Radiative-convective equilibrium in a grey atmosphere

Marco Casari

University of Turin

Complex systems in climate physics, 3 October 2023

Introduction

- Average vertical temperature profile $T(t, z)$ of atmosphere.

Introduction

- Average vertical temperature profile $T(t, z)$ of atmosphere.
- Radiative Transfer Equation (RTE).

Introduction

- Average vertical temperature profile $T(t, z)$ of atmosphere.
- Radiative Transfer Equation (RTE).
- Fluid dynamics equations.

Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial q}{\partial z} \quad . \tag{1}$$

Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial q}{\partial z} \quad . \tag{1}$$

- Radiative-convective equilibrium.

Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial q}{\partial z} \quad . \tag{1}$$

- Radiative-convective equilibrium.
- Grey atmosphere.

Additional hypotheses

- Hypotheses on the planet.

Additional hypotheses

- Hypotheses on the planet.
- Hypotheses on the composition of atmosphere.

Additional hypotheses

- Hypotheses on the planet.
- Hypotheses on the composition of atmosphere.
- Hypotheses on total heat flux.
- Resulting thermodynamic energy equation:

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial}{\partial z}\left(E_{\mathrm{U}} - E_{\mathrm{D}}\right) \quad . \tag{2}$$

Vertical coordinates

- Relation between pressure and altitude:

$$P(z) = P_{\mathrm{g}} \exp\left(-\frac{z - z_{\mathrm{g}}}{z_0}\right) \quad . \tag{3}$$

- Relation between optical depth and pressure:

$$\delta(P) = \frac{\mu_{\mathrm{m}}}{g}(P - P_{\mathrm{TOA}}) \quad . \tag{4}$$

- Relation between optical depth and altitude:

$$\delta(z) = \frac{\mu_{\mathrm{m}}}{g}\left(P_{\mathrm{g}} \exp\left(-\frac{z - z_{\mathrm{g}}}{z_0}\right) - P_{\mathrm{TOA}}\right) \quad . \tag{5}$$

Equations in radiative equilibrium

- RTE for non-scattering medium in LTE:

$$\frac{1}{\mu}\frac{\partial L}{\partial z} = B_\nu - L \quad . \tag{6}$$

Equations in radiative equilibrium

- RTE for non-scattering medium in LTE:

$$\frac{1}{\mu}\frac{\partial L}{\partial z} = B_\nu - L \quad . \tag{6}$$

- Integration over frequency and solid angle.
- Diffusion approximation: $\delta' = D\delta$.

Equations in radiative equilibrium

- RTE for non-scattering medium in LTE:

$$\frac{1}{\mu}\frac{\partial L}{\partial z} = B_\nu - L \quad . \tag{6}$$

- Integration over frequency and solid angle.
- Diffusion approximation: $\delta' = D\delta$.
- Equations for irradiances:

$$-\frac{\partial}{\partial \delta'} E_{\mathrm{U}}(t,\delta') = \sigma T(t,\delta')^4 - E_{\mathrm{U}}(t,\delta') \quad , \tag{7}$$

$$\frac{\partial}{\partial \delta'} E_{\mathrm{D}}(t,\delta') = \sigma T(t,\delta')^4 - E_{\mathrm{D}}(t,\delta') \quad . \tag{8}$$

Initial Value Problem

- Steady state.

Initial Value Problem

- Steady state.
- Initial conditions on irradiances:

$$E_{\mathrm{D}}(0) = 0 \quad , \tag{9}$$

$$E_{\mathrm{U}}(0) = S_{\mathrm{t}} \quad . \tag{10}$$

Initial Value Problem

- Steady state.
- Initial conditions on irradiances:

$$E_{\mathrm{D}}(0) = 0 \quad , \tag{9}$$

$$E_{\mathrm{U}}(0) = S_{\mathrm{t}} \quad . \tag{10}$$

- Additional relations:

$$\frac{\mathrm{d}}{\mathrm{d}\delta'}\left(E_{\mathrm{U}}(\delta') - E_{\mathrm{D}}(\delta')\right) = 0 \quad , \tag{11}$$

$$E_{\mathrm{U}}(\delta') - E_{\mathrm{D}}(\delta') = S_{\mathrm{t}} \quad . \tag{12}$$

Initial Value Problem

- Steady state.
- Initial conditions on irradiances:

$$E_\mathrm{D}(0) = 0 \quad , \tag{9}$$

$$E_\mathrm{U}(0) = S_\mathrm{t} \quad . \tag{10}$$

- Additional relations:

$$\frac{\mathrm{d}}{\mathrm{d}\delta'}\left(E_\mathrm{U}(\delta') - E_\mathrm{D}(\delta')\right) = 0 \quad , \tag{11}$$

$$E_\mathrm{U}(\delta') - E_\mathrm{D}(\delta') = S_\mathrm{t} \quad . \tag{12}$$

- Initial condition on temperature:

$$T(0) = \left(\frac{S_\mathrm{t}}{2\sigma}\right)^{\frac{1}{4}} \quad . \tag{13}$$

Analytical solution

- Temperature:

$$T(\delta) = \left(\frac{S_t}{2\sigma}(1 + D\delta) \right)^{\frac{1}{4}} \quad . \tag{14}$$

- Irradiances:

$$E_U(\delta) = \frac{S_t}{2}(2 + D\delta) \quad , \tag{15}$$

$$E_D(\delta) = \frac{S_t}{2} D\delta \quad . \tag{16}$$

Numerical solution

- Normalisation:

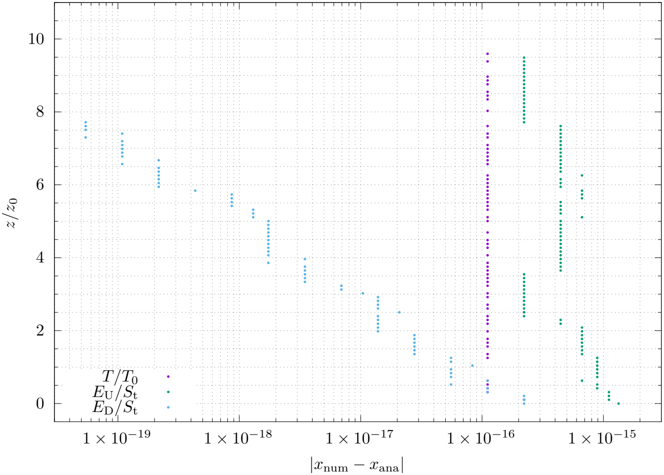
$$Y_0 = \frac{T^4}{T_0^4} \quad , \quad Y_1 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \quad , \quad Y_2 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \tag{17}$$

- Resulting system of ODEs:

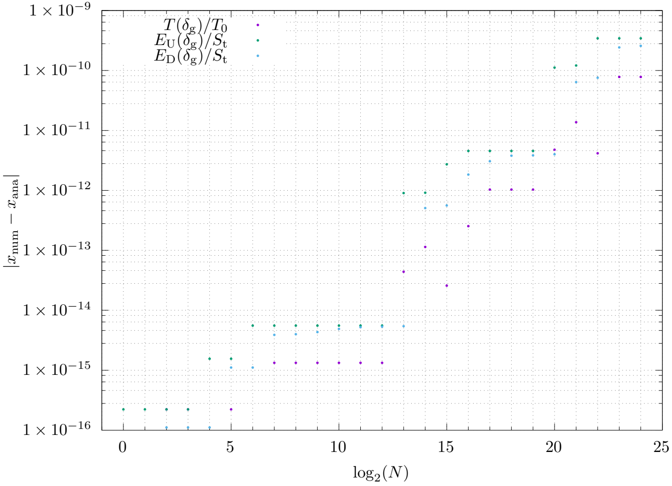
$$\begin{cases} \frac{\mathrm{d}Y_0}{\mathrm{d}\delta} = \frac{D}{2} \\ \frac{\mathrm{d}Y_1}{\mathrm{d}\delta} = D(Y_1 - Y_0) \\ \frac{\mathrm{d}Y_2}{\mathrm{d}\delta} = D(Y_0 - Y_2) \end{cases} \quad . \tag{18}$$

- Runge-Kutta method of order 4.
- Non-uniform step size.

Errors of numerical solutions



Stability analysis



Time integration

- Iterative procedure.

Time integration

- Iterative procedure.
- Additional normalisation:

$$Y_3 = \frac{T}{T_0} \quad . \tag{19}$$

Time integration

- Iterative procedure.
- Additional normalisation:

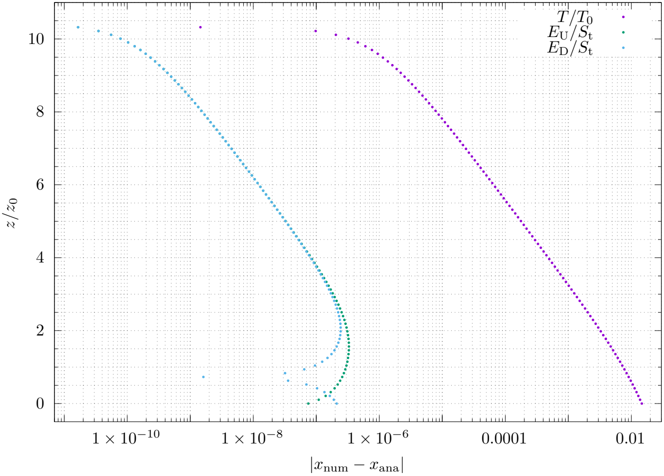
$$Y_3 = \frac{T}{T_0} \quad . \tag{19}$$

- Resulting system of PDEs:

$$\begin{cases} \frac{\partial}{\partial t} Y_3(t,\delta) = \frac{\mu_\mathrm{m} S_\mathrm{t}}{c_P T_0} \frac{\partial}{\partial \delta}(Y_1(t,\delta) - Y_2(t,\delta)) \\ \frac{\partial}{\partial \delta} Y_1(t,\delta) = D(Y_1(t,\delta) - Y_3(t,\delta)^4) \\ \frac{\partial}{\partial \delta} Y_2(t,\delta) = D(Y_3(t,\delta)^4 - Y_2(t,\delta)) \end{cases} \quad . \tag{20}$$

- Euler method for temporal integration.
- Constant time step.

Errors of time integration at steady state



Radiative-convective equilibrium

- Lapse rate at steady state.

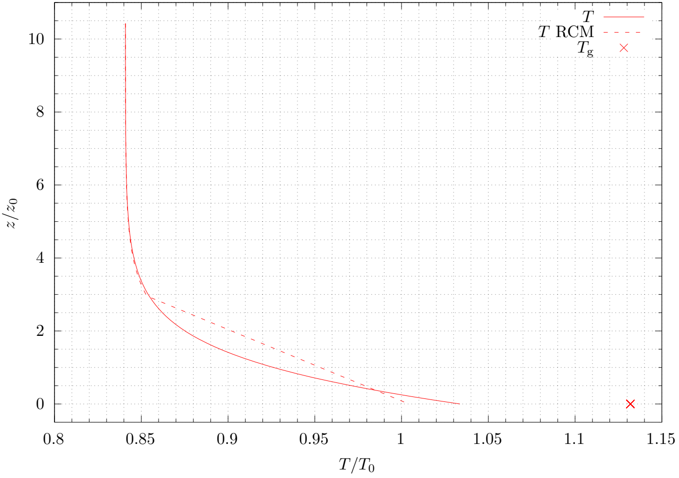
Radiative-convective equilibrium

- Lapse rate at steady state.
- Convection faster than radiative processes.

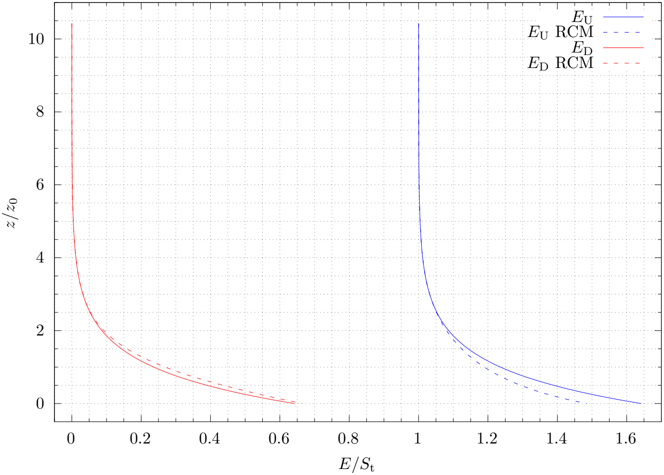
Radiative-convective equilibrium

- Lapse rate at steady state.
- Convection faster than radiative processes.
- Convective adjustment if $-\frac{\partial T}{\partial z} > \Gamma_0$.

Temperature plot



Irradiances plot



Conclusion

- Vertical temperature profile of a grey atmosphere described in general by a system of PDEs and at the steady state by a system of ODEs.
- Negligible errors for ODEs numerical solutions.
- Possible improvements on description of physical phenomena.
- Possible improvements on precision of PDEs numerical solutions.