DIỄN ĐÀN TOÁN HỌC VIỆT NAM
www.maths.vn
----oOo----

Tuyển tập

Bất đẳng thức

Volume 1

Biên tập: **Võ Quốc Bá Cẩn**

Tác giả các bài toán: **Trần Quốc Luật**

Thành viên tham gia giải bài:

1. **Võ Quốc Bá Cẩn** (nothing)
2. **Ngô Đức Lộc** (Honey_suck)
3. **Trần Quốc Anh** (nhocnhoc)
4. Seasky
5. Materazzi

Tran Quoc Luat's Inequalities

Vo Quoc Ba Can - Pham Thi Hang

December 26, 2008

Preface

"Life is good for only two things, discovering mathematics and teaching mathematics."

S. Poisson

Bat dang thuc la mot trong linh vuc hay va kho. Hien nay, co kha nhieu nguoi quan tam den no boi no thuc su rat don gian, quyen ru va ban khong can phai "hoc vet" nhieu dinh ly de co the giai duoc chung. Khi hoc bat dang thuc, hai dieu cuon hut chung ta nhat chinh la sang tao va giai bat dang thuc. Nham muc dich kich thich su sang tao cua hoc sinh sinh vien nuoc nha, dien dan mathsvn da co mot so topic sang tao bat dang thuc danh rieng cho cac ca nhan tren dien dan. Tuy nhien, cac topic do con roi rac nen ta can mot su tong hop lai thong nhat hon de cho ban doc tien theo doi, do la li do ra doi cua quyen sach nay. Quyen sach duoc trinh bay trong phan chinh bang tieng Anh voi muc dich giup chung ta ren luyen them ngoai ngu va co the gioi thieu no den cac ban trong va ngoai nuoc. Mac du da co gang bien soan nhung sai sot la dieu khong the tranh khoi, rat mong nhan duoc su gop y cua ban doc gan xa. Moi su dong gop y kien xin duoc gui ve tac gia theo: babylearnmath@yahoo.com. Xin chan than cam on!

Vo Quoc Ba Can

Chapter 1

Problems

"Each problem that I solved became a rule, which served afterwards to solve other problems."

R. Descartes

1. Given a triangle ABC with the perimeter is $2p$. Prove that the following inequality holds
$$\frac{a}{p-a}+\frac{b}{p-b}+\frac{c}{p-c}\geq\sqrt{\frac{b+c}{p-a}}+\sqrt{\frac{c+a}{p-b}}+\sqrt{\frac{a+b}{p-c}}.$$

2. Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2+abc=4$. Prove that the following inequality holds
$$a^2+b^2+c^2\geq a^2b^2+b^2c^2+c^2a^2.$$

3. Show that for any positive real numbers $a,b,c,$ we have
$$a^3+b^3+c^3+6abc\geq\sqrt[3]{abc}(a+b+c)^2.$$

4. Let a,b,c be nonnegative real numbers with sum 1. Determine the maximum and minimum values of
$$P(a,b,c)=(1+ab)^2+(1+bc)^2+(1+ca)^2.$$

5. Let a,b,c be nonnegative real numbers with sum 1. Determine the maximum and minimum values of
$$P(a,b,c)=(1-4ab)^2+(1-4bc)^2+(1-4ca)^2.$$

6. Let a,b,c be positive real numbers. Prove that
$$\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)^2\geq 4(ab+bc+ca)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

7. Let a,b,c be the side of a triangle. Show that

$$\sum_{cyc}(a+b)(a+c)\sqrt{b+c-a} \ge 4(a+b+c)\sqrt{(b+c-a)(c+a-b)(a+b-c)}.$$

8. Given a triangle with sides a,b,c satisfying $a^2+b^2+c^2=3$. Show that

$$\frac{a+b}{\sqrt{a+b-c}}+\frac{b+c}{\sqrt{b+c-a}}+\frac{c+a}{\sqrt{c+a-b}} \ge 6.$$

9. Given a triangle with sides a,b,c satisfying $a^2+b^2+c^2=3$. Show that

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{c+a-b}}+\frac{c}{\sqrt{a+b-c}} \ge 3.$$

10. Show that if a,b,c are positive real numbers, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge 1+\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}.$$

11. Show that if a,b,c are positive real numbers, then

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+a}\right)^2 \ge \frac{3}{4}+\frac{a^2b+b^2c+c^2a-3abc}{(a+b)(b+c)(c+a)}.$$

12. Let a,b,c be positive real numbers. Prove that

$$\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{8a^2b^2c^2} \ge \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2.$$

13. Let a,b,c be positive real numbers. Prove the inequality

$$\frac{(b+c)^2}{a(b+c+2a)}+\frac{(c+a)^2}{b(c+a+2b)}+\frac{(a+b)^2}{c(a+b+2c)} \ge 3.$$

14. Let a,b,c be positive real numbers. Prove the inequality

$$\frac{(b+c)^2}{a(b+c+2a)}+\frac{(c+a)^2}{b(c+a+2b)}+\frac{(a+b)^2}{c(a+b+2c)} \ge 2\left(\frac{b+c}{b+c+2a}+\frac{c+a}{c+a+2b}+\frac{a+b}{a+b+2c}\right).$$

15. Let a,b,c be positive real numbers. Prove that

$$\frac{(b+c)^2}{a(b+c+2a)}+\frac{(c+a)^2}{b(c+a+2b)}+\frac{(a+b)^2}{c(a+b+2c)} \ge 2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

16. Let a,b,c be positive real numbers. Prove that

$$a^3b^3+b^3c^3+c^3a^3 \ge (b+c-a)(c+a-b)(a+b-c)(a^3+b^3+c^3).$$

17. If a,b,c are positive real numbers such that $abc=1$, show that we have the following inequality
$$a^3+b^3+c^3 \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}.$$

18. Given nonnegative real numbers a,b,c such that $ab+bc+ca+abc=4$. Prove that
$$a^2+b^2+c^2+2(a+b+c)+3abc \geq 4(ab+bc+ca).$$

19. Let a,b,c be real numbers with $\min\{a,b,c\} \geq \frac{3}{4}$ and $ab+bc+ca=3$. Prove that
$$a^3+b^3+c^3+9abc \geq 12.$$

20. Let a,b,c be positive real numbers such that $a^2b^2+b^2c^2+c^2a^2=1$. Prove that
$$(a^2+b^2+c^2)^2+abc\sqrt{(a^2+b^2+c^2)^3} \geq 4.$$

21. Show that if a,b,c are positive real numbers, the following inequality holds
$$(a+b+c)^2(ab+bc+ca)^2+(ab+bc+ca)^3 \geq 4abc(a+b+c)^3.$$

22. Let a,b,c be real numbers from the interval $[3,4]$. Prove that
$$(a+b+c)\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right) \geq 3(a^2+b^2+c^2).$$

23. Given ABC is a triangle. Prove that
$$8\cos^2 A\cos^2 B\cos^2 C+\cos 2A\cos 2B\cos 2C \geq 0.$$

24. Let a,b,c be positive real numbers such that $a+b+c=3$ and $ab+bc+ca \geq 2\max\{ab,bc,ca\}$. Prove that
$$a^2+b^2+c^2 \geq a^2b^2+b^2c^2+c^2a^2.$$

Chapter 2

Solutions

"Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?"

P. Halmos, *I Want to be a Mathematician*

Problem 2.1 *Given a triangle ABC with the perimeter is* $2p$. *Prove that the following inequality holds*

$$\frac{a}{p-a}+\frac{b}{p-b}+\frac{c}{p-c} \geq \sqrt{\frac{b+c}{p-a}}+\sqrt{\frac{c+a}{p-b}}+\sqrt{\frac{a+b}{p-c}}.$$

Solution. Setting $x=p-a, y=p-b$ and $z=p-c$, then $a=y+z, b=z+x$ and $c=x+y$. The original inequality becomes

$$\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z} \geq \sqrt{2+\frac{y+z}{x}}+\sqrt{2+\frac{z+x}{y}}+\sqrt{2+\frac{x+y}{z}}.$$

By **AM-GM Inequality**, we have

$$4\sqrt{2+\frac{y+z}{x}} \leq 2+\frac{y+z}{x}+4=\frac{y+z}{x}+6.$$

It follows that

$$4\left(\sqrt{2+\frac{y+z}{x}}+\sqrt{2+\frac{z+x}{y}}+\sqrt{2+\frac{x+y}{z}}\right) \leq \frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}+18.$$

We have to prove

$$4\left(\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\right) \geq \frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}+18,$$

or equivalently,

$$\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\geq 6,$$

which is obviously true by **AM-GM Inequality**.
Equality holds if and only if $a=b=c$. □

Problem 2.2 *Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2+abc=4$. Prove that the following inequality holds*

$$a^2+b^2+c^2\geq a^2b^2+b^2c^2+c^2a^2.$$

Solution. From the given condition, we see that there exist $x,y,z\geq 0$ such that

$$a=\frac{2x}{\sqrt{(x+y)(x+z)}},\quad b=\frac{2y}{\sqrt{(y+z)(y+x)}},\quad c=\frac{2z}{\sqrt{(z+x)(z+y)}}.$$

Using this substitution, we may write our inequality as

$$\sum_{cyc}\frac{x^2}{(x+y)(x+z)}\geq\sum_{cyc}\frac{4y^2z^2}{(y+z)^2(x+y)(x+z)},$$

which is obviously true because

$$\sum_{cyc}\frac{4y^2z^2}{(y+z)^2(x+y)(x+z)}\leq\sum_{cyc}\frac{yz(y+z)^2}{(y+z)^2(x+y)(x+z)}=\sum_{cyc}\frac{yz}{(x+y)(x+z)},$$

and

$$\sum_{cyc}\frac{x^2}{(x+y)(x+z)}=\sum_{cyc}\frac{yz}{(x+y)(x+z)}.$$

Our proof is completed. Equality holds if and only if $a=b=c=1$ or $a=b=\sqrt{2},c=0$ and its cyclic permutations. □

Problem 2.3 *Show that for any positive real numbers a,b,c, we have*

$$a^3+b^3+c^3+6abc\geq\sqrt[3]{abc}(a+b+c)^2.$$

Solution 1. According to the **AM-GM Inequality**, we have the following estimation

$$6\sqrt[3]{abc}(a+b+c)^2\leq(a+b+c)^3+9\sqrt[3]{a^2b^2c^2}(a+b+c),$$

which leads us to prove the sharper inequality

$$6(a^3+b^3+c^3+6abc)\geq(a+b+c)^3+9\sqrt[3]{a^2b^2c^2}(a+b+c),$$

or

$$5(a^3+b^3+c^3)-3\sum_{cyc}ab(a+b)+30abc\geq 9\sqrt[3]{a^2b^2c^2}(a+b+c),$$

From **Schur's Inequality** for third degree, we have

$$3(a^3+b^3+c^3)-3\sum_{cyc}ab(a+b)+9abc\geq 0,$$

and we deduce our inequality to

$$2(a^3+b^3+c^3)+21abc \ge 9\sqrt[3]{a^2b^2c^2}(a+b+c),$$

Again, the **Schur's Inequality** for third degree shows that

$$4(a^3+b^3+c^3)+15abc \ge (a+b+c)^3,$$

and with this inequality, we finally come up with

$$(a+b+c)^3-15abc+42abc \ge 18\sqrt[3]{a^2b^2c^2}(a+b+c),$$

$$(a+b+c)^3+27abc \ge 18\sqrt[3]{a^2b^2c^2}(a+b+c),$$

By **AM-GM Inequality**, we have that

$$\begin{aligned} 2(a+b+c)^3+54abc &= (a+b+c)^3+\left[(a+b+c)^3+27abc+27abc\right] \\ &\ge (a+b+c)^3+27\sqrt[3]{a^2b^2c^2}(a+b+c) \\ &\ge 9\sqrt[3]{a^2b^2c^2}(a+b+c)+27\sqrt[3]{a^2b^2c^2}(a+b+c) \\ &= 36\sqrt[3]{a^2b^2c^2}(a+b+c). \end{aligned}$$

It shows that

$$(a+b+c)^3+27abc \ge 18\sqrt[3]{a^2b^2c^2}(a+b+c),$$

which completes our proof. Equality holds if and only if $a=b=c$. □

Solution 2 (by Seasky). Since the inequality being homogeneous, we can suppose without loss of generality that $abc=1$. In this case, the inequality can be rewitten in the form

$$P(a,b,c)=a^3+b^3+c^3+6-(a+b+c)^2 \ge 0.$$

We will now use mixing variables method to solve this inequality. Assuming $a \ge b \ge c$, we claim that

$$P(a,b,c) \ge P(t,t,c),$$

where $t=\sqrt{ab} \ge 1$.
Indeed, we have

$$\begin{aligned} & P(a,b,c)-P(t,t,c) = \\ = \;& \left(a^3+b^3-2ab\sqrt{ab}\right)-(a^2+b^2-2ab)-2c\left(a+b-2\sqrt{ab}\right) \\ = \;& (a-b)^2(a+b)+ab\left(\sqrt{a}-\sqrt{b}\right)^2-(a-b)^2-2c\left(\sqrt{a}-\sqrt{b}\right)^2 \\ = \;& \left(\sqrt{a}-\sqrt{b}\right)^2\left[\left(\sqrt{a}+\sqrt{b}\right)^2(a+b-1)+ab-2c\right] \ge 0, \end{aligned}$$

because

$$\left(\sqrt{a}+\sqrt{b}\right)^2(a+b-1)+ab-2c \ge 4(2-1)+1-2=3>0.$$

So, the above statement holds and all we have to do is to prove that $P(t,t,c) \geq 0$, which is equivalent to each of the following inequalities

$$2t^3 + c^3 + 6 \geq (2t + c)^2,$$

$$2t^3 + \frac{1}{t^6} + 6 \geq \left(2t + \frac{1}{t^2}\right)^2,$$

$$2t^9 + 6t^6 + 1 \geq t^2(2t^3 + 1)^2,$$

$$2t^9 + 6t^6 + 1 \geq 4t^8 + 4t^5 + t^2,$$

$$2t^9 - 4t^8 + 6t^6 - 4t^5 - t^2 + 1 \geq 0,$$

$$(t-1)^2\left(2t^7 - 2t^5 + 2t^4 + 2t^3 + 2t^2 + 2t + 1\right) \geq 0,$$

which is obivously true because $t \geq 1$. $\square$

Problem 2.4 *Let a,b,c be nonnegative real numbers with sum 1. Determine the maximum and minimum values of*

$$P(a,b,c) = (1+ab)^2 + (1+bc)^2 + (1+ca)^2.$$

Solution. It is clear that $\min P = 3$ with equality attains when $a = 1, b = c = 0$ and its cyclic permutions. Now, let us find $\max P$. We claim that $\max P = \frac{100}{27}$ attains when $a = b = c = \frac{1}{3}$, or

$$(1+ab)^2 + (1+bc)^2 + (1+ca)^2 \leq \frac{100}{27},$$

$$a^2b^2 + b^2c^2 + c^2a^2 + 2(ab+bc+ca) \leq \frac{19}{27},$$

$$(ab+bc+ca)^2 + 2(ab+bc+ca) - 2abc \leq \frac{19}{27},$$

$$(ab+bc+ca+1)^2 - 2abc \leq \frac{46}{27}.$$

According to **AM-GM Inequality**, we have

$$(ab+bc+ca+1)^2 \leq \frac{4}{3}(ab+bc+ca+1).$$

And we deduce the inequality to

$$\frac{4}{3}(ab+bc+ca+1) - 2abc \leq \frac{46}{27},$$

$$4(ab+bc+ca) - 6abc \leq \frac{10}{9},$$

which is obviously true by **Schur's Inequality** for third degree,

$$4(ab+bc+ca) - 6abc \leq (1+9abc) - 6abc = 1 + 3abc \leq 1 + 3 \cdot \frac{1}{27} = \frac{10}{9}.$$

With the above solution, we have the conclusion for the requirement is $\min P = 3$ and $\max P = \frac{100}{27}$. $\square$

Problem 2.5 *Let a,b,c be nonnegative real numbers with sum 1. Determine the maximum and minimum values of*

$$P(a,b,c) = (1-4ab)^2 + (1-4bc)^2 + (1-4ca)^2.$$

Solution (by Honey_suck). Notice that

$$1 \geq 1-4ab \geq 1-(a+b)^2 \geq 1-(a+b+c)^2 = 0.$$

Hence

$$(1-4ab)^2 \leq 1,$$

and it follows that

$$P(a,b,c) \leq 1+1+1 = 3,$$

which equality holds when $a=1, b=c=0$ and its cyclic permutions. And we conclude that $\max P = 3$.
Moreover,applying **Cauchy Schwarz Inequality** and **AM-GM Inequality**, we have

$$\begin{aligned}
(1-4ab)^2 + (1-4bc)^2 + (1-4ca)^2 &\geq \frac{1}{3}(1-4ab+1-4bc+1-4ca)^2 \\
&= \frac{1}{3}\left[3-4(ab+bc+ca)\right]^2 \\
&\geq \frac{1}{3}\left(3-4\cdot\frac{1}{3}\right)^2 = \frac{25}{27},
\end{aligned}$$

with equality holds when $a=b=c=\frac{1}{3}$. And we conclude that $\min P = \frac{25}{27}$.
This completes the proof. $\square$

Problem 2.6 *Let a,b,c be positive real numbers. Prove that*

$$\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)^2 \geq 4(ab+bc+ca)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Solution 1. We can rewite the inequality as

$$\left[\sum_{cyc} ab(a+b)\right]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

Assuming that $a \geq b \geq c$, then using **AM-GM Inequality**, we have that

$$\begin{aligned}
&4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2) = \\
&= \frac{16(a+b)^2(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)}{4(a+b)^2} \\
&\leq \frac{\left[(a+b)^2(ab+bc+ca)+4(a^2b^2+b^2c^2+c^2a^2)\right]^2}{4(a+b)^2}.
\end{aligned}$$

It suffices to show that

$$2ab(a+b)+2bc(b+c)+2ca(c+a) \geq \frac{(a+b)^2(ab+bc+ca)+4(a^2b^2+b^2c^2+c^2a^2)}{a+b},$$

$$2ab(a+b)+2c^2(a+b)+2c(a^2+b^2) \geq (a+b)(ab+bc+ca)+\frac{4(a^2b^2+b^2c^2+c^2a^2)}{a+b},$$

$$ab(a+b)+2c^2(a+b)+c(a-b)^2 \geq \frac{4a^2b^2}{a+b}+\frac{4c^2(a^2+b^2)}{a+b},$$

$$ab\left(a+b-\frac{4ab}{a+b}\right)+c(a-b)^2 \geq 2c^2\left[\frac{2(a^2+b^2)}{a+b}-a-b\right],$$

$$\frac{ab(a-b)^2}{a+b}+c(a-b)^2 \geq \frac{2c^2(a-b)^2}{a+b},$$

$$\frac{ab}{a+b}+c \geq \frac{2c^2}{a+b},$$

which is obviously true because

$$\frac{2c^2}{a+b} \leq \frac{2c^2}{c+c} = c \leq c+\frac{ab}{a+b}.$$

The proof is completed and we have equality iff $a=b=c$. □

Solution 2. Similar to solution 1, we need to prove

$$\left[\sum_{cyc} ab(a+b)\right]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

For all real numbers m,n,p,x,y,z, we have the following interesing identity of **Lagrange**

$$(x^2+y^2+z^2)(m^2+n^2+p^2)-(mx+ny+pz)^2 = (my-nx)^2+(nz-py)^2+(px-mz)^2.$$

Now, applying this identity with $x=\sqrt{ab}, y=\sqrt{bc}, z=\sqrt{ca}, m=(a+b)\sqrt{ab}, n=(b+c)\sqrt{bc}$, and $p=(c+a)\sqrt{ca}$, we obtain

$$(ab+bc+ca)\left[\sum_{cyc} ab(a+b)^2\right]-\left[\sum_{cyc} ab(a+b)\right]^2 = abc\sum_{cyc} c(a-b)^2.$$

Moreover,

$$\sum_{cyc} ab(a+b)^2-4(a^2b^2+b^2c^2+c^2a^2) = \sum_{cyc} ab(a-b)^2,$$

So, we may rewrite our inequality as

$$(ab+bc+ca)\left[\sum_{cyc} ab(a+b)^2-4\sum_{cyc} a^2b^2\right] \geq$$

$$\geq \quad (ab+bc+ca)\left[\sum_{cyc} ab(a+b)^2\right]-\left[\sum_{cyc} ab(a+b)\right]^2,$$

$$(ab+bc+ca)\sum_{cyc} ab(a-b)^2 \geq abc\sum_{cyc} c(a-b)^2,$$

$$\sum_{cyc} a^2b^2(a-b)^2+abc\sum_{cyc}(a+b-c)(a-b)^2 \geq 0,$$

$$\sum_{cyc} a^2b^2(a-b)^2 + 2abc\sum_{cyc} a(a-b)(a-c) \geq 0,$$

which is obviously true by **Schur's Inequality** for third degree. $\square$

Problem 2.7 *Let a,b,c be the side of a triangle. Show that*

$$\sum_{cyc}(a+b)(a+c)\sqrt{b+c-a} \geq 4(a+b+c)\sqrt{(b+c-a)(c+a-b)(a+b-c)}.$$

Solution. The inequality can be rewritten in the form

$$\sum_{cyc} \frac{(a+b)(a+c)}{\sqrt{(c+a-b)(a+b-c)}} \geq 4(a+b+c),$$

which is obviously true because

$$\begin{aligned}
\sum_{cyc} \frac{(a+b)(a+c)}{\sqrt{(c+a-b)(a+b-c)}} &= \sum_{cyc} \frac{(a+b)(a+c)}{\sqrt{a^2-(b-c)^2}} \\
&\geq \sum_{cyc} \frac{(a+b)(a+c)}{a} \\
&= 3(a+b+c) + \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) \\
&\geq 4(a+b+c),
\end{aligned}$$

where the last inequality is valid because

$$\begin{aligned}
\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} &= a\left(\frac{b}{2c} + \frac{c}{2b}\right) + b\left(\frac{c}{2a} + \frac{a}{2c}\right) + c\left(\frac{a}{2b} + \frac{b}{2a}\right) \\
&\geq a+b+c.
\end{aligned}$$

Equality holds iff $a=b=c$. $\square$

Problem 2.8 *Given a triangle with sides a,b,c satisfying $a^2+b^2+c^2=3$. Show that*

$$\frac{a+b}{\sqrt{a+b-c}} + \frac{b+c}{\sqrt{b+c-a}} + \frac{c+a}{\sqrt{c+a-b}} \geq 6.$$

Solution. Firstly, to prove the original inequality, we will show that[1]

$$ab+bc+ca \geq \frac{4a^3(b+c-a)}{(b+c)^2} + \frac{4b^3(c+a-b)}{(c+a)^2} + \frac{4c^3(a+b-c)}{(a+b)^2},$$

$$\sum_{cyc}\left[a^2 - \frac{4a^3(b+c-a)}{(b+c)^2}\right] \geq a^2+b^2+c^2-ab-bc-ca,$$

[1] We may prove this statement easily by using tangent line technique, the readers can try it! In here, we present a nonstandard proof for it, this proof seems to be complicated but it is nice about its idea.

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}+\frac{c^2(2c-a-b)^2}{(a+b)^2}\geq a^2+b^2+c^2-ab-bc-ca,$$

Without loss of generality, we may assume that $a \geq b \geq c$, then

$$\frac{a^2}{(b+c)^2}\geq\frac{b^2}{(c+a)^2}\quad\text{and}\quad(2a-b-c)^2\geq(2b-c-a)^2.$$

Thus, using **Chebyshev's Inequality**, we have

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}\geq\frac{1}{2}\left[\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right].\qquad(1)$$

Notice that

$$\frac{1}{4}\left[(2a-b-c)^2+(2b-c-a)^2\right]-(a^2-ab+b^2)\geq\frac{1}{8}(a+b-2c)^2-\frac{1}{4}(a+b)^2,\qquad(2)$$

And

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}\geq\frac{2(a+b)^2}{(a+b+2c)^2}\geq\frac{1}{2},$$

which yieds that

$$\begin{aligned}&\frac{1}{2}\left[\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}-\frac{1}{2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right]\geq\\ \geq\ &\frac{1}{2}\left[\frac{2(a+b)^2}{(a+b+2c)^2}-\frac{1}{2}\right]\left[(2a-b-c)^2+(2b-c-a)^2\right]\\ \geq\ &\frac{1}{4}\left[\frac{2(a+b)^2}{(a+b+2c)^2}-\frac{1}{2}\right](a+b-2c)^2.\qquad(3)\end{aligned}$$

From $(1), (2)$ and (3), we obtain

$$\frac{a^2(2a-b-c)^2}{(b+c)^2}+\frac{b^2(2b-c-a)^2}{(c+a)^2}-(a^2-ab+b^2)\geq\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}-\frac{1}{4}(a+b)^2.$$

Using this inequality, we have to prove

$$\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}-\frac{1}{4}(a+b)^2+\frac{c^2(a+b-2c)^2}{(a+b)^2}\geq c^2-c(a+b),$$

$$\frac{(a+b)^2(a+b-2c)^2}{2(a+b+2c)^2}+\frac{c^2(a+b-2c)^2}{(a+b)^2}\geq\frac{1}{4}(a+b-2c)^2,$$

$$\frac{(a+b)^2}{2(a+b+2c)^2}+\frac{c^2}{(a+b)^2}\geq\frac{1}{4},$$

which can be easily checked. Thus, the above statement is proved.
Now, turning back to our problem, using **Holder Inequality**, we have

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2\left[\sum_{cyc}\frac{a^3(b+c-a)}{(b+c)^2}\right]\geq\left(\sum_{cyc}a\right)^3.$$

It follows that

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2 \geq \frac{\left(\sum_{cyc} a\right)^3}{\sum_{cyc}\frac{a^3(b+c-a)}{(b+c)^2}} \geq \frac{4\left(\sum_{cyc} a\right)^3}{\sum_{cyc} ab}.$$

Moreover, by **AM-GM Inequality**, we have

$$\begin{aligned}\sum_{cyc} ab &= \sqrt{\frac{1}{3}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)} \\ &\leq \sqrt{\frac{1}{3}\left(\frac{\sum_{cyc} ab+\sum_{cyc} ab+\sum_{cyc} a^2}{3}\right)^3} = \frac{\left(\sum_{cyc} a\right)^3}{9}.\end{aligned}$$

Hence

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{b+c-a}}\right)^2 \geq \frac{4\left(\sum_{cyc} a\right)^3}{\sum_{cyc} ab} \geq 36.$$

Our proof is completed. Equality holds if and only if $a=b=c=1$. □

Problem 2.9 *Given a triangle with sides a,b,c satisfying $a^2+b^2+c^2=3$. Show that*

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{c+a-b}}+\frac{c}{\sqrt{a+b-c}} \geq 3.$$

Solution (by Materazzi). Applying **Holder Inequality**, we obtain

$$\left(\sum_{cyc}\frac{a}{\sqrt{b+c-a}}\right)^2\left[\sum_{cyc} a(b+c-a)\right] \geq (a+b+c)^3.$$

And we deduce our inequality to show that

$$(a+b+c)^3 \geq 9\sum_{cyc} a(b+c-a),$$

$$(a+b+c)^3 \geq 9\left[2(ab+bc+ca)-3\right].$$

Setting $p=a+b+c$, then it $\sqrt{3} \leq p \leq 3$ and $2(ab+bc+ca)=p^2-3$. Thus, we can rewrite the above inequality as

$$p^3 \geq 9(p^2-6),$$

$$p^3-9p^2+54 \geq 0,$$

$$(3-p)(18+6p-p^2) \geq 0,$$

which is obviously true. Equality holds if and only if $a=b=c=1$. □

Problem 2.10 *Show that if a, b, c are positive real numbers, then*

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \geq 1+\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}.$$

Solution. The given inequality is equivalent to

$$\sum_{cyc}\frac{a^2}{(a+b)^2}+2\sum_{cyc}\frac{ab}{(a+b)(b+c)} \geq 1+2\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}+\frac{2abc}{(a+b)(b+c)(c+a)},$$

Using the known inequality[2]

$$\sum_{cyc}\frac{a^2}{(a+b)^2} \geq 1-\frac{2abc}{(a+b)(b+c)(c+a)},$$

we can deduce it to

$$\sum_{cyc}\frac{ab}{(a+b)(b+c)} \geq \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}+\frac{2abc}{(a+b)(b+c)(c+a)},$$

$$a^2b+b^2c+c^2a+abc \geq \sqrt{2abc(a+b)(b+c)(c+a)}.$$

Now, we assume that $c=\min\{a,b,c\}$, applying **AM-GM Inequality**, we have

$$\begin{aligned}
a^2b+b^2c+c^2a+abc &= a(ab+c^2)+bc(a+b) \\
&= \frac{a(a+c)(b+c)}{2}+bc(a+b)+\frac{a(a-c)(b-c)}{2} \\
&\geq \frac{a(a+c)(b+c)}{2}+bc(a+b) \\
&\geq 2\sqrt{\frac{a(a+c)(b+c)}{2}\cdot bc(a+b)} \\
&= \sqrt{2abc(a+b)(b+c)(c+a)}.
\end{aligned}$$

Our proof is completed. Equality holds if and only if $a=b=c$. □

Problem 2.11 *Show that if a, b, c are positive real numbers, then*

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+a}\right)^2 \geq \frac{3}{4}+\frac{a^2b+b^2c+c^2a-3abc}{(a+b)(b+c)(c+a)}.$$

Solution. We have

$$\frac{a^2}{(a+b)^2}=\frac{a}{a+b}-\frac{ab}{(a+b)^2},$$

and

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}=1+\frac{a^2b+b^2c+c^2a+abc}{(a+b)(b+c)(c+a)}.$$

[2] The proof will be left to the readers

Hence, the given inequality can be rewritten as

$$\frac{1}{4}+\frac{4abc}{(a+b)(b+c)(c+a)} \geq \frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2},$$

$$\left(\frac{a-b}{a+b}\right)^2+\left(\frac{b-c}{b+c}\right)^2+\left(\frac{c-a}{c+a}\right)^2 \geq 2-\frac{16abc}{(a+b)(b+c)(c+a)}.$$

Now, notice that

$$\begin{aligned}
& 2-\frac{16abc}{(a+b)(b+c)(c+a)} = \\
= \; & \frac{2[(a+b)(b+c)(c+a)-8abc]}{(a+b)(b+c)(c+a)} \\
= \; & \frac{2[(b+c)(a-b)(a-c)+(c+a)(b-c)(b-a)+(a+b)(c-a)(c-b)]}{(a+b)(b+c)(c+a)} \\
= \; & 2\left[\frac{(a-b)(a-c)}{(a+b)(a+c)}+\frac{(b-c)(b-a)}{(b+c)(b+a)}+\frac{(c-a)(c-b)}{(c+a)(c+b)}\right].
\end{aligned}$$

The above inequality is equivalent to

$$\left(\frac{a-b}{a+b}\right)^2+\left(\frac{b-c}{b+c}\right)^2+\left(\frac{c-a}{c+a}\right)^2 \geq 2\left[\frac{(a-b)(a-c)}{(a+b)(a+c)}+\frac{(b-c)(b-a)}{(b+c)(b+a)}+\frac{(c-a)(c-b)}{(c+a)(c+b)}\right],$$

$$\left(\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}\right)^2 \geq 0,$$

which is obviously true. Equality holds if and only if $a=b$ or $b=c$ or $c=a$. □

Problem 2.12 *Let a,b,c be positive real numbers. Prove that*

$$\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{8a^2b^2c^2} \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2.$$

Solution 1. Without loss of generality, we may assume that $a \geq b \geq c$, then we have that

$$\begin{aligned}
(a^2+b^2)(a^2+c^2)-\left[a^2+\frac{(b+c)^2}{4}\right]^2 &= \frac{(b-c)^2(8a^2-b^2-c^2-6bc)}{16} \geq 0, \\
b^2+c^2 &\geq \frac{(b+c)^2}{2}.
\end{aligned}$$

It suffices to prove that

$$\frac{\left[4a^2+(b+c)^2\right](b+c)}{16abc} \geq \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

This inequality is homogeneous, we may assume that $b+c=1$, putting $x=bc$, then $a \geq \frac{1}{2}$, and $\frac{1}{4} \geq x \geq 0$. The above inequality becomes

$$\frac{4a^2+1}{16ax} \geq \frac{a^2+1-2x}{a+x},$$

$$(4a^2+1)(a+x) \geq 16ax(a^2+1-2x),$$
$$32ax^2 - \left(16a^3-4a^2+16a-1\right)x + a\left(4a^2+1\right) \geq 0,$$
$$2a(4x-1)^2 + 2a(8x-1) - \left(16a^3-4a^2+16a-1\right)x + a\left(4a^2+1\right) \geq 0,$$
$$2a(4x-1)^2 + (1+4a^2-16a^3)x + a(4a^2-1) \geq 0,$$
$$2a(4x-1)^2 - (2a-1)(8a^2+2a+1)x + a(4a^2-1) \geq 0,$$

which is true because

$$\begin{aligned} -4(2a-1)(8a^2+2a+1)x + 4a(4a^2-1) &\geq 4a(4a^2-1) - (2a-1)(8a^2+2a+1) \\ &= (2a-1)^2 \geq 0. \end{aligned}$$

Our proof is completed. Equality holds if and only if $a = b = c$. □

Solution 2. Put $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$, then the above inequality becomes

$$(x+y+z)^2(x^2+y^2)(y^2+z^2)(z^2+x^2) \geq 8(x^2y^2+y^2z^2+z^2x^2)^2.$$

Notice that

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) = (x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2) - x^2y^2z^2.$$

Thus, we can rewrite the above inequality as

$$(x^2y^2+y^2z^2+z^2x^2)\left[(x+y+z)^2(x^2+y^2+z^2) - 8(x^2y^2+y^2z^2+z^2x^2)\right] \geq x^2y^2z^2(x+y+z)^2.$$

Now, we see that

$$\begin{aligned} & (x+y+z)^2(x^2+y^2+z^2) - 8(x^2y^2+y^2z^2+z^2x^2) \\ = & \sum_{cyc} x^4 + 2\sum_{cyc} xy(x^2+y^2) + 2xyz\sum_{cyc} x - 6\sum_{cyc} x^2y^2 \\ = & \sum_{cyc} x^2(x-y)(x-z) + 3\sum_{cyc} xy(x-y)^2 + xyz\sum_{cyc} x \geq xyz\sum_{cyc} x. \end{aligned}$$

It suffices to prove that

$$xyz(x^2y^2+y^2z^2+z^2x^2)(x+y+z) \geq x^2y^2z^2(x+y+z)^2,$$
$$x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z),$$

which is obviously true by **AM-GM Inequality**. □

Solution 3. Similar to **solution 2**, we need to prove that

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)(x+y+z)^2 \geq 8(x^2y^2+y^2z^2+z^2x^2)^2.$$

By **AM-GM Inequality**, we have

$$\begin{aligned} (x+y+z)^2 &= x^2+y^2+z^2+2xy+2yz+2zx \\ &\geq x^2+y^2+z^2 + \frac{4x^2y^2}{x^2+y^2} + \frac{4y^2z^2}{y^2+z^2} + \frac{4z^2x^2}{z^2+x^2}. \end{aligned}$$

Hence, it suffices to prove that

$$(m+n)(n+p)(p+m)\left(m+n+p+\frac{4mn}{m+n}+\frac{4np}{n+p}+\frac{4pm}{p+m}\right)\geq 8(mn+np+pm)^2,$$

where $m=a^2, n=b^2, p=c^2$.
This inequality is equivalent with

$$mn(m-n)^2+np(n-p)^2+pm(p-m)^2\geq 0,$$

which is obviously true. □

Solution 4. [3]Again, we will give the solution to the inequality

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)\,(x+y+z)^2\geq 8(x^2y^2+y^2z^2+z^2x^2)^2.$$

Assuming that $x\geq y\geq z$, then by **Cauchy Schwarz Inequality**, we have

$$(x^2+z^2)(y^2+z^2)\geq (xy+z^2)^2.$$

Hence, it suffices to prove that

$$(x^2+y^2)(xy+z^2)^2(x+y+z)^2\geq 8(x^2y^2+y^2z^2+z^2x^2)^2,$$

$$\sqrt{2(x^2+y^2)}(xy+z^2)(x+y+z)\geq 4(x^2y^2+y^2z^2+z^2x^2),$$

We have

$$\begin{aligned}\sqrt{2(x^2+y^2)} &= x+y+\frac{(x-y)^2}{x+y+\sqrt{2(x^2+y^2)}}\geq x+y+\frac{(x-y)^2}{x+y+\sqrt{2}(x+y)}\\ &= x+y+\frac{\left(\sqrt{2}-1\right)(x-y)^2}{x+y}\geq x+y+\frac{3(x-y)^2}{8(x+y)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sqrt{2(x^2+y^2)}(xy+z^2)(x+y+z)\\ =\;& xy(x+y)\sqrt{2(x^2+y^2)}+z(x+z)(y+z)\sqrt{2(x^2+y^2)}\\ \geq\;& xy(x+y)\sqrt{2(x^2+y^2)}+z(x+y)(x+z)(y+z)+\frac{3z^2(x-y)^2}{8}+\frac{3(x-y)^2xyz}{8(x+y)}.\end{aligned}$$

It suffices to prove that

$$xy(x+y)\sqrt{2(x^2+y^2)}+z(x+y)(x+z)(y+z)+\frac{3z^2(x-y)^2}{8}+\frac{3(x-y)^2xyz}{8(x+y)}\geq 4(x^2y^2+y^2z^2+z^2x^2),$$

$$xy\left[(x+y)\sqrt{2(x^2+y^2)}-4xy\right]+xyz\left[x+y+\frac{3(x-y)^2}{8(x+y)}\right]-\frac{\left(21x^2-10xy+21y^2\right)z^2}{8}+z^3(x+y)\geq 0.$$

[3]This proof seems to be the most complicated but the idea is very interesting.

We have

$$xy\left[(x+y)\sqrt{2(x^2+y^2)}-4xy\right] \ge xz\left[(x+y)\sqrt{2(x^2+y^2)}-4xy\right].$$

Hence, it suffices to prove that

$$f(z)=x\left[(x+y)\sqrt{2(x^2+y^2)}-4xy\right]+xy\left[x+y+\frac{3(x-y)^2}{8(x+y)}\right]-\frac{\left(21x^2-10xy+21y^2\right)z}{8}+z^2(x+y)\ge 0.$$

Also, we have

$$f(z)-f(y)=\frac{1}{8}(y-z)\left(21x^2+13y^2-18xy-8xz-8yz\right)\ge 0.$$

Therefore

$$\begin{aligned} f(z) &\ge f(y)=x\left[(x+y)\sqrt{2(x^2+y^2)}-4xy\right]-\frac{y(10x+13y)(x-y)^2}{8(x+y)} \\ &= \frac{2x(x-y)^2(x^2+y^2+4xy)}{(x+y)\sqrt{2(x^2+y^2)}+4xy}-\frac{y(10x+13y)(x-y)^2}{8(x+y)}\ge 0, \end{aligned}$$

since

$$\begin{aligned} \frac{2x(x^2+y^2+4xy)}{(x+y)\sqrt{2(x^2+y^2)}+4xy}-\frac{y(10x+13y)}{8(x+y)} &\ge \frac{2x(x^2+y^2+4xy)}{2(x^2+y^2)+4xy}-\frac{y(10x+13y)}{8(x+y)} \\ &= \frac{8x^3+22x^2y-15xy^2-13y^3}{8(a+b)^2}\ge 0. \end{aligned}$$

Thus, our inequality is proved. □

Solution 5 (by Gabriel Dospinescu). We rewrite the inequality in the form

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\ge 8(a^2+b^2+c^2)^2.$$

Setting $a^2+b^2=2x, b^2+c^2=2y$ and $c^2+a^2=2z$, then x,y,z are the side lengths of a triangle and we may rewrite the inequality as

$$\frac{1}{\sqrt{y+z-x}}+\frac{1}{\sqrt{z+x-y}}+\frac{1}{\sqrt{x+y-z}}\ge\frac{x+y+z}{\sqrt{xyz}}.$$

By **Holder Inequality**, we have

$$\left(\sum_{cyc}\frac{1}{\sqrt{y+z-x}}\right)^2\left[\sum_{cyc}x^3(y+z-x)\right]\ge(x+y+z)^3.$$

It suffices to show that

$$xyz(x+y+z)\ge\sum_{cyc}x^3(y+z-x),$$

which is just **Schur's Inequality** for fourth degree.

This completes the proof. □

Problem 2.13 *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{(b+c)^2}{a(b+c+2a)}+\frac{(c+a)^2}{b(c+a+2b)}+\frac{(a+b)^2}{c(a+b+2c)} \geq 3.$$

Solution 1. After using **AM-GM Inequality**, it suffices to prove that

$$(a+b)^2(b+c)^2(c+a)^2 \geq abc(a+b+2c)(b+c+2a)(c+a+2b).$$

Now, applying **Cauchy Schwarz Inequality** and **AM-GM Inequality**, we obtain

$$\begin{aligned}(b+c)^2(a+b)(a+c) &\geq (b+c)^2\left(a+\sqrt{bc}\right)^2 \\ &= bc\left[\frac{a(b+c)}{\sqrt{bc}}+b+c\right]^2 \\ &\geq bc(2a+b+c)^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(a+b)^2(c+a)(c+b) &\geq ab(a+b+2c)^2, \\ (c+a)^2(b+c)(b+a) &\geq ca(c+a+2b)^2.\end{aligned}$$

Multiplying these inequalities and taking the square root, we get the result. Equality holds if and only if $a=b=c$. □

Solution 2. We need to prove the inequality

$$(a+b)^2(b+c)^2(c+a)^2 \geq abc(a+b+2c)(b+c+2a)(c+a+2b).$$

According to the **AM-GM Inequality**, we have

$$\begin{aligned}abc(a+b+2c)(b+c+2a)(c+a+2b) &\leq \frac{64}{27}abc(a+b+c)^3 \\ &\leq \frac{64}{81}(a+b+c)^2(ab+bc+ca)^2.\end{aligned}$$

And we deduce the problem to

$$(a+b)^2(b+c)^2(c+a)^2 \geq \frac{64}{81}(a+b+c)^2(ab+bc+ca)^2,$$

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca),$$

$$ab(a+b)+bc(b+c)+ca(c+a) \geq 6abc,$$

which is obviously true by **AM-GM Inequality**. □

Solution 3 (by Materazzi). We will try to write the inequality as the sum of squares, which shows that the original inequality is valid. Indeed, we have

$$\begin{aligned}\sum_{cyc}\frac{(b+c)^2}{a(2a+b+c)}-3 &= \sum_{cyc}\frac{(b+c)^2-a(2a+b+c)}{a(2a+b+c)}\\ &= (a+b+c)\sum_{cyc}\frac{b+c-2a}{a(2a+b+c)}\\ &= (a+b+c)\sum_{cyc}\left[\frac{c-a}{a(2a+b+c)}-\frac{a-b}{a(2a+b+c)}\right]\\ &= (a+b+c)\sum_{cyc}\left[\frac{a-b}{b(2b+c+a)}-\frac{a-b}{a(2a+b+c)}\right]\\ &= (a+b+c)\sum_{cyc}\frac{(a-b)^2(2a+2b+c)}{ab(2a+b+c)(2b+c+a)},\end{aligned}$$

which is obviously nonnegative. □

Solution 4. We have the following identity

$$\frac{4(b+c)^2}{a(2a+b+c)}+\frac{27a}{a+b+c}-13=\frac{(7a+4b+4c)(2a-b-c)^2}{a(a+b+c)(2a+b+c)}\geq 0,$$

which yields that

$$\frac{(b+c)^2}{a(2a+b+c)}\geq\frac{13}{4}-\frac{27}{4}\cdot\frac{a}{a+b+c}.$$

It follows that

$$\sum_{cyc}\frac{(b+c)^2}{a(2a+b+c)}\geq\frac{39}{4}-\frac{27}{4}\left(\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}\right)=3.$$

□

Problem 2.14 *Let a,b,c be positive real numbers. Prove the inequality*

$$\frac{(b+c)^2}{a(b+c+2a)}+\frac{(c+a)^2}{b(c+a+2b)}+\frac{(a+b)^2}{c(a+b+2c)}\geq 2\left(\frac{b+c}{b+c+2a}+\frac{c+a}{c+a+2b}+\frac{a+b}{a+b+2c}\right).$$

Solution 1. We may write our inequality as

$$\sum_{cyc}\left[\frac{(b+c)^2}{a(b+c+2a)}-\frac{2(b+c)}{b+c+2a}\right]\geq 0,$$

$$\sum_{cyc}(b+c-2a)\cdot\frac{b+c}{a(2a+b+c)}\geq 0.$$

Assuming without loss of generality that $a\geq b\geq c$, then we can easily check that

$$b+c-2a\leq c+a-2b\leq a+b-2c,$$

and
$$\frac{b+c}{a(2a+b+c)} \le \frac{c+a}{b(2b+c+a)} \le \frac{a+b}{c(2c+a+b)}.$$
Hence, we may apply the **Chebyshev's Inequality** as follow
$$\begin{aligned}\sum_{cyc}(b+c-2a)\cdot\frac{b+c}{a(2a+b+c)} &\ge \frac{1}{3}\left[\sum_{cyc}(b+c-2a)\right]\left[\sum_{cyc}\frac{b+c}{a(2a+b+c)}\right]\\ &= 0.\end{aligned}$$
This completes the proof. Equality holds if and only if $a=b=c$. □

Solution 2 (by Honey_suck). We have a notice that
$$\begin{aligned}\sum_{cyc}\frac{(b+c)(b+c-2a)}{a(2a+b+c)} &= \sum_{cyc}\left[\frac{(b+c)(c-a)}{a(2a+b+c)} - \frac{(b+c)(a-b)}{a(2a+b+c)}\right]\\ &= \sum_{cyc}\left[\frac{(c+a)(a-b)}{b(2b+c+a)} - \frac{(b+c)(a-b)}{a(2a+b+c)}\right]\\ &= \sum_{cyc}\frac{(a-b)^2(2a^2+2b^2+c^2+3ab+3bc+3ca)}{ab(2a+b+c)(2b+c+a)},\end{aligned}$$
which is obviously nonnegative. □

Solution 3. Again, we notice that
$$\frac{(b+c)(b+c-2a)}{a(2a+b+c)} - \frac{3(b+c-2a)}{2(a+b+c)} = \frac{(3a+2b+2c)(2a-b-c)^2}{2a(a+b+c)(2a+b+c)} \ge 0.$$
It follows that
$$\sum_{cyc}\frac{(b+c)(b+c-2a)}{a(2a+b+c)} \ge \frac{3}{2}\sum_{cyc}\frac{b+c-2a}{a+b+c} = 0.$$
□

Problem 2.15 *Let a,b,c be positive real numbers. Prove that*
$$\frac{(b+c)^2}{a(b+c+2a)} + \frac{(c+a)^2}{b(c+a+2b)} + \frac{(a+b)^2}{c(a+b+2c)} \ge 2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$
Solution. We see that
$$\frac{(b+c)^2}{a(b+c+2a)} = \frac{b+c}{a} + \frac{4a}{2a+b+c} - 2.$$
Hence, we may write the inequality in the form
$$\sum_{cyc}\frac{b+c}{a} + 4\sum_{cyc}\frac{a}{2a+b+c} \ge 2\sum_{cyc}\frac{a}{b+c} + 6.$$
Now, using **Cauchy Schwarz Inequality**, we have
$$\sum_{cyc}\frac{b+c}{a} = \sum_{cyc}\left(\frac{a}{b} + \frac{a}{c}\right) \ge \sum_{cyc}\frac{4a}{b+c}.$$

It suffices to show that

$$2\sum_{cyc}\frac{a}{b+c}+4\sum_{cyc}\frac{a}{2a+b+c}\geq 6,$$

$$\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{2a}{2a+b+c}\geq 3,$$

which is obviously true because

$$\begin{aligned}
\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{2a}{2a+b+c} &= \sum_{cyc}a\left(\frac{1}{b+c}+\frac{1}{\frac{2a+b+c}{2}}\right)\\
&\geq \sum_{cyc}\frac{4a}{b+c+\frac{2a+b+c}{2}}\\
&= 8\sum_{cyc}\frac{a}{2a+3b+3c}\\
&\geq \frac{8(a+b+c)^2}{\sum_{cyc}a(2a+3b+3c)}\\
&= \frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)},
\end{aligned}$$

and

$$4(a+b+c)^2-3(a^2+b^2+c^2)-9(ab+bc+ca)=a^2+b^2+c^2-ab-bc-ca\geq 0.$$

Equality holds if and only if $a=b=c$. $\square$

Problem 2.16 *Let a,b,c be positive real numbers. Prove that*

$$a^3b^3+b^3c^3+c^3a^3\geq (b+c-a)(c+a-b)(a+b-c)(a^3+b^3+c^3).$$

Solution 1. By **Schur's Inequality** for third degree, we have

$$a^3b^3+b^3c^3+c^3a^3\geq abc\left[\sum_{cyc}ab(a+b)-3abc\right].$$

It suffices to prove that

$$abc\left[\sum_{cyc}ab(a+b)-3abc\right]\geq (b+c-a)(c+a-b)(a+b-c)(a^3+b^3+c^3),$$

which is equivalent to each of the following inequalities

$$\frac{\sum_{cyc}ab(a+b)-3abc}{a^3+b^3+c^3}\geq\frac{(b+c-a)(c+a-b)(a+b-c)}{abc},$$

$$1-\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}\geq 1-\frac{\sum_{cyc}ab(a+b)-3abc}{a^3+b^3+c^3},$$

$$\frac{\sum_{cyc} a(a-b)(a-c)}{abc} \geq \frac{\sum_{cyc} a(a-b)(a-c)}{a^3+b^3+c^3},$$
$$(a^3+b^3+c^3-abc)\sum_{cyc} a(a-b)(a-c) \geq 0,$$

which is obviously true by **AM-GM Inequality** and **Schur's Inequality** for third degree. □

Solution 2. Without loss of generality, we may assume that $a \geq b \geq c$, then we have 2 cases

Case 1. If $b+c \leq a$, then the inequality is trivial since

$$(b+c-a)(c+a-b)(a+b-c) \leq 0.$$

Case 2. If $b+c \geq a$, then we have

$$\begin{aligned}(c+a-b)(a+b-c) &= a^2-(b-c)^2 \leq a^2,\\ b^3c^3-a^2bc(b+c-a)^2 &= bc\left[bc+a(b+c-a)\right](a-b)(a-c) \geq 0.\end{aligned}$$

It suffices to prove that

$$a^3(b^3+c^3)+a^2bc(b+c-a)^2 \geq a^2(b+c-a)(a^3+b^3+c^3),$$
$$a(b^3+c^3)+bc(b+c-a)^2 \geq (b+c-a)(a^3+b^3+c^3).$$

Now, since this inequality is homogeneous, we can assume that $b+c=1$ and put $x=bc$, then $1 \geq a \geq \frac{1}{2}$ and $\frac{1}{4} \geq x \geq a(1-a)$. The above inequality becomes

$$a(1-3x)+x(1-a)^2 \geq (1-a)(a^3+1-3x),$$
$$f(x) = (4-8a+a^2)x+a^4-a^3+2a-1 \geq 0.$$

We see that $f(x)$ is a linear function of x, thus

$$f(x) \geq \min\left\{f\left(\frac{1}{4}\right), f\left(a(1-a)\right)\right\}.$$

But, we have

$$f\left(\frac{1}{4}\right) = \frac{a^2(2a-1)^2}{4} \geq 0, \quad \text{and} \quad f\left(a(1-a)\right) = (2a-1)^3 \geq 0.$$

Thus, our proof is completed. Equality holds if and only if $a=b=c$. □

Solution 3 (by nhocnhoc). By expanding, we see that the given inequality is equivalent to

$$\sum_{cyc} a^6+3\sum_{cyc} a^3b^3+2abc\sum_{cyc} a^3 \geq \sum_{cyc} ab(a^4+b^4)+\sum_{cyc} a^2b^2(a^2+b^2)+abc\sum_{cyc} ab(a+b),$$

$$\sum_{cyc}\left[\frac{a^6+b^6}{2}+3a^3b^3-ab(a^4+b^4)-a^2b^2(a^2+b^2)\right]+abc\sum_{cyc}\left[a^3+b^3-ab(a+b)\right] \geq 0,$$

$$\frac{1}{2}\sum_{cyc}(a^4+b^4-3a^2b^2)(a-b)^2+abc\sum_{cyc}(a-b)^2(a+b) \geq 0,$$

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0,$$

where

$$\begin{aligned} S_a &= b^4+c^4-3b^2c^2+2abc(b+c), \\ S_b &= c^4+a^4-3c^2a^2+2abc(c+a), \\ S_c &= a^4+b^4-3a^2b^2+2abc(a+b). \end{aligned}$$

Without loss of generality, we may assume that $a \geq b \geq c$, then it is easy to check that $S_a, S_b \geq 0$. Moreover, we have

$$\begin{aligned} S_b+S_c &= 2a^4+b^4-3a^2b^2+c^4+2abc(2a+b+c)-3a^2c^2 \\ &\geq 2a^4+b^4-3a^2b^2+c^4+2ac^2(2a+b+c)-3a^2c^2 \\ &= 2a^4+b^4-3a^2b^2+c^4+ac^2(a+2b+2c) \\ &> 2a^4+b^4-3a^2b^2 = (a^2-b^2)(2a^2-b^2) \geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 &\geq S_b(c-a)^2+S_c(a-b)^2 \\ &\geq S_b(c-a)^2-S_b(a-b)^2 \\ &= S_b(b-c)(2a-b-c) \geq 0. \end{aligned}$$

This completes the proof. $\square$

Problem 2.17 *If a,b,c are positive real numbers such that $abc = 1$, show that we have the following inequality*

$$a^3+b^3+c^3 \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}.$$

Solution. By **GM-HM Inequality**, we have that

$$\begin{aligned} \frac{4a}{b+c}+\frac{4b}{c+a}+\frac{4c}{a+b}+6 &\leq a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{c}+\frac{1}{a}\right)+c\left(\frac{1}{a}+\frac{1}{b}\right)+6 \\ &= a^2(b+c)+b^2(c+a)+c^2(a+b)+6abc \\ &\leq 2a^2(b+c)+2b^2(c+a)+2c^2(a+b) \\ &= 2ab(a+b)+2bc(b+c)+2ca(c+a) \\ &\leq 2(a^3+b^3)+2(b^3+c^3)+2(c^3+a^3) \\ &= 4(a^3+b^3+c^3). \end{aligned}$$

Hence

$$a^3+b^3+c^3 \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}.$$

Our proof is completed. Equality holds if and only if $a=b=c=1$.

Remark 1 *We can prove that the stronger inequality holds*

$$a^3+b^3+c^3-3 \geq \left(6+4\sqrt{2}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right).$$

Indeed, this inequality is equivalent to each of the following

$$\frac{a^3+b^3+c^3}{abc}-3 \geq \left(6+4\sqrt{2}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right),$$

$$\frac{(a+b+c)\sum\limits_{cyc}(a-b)(a-c)}{abc} \geq \left(3+2\sqrt{2}\right)\sum_{cyc}\frac{(2a+b+c)(a-b)(a-c)}{(a+b)(b+c)(c+a)},$$

$$X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) \geq 0,$$

where

$$\begin{aligned}
X &= \frac{(a+b+c)(a+b)(b+c)(c+a)}{abc}-\left(3+2\sqrt{2}\right)(2a+b+c) \\
&= \frac{(a+b+c)(b+c)}{a}\left[\frac{a(a+b+c)}{bc}+1\right]-\left(3+2\sqrt{2}\right)(2a+b+c) \\
&\geq \frac{(a+b+c)(b+c)}{a}\left[\frac{4a(a+b+c)}{(b+c)^2}+1\right]-\left(3+2\sqrt{2}\right)(2a+b+c) \\
&= (2a+b+c)\left[\frac{(a+b+c)(2a+b+c)}{a(b+c)}-3-2\sqrt{2}\right] \geq 0,
\end{aligned}$$

and Y,Z are similar.
Now, assume that $a \geq b \geq c$, then we see that $Z \geq Y \geq X$, thus

$$\begin{aligned}
X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) &\geq Y(b-c)(b-a)+Z(c-a)(c-b) \\
&\geq Y(b-c)(b-a)+Y(c-a)(c-b) \\
&= Y(b-c)^2 \geq 0.
\end{aligned}$$

□

Problem 2.18 *Given nonnegative real numbers a,b,c such that $ab+bc+ca+abc=4$. Prove that*

$$a^2+b^2+c^2+2(a+b+c)+3abc \geq 4(ab+bc+ca).$$

Solution. Setting $p=a+b+c, q=ab+bc+ca$ and $r=abc$. The given condition gives us $q+r=4$, and we have to prove

$$p^2-2q+2p+3(4-q) \geq 4q,$$
$$p^2+2p+12 \geq 9q.$$

If $p \geq 4$, then it is trivial because

$$p^2+2p+12 \geq 16+8+12 = 36 \geq 9q.$$

If $4 \geq p \geq 3$,[4] applying **Schur's Inequality** for third degree, we have

$$p^3-4pq+9r \geq 0,$$

[4] In here, we have $p \geq 3$ because $\frac{p^2}{3}+\frac{p^3}{27} \geq q+r=4$, which gives us $p \geq 3$.

which yields that

$$p^3 - 4pq + 9(4-q) \ge 0,$$

or

$$q \le \frac{p^3+36}{4p+9}.$$

It follows that

$$\begin{aligned} p^2+2p+12-9q &\ge p^2+2p+12-9\cdot\frac{p^3+36}{4p+9} \\ &= \frac{(5p+18)(4-p)(p-3)}{4p+9} \ge 0. \end{aligned}$$

This completes our proof. Equality holds if and only if $a=b=c=1$ or $a=b=2, c=0$ and its cyclic permutations. □

Problem 2.19 *Let a,b,c be real numbers with $\min\{a,b,c\} \ge \frac{3}{4}$ and $ab+bc+ca=3$. Prove that*

$$a^3+b^3+c^3+9abc \ge 12.$$

Solution. Notice that from the given condition, we have

$$a^2 \ge \frac{9}{16} = \frac{3}{16}(ab+bc+ca) \ge \frac{3}{16}a(b+c) > \frac{1}{8}a(b+c).$$

It follows that $8a > b+c$. Similarly, we have $8b > c+a$ and $8c > a+b$.
Now, using **AM-GM Inequality**, we obtain

$$\begin{aligned} 36 &= 4(ab+bc+ca)\sqrt{3(ab+bc+ca)} \\ &\le 2(ab+bc+ca)\left[a+b+c+\frac{3(ab+bc+ca)}{a+b+c}\right]. \end{aligned}$$

And we deduce the inequality to

$$3\sum_{cyc}a^3+27abc \ge 2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+\frac{6(ab+bc+ca)^2}{a+b+c},$$

$$3\sum_{cyc}a^3+27abc-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) \ge \frac{6(ab+bc+ca)^2}{a+b+c}-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right),$$

We have

$$3\sum_{cyc}a^3+27abc-4\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) = \sum_{cyc}(3a-b-c)(a-b)(a-c),$$

and

$$\frac{6(ab+bc+ca)^2}{a+b+c}-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) = -\frac{2(ab+bc+ca)}{a+b+c}\sum_{cyc}(a-b)(a-c).$$

Hence, we may write the above inequality as

$$X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) \geq 0,$$

where

$$\begin{aligned} X &= \frac{4(ab+bc+ca)}{a+b+c}+6a-2b-2c+\frac{(b-c)^2}{a+b+c} \\ &= \frac{4a(b+c)+(b+c)^2}{a+b+c}+6a-2b-2c \\ &= \frac{6a^2+(b+c)(8a-b-c)}{a+b+c} > 0, \end{aligned}$$

and Y,Z are similar.
We will now assume that $a \geq b \geq c$, then

$$\begin{aligned} X-Y &= \frac{4a(b+c)+(b+c)^2-4b(c+a)-(c+a)^2}{a+b+c}+8(a-b) \\ &= 8(a-b)-\frac{(a-b)(a+b-2c)}{a+b+c} = \frac{(a-b)(7a+7b+10c)}{a+b+c} \geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) &\geq X(a-b)(a-c)+Y(b-c)(b-a) \\ &\geq Y(a-b)(a-c)+Y(b-c)(b-a) \\ &= Y(a-b)^2 \geq 0. \end{aligned}$$

This completes our proof. Equality holds if and only if $a=b=c=1$. $\square$

Problem 2.20 *Let a,b,c be positive real numbers such that $a^2b^2+b^2c^2+c^2a^2=1$. Prove that*

$$(a^2+b^2+c^2)^2+abc\sqrt{(a^2+b^2+c^2)^3} \geq 4.$$

Solution. According to **AM-GM Inequality** and **Cauchy Schwarz Inequality**, we have

$$\begin{aligned} \sqrt{(a^2+b^2+c^2)^3} &\geq \sqrt{3(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)} \\ &= \sqrt{3(a^2+b^2+c^2)} \geq a+b+c. \end{aligned}$$

It suffices to show that

$$(a^2+b^2+c^2)^2+abc(a+b+c) \geq 4,$$
$$a^4+b^4+c^4+abc(a+b+c) \geq 2(a^2b^2+b^2c^2+c^2a^2),$$
$$\sum_{cyc} a^2(a-b)(a-c)+\sum_{cyc} ab(a-b)^2 \geq 0,$$

which is obviously true by **Schur's Inequality** for fourth degree. Equality holds if and only if $a=b=c=\frac{1}{\sqrt[4]{3}}$ or $a=b=1, c=0$ and its cyclic permutations. $\square$

Problem 2.21 *Show that if a,b,c are positive real numbers, the following inequality holds*

$$(a+b+c)^2(ab+bc+ca)^2+(ab+bc+ca)^3 \geq 4abc(a+b+c)^3.$$

Solution. Setting $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$, then we may write the above inequality in the form

$$(x+y+z)^2(xy+yz+zx)^2+xyz(x+y+z)^3 \geq 4(xy+yz+zx)^3.$$

Using **AM-GM Inequality**, we have

$$xyz(x+y+z)^3 \geq 3xyz(x+y+z)(xy+yz+zx),$$

and we can deduce the inequality to

$$(x+y+z)^2(xy+yz+zx)+3xyz(x+y+z) \geq 4(xy+yz+zx)^2,$$

which can be easily simplied to

$$xy(x-y)^2+yz(y-z)^2+zx(z-x)^2 \geq 0,$$

which is obviously true and this completes our proof. Equality holds if and only if $a=b=c$. □

Problem 2.22 *Let a,b,c be real numbers from the interval $[3,4]$. Prove that*

$$(a+b+c)\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right) \geq 3(a^2+b^2+c^2).$$

Solution 1. The given condition shows that a,b,c are the side lengths of a triangle, hence we may put $a=y+z, b=z+x$ and $c=x+y$ where $x,y,z>0$. The above inequality becomes

$$(x+y+z)\left[\sum_{cyc}\frac{(x+y)(x+z)}{y+z}\right] \geq 3(x^2+y^2+z^2+xy+yz+zx).$$

We have

$$\begin{aligned}(x+y+z)\left[\sum_{cyc}\frac{(x+y)(x+z)}{y+z}\right] &= \sum_{cyc}\frac{x(x+y)(x+z)}{y+z}+\sum_{cyc}(x+y)(x+z)\\ &= \sum_{cyc}\frac{x(x^2+yz)}{y+z}+2\sum_{cyc}x^2+3\sum_{cyc}xy.\end{aligned}$$

From this, we may rewrite our inequality as

$$\frac{x(x^2+yz)}{y+z}+\frac{y(y^2+zx)}{z+x}+\frac{z(z^2+xy)}{x+y} \geq x^2+y^2+z^2,$$

$$\frac{x(x-y)(x-z)}{y+z}+\frac{y(y-z)(y-x)}{z+x}+\frac{z(z-x)(z-y)}{x+y} \geq 0,$$

which is obviously true by **Vornicu Schur Inequality.**
Equality holds if and only if $a=b=c$. □

Solution 2 (by nhocnhoc). We will rewite the inequality as

$$\frac{a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)}{abc} \geq \frac{3(a^2+b^2+c^2)-(a+b+c)^2}{a+b+c},$$

$$\left(\frac{a}{bc}-\frac{2}{a+b+c}\right)(b-c)^2+\left(\frac{b}{ca}-\frac{2}{a+b+c}\right)(c-a)^2+\left(\frac{c}{ab}-\frac{2}{a+b+c}\right)(a-b)^2 \geq 0,$$

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0,$$

where

$$S_a=\frac{a}{bc}-\frac{2}{a+b+c}, \quad S_b=\frac{b}{ca}-\frac{2}{a+b+c}, \quad S_c=\frac{c}{ab}-\frac{2}{a+b+c}.$$

Without loss of generality, we may assume that $a \geq b \geq c$. It is easy to see that $S_a \geq S_b \geq S_c$. Moreover, we have

$$S_b+S_c=\frac{b}{ca}+\frac{c}{ab}-\frac{4}{a+b+c} \geq \frac{2}{a}-\frac{4}{a+b+c}=\frac{2(b+c-a)}{a+b+c}>0$$

since $b+c>4 \geq a$.
It follows that $S_a \geq S_b \geq 0$. And we obtain

$$\begin{aligned} S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 &\geq S_b(c-a)^2+S_c(a-b)^2 \\ &\geq S_b(c-a)^2-S_b(a-b)^2 \\ &= S_b(b-c)(2a-b-c) \geq 0. \end{aligned}$$

This completes the proof. □

Problem 2.23 *Given ABC is a triangle. Prove that*

$$8\cos^2 A\cos^2 B\cos^2 C+\cos 2A\cos 2B\cos 2C \geq 0.$$

Solution (by Honey_suck). Since $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$ and $\cos A\cos B\cos C \leq \frac{1}{8}$, it follows that

$$\cos^2 A+\cos^2 B+\cos^2 C \geq \frac{3}{4},$$

and

$$(1-\cos^2 A-\cos^2 B-\cos^2 C)^2=4\cos^2 A\cos^2 B\cos^2 C.$$

Setting $a=\cos^2 A, b=\cos^2 B$ and $c=\cos^2 C$, then $a+b+c \geq \frac{3}{4}$ and $(a+b+c-1)^2=4abc$. We need to prove that

$$8abc+(2a-1)(2b-1)(2c-1) \geq 0,$$

$$16abc+2(a+b+c) \geq 4(ab+bc+ca)+1,$$

$$4(a+b+c-1)^2+2(a+b+c) \geq 4(ab+bc+ca)+1.$$

By **Schur's Inequality** for third degree, we have

$$\begin{aligned} 4(ab+bc+ca) &\leq \frac{(a+b+c)^3+9abc}{a+b+c} \\ &= \frac{4(a+b+c)^3+9(a+b+c-1)^2}{4(a+b+c)}. \end{aligned}$$

It suffices to show that

$$4(a+b+c-1)^2+2(a+b+c) \geq \frac{4(a+b+c)^3+9(a+b+c-1)^2}{4(a+b+c)}+1,$$

$$4(p-1)^2+2p \geq \frac{4p^3+9(p-1)^2}{4p}+1, \ (p=a+b+c)$$

$$\frac{3(4p-3)(p-1)^2}{4p} \geq 0,$$

which is obviously true since $p \geq \frac{3}{4}$. $\square$

Problem 2.24 *Let a,b,c be positive real numbers such that $a+b+c=3$ and $ab+bc+ca \geq 2\max\{ab,bc,ca\}$. Prove that*

$$a^2+b^2+c^2 \geq a^2b^2+b^2c^2+c^2a^2.$$

Solution. The desired inequality is equivalent to each of the following inequalities

$$(a+b+c)^2(a^2+b^2+c^2) \geq 9(a^2b^2+b^2c^2+c^2a^2),$$

$$(a+b+c)^2 \geq \frac{9(a^2b^2+b^2c^2+c^2a^2)}{a^2+b^2+c^2},$$

$$3(a^2+b^2+c^2)-\frac{9(a^2b^2+b^2c^2+c^2a^2)}{a^2+b^2+c^2} \geq 3(a^2+b^2+c^2)-(a+b+c)^2,$$

$$\frac{3\sum_{cyc}(a^2-b^2)(a^2-c^2)}{a^2+b^2+c^2} \geq 2\sum_{cyc}(a-b)(a-c),$$

$$X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) \geq 0,$$

where

$$\begin{aligned} X &= 3(a+b)(a+c)-2(a^2+b^2+c^2)+2(b-c)^2 \\ &= a^2+3a(b+c)-bc > ab+bc+ca-2bc \\ &\geq ab+bc+ca-2\max\{ab,bc,ca\} \geq 0, \end{aligned}$$

and Y,Z are similar.
Now, we assume that $a \geq b \geq c$, then

$$X-Y=(a-b)(a+b+4c) \geq 0.$$

It follows that

$$\begin{aligned} X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b) &\geq X(a-b)(a-c)+Y(b-c)(b-a) \\ &\geq Y(a-b)(a-c)+Y(b-c)(b-a) \\ &= Y(a-b)^2 \geq 0. \end{aligned}$$

This completes our proof. Equality holds if and only if $a=b=c$. $\square$