Some geometric inequalities

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Problem 1. *With arbitrary triangle ABC inscribed (O), incenter I and an arbitrary point M in small arc BC. Prove that $MA + 2OI \geq MB + MC \geq MA - 2OI$.*

Proof. By the projection of vectors we have

$$MA^2 = 2\overrightarrow{MO}.\overrightarrow{MA} \Rightarrow MA = 2\overrightarrow{MO}.\frac{\overrightarrow{MA}}{MA}$$

similar

$$MB = 2\overrightarrow{MO}.\frac{\overrightarrow{MB}}{MB}, MC = 2\overrightarrow{MO}.\frac{\overrightarrow{MC}}{MC}$$

From them we have

$$MB + MC - MA = 2\overrightarrow{MO}.(\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}) \quad (1)$$

By Cauchy-Swart inequality we have

$$-MO|\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}| \leq \overrightarrow{MO} \cdot (\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}) \leq MO|\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}| \quad (2)$$

We have $MO = R$, we will calculate $|\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}|$

$$\begin{aligned}
&|\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}|^2 \\
&= 3 + 2(\frac{\overrightarrow{MB}}{MB}.\frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MB}}{MB}.\frac{\overrightarrow{MA}}{MA} - \frac{\overrightarrow{MC}}{MC}.\frac{\overrightarrow{MA}}{MA}) \\
&= 3 + 2(\cos(\overrightarrow{MB}, \overrightarrow{MC}) - \cos(\overrightarrow{MB}, \overrightarrow{MA}) - \cos(\overrightarrow{MC}, \overrightarrow{MA})) \\
&= 3 - 2(\cos A + \cos B + \cos C) \text{ (Because } M \text{ in small arc } BC) \\
&= 3 - 2\frac{R + r}{R} \text{ (Here we use the well-know equality: } \cos A + \cos B + \cos C = \frac{R + r}{R} \text{)} \\
&= \frac{R^2 - 2Rr}{R^2} \\
&= \frac{OI^2}{R^2} \text{ (Here we use the Euler's formula)}
\end{aligned}$$

From this we have

$$MO|\frac{\overrightarrow{MB}}{MB} + \frac{\overrightarrow{MC}}{MC} - \frac{\overrightarrow{MA}}{MA}| = OI \quad (3)$$

Thus from $(1), (2), (3)$ we have the inequality $MA + 2OI \geq MB + MC \geq MA - 2OI$ and equality when ABC is equaliteral triangle. We completed the solution. □

Problem 2. *Given are $\triangle ABC$ orthorcenter H, circumradius R, with any M on plane find minimum value of sum*

$$MA^3 + MB^3 + MC^3 - \frac{3}{2}R \cdot MH^2$$

Proof. By AM-GM inequality we have

$$\frac{MA^3}{R} + \frac{R^2 + MA^2}{2} \geq \frac{MA^3}{R} + R.MA \geq 2MA^2$$

$$\Rightarrow \frac{MA^3}{R} \geq \frac{3}{2}MA^2 - \frac{R^2}{2}$$

Similar we have

$$\frac{MB^3}{R} \geq \frac{3}{2}MB^2 - \frac{R^2}{2}, \frac{MC^3}{R} \geq \frac{3}{2}MC^2 - \frac{R^2}{2}$$

Thus

$$\frac{MA^3 + MB^3 + MC^3}{R} \geq \frac{3}{2}(MA^2 + MB^2 + MC^2) - \frac{3}{2}R^2 \quad (1)$$

Called O is circumcenter of ABC

$$\begin{aligned}
&MA^2 + MB^2 + MC^2 \\
&= (\overrightarrow{MO} + \overrightarrow{OA})^2 + (\overrightarrow{MO} + \overrightarrow{OB})^2 + (\overrightarrow{MO} + \overrightarrow{OC})^2 \\
&= 3MO^2 + 2\overrightarrow{MO}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + 3R^2 \\
&= 3MO^2 + 2\overrightarrow{MO}.\overrightarrow{OH} + 3R^2 \\
&= 3MO^2 - (OM^2 + OH^2 - MH^2) + 3R^2 \\
&= 2MO^2 - OH^2 + MH^2 + 3R^2 \\
&\geq 3R^2 - OH^2 + MH^2 \quad (2)
\end{aligned}$$

(Here we use familiar equal of vector $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$)
Form $(1), (2)$ we have

$$\frac{MA^3 + MB^+ MC^3}{R} \geq \frac{3}{2}(3R^2 - OH^2 + MH^2) - \frac{3}{2}R^2$$

$$\Rightarrow MA^3 + MB^3 + MC^3 - \frac{3}{2}R.MH^2 \geq 3R^3 - \frac{3}{2}R.OH^2 = const$$

Easily seen equal when $M \equiv O$
Thus we have $MA^3 + MB^3 + MC^3 - \frac{3}{2}R.MH^2$ has minimum value iff $M \equiv O$ □

Problem 3. *Given are $\triangle ABC$ with sides a, b, c and $\triangle A'B'C'$ with sides a', b', c' and area S'. With any M on plane prove that*

$$\frac{a'^2}{a}MA + \frac{b'^2}{b}MB + \frac{c'^2}{c}MC \geq 4S'$$

Proof. We well know the inequality:
Given triangle ABC and $\forall x, y, z > 0$ then

$$\frac{(x+y+z)^2}{4} \geq yz\sin^2 A + zx\sin^2 B + xy\sin^2 C$$

we can replace $yz \to x, zx \to y, xy \to z$ we will get the inequality:

$$\frac{1}{4}(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}})^2 \geq x\sin^2 A + y\sin^2 B + z\sin^2 C(*)$$

Now we let

$$x = \frac{MBMC}{bc.a'^2} = \frac{MBMC}{bc.4R'^2\sin^2 A'}$$
$$y = \frac{MCMA}{ca.b'^2} = \frac{MCMA}{ca.4R'^2\sin^2 B'}$$
$$z = \frac{MAMB}{ab.c'^2} = \frac{MAMB}{ab.4R'^2\sin^2 C'}$$

Thus we will have:

$$\sqrt{\frac{yz}{x}} = \sqrt{\frac{\frac{MCMA}{ca.b'^2}.\frac{MAMB}{ab.c'^2}}{x = \frac{MBMC}{bc.a'^2}}} = \frac{a'}{a.b'c'}MA$$

similar we have

$$\sqrt{\frac{zx}{y}} = \frac{b'}{b.c'a'}MB, \sqrt{\frac{xy}{z}} = \frac{c'}{c.a'b'}MC$$

and using inequality $(*)$ for triangle $A'B'C'$ and x, y, z as above we will get

$$\frac{1}{4}(\sum_{a,b,c}\frac{a'}{a.b'c'}MA)^2 \geq \sum_{a,b,c}\frac{MBMC}{bc.4R'^2\sin^2 A'}\sin^2 A' = \frac{1}{4R'^2}(\sum_{a,b,c}\frac{MBMC}{bc})$$

If we use well know inequality

$$\frac{MB.MC}{bc} + \frac{MC.MA}{ca} + \frac{MA.MB}{ab} \geq 1$$

then we get the consequence inequality:

$$\frac{1}{4}(\sum_{a,b,c}\frac{a'}{a.b'c'}MA)^2 \geq \frac{1}{4R'^2}$$
$$\Leftrightarrow \frac{a'}{a.b'c'}MA + \frac{b'}{b.c'a'}MB + \frac{c'}{c.a'b'}MC \geq \frac{1}{R'}$$
$$\Leftrightarrow \frac{a'^2}{a}MA + \frac{b'^2}{b}MB + \frac{c'^2}{c}MC \geq \frac{a'b'c'}{R'} = 4S'$$

Easily seen we have equal when $\triangle A'B'C' \sim \triangle ABC$ and $M \equiv H$(Orthocenter of triangle ABC).
$\square$

Remark. This inequality have some nice applycations

- If $\triangle A'B'C' \equiv \triangle ABC$ we get the well know inequality $aMA + bMB + cMC \geq 4S$.

- If $\triangle A'B'C' \equiv \triangle J_aJ_bJ_c$ with J_a, J_b, J_c are three excenter of triangle ABC with noitice $a' = 4R\cos\dfrac{A}{2}, b' = 4R\cos\dfrac{B}{2}, c' = 4R\cos\dfrac{C}{2}$ and $S' = \dfrac{2SR}{r}$ we will get the nice inequality
$$\cot\frac{A}{2}MA + \cot\frac{B}{2}MB + \cot\frac{C}{2}MC \geq a + b + c.$$

- If $\triangle A'B'C' \equiv \triangle BCA$ we will get the non symmetry inequality
$$\frac{b^2}{a}MA + \frac{c^2}{b}MB + \frac{a^2}{c}MA \geq 4S$$

There are some nice other inequality is a consequence of this inequatlity.

Problem 4. *Let M be an arbitrary point inside equaliteral triangle ABC.Find min value of*
$$\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC}$$

Proof. We can assume ABC is an equaliteral triangle with side 1,let barycentric coordinates of M is $(x, y, z), x + y + z = 1$ because M inside triangle $\Rightarrow x, y, z > 0$ By distance formula we have $MA^2 = \dfrac{y+z}{x+y+z}a^2 - \dfrac{yz+zx+xy}{(x+y+z)^2}a^2 =$ by $x + y + z = 1$ and $a = 1 \Rightarrow MA = \sqrt{y^2 + yz + z^2}$, similarly $MB = \sqrt{z^2 + zx + x^2}, MC = \sqrt{x^2 + xy + y^2}$ therefore we need find min value of
$$\frac{1}{\sqrt{y^2+yz+z^2}} + \frac{1}{\sqrt{z^2+zx+x^2}} + \frac{1}{\sqrt{x^2+xy+y^2}}$$

when $x, y, z > 0, x + y + z = 1$, we will solve it with Lagrange multipliers
WLOG $0 \leq x \leq y \leq z < 1$
Case 1: $x = 0$ we have to prove
$$f(y,z) = \frac{1}{\sqrt{y^2+yz+z^2}} + \frac{1}{y} + \frac{1}{z} \geq 4 + \frac{2\sqrt{3}}{3}$$

indeed
$$f(y,z) \geq f(\sqrt{yz}, \sqrt{yz}) \geq f(\frac{y+z}{2}, \frac{y+z}{2}) = 4 + \frac{2\sqrt{3}}{3}$$
Case 2: $0 < x \leq y \leq z < 1$
$$F(x,y,z,\lambda) = \sum \frac{1}{\sqrt{x^2+xy+y^2}} + \lambda(x+y+z-1)$$

$\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial z} = 0$, then

$$-\frac{1}{2}\left[\frac{2x+y}{\sqrt{(x^2+xy+y^2)^3}}+\frac{2x+z}{\sqrt{(z^2+zx+x^2)^3}}\right]+\lambda=0$$

$$-\frac{1}{2}\left[\frac{2y+x}{\sqrt{(x^2+xy+y^2)^3}}+\frac{2y+z}{\sqrt{(y^2+yz+z^2)^3}}\right]+\lambda=0$$

$$-\frac{1}{2}\left[\frac{2z+x}{\sqrt{(z^2+zx+x^2)^3}}+\frac{2z+y}{\sqrt{(y^2+yz+z^2)^3}}\right]+\lambda=0$$

Adding

$$\lambda=\frac{1}{2}\left[\frac{x+y}{\sqrt{(x^2+xy+y^2)^3}}+\frac{y+z}{\sqrt{(y^2+yz+z^2)^3}}+\frac{z+x}{\sqrt{(z^2+zx+x^2)^3}}\right]$$

Inserting λ in first we get

$$x\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(y^2+yz+z^2)^3}}$$

Similarly

$$y\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(z^2+zx+x^2)^3}}$$

$$z\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(x^2+xy+z^2)^3}}$$

Hence

$$y^2(z^2+zx+x^2)^3=z^2(x^2+xy+y^2)^3$$

We put $y = ax, z = bx$, where $1 \le a \le b$

$$a^2(b^2+b+1)^3=b^2(a^2+a+1)^3$$

we get $a = b$ Hence $y = z$ as necessary for critical points in the interior of the region $0 < x, y, z < 1$
We have to prove

$$\frac{2}{\sqrt{x^2+xy+y^2}}+\frac{1}{y\sqrt{3}}\ge 4+\frac{2\sqrt{3}}{3}$$

where $x + 2y = 1$

$$g(y)=\frac{2}{\sqrt{3y^2-3y+1}}+\frac{1}{y\sqrt{3}}\ge 4+\frac{2\sqrt{3}}{3}$$

where

$$\frac{1}{3}\le y\le\frac{1}{2}$$

since $x \le y \le z$ By differentiation it is easily checked that the absolute minimum of $g(y)$ on $\left[\frac{1}{3}, \frac{1}{2}\right]$ is $4 + \frac{2\sqrt{3}}{3} = g(1/2)$.

Thus $\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC}$ get min value $\Leftrightarrow M(\frac{1}{2}, \frac{1}{2}, 0)$and others permutation $\Leftrightarrow M$ concur three midpoint of three side. □

Problem 5. *Suppose a, b, c are sidelengths of a triangle and m_a, m_b, m_c are its medians. Prove the inequality*

$$\frac{m_a}{a^2} + \frac{m_b}{b^2} + \frac{m_c}{c^2} \ge \frac{\sqrt{3}(a^2+b^2+c^2)}{2abc}$$

Proof. This inequality equivalent

$$(\frac{m_a bc}{a} + \frac{m_b ca}{b} + \frac{m_c ab}{c})^2 \ge \frac{3}{4}(a^2+b^2+c^2)^2$$

we have

$$(\frac{m_a bc}{a} + \frac{m_b ca}{b} + \frac{m_c ab}{c})^2 \ge 3(\sum \frac{(m_b ca).(m_c ab)}{bc}) = 3(\sum a^2 m_b m_c)$$

we will prove

$$3(\sum a^2 m_b m_c) \ge \frac{3}{4}(a^2+b^2+c^2)^2 \Leftrightarrow 4(\sum a^2 m_b m_c) \ge (a^2+b^2+c^2)^2$$

Indeed turn into triangle with three side m_a, m_b, m_c we need prove:

$$\sum 4m_a^2 \frac{3}{4} b \frac{3}{4} c \ge (m_a^2 + m_b^2 + m_c^2)^2 \Leftrightarrow \sum (2(b^2+c^2) - a^2)bc \ge (a^2+b^2+c^2)^2$$

by equivalent tranformation we have

$$\Leftrightarrow \sum \frac{1}{2}(a^2 - (b-c)^2)(b-c)^2 \ge 0$$

which is true because $a > |b-c|, b > |c-a|, c > |a-b|$ with any triangle ABC. □

Problem 6. *Let triangle ABC and X, Y, Z are arbitrary points on segment BC, CA, AB. Prove that*

$$\frac{1}{S_{AYZ}} + \frac{1}{S_{BZX}} + \frac{1}{S_{CXY}} \ge \frac{3}{S_{XYZ}}$$

Lemma 6.1. *Let $a, b, c > 0$ be positive real numbers then*

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \ge \frac{3}{1+abc}$$

Proof. We have

$$(1+abc)(\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)}) + 3 = \sum \frac{1+a}{a(1+b)} + \sum \frac{b(c+1)}{1+b} \ge \frac{3}{\sqrt[3]{abc}} + 3\sqrt[3]{abc} \ge 6$$

So we are done.In fact the ineq could be better and stronger as

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \ge \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})}$$

□

Proof. We let $\dfrac{BX}{BC} = x, \dfrac{CY}{CA} = y, \dfrac{AZ}{AB} = z, S_{ABC} = S, 0 < x, y, z < 1$ we will have

$$\frac{S_{AYZ}}{S} = z(1-y), \frac{S_{BZX}}{S} = x(1-z), \frac{S_{CXY}}{S} = y(1-x)$$

Thus we have

$$S_{XYZ} = S - S_{AYZ} - S_{BZX} - S_{CXY} = S - S(z(1-y) + x(1-z) + y(1-x)) = S(xyz - (x-1)(y-1)(x-1))$$

Thus we need to prove

$$\frac{1}{S_{AYZ}} + \frac{1}{S_{BZX}} + \frac{1}{S_{CXY}} \geq \frac{3}{S_{XYZ}}$$

$$\Leftrightarrow \frac{S}{z(1-y)} + \frac{S}{x(1-z)} + \frac{S}{y(1-x)} \geq \frac{3S}{xyz + (1-x)(1-y)(1-z)}$$

$$\Leftrightarrow \frac{xy}{(1-y)} + \frac{yz}{(1-z)} + \frac{zx}{(1-x)} \geq \frac{3}{1 + \dfrac{(1-x)(1-y)(1-z)}{xyz}}$$

Now let $\dfrac{1-x}{x} = a > 0, \dfrac{1-y}{y} = b > 0, \dfrac{1-z}{z} = c > 0$ we will get $\dfrac{1}{(a+1)b} + \dfrac{1}{(b+1)c} + \dfrac{1}{(c+1)a} \geq \dfrac{3}{1+abc}$ now replace $a \to b, b \to c, c \to a$ we will get our above lemma. □

Problem 7. *Given two triangles ABC and $A'B'C'$ with ares S, S' resp prove that*

$$aa' + bb' + bb' \geq 4\sqrt{3SS'}$$

Proof.

$$(\sum aa')^2 \geq 3(\sum bb'cc') = 12SS'(\sum \frac{1}{\sin A \sin A'}) = 24SS' \sum \frac{1}{\cos(A-A') - \cos(A+A')} \geq$$

$$\geq 24SS' \sum \frac{1}{1 - \cos(A+A')} = 12SS' \sum \frac{1}{\sin^2 \dfrac{A+A'}{2}} \geq 48SS' \Rightarrow aa' + bb' + cc' \geq 4\sqrt{3SS'}$$

In the last inequality we easily seen $\sum \dfrac{A+A'}{2} = \pi$ thus they are three angle of a triangle therefore apply the inequality $\sum \dfrac{1}{\sin^2 A} \geq 4$ for them,we are done. □