On Casey inequality

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Casey's theorem is one of famous theorem of geometry, we can see it in [3,4]. Ptolemy's theorem (see in [2]) can be considered as special case of Casey's theorem but Ptolemy inequality (see in [3]) can be considered as an extension of Ptolemy's theorem. Now we will show an extension of Ptolemy inequality. We begin with Casey's theorem.

Theorem 1 (Casey's theorem)**.** *Four circles c_1, c_2, c_3, and c_4 are tangent to a fifth circle or a straight line iff*

$$T_{(12)}T_{(34)} \pm T_{(13)}T_{(42)} \pm T_{(14)}T_{(23)} = 0.$$

where $T_{(ij)}$ is the length of a common tangent to circles i and j.

We can see a nice corollary which we call by "a part of Casey's theorem"

Theorem 2 (Casey's theorem)**.** *Let ABC be a triangle inscribed circle (O). The circle (I) touch to (O) at a point in arc $\overset{\frown}{BC}$ which does not contain A. From A, B, C draw the tangents AA', BB', CC' to (I) $(A', B', C' \in (I))$, respectively. Prove that $aAA' = bBB' + bCC'$. With a, b, c are the sides of triangle ABC, respectively.*

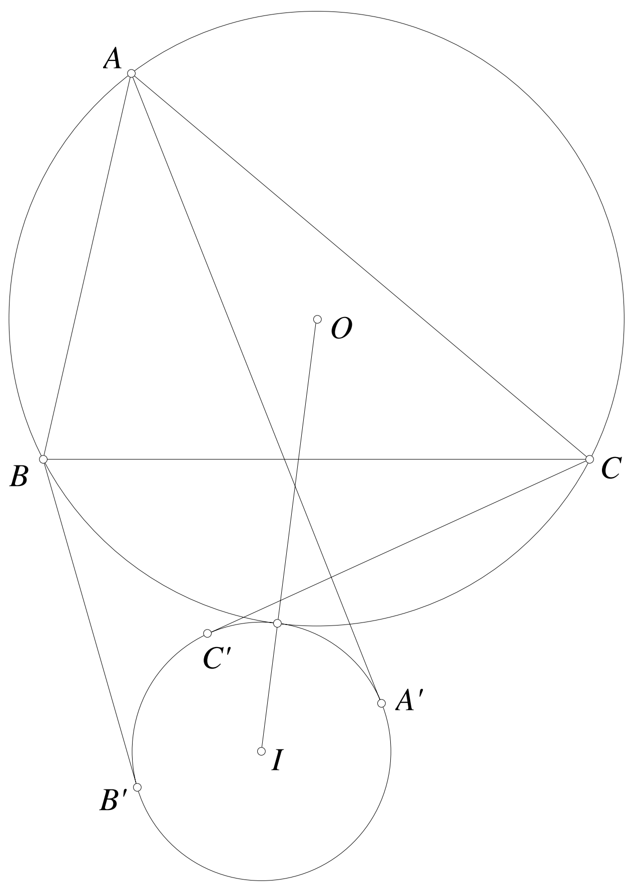

Figure 1.

The folowing theorem is main theorem of this article, it consider as an extension of Ptolemy's inequality. We will call it by Casey's inequality

Theorem 3 (Casey inequality)**.** *Let ABC be a triangle inscribed circle (O). (I) is an arbitrary circle. From A, B, C draw the tangents AA', BB', CC' to (I) $(A', B', C' \in (I))$, respectively. Prove that*

1/ If $(I) \cap (O) = \varnothing$ then $a \cdot AA', b \cdot BB', c \cdot CC'$ are three side of a triangle.

2/ If $(I) \cap (O) \neq \varnothing$ as following

(I) intersects the arc $\overset{\frown}{BC}$ which does not contain A then $aAA' \geq bBB' + cCC'$

(I) intersects the arc $\overset{\frown}{CA}$ which does not contain B then $bBB' \geq cCC' + aAA'$

(I) intersects the arc $\overset{\frown}{AB}$ which does not contain C then $cCC' \geq aAA' + bBB'$

Equality holds iff circle (I) tangents to (O).

Proof. 1/ If $(I) \cap (O) = \varnothing$. Assume that radius of (I) is r, draw circle (I, r') (circle center I and radius r') touch (O)at a point in arc $\overset{\frown}{BC}$ which does not contain A. Easily seen $r' \geq r$. Draw the tangents AA'', BB'', CC'' of (I, r') $(A'', B'', C'' \in (I, r'))$, respectively. Apply Pythagoras' theorem we have $AA'^2 + r^2 = IA^2, AA''^2 + r'^2 = IA^2$. Therefore $AA'^2 = AA''^2 + r'^2 - r^2$ and analogously then $BB'^2 = BB''^2 + r'^2 - r^2, CC'^2 = CC''^2 + r'^2 - r^2 \quad (1)$

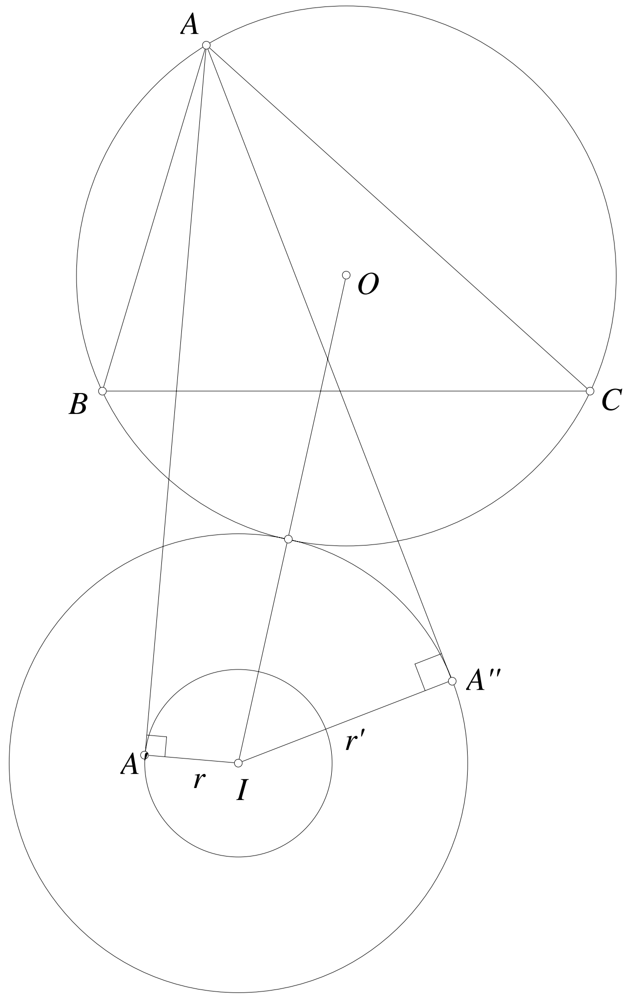

Figure 2.

From theorem 2, square both two sides we get $a^2AA''^2 = b^2BB''^2 + c^2CC''^2 + 2bcBB''CC'' \quad (2)$

Now if we prove that $bBB' + cCC' \geq aAA' \geq |bBB' - cCC'|$ then $a \cdot AA', b \cdot BB', c \cdot CC'$ will be three sides of a triangle. Indeed, the inequality $bBB' + cCC' \geq aAA'$ are equivalent to

$$b^2BB'^2 + c^2CC'^2 + 2bcBB'CC' \geq a^2AA'^2$$
$$b^2(BB''^2 + r'^2 - r^2) + c^2(CC''^2 + r'^2 - r^2) + 2bcBB'CC' \geq a^2AA'^2 \text{ (Get from (1))}$$
$$(b^2 + c^2 - a^2)(r'^2 - r^2) - 2bcBB''CC'' + 2bcBB'CC' \geq 0 \text{ (Get from (2))}$$
$$2bc\cos A(r'^2 - r^2) - 2bcBB''CC'' + 2bc\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq 0 \text{ (Get from (1))}$$
$$\cos A(r'^2 - r^2) - BB''CC'' + \sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq 0$$

The last inequality is true $\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq BB''CC'' + r'^2 - r^2$ because of Cauchy-Schwarz inequality, note that the last inequality is true because $\cos A(r'^2 - r^2) + r'^2 - r^2 \geq 0$ from $r' \geq r$ and $(1 + \cos A) \geq 0$. We are done.

Now the inequality $aAA' \geq |bBB' - cCC'|$ is equivalent to $b^2BB'^2 + c^2CC'^2 - 2bbBB'CC' \leq a^2AA'^2$.

Use analogous transforms as above we must prove that

$$\cos A(r'^2 - r^2) - BB''CC'' - \sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq 0$$

Because $-\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \leq -BB''CC'' - (r'^2 - r^2)$ therefore $LHS \leq \cos A(r'^2 - r^2) - r'^2 - r^2 - 2BB''CC'' < 0$ which is true inequality.

The cases (I, r') touch are CA which does not contain B and the arc $\overset{\frown}{AB}$ which does not contain C we prove analogously. We are done part 1/.

2/ If $(I) \cap (O) \neq \varnothing$. Assume (I, r) intersect arc $\overset{\frown}{BC}$ which does not contain A. Draw (I, r'') touch arc $\overset{\frown}{BC}$ which does not contain A. Easily seen $r'' \leq r$. Draw the tangents AA'', BB'', CC'' of (I, r'') $(A'', B'', C'' \in (I, r''))$, respectively. Analogous, apply Pythagoras' theorem as in (1), we get the equalities

$$AA'^2 = AA''^2 + r''^2 - r^2, BB'^2 = BB''^2 + r''^2 - r^2, CC'^2 = CC''^2 + r''^2 - r^2$$

Or

$$AA''^2 = AA'^2 + r^2 - r''^2, BB''^2 = BB'^2 + r^2 - r''^2, CC''^2 = CC'^2 + r^2 - r''^2 \quad (3)$$

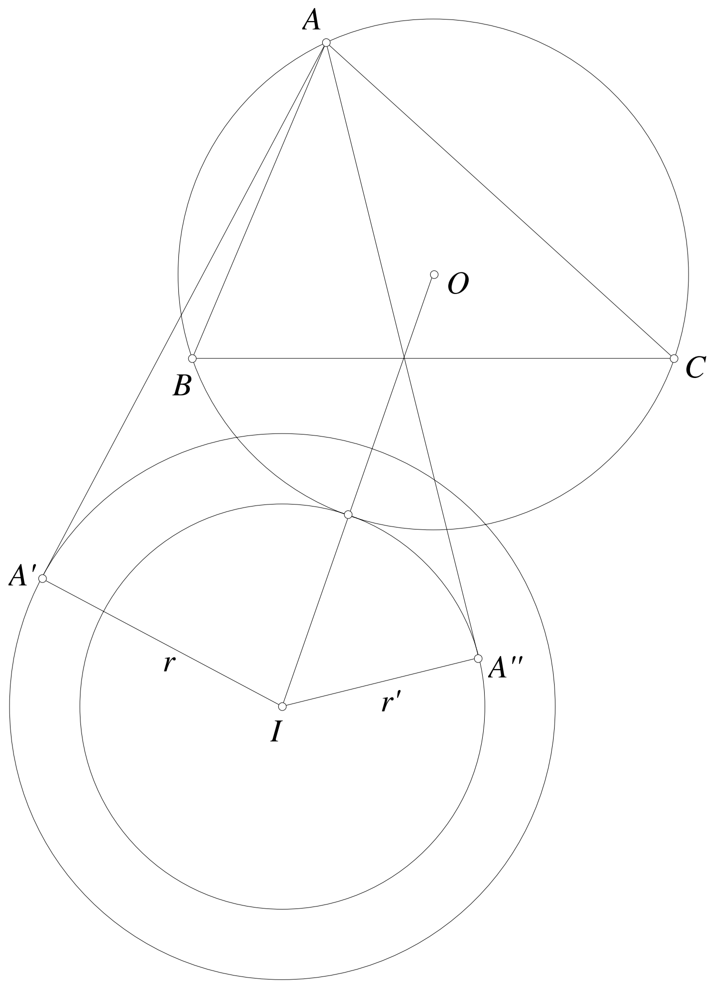

Figure 3.

Use theorem 2 and (3) with analogous transforms the inequality is equivalent to

$$\cos A(r''^2 - r^2) - BB''CC'' + BB'CC' \leq 0 \quad (4)$$

Note that $BB''CC'' = \sqrt{(BB'^2 + r^2 - r''^2)(CC'^2 + r^2 - r''^2)} \geq BB'CC' + r^2 - r''^2$

So that $LHS \leq \cos A(r''^2 - r^2) - (r^2 - r''^2) = (r''^2 - r^2)(1 + \cos A) \leq 0$. which is true because $r'' \leq r, 1 + \cos A \geq 0$.

The cases (I, r'') touch arc $\overparen{CA}$ which does not contain B and the arc $\overparen{AB}$ which does not contain C we prove analogously. We are done part 2/. □

References

[1] http://mathworld.wolfram.com/PtolemyInequality.html

[2] http://mathworld.wolfram.com/PtolemysTheorem.html

[3] Roger A. Johnson, *Advanced Euclidean Geometry* Dover Publications (August 31, 2007)

[4] http://mathworld.wolfram.com/CaseysTheorem.html