High school for gifted student - HSGS

First year project

Some geometrical problems proposed for IMO team

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Preface

During the time of training for Saudi Arabia Mathematics Olympiad Team in 2015, I accumulated a number of interesting geometric problem for the pupils in training for IMO exam. These problems are published in IMO Shortlist, or some of which were suggested myself (not written reference or written expanding). These problems were classified according to three degrees in a IMO exam: easy, medium and difficult with star signed. Most of these problems based on the very basic knowledge of the plane geometry. These are the congruent triangles, the similar triangles and cyclic quadrileteral. And in the same time, we apply the knowledge about the power of point with respect to the circle, radical axis and harmonic range. You all (from beginners or who are proficiently with Olympic exam) could find very useful things in these problems. Good luck and successful to my dear students.

Jeddah, summer 2015

Tran Quang Hung.

Chapter 1

Some fundamental concepts

1.1 Cyclic quadrilateral

Cyclic quadrilateral is simple configuration of geometry. When we have four points lie on circle (concyclic points) and they creat a convex quarilateral then we have a cyclic quadrilateral. Almost the geometric problems in olympiad using cyclic quadrilateral. So we will give overview about cyclic quadrilateral

1.2 Necessary and sufficient conditions of a cyclic quadrial-teral

Let $ABCD$ be a convex quadrilateral with AB intersects CD at E. AD intersects BC at F. AC intersects BD at G. We have the following conditions are equivalent

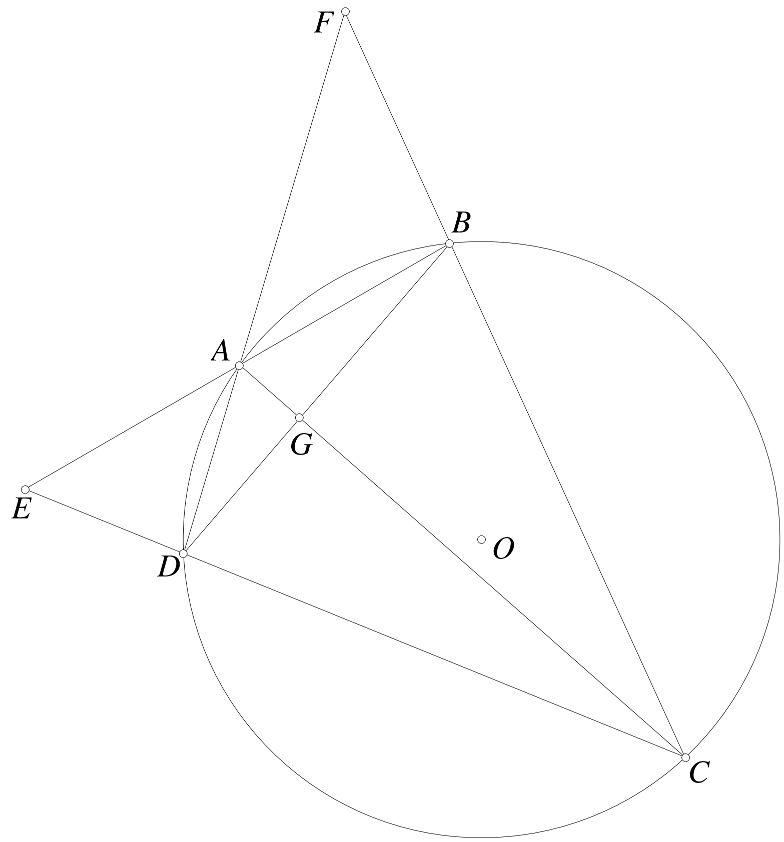

Figure 1.

1) Quadrilateral $ABCD$ is cyclic.
2) $\angle ABC = \angle ADC$ (Property of adjacent angle)
3) $\angle ABC + \angle ADC = 180°$ (Property opposite angle)
4) $\angle FBA = \angle ADC$ (Property exterior angle)
5) $EA.EB = ED.EC$ (Metric relation)
6) $FB.FC = FA.FD$ (Metric relation)
7) $GA.GC = GB.GD$ (Metric relation)

1.3 The extension of metric relation, power of the point with respect to a circle

Let $ABCD$ be cyclic quadrilateral inscribed in circle (O, R). AB intersects CD at E. AD intersects BC at F. AC intersects BD at G. We have the following equality

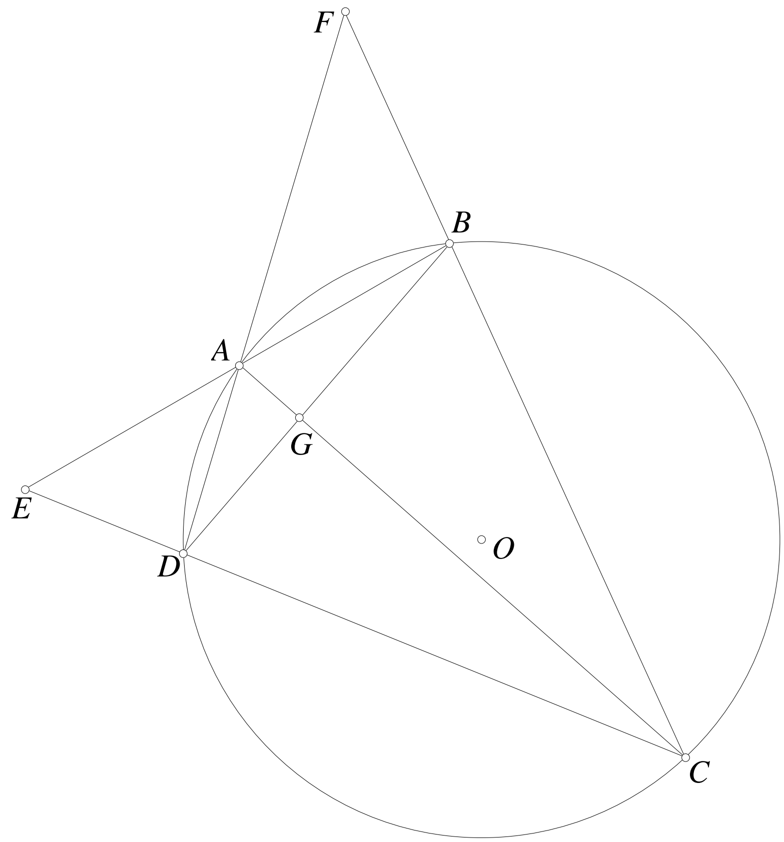

Figure 2.

1) $EA.EB = ED.EC = OE^2 - R^2$
2) $FB.FC = FA.FD = OF^2 - R^2$
3) $GA.GC = GB.GD = R^2 - OG^2$

1.4 Degenerate to the tangents

Let ABC be a triangle inscribed in circle (O). T is a point lie on line BC externally the segment BC. We have the following conditions are equivalent

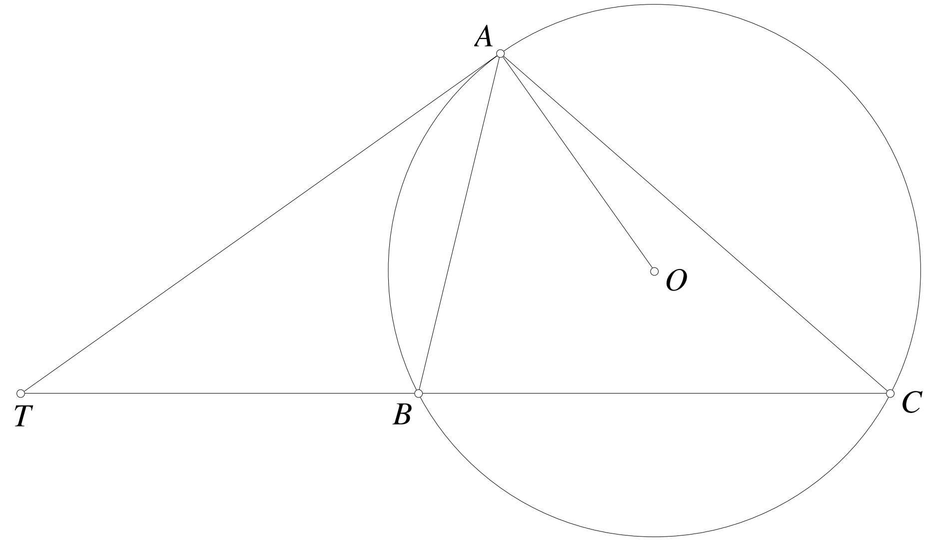

Figure 3.

1) TA is tangent of (O)
2) $\angle TAB = \angle ACB$ (Property angle of tangents)
3) $TA^2 = TB.TC$ (Metric relation of tangents)
4) $\dfrac{TB}{TC} = \dfrac{AB^2}{AC^2}$ (Metric relation of tangents)

Remark. Three sections give us the overview about necessary and sufficient conditions of a cyclic quadrialteral, the metric relation in cyclic quadrialteral (power of the points) and properties of tangents. We usually called the properties about angles by the terminology "angle chasing". Angle chasing is really the most important properties of cyclic quadrilateral. But according to our, the problems of cyclic quadrilateral is subjective if they have the both properties and metric relation in them. Now we give some propblems

Chapter 2

Problems training

2.1 First day

Problem 1 (Own)**.** Let ABC be a triangle inscribed in circle (O). Its incircle (I) touches BC at
★ D. AI cuts (O) again at E. ED cuts (O) again at G. Prove that $\angle AGI = 90°$.

Problem 2 (Own)**.** Let ABC be a triangle inscribed in circle (O). Its incircle (I) touches BC at D. AI cuts (O) again at E. The line passes through I which is perpendicular to OI, intersect ED, AO
★★ at M, N, reps. Prove that I is midpoint of MN.

Problem 3 (Own)**.** Let ABC be an acute triangle, its altitudes are concurrent at orthocenter H. The line passes through H which is perpendicular to Euler line of ABC, intersec AB, AC, DE, DF
★★ at M, N, P, Q, resp. Prove that $MN = 2PQ$.

Problem 4 (Own)**.** Let ABC be an acute triangle with altitude AD, orthocenter H and circumcenter
★ O. F lies on AB such that DF is perpendicular to OD. Prove that $\angle FHD = \angle B$.

2.2 Second day

Problem 5 (David Monk)**.** Let ABC be a triangle right at A. M is midpoint of BC. Points E, F lie on line CA, AB such that E, M, F are collinear. Point P lies on segment EF such that two segment MP and EF have the same midpoint N. T is projection of P on BC. Prove that AN is bisector of $\angle MAT$.
⋆

Problem 6 (David Monk)**.** Let ABC be a triangle and its incircle touches BC, CA, AB at D, E, F, reps. AD cuts (I) again at G. H lies on line EF such that $GH \perp AD$. Prove that $AH \parallel BC$.
⋆⋆

Problem 7 (David Monk)**.** Let ABC be a triangle and its incircle touches BC, CA, AB at D, E, F, reps. Let point H lie on line DE such that $AH \parallel EF$. Prove that BH bisects segment EF.
⋆

Problem 8 (David Monk)**.** Let $ABCD$ be cyclic quadrilateral. AD cuts BC at E. d is the line passing through E and that is parallel to CD. Let p, q be distance from A, B to d and r be distance from E to AB. Prove that $p.q = r^2$.
⋆

Problem 9 (David Monk)**.** Let $ABCD$ be cyclic quadrilateral. AC cuts BD at E. M, N are midpoints of CD, AB such that $\angle AMD = \angle CNB$. Prove that $\angle EMC = \angle ABC$.
⋆⋆

Problem 10 (Russia 2015)**.** Let ABC be acute triangle with altitude AH. M is midpoint of BC. Points E, F lie on CA, AB such that ME, MF are perpendicular to AB, AC, resp. BC cuts circumcircle of triangle MEF again at N. Prove that $BH = NC$.
⋆⋆

Problem 11 (Russia 2015)**.** Let ABC be a triangle inscribed in circle (O). Tangent of (O) at A intersects BC at D. I is incenter of ABC. Bisector of $\angle D$ cuts IB, IC at P, Q, resp. M is midpoint of arc BC that contain A of circle (O). Prove that line IM bisects segment PQ.
⋆⋆

Problem 12 (Russia 2015)**.** Let ABC be a triangle inscribed circle (O) with altitude AH, centroid G. Ray GH intersects (O) at D. Prove that AB is tangent to circumcircle of triangle BDH.
⋆

Problem 13 (Balkan shortlist 2009)**.** Let ABC be a triangle inscribed in circle (O) with orthocenter H. K is projection of H on tangent at A of (O). L is projection of H on symmedian from A. Prove that KL bisects segment BC.
⋆⋆⋆

Problem 14 (Own)**.** Let ABC be acute triangle with orthocenter H and M is midpoint of BC. P is a point on HM. E, F are projection of P on side CA, AB. Prove that the tangents at E, F of circle diameter AP intersect on perpendicular bisector of BC.
⋆⋆⋆

Problem 15 (Own, extension of IMO 2010 P2)**.** Let ABC be a triangle inscribed in circle (O) with incenter I. AI cuts (O) again at D. E is a point on segment BC. M is midpoint of IE. P lies on line DM such that PI is perpendicular to OI. Q is symmetric of P through I. Assume that Q is inside triangle ABC. Prove that AI is bisector of $\angle QAE$.
⋆⋆⋆

Problem 16 (AoPS)**.** Let ABC be a triangle and two point P, Q such that $\dfrac{AP}{AQ} = \dfrac{BP}{BQ} = \dfrac{CP}{CQ}$. Prove that PQ passes through circumcenter of triangle ABC.
⋆⋆

2.3 Third day

Problem 17 (IMO Shortlist 2014 G3)**.** Let ABC be a triangle inscribed circle (O). M is midpoint of arc BC which does not contain A. Perpendicular bisector of AB, AC cut circle dimaeter AM at P, Q, reps outside $\angle BAC$. PQ cuts perpendicular bisector of AM at R. Prove that $AR \parallel BC$. ★★★

Problem 18 (Own, extension of Iran 2012)**.** Let the triangle ABC $(AB < AC)$ inscribed in the circle (O). The bisector of the angle $\angle BAC$ cuts (O) again at D. E is symmetry of D through O. F is the point on the chord BD not contain A, C of (O). FE cuts BC at G. H belongs to AF such that $GH \parallel AD$. Prove that HG is the bisector of the angle $\angle BHC$. ★★★

Problem 19 (Own, extesnion of IMO 2014 P4)**.** Let the triangle ABC and the points P, Q are lying on BC such that $AP = AQ$. The circumcircle of the triangle APB cuts CA again at E. The circumcircle of the triangle AQC cuts AB again at F. Get the points M, N are lying on opposite rays of PE, QF such that $PM.QN = PE.QF$.

a) Prove that BN and CM are intersected at R on the circle (O) circumcircle of the triangle ABC.

b) Call by K the circumcircle center of the triangle RMN. Prove that AK perpendicular with BC. ★★★

Problem 20 (AoPS)**.** Given an acute triangle ABC inscribed in the circle (O). The altitudes BE, CF of triangle ABC intersects each other at H. The line AH meets the circle (O) at D which differs from A. The line DE meets (O) at G which differs from D. Show that BG bisects the segment EF. ★★

Problem 21 (Extension of IMO Shortlist 2014 G3)**.** Let ABC be a triangle inscribed circle (O). D is midpoint of arc BC which does not contain A. P is a point on perpendicular bisector of AD. M, N lies on circle diameter AD and outside triangle such that $PM \perp AC, PN \perp AB$. MN intersects perpendicular bisector of AD at R. Prove that $AR \perp PD$. ★★

Problem 22 (AoPS)**.** Let the acute triangle ABC inscribed in the circle (O). AD, BE, CF are the altitudes of the triangle ABC and converge at H. D, E, F are on BC, CA, AB respectively. Call by AG the diameter of the circle (O). AG intersect EF, BC at X, Y respectively. The intersection of AD and the tangent of the circle (O) at G is Z. Prove that $HX \parallel YZ$. ★

Problem 23 (Own, extension of problem 20)**.** Given an acute triangle ABC inscribed in the circle (O). Point P belongs to the minor arc $\overset{\frown}{BC}$ so that if Q is symmetric to P with respect to BC then Q will be inside the triangle ABC. The lines QB, QC intersect the lines CA, AB at E, F respectively. The line PE meets the circle (O) at R which differs from P. Demonstrate that BR bisects segment EF. ★★★

2.4 Fourth day

Problem 24 (IMO Shortlist 2006 G4)**.** Let ABC be a triangle with symmedian BE, CF. Let M, N be midpoints of BE, CF. Prove that BN, CM and perpendicular bisector of BC are concurrent. ⋆⋆

Problem 25 (Inspire from Serbia 2013 and Iran 2015)**.** Let ABC be a triangle inscribed in circle (O). X is a point on minor arc BC such that it E, F are projection of X on IB, IC then midpoint of EF lies on perpendicular bisector of BC. Let J be $A-$ excenter of triangle ABC. Prove that XJ passes through midpoint of major arc BC. ⋆⋆⋆

Problem 26 (AoPS)**.** Let ABC be a triangle with incircle (I) touches BC, CA, AB at D, E, F. H is orthocenter of triangle ABC and K is projection of D on EF. Prove that $\angle IKD = \angle DKH$. ⋆⋆⋆

Problem 27 (Own)**.** Let ABC be a triangle its incenter is I. E, F are reflection of I through CA, AB. EF intersets IB, IC at P, Q. Perpendicular bisector of PQ cuts BC at R. Prove that $CR.CA = BR.BA$. ⋆⋆

2.5 Fifth day

2.5.1 IMO+

Problem 28 (IMO Shortlist 2014 G4)**.** Let ABC be a triangle inscribed in circle (O). P is a point on arc $\overset{\frown}{BC}$ which does not contain A. M is a point divide segment AP in a constant ratio. Circumcircle of triangle MPB and MAC intersect again at point Q. Prove that Q always lies on a fixed circle when P moves. $\star\star$

Problem 29 (AopS)**.** Let ABC be a triangle inscribed in circle (O). Its incircle (I) touches BC, CA, AB at D, E, F, reps. K is projection of D on EF. AK cuts (O) again at G. Prove that GD is bisector of $\angle BGC$. $\star\star$

Problem 30 (Ta Hong Son)**.** Let M, N be two points interior to the circle (O) such that O is the midpoint of MN. Let S be an arbitrary point lies on (O), and E, F the second intersections of the lines SM, SN with (O), respectively. The tangents at E, F with respect to the circle (O) intersect each other at I. Prove that the perpendicular bisector of the segment MN passes through the midpoint of SI. $\star\star$

Problem 31 (Own, extension of problem 30)**.** Let ABC be a triangle inscribed in circle (O) and M is midpoint of BC. P, Q lie on BC and P, Q are symmetric through M. AP, AQ cut (O) again at E, F. Tangents at E, F of (O) intersect at I. K is projection of A on OM and L is projection of O on AM. Prove that KL bisects AI. $\star\star$

Problem 32 (Own, extension of problem 29)**.** Let ABC be a triangle inscribed in circle (O). Let P, Q be two isogonal conjugate points on bisector of $\angle BAC$. E, F are projection of P on CA, AB. D is projection of Q on BC. K is projection of D on EF. EF cuts BC at G. AK cuts (O) again at L. Prove that line GL always passes through fixed point when P, Q move. $\star\star\star$

2.5.2 IMO

Problem 33 (IMO 2009 P2)**.** Let ABC be a triangle with points E, F lie on CA, AB, resp. O is circumcenter of triangle ABC. Let M, N, P be the midpoints of segments BE, CF, EF, resp. Prove that circumcircle of triangle MPN is tangent to EF iff $OE = OF$. $\star\star$

Problem 34 (IMO 2010 P4)**.** Let ABC be a triangle inscriebd in circle (O). Tangent at A of (O) cuts BC at T. P is a point inside (O). PA, PB, PC cut (O) again at D, E, F, resp. Prove that $DE = DF$ iff $TA = TP$. $\star$

Problem 35 (VMO 2013)**.** Let ABC be a triangle with incircle (I) touches BC, CA, AB at D, E, F, reps. Let G, H be symmetric point of E, F through I. Line GH cuts IB, IC at P, Q. Assume that B, C are fixed point and A changes such that ratio $\dfrac{AB}{AC}$ is constant. Prove that perpendicular bisector of PQ always passes through a fixed point. $\star\star$

2.6 Sixth day

2.6.1 IMO+

Problem 36 (IMO Shortlist 2009 G7)**.** Let $ABCD$ be quadrilateral with AB cuts CD at S. Let H, K be orthocenters of triangles SAD, SBC and M, N are ninepoint center of triangles SAD, SBC. Prove that the line passes throug M are perpendicular to BC and the line passes through N are perpendicular to AD intersect on HK.
⋆⋆⋆

Problem 37 (Own, extension of problem 26)**.** Let ABC be a triangle inscribed in circle (O). Let P, Q be two isogonal conjugate points on bisector of $\angle BAC$. E, F are projection of P on CA, AB. D is projection of Q on BC. K is projection of D on EF. J is reflection of P through DK. Prove that line JK always passes through fixed point when P, Q move.
⋆⋆⋆

Problem 38 (Kostas Vittas)**.** Let ABC be isocesles triangle with $AB = AC$. (K) is the circle passing through A, B. (L) is the circles passing through A, C. The line passes through A is parallel to BC, that intersect $(K), (L)$ again at M, N, reps. Prove that the line passes throug K are perpendicular to BN and the line passes through L are perpendicular to CM intersect on perpendicular bisector of BC.
⋆⋆

Problem 39 (Kostas Vittas)**.** Let AD be altitude of triangle ABC. Circle (K) diameter AD cut CA, AB again at E, F. Tangents from E, F of (K) cut BC at M, N. Let EB, EN cut FC, FM at P, Q, reps. Prove that line PQ bisects segment BC.
⋆⋆

Problem 40 (Own, extension of Kostas Vittas's problem)**.** Let D be a point on altitude of triangle ABC. Circle (K) diameter AD cut CA, AB again at E, F. Tangents from E, F of (K) cut BC at M, N. Let EB, EN cut FC, FM at P, Q, reps. Prove that PQ always passes through a fixed point when D moves.
⋆⋆⋆

2.6.2 IMO

Problem 41 (David Monk)**.** Let ABC be a triangle inscribed in circle (O), orthocenter H. Tangent at A of (O) intersect BC at T. D is symmetric of O through A. E is midpoint of AH. Prove that four points A, D, T, E are concylic.
⋆

Problem 42 (IMO 2012 P1)**.** Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.
⋆

Problem 43 (IMO 2012 P5)**.** Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.Show that $MK = ML$.
⋆⋆

Problem 44 (IMO Shortlist 2012 G2)**.** Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD.
$\star$ Prove that D, H, F, G are concyclic.

2.7 Seventh day

2.7.1 IMO+

2.7.2 IMO

2.8 Eighth day

2.8.1 IMO+

Problem 45 (ELMO 2015, Problem 3 (Shortlist G3))**.** Let ω be a circle and C a point outside it; distinct points A and B are selected on ω so that CA and CB are tangent to ω. Let X be the reflection of A across the point B, and denote by γ the circumcircle of triangle BXC. Suppose γ and ω meet at $D \neq B$ and line CD intersects ω at $E \neq D$. Prove that line EX is tangent to the circle γ. ⋆⋆

Problem 46 (Own, extension of IMO Shortlist 2007 G7)**.** Let ABC be acute triangle inscribed in circle (O) with incenter I, altitude AD and circumradius R. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Let M be projection of B on IK. AD cuts (O) again at N. Assume that $IK \parallel AB$. Prove that $MN \parallel ID$. ⋆⋆⋆

Problem 47 (Own, extension of IMO Shortlist 2006 G1)**.** Given are a triangle ABC. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle iff $CA + CB = 3AB$ or $CA = CB$. ⋆⋆⋆

Problem 48 (Own, inspire from IMO Shortlist 2005 G7)**.** Let ABC be a triangle inscribed in circle (O) and incenter I. Circle excircle (L) of vertex C touches AB at M. MI cuts BC at N. P is projeciton of C on JB. Prove that AI and PN intersect on (O). ⋆⋆⋆

2.8.2 IMO

Problem 49 (IMO Shortlist 2010 G1)**.** Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$. ⋆

Problem 50 (IMO 2005 P2)**.** Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent. ⋆

Problem 51 (IMO 2007 P1)**.** In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area. ⋆

Problem 52 (IMO Shortlist 2008 G1)**.** Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic. ⋆

2.9 Ninth day

2.9.1 IMO+

Problem 53 (IMO Shortlist 2006 G2). Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\dfrac{AK}{KB} = \dfrac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$.
★★ Prove that the points P, Q, B and C are concyclic.

Problem 54 (IMO Shortlist 2006 G3). Consider a convex pentagon $ABCDE$ such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. Let P be the point of intersection of the lines
★★ BD and CE. Prove that the line AP passes through the midpoint of the side CD.

Problem 55 (IMO Shortlist 2006 G7). Let ABC be a triangle inscribed in circle (O) with incenter I. BI, IC cut (O) again at E, F. M, N are midpoints of CA, AB. Let ℓ be the common tangents of
★★★ circle diameter ME, NF such that M, N and I lie on opposite side of ℓ. Prove that $\ell \parallel BC$.

Problem 56 (China TST 2015 day 1 P1). The circle Γ through A of triangle ABC meets sides AB, AC at E,F respectively, and circumcircle of ABC at P. Prove that reflection of P across EF
★★ is on BC if and only if Γ passes through O the circumcentre of ABC.

Problem 57 (Centro American Math Olympiad 2015 P3). Let $ABCD$ be a cyclic quadrilateral with $AB < CD$, and let P be the point of intersection of the lines AD and BC.The circumcircle of the triangle PCD intersects the line AB at the points Q and R. Let S and T be the points where the tangents from P to the circumcircle of $ABCD$ touch that circle. Prove that $QRST$ is a cyclic
★ quadrilateral.

Problem 58 (Own, extension of JBMO 2015 P3). Let ABC be a triangle with median AM. P is a point on BC. Let E, F be the points such that $CE \perp BC, PE \perp AC$ and $BF \perp BC, PF \perp AB$. Let Q be symmetric of P through M. AQ cuts EF at D.

a) Prove that $\angle BDC = \angle EQF$.

★★★ b) Prove that D always lies on a fixed circle when P moves.

Problem 59 (Own, inspire from ELMO 2015 Problem 3). Let (O) be a circle and C a point outside it; distinct points A and B are selected on (O) so that CA and CB are tangent to (O). The line passes through C that intersects (O) at M, N. Denote by (K) the circumcircle of triangle CAN.
★★★ AB cuts (K) again at P. PM cuts (K) and (O) again at Q, R, reps. Prove that RA bisects BQ.

Problem 60 (Russia 2014). Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK.
★★ If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

Problem 61 (Russia 2013). Squares $CAKL$ and $CBMN$ are constructed on the sides of acute-angled triangle ABC, outside of the triangle. Line CN intersects line segment AK at X, while line CL intersects line segment BM at Y. Point P, lying inside triangle ABC, is an intersection of the circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that angle
★★ $\angle ACS = \angle BCP$.

Problem 62 (IMO Shortlist 2013 G2)**.** Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$. ⋆⋆

Problem 63 (AoPS)**.** Let ABC be a triangle inscribed in circle (O). Tangents at B, C of (O) intersect at T. Bisector BE, CF intersect at I. Prove that IT bisects segment EF. ⋆⋆

2.9.2 IMO

Problem 64 (IMO 2013 P4)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogoously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear. ⋆

Problem 65 (IMO Shortlist 2004 G1)**.** Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC. ⋆

Problem 66 (IMO Shortlist 2003 G1)**.** Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC. ⋆

Problem 67 (IMO Shortlist 2001 G1)**.** Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent. ⋆

Problem 68 (Russia 2014)**.** Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic. ⋆⋆

2.10 Tenth day

2.10.1 IMO+

Problem 69 (IMO Shortlist 2013 G4)**.** Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

⋆⋆

Problem 70 (Extension of IMO Shortlist 2013 G4)**.** Let ABC be a triangle bisector AD. (K) is the circle passing through A, D and is tangent to AB. E is reflection of A through CK. DE cuts AC at F. Prove that $BA = BF$.

⋆⋆

Problem 71 (Own, inspire from problem in AoPS)**.** Let ABC be a triangle bisector AD. (K) is the circle passing through A, B and is tangent to AD. M is midpoint of AD. MB cuts (K) again at N. Prove that $NA \perp NC$.

⋆⋆

Problem 72 (Own, extension IMO 2014 P4)**.** Let ABC be a triangle and the points P, Q are lying on BC such that $AP = AQ$. The circumcircle of the triangle APB cuts CA again at E. The circumcircle of the triangle AQC cuts AB again at F. Get the points M, N on the opposite ray of PA, QA such that $PM.QN = PE.QF$. Prove that BN and CM are always intersected each other on a fixed circle when M, N are moving.

Problem 73 (Own, extension IMO 2014 P4)**.** Let ABC be a triangle and the points P, Q lying on the edge BC such that $AP = AQ$. The circumcircle of the triangle APB cut CA again at E. The circumcircle center of the triangle AQC cut AB again at F. Get the points M, N are lying on opposite ray of PA, QA such that $PM.QN = PE.QF$.

a) Prove that BN and CM are always intersected at R on the circle (K) fixed when M, N is moving.

b) Call by L the circumcircle center of the triangle RMN. Prove that AL perpendicular with BC.

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Problem 74 (Own, extension of IMO Shortlist 2012 G4)**.** Let ABC be a triangle with circumcenter O and bisector AD. Let E lie on OA such that $DE \perp BC$. AD cuts circumcircle of triangle BEC such that F is outside triangle ABC. Assume that B, C are fixed and A change such that ratio $\dfrac{AB}{AC}$ is constant. Prove that F always lies on a fixed line when A moves.

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Problem 75 (Own, extension of IMO 1998 P1)**.** Let (I) be the incircle of triangle ABC. Let K, L and M be the points of tangency of the incircle of ABC with AB, BC and CA, respectively. The lines MK and ML intersect the line passing through B and is parallel to KL at the points Q and R, resp. Circle diameter QR cut (I) at S, T. Prove that ST bisects the segment KL.

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2.10.2 IMO

Problem 76 (IMO 2014 P4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. ⋆

Problem 77 (IMO 2012 P1). Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST. ⋆

Problem 78 (IMO 2010 P2). Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ. ⋆⋆

Problem 79 (Russia 2012). The inscribed circle ω of the non-isosceles acute-angled triangle ABC touches the side BC at the point D. Suppose that I and O are the centres of inscribed circle and circumcircle of triangle ABC respectively. The circumcircle of triangle ADI intersects AO at the points A and E. Prove that AE is equal to the radius r of ω. ⋆

Problem 80 (Russia 2012). Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the feet of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle. ⋆⋆

2.11 Eleventh day

2.11.1 IMO+

Problem 81 (Own Oman)**.** Let ABC be isocesles triangle with $AB = AC$. P is a point inside triangle such that $\angle PBC = \angle PCA$. H is projection of P on BC and AP cuts circumcricle of triangle PBC again at Q. Prove that QH always passes through a fixed point when P moves. ⋆⋆

Problem 82 (Russia 2012)**.** The point E is the midpoint of the segment connecting the orthocentre of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B' respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcentre and the incentre of triangle ABC. ⋆⋆⋆

Problem 83 (Russia 2011)**.** Let ABC be a triangle with $AB < AC$. Let q be quarter perimeter of triangle ABC. X, Y are the points on ray AB, AC such that $AX = AY = q$. Assume that segments XY and BC intersect at M. Prove that perimeter of triangle ABM is equal to q ⋆⋆⋆

Problem 84 (Own, extension of IMO shortlist 2012 G6)**.** Let ABC be a triangle. The point D, E, F lies on side BC, CA, AB of triangle ABC such that $BF + CE + BC = CD + AF + CA = AE + BD + AB$. Circumcircle of triangle AEF, BFD, CED have a common point P. Prove that P always lies on a fixed circle when D, E, F moves. ⋆⋆⋆

Problem 85 (Own, extension of IMO shortlist 2012 G6)**.** Let ABC be a triangle. The point D, E, F lies on side BC, CA, AB of triangle ABC such that $BF + CE + BC = CD + AF + CA = AE + BD + AB$. Circumcircle of triangle AEF, BFD, CED have a common point P. Prove that P always lies on a fixed circle when D, E, F moves. ⋆⋆⋆

Problem 86 (Own, xtension of IMO shortlist 1997 P18)**.** Let ABC be an acute triangle and E, F lie on side CA, AB such that $BCEF$ is cyclic quadrilateral. BE cuts CF at H. AH cuts BC at D. The line passes through D and is parallel to EF which intersects CA, AB at M, N, resp. EF cuts BC at G. Prove that circumcircle of triangle GMN always passes throgh a fixed point when E, F move. ⋆⋆

Problem 87 (Inspire from IMO shortlist 2014 G6)**.** Let ABC be a triangle. E, F lie on side CA, AB. Perpendicular bisector of EF cuts BC at D and M is midpoint of E, F. Perpendicular bisector of DM cuts CA, AB at P, Q. Prove that four points A, P, D, Q are concyclic iff BE, CF intersect on circumcircle of triangle AEF. ⋆⋆⋆

2.11.2 IMO

Problem 88 (IMO 1998 P1)**.** A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove
$\star$ that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

Problem 89 (IMO 2000 P1)**.** Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that
$\star$ $EP = EQ$.

Problem 90 (IMO 1995 P1)**.** Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines
$\star$ AM, DN, XY are concurrent.

Problem 91 (IMO 1996 P2)**.** Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE
$\star\star$ meet at a point.

Problem 92 (IMO 1994 P2)**.** Let ABC be an isosceles triangle with $AB = AC$. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that
$\star$ E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Problem 93 (IMO 1987 P2)**.** In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle
$\star$ ABC have equal areas.

Problem 94 (IMO 1985 P1)**.** A circle has center on the side AB of the cyclic quadrilateral $ABCD$.
$\star$ The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Problem 95 (IMO 1985 P5)**.** A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B).
$\star\star$ Prove that $\angle OMB = 90°$.

2.12 Twelfth day

2.12.1 IMO+

Problem 96 (Own, extension IMO 1996 P2)**.** Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC = \angle BPC - \angle BAC.$$

Let D, E, F be the incenters of triangles BPC, CPA, APB, respectively. Show that the lines AD,
$\star\star\star$ BE, CF are concurent.

Problem 97 (Onw, extension of IMO 2009 P2)**.** Let ABC be a triangle with circumcircle O. E, F lie on side CA, AB. M, N, P are midpoint of BE, CF, EF. Let Q be projection of O on EF.

a) Prove that four points M, N, P, Q are concyclic.

$\star\star\star$ b) Prove that reflection of Q through MN lies on ninepoint circle of triangle ABC.

Problem 98 (AoPS)**.** Let ABC be a triangle inscribed in circle (O). P is a point on bisector of $\angle BAC$. PB, PC cut CA, AB at E, F and cut (O) again at M, N. NE cuts MF at Q. Prove that
$\star\star\star$ PQ bisects BC.

Problem 99 (Own, inspire from IMO shortlist 2011 G5)**.** Let ABC be a triangle inscribed in circle (O). I is its incenter. IB, IC cut (O) again at E, F. EF cuts CA, AB at M, N. P is a point such that $PM \parallel IC, PN \parallel IB$. Assume that (O) and B, C are fixed and A moves. Prove that PI always
$\star\star\star$ passes through a fixed point.

Problem 100 (AoPS)**.** Let ABC be a triangle E, F are midpoint of CA, AB. P is an any point. PE, PF cut BC at M, N, resp. NE cuts MF at Q. Let R is the point such that $RB \parallel PE$ and
$\star$ $RQ \parallel PF$. Prove that PQ bisects segment AR.

Problem 101 (AoPS)**.** Let CD be a median of triangle ABC. Perpendicular bisectors of AC and BC intersect CD at A_c, B_c. AA_c and BB_c intersect at Z. X and Y are defnied similarly. Let O be
$\star\star\star$ the circumcentre of triangle ABC. Prove X, Y, Z, O are concyclic.

Problem 102 (Russia 2009)**.** Let be given a triangle ABC and its internal angle bisector BD $(D \in AC)$. The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with
$\star\star$ diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

2.12.2 IMO

Problem 103 (Russia 2013). Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the ⋆ midpoint of BC.

Problem 104 (Russia 2011). Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that ⋆ the intersection of the heights of the triangle POQ lies on line AC.

Problem 105 (Russia 2011). Given is an acute triangle ABC. Its heights BB_1 and CC_1 are extended past points B_1 and C_1. On these extensions, points P and Q are chosen, such that angle PAQ is ⋆ right. Let AF be a height of triangle APQ. Prove that angle BFC is a right angle.

Problem 106 (Russia 2011). On side BC of parallelogram $ABCD$ (A is acute) lies point T so that triangle ATD is an acute triangle. Let O_1, O_2, and O_3 be the circumcenters of triangles ABT, DAT, ⋆ and CDT respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line AD.

Problem 107 (Russia 2010). Let O be the circumcentre of the acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcentre of triangle OPQ. Prove that ⋆⋆ $\angle BAS = \angle CAM$.

Problem 108 (Russia 2010). Into triangle ABC gives point K lies on bisector of $\angle BAC$. Line CK intersect circumcircle ω of triangle ABC at $M \neq C$. Circle Ω passes through A, touch CM at K ⋆ and intersect segment AB at $P \neq A$ and ω at $Q \neq A$. Prove, that P, Q, M lies at one line.

Problem 109 (Russia 2008). In a scalene triangle ABC the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, ⋆ while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel.

2.13 Thirteenth day

Problem 110 (Own, extension of IMO shortlist 2009 G4). Let ABC be a triangle. (K) is the circle passing through B, C. (K) cuts CA, AB again at E, F. BE cuts CF at H. M, N, P are midpoints of AH, BE, EF. AN cuts CF at Q. R is symmetric of F through Q. Prove that the line connecting midpoint of MQ and PR always passes through a fixed point when (K) moves.

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Problem 111 (Onw, extension of IMO shortlist 2012 G2). Let $ABCD$ be a cyclic quadrilateral inscribed in circle (O) and whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G, L be the point such that $ECGD, EBLA$ is a parallelogram, and let H, K be the images of E, B under reflection in AD, AC, resp. DG cuts (O) again at N. CK cuts AN at P. DH cuts BN at Q. Prove that PQ and GL intersect on circumcircle of triangle DGH.

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Problem 112 (Own, extension of IMO shortlist 2012 G2). Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic and Simson line of H with respect to triangle FGD are perpendicular to BC.

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Problem 113 (Own, extension of IMO shortlist 2007 G3). Let $ABCD$ be trapezoid with $AB \parallel CD$. Assume that there are the points E, F on side CD, CB such that $\angle DEF = \angle AEB$. AC cuts BD at G. Prove that GE is tangent to circumcirle of triangle CEF.

★★★

Problem 114 (Own, inspire from IMO shortlist 2011 G3). Let ABC be a triangle and P is a point inside it. Let Perpendicular bisector of PC, PB cut CA, AB at M, N. Q is relfection of P through MN. Prove that radical axis of two pedal circles of points P, Q bisects segment PQ.

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