# Extension of a geometric problem in shortlist 2012

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**Abstract**

This article turns around a mice geometric problem in shortlist 2012 by using pure geometry tools.

The following problem was proposed in shortlist 2012.

**Problem 1.** Let $ABC$ be an acute triangle and its altitudes $AD, BE, CF$. Denote $K, L$ by incenters of triangles $BFD, CDE$. Let $P, Q$ be a circumcenters of triangles $ABK, ACL$. Prove that $PQ \parallel KL$.

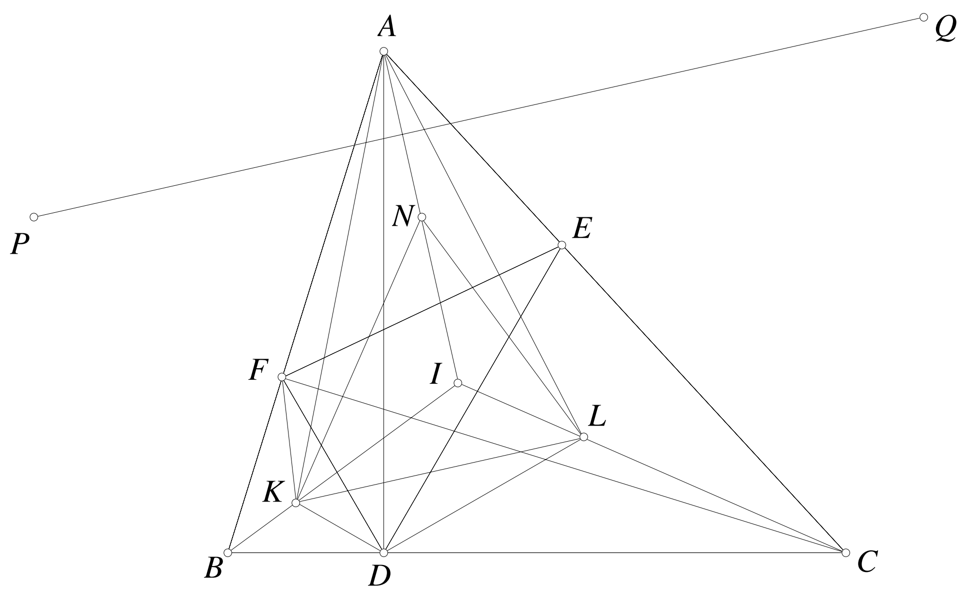


Figure 1.

**Solution.** It is easy to be seen that triangles $\triangle DFB \sim \triangle DCE$. As well-known, $K, L$ are incenters of those triangle, we imply that $\triangle DKF \sim \triangle DLC$. From this similar pair follows $\triangle DKL \sim \triangle DFC$. Therefore $\angle DKL = \angle DFC = \angle DAC$. Since that, we have $\angle BKL = \angle BKD + \angle DKL = 90^\circ + \dfrac{\angle BFD}{2} + \angle DFC = 90^\circ + \dfrac{\angle ACB}{2} + 90^\circ - \angle ACB = 180^\circ - \dfrac{\angle ACB}{2} = 180^\circ - \angle LCB$ we deduce that the quadrilateral $BKLC$ is inscribed in a circle. Similarly, if $N$ is incenter of triangle $AEF$ then quadrilaterals $ANKB$ and $ANLC$ are concyclic. Hence $AN$ is a chord of the circle $(P)$ circumscribed about triangle $ABK$ and the circle $(Q)$ circumscribed about $ACL$. Therefore $PQ \perp AN$. It is obvious that $BK, CL, AN$ are concurrent at $I$ where $I$ is the incenter of triangle $ABC$. From that, we have external angles $\angle ILN = \angle NAC = \angle NAC = \angle IKN$. Analogously, $\angle INK = \angle ILK, \angle INL = \angle IKL$ infers that $I$ is an orthocenter of triangle $KLN$. Therefore $PQ \perp AN \equiv AI \perp KL$ follows $PQ \parallel KL$. This completes the proof. □

**Comment.** Proving a cyclic quadrilateral $KBCL$ play an important role on the solution. On the solution above, the similar triangles having a common vertex was used effectively and clearly. Then, we do not need to draw any auxiliary figure. This solutions based of the idea of Tran Dang Phuc

- my old students. Furthermore, some different ways were proposed to show that the quadrilateral $KBCL$ was cyclic on [1] and on original. From exploiting around this method, we get the following problem.

**Problem 2.** Let $ABC$ be a triangle and $AC > AB$. The angle bisector of $\angle BAC$ intersects $BC$ at $D$. $E$ be a point which lies between $B$ and $D$ such that $\dfrac{ED}{EA} = \dfrac{AC - AB}{AC + AB}$. Denote $K, L$ by incenters of triangles $EAB, EAC$. Prove that the quadrilateral $KBCL$ is inscribed in a circle.

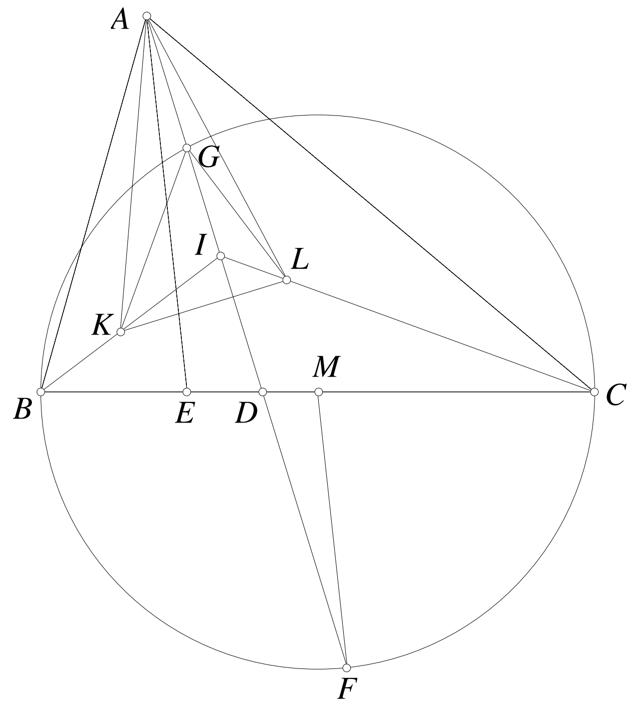


Figure 2.

**Solution.** Denote $M$ by a midpoint of segment $BC$. Let $F$ be a point which lies on the circle with diameter $BC$ and outside triangle $ABC$ such that $MF \parallel AE$. It is easy to prove $DM = MB - DB = \dfrac{BC}{2} - \dfrac{AB.BC}{AB + AC} = \dfrac{BC(AC - AB)}{2(AB + AC)} = \dfrac{MF(AC - AB)}{AB + AC}$. Therefore, we have $\dfrac{ED}{EA} = \dfrac{MD}{MF}$. We could point out easily that $\triangle AED \sim \triangle MFD$. From this follows $A, D, F$ are collinear. Let $AF$ meet the circle with diameter $BC$ at $G$ which is differ from $F$. It is clear that $\angle EAD = \angle DFM = \angle DGM$.

The sum of angles in both triangles $EAD$ and $GMD$ is $360°$. On the other hands, $\angle EDG + \angle GDM = 180°$ we imply that $\angle AED + \angle DMG + 2\angle DGM = 180°$. Note that $\angle DMG = 2\angle MGC$, hence $2(\angle DGM + \angle MGC) = 180° - \angle AED$ or $\angle DGC = 90° - \dfrac{AED}{2}$. Note that $L$ be a center of the incircle of triangle $AEC$, thus $\angle AGC = 180° - \angle DGC = 90° + \dfrac{\angle AED}{2} = \angle ALC$. So, we deduce that the quadrilateral $AGLC$ is concyclic. Analogously, the quadrilateral $AGKB$ is inscribed in a circle.

Note that the internal angle bisectors $AD, BK, CL$ are concurrent at incenter $I$. From two concyclic quadrilaterals $AGCL$ and $AGKB$ follows $IK.IB = IG.IA = IL.IC$. Therefore, the quadrilateral $BKLC$ is concyclic. This completes the proof. $\square$

**Comment.** $E$ satisfied $\dfrac{ED}{EA} = \dfrac{AC - AB}{AC + AB}$ is the most interesting point of this problem. We could see that the condition is solved ingeniously by drawing point $F$ on the circle with diameter $BC$. Basing on the idea of the problem on shortlist, we present the following problem, which was proposed on HUS High school for Gifted Students contest (2013, Round 1, Day 2) [2].

**Problem 3.** Let $ABC$ be a triangle such that $AC > AB$. Angle bisector of $\angle BAC$ intersects $BC$ at $D$. Point $E$ lies between $B, D$ such that $\dfrac{ED}{EA} = \dfrac{AC - AB}{AC + AB}$. Denote $K, L$ by incenters of triangles $EAB, EAC$ respectively. Let $P, Q$ be circumcircles of triangles $KAB, LAC$ in turn. Prove that $PQ$ is parallel to $KL$.

The first proof could be used to solve the problem 2, as follows

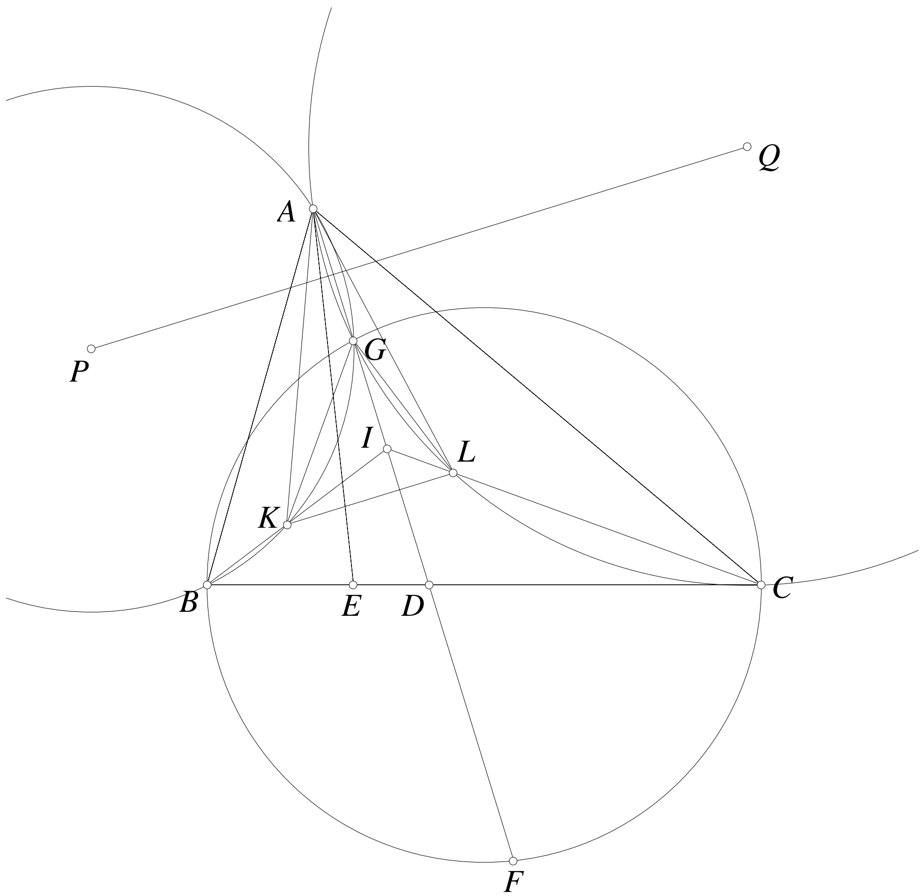


Figure 3.

**Solution 1.** By an construction analogous to the proof of problem 2, we get concyclic quadrilaterals $AGKB, AGLC, BKLC$. Therefore

$$\angle IKL + \angle GLK = \angle ICB + (\angle IBC + \angle GAC) = \frac{\angle ABC + \angle BAC + \angle BCA}{2} = 90^\circ.$$

Or we could say that $IK \perp GL$, similarly $IL \perp GK$. So, we infers $AG \equiv IG \perp KL$. Note that two circles $(P), (Q)$ intersect each other at $A, G$. We get $AG \perp PQ$. Therefore, from properties above, it is easily to be seen that $PQ \parallel KL$. This concludes the proof. $\square$

However, the two following proofs are quite brief. Those solutions infer the problem immediately, so we have to prove a Lemma.

**Lemma 3.1.** Let $ABC$ be a triangle inscribed in circle $(O)$ and $I$ be incenter. $AI$ intersects $(O)$ at $D$ which differs from $A$. Show that $D$ is a circumcenter of triangle $IBC$ and $\dfrac{DI}{DA} = \dfrac{BC}{AB+AC}$.

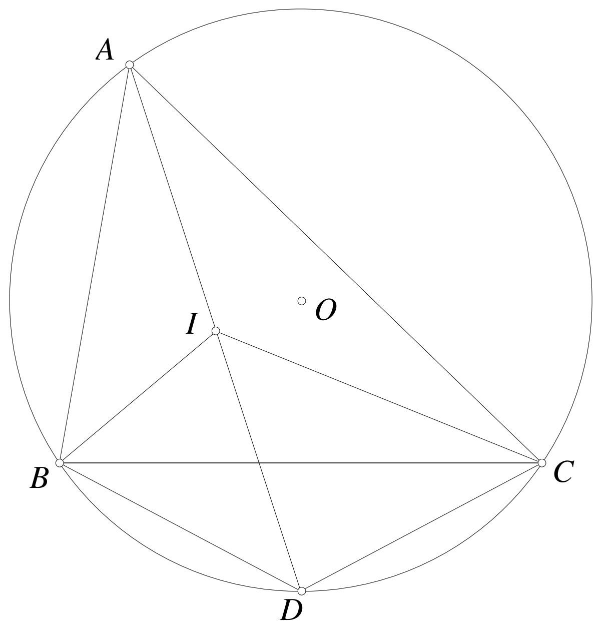


Figure 4.

**Proof.** We have $\angle BID = \angle IBA + \angle IAB = \angle IAC + \angle IBC = \angle CBD + \angle IBC = \angle IBD$. Then $BID$ is an isosceles triangle at $D$. Analogously, $CID$ is an isosceles triangle at $D$. T $DI = DB = DC$. Applying Ptolemy theorem with respect to the quadrilateral $ABDC$, we get $DB.CA + DC.AB = DA.AB$ or $DI(AB + AC) = DA.BC$. Therefore, $\dfrac{DI}{DA} = \dfrac{BC}{AB+AC}$. □

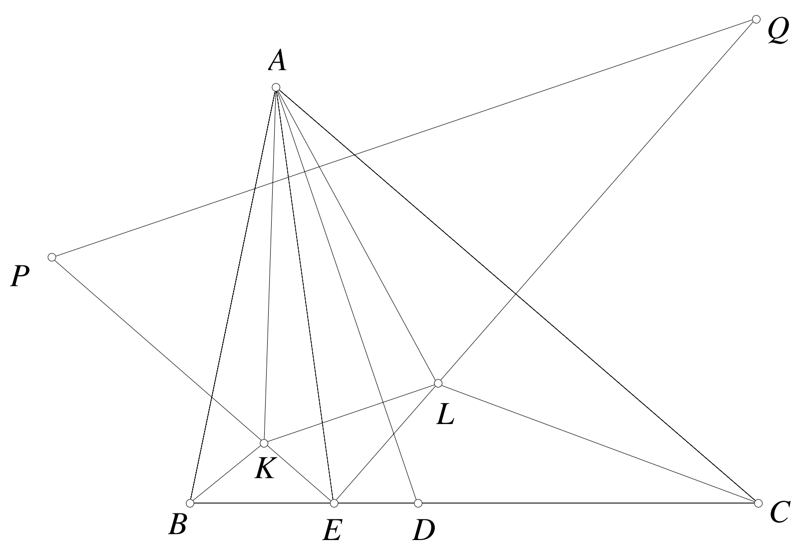


Figure 5.

**Solution 2.** From Lemma, it is easily to be seen that $\dfrac{PK}{PE} = \dfrac{AB}{EA+EB}$ and $\dfrac{QL}{QE} = \dfrac{AC}{EA+EC}$. Therefore, we have to show that

$$\frac{AB}{EA+EB} = \frac{AC}{EA+EC}$$
$$\Leftrightarrow \frac{AB}{EA+DB-ED} = \frac{AC}{EA+DC+ED}$$
$$\Leftrightarrow AB(EA+DC+ED) = AC(EA+DB-ED)$$
$$\Leftrightarrow AB(EA+ED) = AC(EA-ED)$$
$$\Leftrightarrow AB(1+\frac{ED}{EA}) = AC(1-\frac{ED}{EA})$$
$$\Leftrightarrow AB(1+\frac{AC-AB}{AB+AC}) = AC(1-\frac{AC-AB}{AC+AB})$$
$$\Leftrightarrow AB.\frac{2AC}{AB+AC} = AC.\frac{2AB}{AB+AC} \text{ (always true).}$$

This completes the proof. □

**Comment.** Exploiting different properties of the concyclic quadrilateral $KBCL$ on problem 1 could generate another interesting problems, especially as the following problem.

**Problem 4.** Let $ABC$ be an acute triangle and $AD, BE, CF$ be altitudes. Denote $(X), (Y), (Z)$ by circles inscribed in triangles $AEF, BFD, CDE$. Let $d_a$ be a common tangent line which is different from $BC$ of $(Y), (Z)$. Analogously, we have $d_b, d_c$. Prove that $d_a, d_b, d_c$ are concurrent.

We will present the extension of the problem basing on the relative position of orthocenter.

**Problem 5.** Given an acute triangle $ABC$ and its altitudes $AD, BE, CF$. Denote $K, L$ by centers of incircles of triangles $DBE, DCF$. Let $P, Q$ be circumcenters of triangles $HBK, HCL$. Show that $PQ \parallel KL$.

Another way to extend excenters was proposed as follows.

**Problem 6.** Given an acute triangle $ABC$ and its altitudes $AD, BE, CF$. Denote $K, L$ by excenters with respect to vertex $D$ of triangles $BFD, CDE$. Let $P, Q$ be centers of circumcircles of triangles $ABK, ACL$. Prove that $PQ \parallel KL$.

**Problem 7.** Given an acute triangle $ABC$ and its altitudes $AD, BE, CF$. Denote $K, L$ by a center of excircle with respect to vertex $D$ of triangle $DBE, DCF$. Let $P, Q$ be circumcircle of triangles $HBK, HCL$. Determine $PQ \parallel KL$.

On the other hands, we could extend the problem completely basing on the cyclic quadrilateral $BKLC$ as follows.

**Problem 8.** Given a triangle $ABC$ and its incenter $I$. A circle $(K)$ passing through $B, C$ intersects $IC, IB$ at $E, F$ respectively. Denote $P, Q$ by circumcenters of triangles $ACE, ABF$ respectively. Show that $PQ \parallel EF$.

The reader is referred to the problems above which are solved easily basing on the idea in this article.

## References

[1] IMO Shortlist 2012, Geometry 3
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=3160579

[2] Tran Quang Hung, Collection of problems from HUS High Shool for Gifted Student contest, 2013.

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