{(18+25k)^2+1}{25}$ cannot be a perfect square.

62. $a,b,c\in\mathbb{R}^+$ and $a^2+b^2+c^2=48$. Prove that

$$a^2\sqrt{2b^3+16}+b^2\sqrt{2c^3+16}+c^2\sqrt{2a^3+16}\leq 24^2$$

63. Find all integer solutions to the equation $x^2=y^2(x+y^4+2y^2)$.

$x^2-x(y^2)-(y^6+2y^4)=0, \Delta=y^4(4y^2+9)=l^2 \Rightarrow 4y^2+9=m^2 \iff (m-2y)(m+2y)=9.$

$(x,y)=(0,0)$ is a solution. Otherwise $y\neq 0$ and $m>3$ or $m<-3$, m odd. So $m\in\{-7,-5,5,7\}$.

$m\in\{-7,7\}$ is impossible. $m\in\{-5,5\}\Rightarrow y\in\{-2,2\}$, which gives $(x,y)=(12,2),(-8,2),(12,-2),(-8,-2)$

64. (2019 Romania JBMO TST 2.2)If x,y and z are real numbers such that $x^2+y^2+z^2=2$, prove that $x+y+z\leq xyz+2$.

65. (2018 Romania JBMO TST 4.1) Determine the prime numbers p for which the number $a=7^p-p-16$ is a perfect square.

For $p=2$, $x^2=31$ which is not possible. Let p be odd. Then $7^p-p-16\equiv(-1)^p-p\equiv-1-p\equiv 0,1(\text{mod } 4)\Rightarrow p\equiv 2,3(\text{mod } 4)$. Since p is odd $p\equiv 3(\text{mod } 4)$ (1) . $x^2\equiv 7^p-p-16\equiv-9(\text{mod p})\Rightarrow p|x^2+3^2$ (2). Since (1) and (2) we have $p|9$ which implies $p=3$. For $p=3$, expression equals to 18^2.

66. (Second Romanian JBMO TST 2016) a,b,c>0 and $abc\geq 1$.Prove that:

$\frac{1}{a^3+2b^3+6}+\frac{1}{b^3+2c^3+6}+\frac{1}{c^3+2a^3+6}\leq\frac{1}{3}$

$\sum\frac{1}{a^3+2b^3+6}=\sum\frac{1}{a^3++b^3+b^3+6}\leq\sum\frac{1}{3ab^2+6}=\sum\frac{1}{3(ab^2+2)}$ so it suffices to prove $\sum\frac{1}{ab^2+2}\leq 1 \iff \sum\frac{2}{ab^2+2}\leq 2 \iff \sum\frac{ab^2}{ab^2+2}\geq 1$ but $2\leq 2abc$ so it suffices to prove $\sum\frac{ab^2}{ab^2+2abc}\geq 1 \iff \sum\frac{b}{b+2c}\geq 1 \iff \sum\frac{b^2}{b^2+2bc}\geq 1$ which is just C-S (cause $\sum\frac{b^2}{b^2+2bc}\geq\frac{(a+b+c)^2}{(a+b+c)^2}=1$)