

# Radiative-convective equilibrium in a grey atmosphere

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## Abstract

Radiative-convective models provide an intermediate complexity approach to the simulation of climate. These models evaluate the atmospheric temperature profile averaged over all latitudes and longitudes, which is function of time and altitude. Physical processes which determine the energy exchange in the model are absorption, transmission, reflection of electromagnetic radiation and convection of fluid. In this work a radiative-convective model is used to derive the temperature of an atmosphere where the optical depth is constant with respect to the frequency of radiation. The resulting temperature profile is compared with the analytical solution provided under the condition of radiative equilibrium.

## 1 Introduction

Climate dynamics of a planet can be studied with models of varying complexity. One of the quantities analysed is the temporal and spatial distribution of temperature in the planetary atmosphere, which is the result of the heat exchange between different processes. Electromagnetic (EM) radiation is emitted, absorbed and scattered by the chemical species distributed in the atmosphere and by the planetary surface. Moreover, the atmosphere receives EM radiation by other celestial bodies, e.g. stars. Radiative processes are described by the Radiative Transfer Equation (RTE). Local temperature differences generate motion of fluid parcels, hence convection. A rough planetary surface can hinder horizontal heat transport. Fluid dynamics equations are needed to represent these processes.

Fluid parcels are treated as open systems and the temperature distribution is obtained by a suitable thermodynamic energy equation, coupled with

the equations of the processes occurring in the atmosphere. Even by limiting the analysis to radiative processes and convection, which are the main drivers of temperature variations, the equations involved are not solvable analytically and are not tractable numerically without simplifications.

A first approximation is to consider the average over all latitudes and longitudes for quantities which depend on spatial position. The resulting atmosphere is represented by plane-parallel layers, identified by their altitude  $z$  ranging from  $z_g$  at ground level to  $z_{\text{TOA}}$  at Top Of the Atmosphere (TOA). With this hypothesis, the differential equation describing the average temperature as function of time  $t$  and altitude  $z$  is

$$\frac{\partial}{\partial t}T(t, z) = -\frac{1}{\varrho c_P} \frac{\partial}{\partial z}q(t, z) \quad , \quad (1)$$

where  $c_P$  is the specific heat at constant pressure of the atmosphere,  $\varrho$  is the average volumetric mass density of the atmosphere and  $q(t, z)$  is the total energy flux due to heat transfer. In general all these quantities are functions of  $t$  and  $z$ , also through  $T$ . Details on the derivation are in **Ramanathan**, where the equation is written initially in terms of volumetric power densities.

A second hypothesis is the radiative-convective equilibrium of the atmosphere. This translates in the existence of a steady state  $\frac{\partial}{\partial t}T(t, z) = 0$  where the planet is in radiative equilibrium, i.e. the total irradiance at TOA is null, and the atmosphere is in convective equilibrium, i.e. fluid parcels are stable with respect to vertical motion.

A model with the previous assumptions is called Radiative-Convective Model (RCM). To simplify further the RTE, in this work the dependence of quantities on the frequency of EM radiation is neglected. An atmosphere with this property is called grey atmosphere.

In the following sections the steady state vertical temperature profile of a grey atmosphere is computed. First the hypothesis of radiative equilibrium is used alone to obtain an analytical solution and use it as validation for a numerical solution. Then convection is taken into account and a simple RCM is implemented starting from the numerical scheme for radiative equilibrium.

## 1.1 Hypotheses and conventions

Some additional hypotheses are assumed to simplify the study. Data on constants are listed in table 1 and where possible, values referred to Earth are used for a prompt comparison with reality.

Some assumptions are made on the planet. It is supposed to have a diurnal cycle, to receive a constant irradiance  $S_0$  and to have a constant Bond albedo  $A$ . These conditions result in a constant irradiance

$$S_t = (1 - A) \frac{S_0}{4} \quad (2)$$

transmitted to the illuminated hemisphere of the planet at TOA from outer space. Gravitational acceleration  $g$  is constant. The atmosphere is supposed to be in hydrostatic equilibrium,

$$dP = -\rho g dz \quad , \quad (3)$$

with  $P$  atmospheric pressure. Specific heat at constant pressure  $c_P$  is assumed constant. Scattering is neglected, hence the attenuation coefficient is equal to the absorption coefficient  $\mu_a(z)$ . Moreover, the absorption coefficient is supposed to depend on  $z$  only through  $\rho$ .

For gases, specific gas constant  $R_m$  is used in thermodynamic relations, which is defined as the gas constant  $R$  divided by the molar mass of the gas. They obey the ideal gas law

$$P = \rho R_m T \quad . \quad (4)$$

The total heat flux  $q(t, z)$  is determined by radiative transfer and atmospheric convection. The RTE is not solved directly by the RCM, instead a two-stream approximation is adopted for the radiation inside atmosphere: components of radiometric quantities in the upward and downward directions are treated separately. Neither the contribution to  $q(t, z)$  due to atmospheric convection is obtained by solving the proper fluid dynamics equations, in

its place a numerical correction is adopted. With these considerations, equation (1) can be rewritten as

$$\frac{\partial}{\partial t} T(t, z) = -\frac{1}{\rho c_P} \frac{\partial}{\partial z} (E_U(t, z) - E_D(t, z)) \quad , \quad (5)$$

where  $E_U(t, z)$  and  $E_D(t, z)$  are the irradiances of upward and downward radiations, respectively.

## 1.2 Vertical coordinates

Altitude is used as vertical coordinate. However, it is convenient to define other vertical coordinates.

A bijective relation relates pressure between altitude  $z$  and pressure  $P$  (cfr. section B.1), hence they can be used interchangeably as vertical coordinates. Pressure decreases with altitude from standard value  $P_g$  at ground level to value  $P_{\text{TOA}}$  at TOA.

To simplify radiative calculations optical depth is used as vertical coordinate, starting from 0 at TOA and increasing downward up to value  $\delta_g$  at ground level. Optical depth is a function of altitude through

$$\delta(z) = \mu_m \int_z^{z_{\text{TOA}}} \rho(z') dz' \quad , \quad (6)$$

where  $\mu_m$  is the mass attenuation coefficient of the atmosphere, which corresponds to the sum of mass attenuation coefficients of the chemical species if they do not interact. This coefficient can be obtained in function of  $\delta_g$  by evaluating equation (9) at the ground level

$$\mu_m = \frac{\delta_g g}{P_g - P_{\text{TOA}}} \quad , \quad (7)$$

or conversely can be used as parameter to derive  $\delta_g$ . A simpler relation exists between  $\delta$  and  $P$  using equation (3) to evaluate the integral in equation (6):

$$\delta(P) = \frac{\mu_m}{g} (P - P_{\text{TOA}}) \quad . \quad (8)$$

Relation (8) is used also in conjunction with equation (22) to derive another formula for optical depth as a function of altitude:

$$\delta(z) = \frac{\mu_m}{g} \left( P_g \exp \left( -\frac{z - z_g}{z_0} \right) - P_{\text{TOA}} \right) \quad , \quad (9)$$

Table 1: Data on constants used in the present work. The middle rule separates standard values on top from arbitrary values chosen for the present work on bottom.

Symbol	Value	Unit	Notes
$A$	0.3		Bond albedo value compatible with various observations, cfr. <b>albedo</b>
$A_g$	0.1		Average surface albedo of Earth, cfr. <b>primer</b>
$c_P$	$1.004 \times 10^3$	J/(K kg)	Specific heat at constant pressure of air, from <b>Catling</b>
$\delta_{\text{TOA}}$	0		Optical depth at TOA, by definition
$D$	1.66		Diffusion coefficient, commonly used value from <b>Catling</b>
$g$	9.80665	m/s <sup>2</sup>	Standard gravitational acceleration of Earth
$P_g$	$1.013250 \times 10^5$	Pa	Standard pressure at ground level, from <b>US1976</b>
$R_m$	$2.8705287 \times 10^2$	J/(K kg)	Specific gas constant of air
$\sigma$	$5.670374419 \times 10^{-8}$	W/(m <sup>2</sup> K <sup>4</sup> )	Stefan-Boltzmann constant
$S_0$	1361.0	W/m <sup>2</sup>	Nominal total solar irradiance, from <b>IAU</b>
$z_g$	0	m	Nominal ground level
$\delta_g$	0.8		Optical depth at ground level of Earth, from <b>Catling</b>
$\mu_m$	$\frac{\delta_g g}{P_g - P_{\text{TOA}}}$		Mass attenuation coefficient of the atmosphere of Earth
$N$	100		Number of atmospheric layers
$P_0$	$1 \times 10^5$	Pa	Arbitrary reference value for pressure
$P_{\text{TOA}}$	3	Pa	Arbitrary TOA pressure of Earth
$z_0$	2000	m	Arbitrary constant set in equation 9 for normalisation
$z_{\text{TOA}}$	55 000	m	Arbitrary TOA altitude of Earth

where ground level is chosen as reference point and constant factor  $z_0$  is introduced to normalise the numerator. Note that  $P_{\text{TOA}}$  can be calculated from a fixed  $z_{\text{TOA}}$  using equation (22), or vice versa.

## 2 Analytical solution in radiative equilibrium

Steady state temperature profile and irradiances are derived analytically in this section.

The additional hypothesis of Local Thermodynamic Equilibrium (LTE) is considered for the atmosphere. With these hypotheses, the RTE becomes

$$\frac{1}{\mu} \frac{\partial L}{\partial z} = B_\nu - L \quad , \quad (10)$$

where  $L(t, z, \theta, \nu)$  is the spectral radiance arriving at altitude  $z$  with angle  $\theta$  with respect to direc-

tion  $\hat{z}$  and  $B_\nu(\nu, T(t, z))$  is Planck's function (cfr. section B.2).

To apply equation (10) to irradiances, two integrations are needed: one over the whole EM spectrum and one over the solid angle corresponding to a hemisphere. The latter can be performed adopting the diffusion approximation (cfr. **Catling**), which have the effect to substitute  $\delta$  with  $\delta' = D\delta$ , where  $D$  is the diffusion coefficient. The resulting equations for irradiances in terms of  $\delta'$  are

$$-\frac{d}{d\delta'} E_U(t, \delta') = \sigma T(t, \delta')^4 - E_U(t, \delta') \quad , \quad (11)$$

$$\frac{d}{d\delta'} E_D(t, \delta') = \sigma T(t, \delta')^4 - E_D(t, \delta') \quad (12)$$

and they are coupled with equation (??) written in terms of  $\delta'$ .

Boundary conditions are  $E_D(0) = 0$  because energy released to atmosphere at TOA by the down-

ward flux is negligible and  $E_U(0)$  is a constant called Outgoing Longwave Radiation (OLR). Since the planet is in radiative equilibrium,

$$E_U(0) = S_t \quad . \quad (13)$$

For the steady state solution, time dependence is dropped and irradiances are related by

$$\frac{d}{d\delta'}(E_U(\delta') - E_D(\delta')) = 0 \quad , \quad (14)$$

which has solution

$$E_U(\delta') - E_D(\delta') = S_t \quad . \quad (15)$$

Temperature is obtained by adding and subtracting equations (11) and (12) and using the conditions provided by equations (14) and (15):

$$2\sigma \frac{d}{d\delta'} T(\delta')^4 = S_t \quad . \quad (16)$$

The solution in terms of  $\delta$  is

$$T(\delta) = \left( \frac{S_t}{2\sigma} (1 + D\delta) \right)^{\frac{1}{4}} \quad , \quad (17)$$

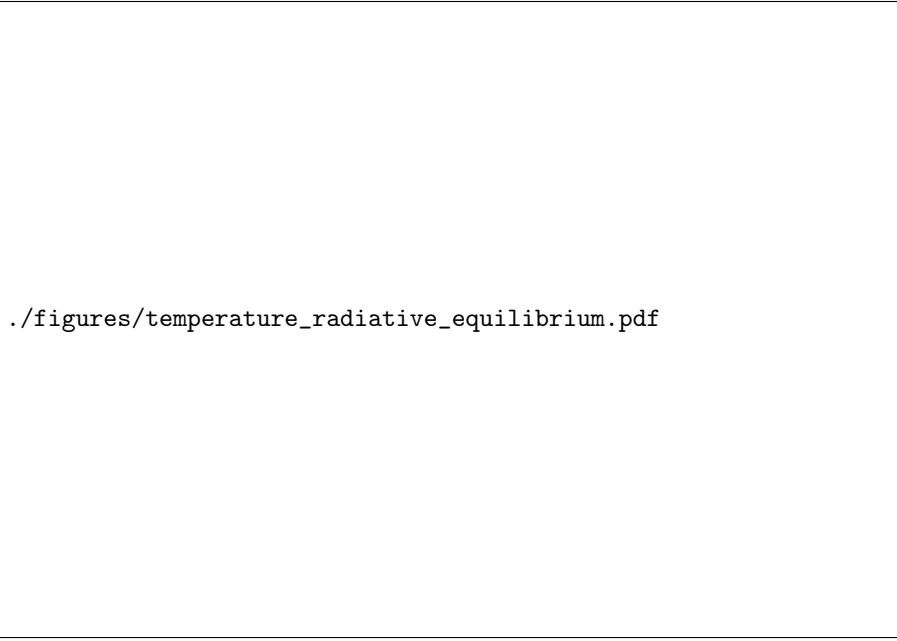
represented in figure 1. The code used to create the plot is ??.

Once the temperature profile is known,  $E_U(\delta)$  and  $E_D(\delta)$  are evaluated with the same procedure used previously for  $T(\delta')$ , resulting in:

$$E_U(\delta) = \frac{S_t}{2} (2 + D\delta) \quad , \quad (18)$$

$$E_D(\delta) = \frac{S_t}{2} D\delta \quad . \quad (19)$$

Irradiances are shown in figure ??.



`./figures/temperature_radiative_equilibrium.pdf`

Figure 1: Steady state vertical temperature profile of a grey atmosphere in radiative equilibrium. Lower altitudes are candidate to present instability due to higher values of the slope.

## A Source code

In this section the C++ code used to obtain the results presented in this work is shown and commented.

First, the parametrization of the vertical coordinate is chosen among three alternatives: altitude  $z$  in m, atmospheric pressure  $P$  in Pa, coordinate  $\sigma = \frac{P - P_{\text{TOA}}}{P_S - P_{\text{TOA}}}$  adimensional, with  $P_{\text{TOA}}$  pressure at the top of the modelised atmosphere and  $P_S$  pressure at the surface. One parametrisation can be written in term of another through a monotonic function (e.g. pressure decreases with altitude, cfr. section B.1). For the initial development the altitude  $z$  is chosen as vertical coordinate because it is more intuitive, moreover plots in **TTAPS-I** are expressed in terms of both  $z$  and  $P$ .

Second, the atmospheric layers are configured. In **TTAPS-I** 20 layers are used and they are numbered from the top of the atmosphere down as it is common in RCMs. The vertical coordinate refers to the center of each layer, with the exception of the last layer which is in direct contact with the surface and needs to be treated separately. Therefore two arrays are needed: one for the point and one with the corresponding layer thicknesses. Values are then assigned as double precision numbers. The value corresponding to the top of the atmosphere is set in a proper variable and a uniform distribution of layer thicknesses is assumed for ease.

## B Mathematical derivations

In this appendix mathematical derivations of some ancillary results and formulae used in the main text are explicitly shown.

### B.1 Relation between pressure and altitude

A general result regarding planetary atmospheres is that atmospheric pressure decreases with increasing altitude. Theoretical relations which approximate this behaviour can be obtained. Hypotheses considered in section 1 are valid.

If density is assumed constant, equation (3) can be solved easily resulting in a linear dependence of pressure  $P$  on altitude  $z$ ,

$$P(z) = P_0 - \rho g(z - z_0) \quad , \quad (20)$$

where  $(z_0, P_0)$  is a reference point inside the atmosphere.

If density is not constant its expression is given by the ideal gas law (cfr. equation (4)) and, assuming constant temperature  $T$ , equation (3) becomes

$$\begin{aligned} dP &= -\frac{Pg}{R_m T} dz \iff \\ \iff \frac{dP}{P} &= -\frac{g}{R_m T} dz \end{aligned} \quad (21)$$

with solution

$$\begin{aligned} \ln(P') \Big|_{P_0}^{P(z)} &= -\frac{g}{R_m T} z' \Big|_{z_0}^z \iff \\ \iff P(z) &= P_0 \exp \left( -\frac{g}{R_m T} (z - z_0) \right) \quad . \end{aligned} \quad (22)$$

This relation is not meaningful, since the aim of the work is to derive the non-constant temperature profile of the atmosphere. However, it can be used inside atmospheric layers where the temperature is considered constant (e.g. stratosphere).

A better approximation assumes non-constant density and constant lapse rate  $\Gamma$ , hence temperature depends linearly on altitude,

$$\Gamma = -\frac{dT}{dz} \iff T(z) = T_0 - \Gamma(z - z_0) \quad , \quad (23)$$

with  $T_0$  temperature corresponding to reference altitude  $z_0$ . Using these assumptions and the den-

sity rewritten through the ideal gas law (4), equation (3) becomes

$$\begin{aligned} dP &= -\frac{Pg}{R_m T} \left( -\frac{dT}{\Gamma} \right) \iff \\ \iff \frac{dP}{P} &= \frac{g}{R_m \Gamma} \frac{dT}{T} \quad , \end{aligned} \quad (24)$$

which has solution

$$\begin{aligned} \ln(P') \Big|_{P_0}^{P(z)} &= \frac{g}{R_m \Gamma} \ln(T') \Big|_{T_0}^{T(z)} \iff \\ \iff P(z) &= P_0 \left( \frac{T_0 - \Gamma(z - z_0)}{T_0} \right)^{\frac{g}{R_m \Gamma}} \quad . \end{aligned} \quad (25)$$

Equation (25) can be used also with a piecewise constant lapse rate in altitude intervals where it is not null. Otherwise, in altitude intervals where lapse rate is null, equation (22) is valid with appropriate boundary conditions to ensure continuity between layers.

### B.2 Radiometric quantities

Refer to **CIE** for more details on quantities reviewed in this section.

Consider electromagnetic radiation emitted by a point source. The total emitted power is called *radiant flux*, with unit W. The density of radiant flux with respect to a solid angle in the direction of emission is called *radiant intensity*, expressed in W/sr. When radiation interacts with a surface, i.e. it gets absorbed, transmitted or reflected, its radiant intensity distributed over the surface is measured through *radiance* in W/(m<sup>2</sup> sr). If the area on which the radiation is incident is expressed through the solid angle it subtends, the integral of radiance over this solid angle is called *irradiance*, expressed in W/m<sup>2</sup>. Note that the coordinate system where the solid angles of radiant intensity and irradiance are defined may not be the same. Radiant flux emitted by a body normalised over the surface of emission is measured by *radiant exitance* in W/m<sup>2</sup>.

All previous quantities can be expressed as densities with respect to the wavelength or the wavenumber and the adjective *spectral* is prefixed to their names. Their units are divided by the respective spectral quantity (e.g. spectral radiance with wavenumber in 1/cm has units W cm/(m<sup>2</sup> sr)).

Spectral radiance of a blackbody is given by Planck's law

$$B_\nu(\nu, T) = 2hc^2\nu^3 \frac{1}{e^{\frac{hc\nu}{k_B T}} - 1} \quad , \quad (26)$$

where  $\nu$  is the wavenumber in unit 1/cm,  $T$  in unit K is the temperature of the emitting body and the other quantities are constants (cfr. table 1). Note that Planck's law has different form when it is expressed in terms of wavelength, due to its definition as density and the resulting change of variables:

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} \quad . \quad (27)$$

If radiance is isotropic, i.e. it is not dependent on the direction of the radiation, the corresponding irradiance is proportional. For instance, if the radiation is absorbed by a hemispheric surface approximated by a blackbody, the spectral irradiance of the surface is

$$\begin{aligned} & \int B_\nu(\nu, T) d\phi \sin(\theta) d\theta \cos(\theta) = \\ & = B_\nu(\nu, T) \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin(\theta) \cos(\theta) d\theta = \\ & = 2\pi B_\nu(\nu, T) \int_0^1 \sin(\theta) d(\sin(\theta)) = \\ & = 2\pi B_\nu(\nu, T) \frac{1}{2} = \\ & = \pi B_\nu(\nu, T) \quad , \end{aligned} \quad (28)$$

where spherical coordinates are used to describe the surface and the term  $\cos(\theta)$  considers the component of radiation along the normal of the infinitesimal solid angle.

### B.3 Radiation attenuation

Details on quantities appearing in this section can be found in **Modest**. Radiation crossing a medium loses energy due to absorption and scattering. The effect of chemical species on these processes is quantified through the *attenuation coefficient* (commonly called *extinction coefficient* in atmospheric sciences), which has different definitions based on the way it is derived (cfr. **Catling**). The attenuation coefficient is the sum of *absorption coefficient*

and *scattering coefficient* which contain information on the attenuation due to the respective physical processes. The *optical depth* (also called *optical thickness*) takes in consideration the amount of substance involved in the absorption.

Ratios between radiant fluxes are related by the conservation of energy: the sum of *internal transmittance* and *internal absorptance* is 1, as well as the sum of *reflectance*, *absorptance* and *transmittance*.

In general these coefficients are functions of wavelength or wavenumber, in which case the prefix *spectral* is adopted. If the medium is a fluid, they depends on temperature and pressure of the medium. Another related quantity is the spectral internal transmittance

$$\tau_1(\nu, s, s_0) = e^{-\delta(\nu, s, s_0)} \quad , \quad (29)$$

where  $\delta(\nu, s, s_0)$  is the spectral optical depth, which depends only on the spectral attenuation coefficient  $\mu(\nu, s')$  of the medium traversed by the radiation from  $s_0$  to  $s$  on the optical path. In the RCM optical paths are straight and form an angle  $\theta$  with the direction normal to the layers, hence the definition of the spectral optical depth becomes:

$$\delta(\nu, s, s_0) = \frac{1}{\cos \theta} \int_{s_0}^s \mu(\nu, s') ds' \quad . \quad (30)$$

If the absorbing species do not interact,  $\mu(\nu, s')$  is simply the sum of the spectral attenuation coefficients of the individual components of the medium.

Moreover, if the medium is homogeneous, in the sense that quantities affecting radiative calculations are not dependent on spatial position (e.g. attenuation coefficients  $\mu$  are constant inside the medium), the spectral attenuation coefficient depends only on the concentration of the absorbing species contained, hence the spectral attenuation coefficient can be rewritten as

$$\mu(\nu, s') = \mu_m(\nu) \rho(s') \quad (31)$$

where  $\mu_m(\nu)$  is the spectral mass attenuation coefficient and  $\rho(s')$  is the volumetric mass density of the absorber.

Names of radiative properties ending with suffix *-ance* are generally used for rough surfaces, while suffix *-ivity* indicates smooth surfaces. In this work the former is adopted. Refer to **Modest** for more information and to the definition of spectral absorptivity in **CIE** for an example of the difference.



## B.4 Quantities commonly used in atmospheric sciences

Earth's surface horizontal profile is not uniform, hence altitude and pressure near ground level could present sudden variations. In models where this is taken into consideration, the sigma coordinate system is commonly used instead, defined by

$$\sigma = \frac{P - P_{\text{TOA}}}{P_g - P_{\text{TOA}}} \quad . \quad (32)$$

To avoid confusion, in this work symbol  $\sigma$  is used for the Stefan-Boltzmann constant (cfr. table 1), except for equation (32).

An alternative quantity evaluated in place of  $T(t, z)$  for a given parcel of fluid is the potential temperature

$$\theta(t, z) = T(t, z) \left( \frac{P_0}{P(z)} \right)^{\frac{R_m}{c_P}} \quad , \quad (33)$$

where  $P_0$  is a reference pressure and quantities  $P(z)$ ,  $R_m$  and  $c_P$  refer to the fluid.

## C Supplementary information

### C.1 Plotting

Software gnuplot is used to generate plots shown in this work. Output values from the simulation are stored in a DAT file.