On two nice geometric problems

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Abstract

This article is about the extension of two geometric problems by using the power of a point and the radial axis tools and pure geometry. One was introduced on the Russian Maths Olympiad and the other occurred on the HNUE[1] High School for Gifted Students contest.

The following problem was proposed on All-Russian Mathematical Olympiad (2013, Grade 9, Day 2, Problem 3) [1]

Problem 1. Squares $CAKL$ and $CBMN$ are constructed on the sides of acuted-angled triangle ABC. Line CN intersects line AK at X. Line CL intersects line BM at Y. Point P, lying inside triangle ABC, is an intersection of circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that $\angle ACS = \angle BCP$.

The following problem was presented on HNUE High School for Gifted Students contest 2014 in Vietnam [2]

Problem 2. Let ABC be not an isosceles triangle at A and $\angle BAC > 45°$. Let O be a circumcenter of triangle ABC. Constructing outside triangle ABC squares $ABKL, ACMN$. Lines AN, AL intersect CM, BK at E, F respectively. Denote P by an intersection of circumcircles of triangles LME and NFK such that P is inside triangle ABC.

a) Prove that E, F, O, P are collinear.

b) Prove that B, C, O, P are concyclic.

Comment. Those are two nice and meaningful geometric problems. We could regard square facts as the similar rectangles or similar parallelograms in general. Regarding to this idea, we are pleased to introduce the generalization of two geometric problems above.

Problem 3. Let ABC be triangle and O be its circumcenter. Constructing outside triangle ABC parallelograms $ABKL, ACMN$ such that $\triangle ABL \sim \triangle CAM$. Lines AN, AL intersect lines CM, BK at E, F respectively. Let P be intersection of circumcircles of triangles LME and NFK and P is inside triangle ABC. Prove that B, C, O, P are concyclic.

Solution. Let line KB intersect line CM at G. Because of $\triangle ABL \sim \triangle CAM$, it is easily seen that $\angle ABK + \angle ACM = 180°$. Therefore G lies on circumcircle (O) of triangle ABC. Clearly, quadrilateral $AFGE$ is parallelogram so $\angle AFG = 180° - \angle FGE = \angle BAC$. We also have inscribed angles $\angle AGF = \angle ACB$ so $\triangle AFG \sim \triangle BAC$. Deducing $AC.FA = AB.FG = AB.AE$. In the same way as similar triangles $\triangle ABL \sim \triangle CAM$, it is easily be seen that $AC.AL = AB.AN$. From that, we obtain $\dfrac{AF}{AL} = \dfrac{AE}{AN}$ or $AL.AE = AF.AN$. Thus A belongs to the radical axis of circumcircles of triangles LME and NFK.

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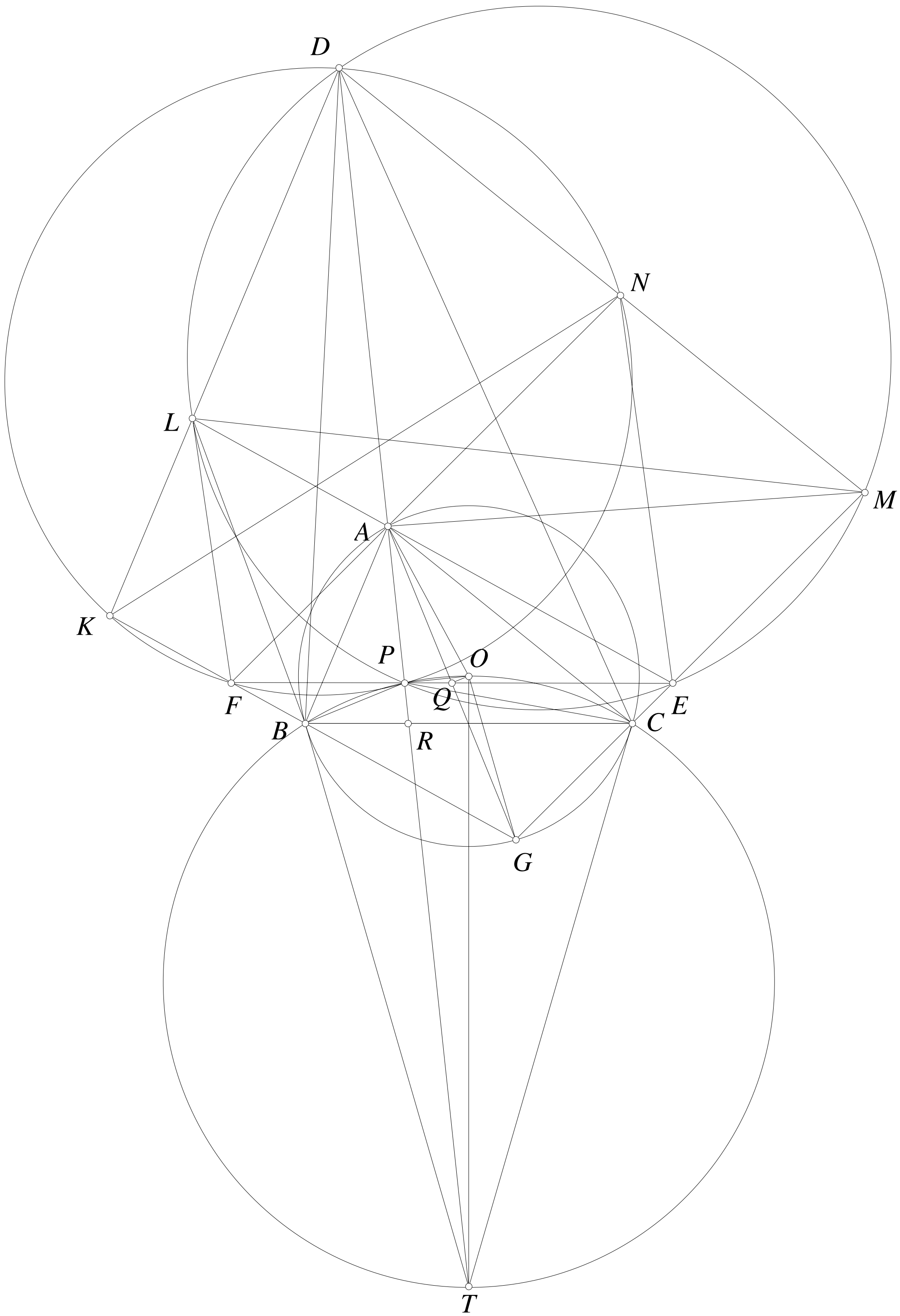

Figure 1.

Let line KL intersects line MN at D, we have $\angle DNF = 180^\circ - \angle ANM = 180^\circ - \angle FKL$ which implies that point D belongs to the circumcirle of triangle NFK. Similarly, D belongs to the circumcircle of triangle LME. Therefore, DP is the radical axis of circumcenters of triangles LME and NFK. We infer point A lies on line DP. Note that $DMEP$ and $DKEP$ are quadrilaterals inscribed in the circles, we get $\angle APF + \angle APE = \angle DME + \angle DKF = 180°$. Thus P lies on EF.

Let Q be a midpoint of AG. Obviously, $\angle AOQ = \frac{1}{2}\angle AOG = \angle ACG = \angle AMC = 180° - \angle DPE = 180° - \angle APQ$. Therefore, the quadrilateral $APQO$ is concyclic, which $OQ \perp AQ$. We imply $AP \perp OP$.

Denote R by an intersection of lines AP and BC, it is easy to prove that $\frac{RB}{RC} = \frac{S_{DAB}}{S_{DAC}} = \frac{S_{LAB}}{S_{LAC}} = \frac{AL.AB}{AN.AC} = \frac{AB^2}{AC^2}$. Consequently, AP is symmedian of triangle ABC. Thus, AP passes through the intersection of tangents to (O) at B, C, we call T. Since $OP \perp AP$, it is plain that points O, P, B, C lie on the circle of diameter OT. The proof is complete. □

Comment. When the parallelograms are squares, we obtain results of those two geometric problems. It could be seen to infer B, C, O, P be concyclic, we have to prove AP is symmemdian as in first problem and point P lies on EF as part a) of the second problem. In this problem, point P is essentially fixed and does not depend on the way of choosing parallelograms because it is the projection of circumcenter O on the $A-$symmedian line. The projection of circumcenter O on the symmedian line is a special point inside triangle and is useful. For instance, it is the center of the homothety taking segment CA to segment AB. We will discover others nice geometric problems if we exploit the extension of the problem. Let's practice the following problems.

Problem 4. Let ABC be not an isosceles triangle at A. Constructing outside triangle ABC similar rectangles $ABKL, ACMN$. Lines AN, AL intersects lines CM, BK at E, F respectively. Let P be an intersection of circumcircles of triangles LME and NFK such that P is inside triangle ABC. Line KN intersect line LM at Q. Prove that $\angle PAB = \angle QAC$.

References

[1] http://www.artofproblemsolving.com/Forum/viewtopic.php?p=3067570

[2] http://diendantoanhoc.net/forum/

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