Radiative-convective equilibrium in a grey atmosphere

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Complex systems in climate physics, 3 October 2023

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- A radiative-convective model is used to study a grey atmosphere.
- Comparison between numerical and analytical solutions is possible in radiative equilibrium.

Introduction

- Average vertical temperature profile $T(t, z)$ of atmosphere.

1. The analysed quantity is the atmospheric temperature profile averaged over all latitudes and longitudes.

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2. RTE describes radiative processes.
3. Fluid dynamics equations describe convective processes.

Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial q}{\partial z} \quad . \tag{1}$$

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1. Thermodynamic energy equation describes average vertical temperature profile.
2. The study is conducted on an atmosphere in radiative-convective equilibrium.
3. Quantities do not depend on the frequency of electromagnetic radiation.

Additional hypotheses

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- Hypotheses on the composition of atmosphere.
- Hypotheses on total heat flux.
- Resulting thermodynamic energy equation:

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2. Hydrostatic equilibrium, constant specific heat at constant pressure, scattering is neglected, absorption coefficient depends only on altitude, constant mass attenutation coefficient, ideal gas.
3. Heat flux determined only by radiative and convective processes, two-stream approximation, numerical correction for convection.

Vertical coordinates

- Relation between pressure and altitude:

$$P(z) = P_{\mathrm{g}} \exp\left(-\frac{z - z_{\mathrm{g}}}{z_0}\right) \quad . \tag{3}$$

- Relation between optical depth and pressure:

$$\delta(P) = \frac{\mu_{\mathrm{m}}}{g}(P - P_{\mathrm{TOA}}) \quad . \tag{4}$$

- Relation between optical depth and altitude:

$$\delta(z) = \frac{\mu_{\mathrm{m}}}{g}\left(P_{\mathrm{g}} \exp\left(-\frac{z - z_{\mathrm{g}}}{z_0}\right) - P_{\mathrm{TOA}}\right) \quad . \tag{5}$$

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1. Obtained from hydrostatic equilibrium and ideal gas law.
2. Obtained from definition of optical depth, hydrostatic equilibrium and hypotheses on attenutation coefficient.
3. Obtained by combining relations (3) and (4).

Equations in radiative equilibrium

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- Equations for irradiances:

$$-\frac{\partial}{\partial \delta'} E_\mathrm{U}(t,\delta') = \sigma T(t,\delta')^4 - E_\mathrm{U}(t,\delta') \quad , \tag{7}$$
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2. To describe irradiances, RTE is integrated over the whole spectrum of radiation and over the solid angle of a hemisphere.
3. Equations for irradiances and temperature are written in terms of δ'.

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- Steady state.

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2. At TOA downward irradiance does not deposit energy and radiative equilibrium fixes upward irradiance, i.e. outgoing longwave radiation.
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4. Initial condition on temperature is obtained by combining all conditions on irradiances.

Analytical solution

- Temperature:

$$T(\delta) = \left(\frac{S_{\mathrm{t}}}{2\sigma}(1 + D\delta)\right)^{\frac{1}{4}} \quad . \tag{14}$$

- Irradiances:

$$E_{\mathrm{U}}(\delta) = \frac{S_{\mathrm{t}}}{2}(2 + D\delta) \quad , \tag{15}$$

$$E_{\mathrm{D}}(\delta) = \frac{S_{\mathrm{t}}}{2}D\delta \quad . \tag{16}$$

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Numerical solution

- Normalisation:

$$Y_0 = \frac{T^4}{T_0^4} \quad , \quad Y_1 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \quad , \quad Y_2 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \tag{17}$$

- Resulting system of ODEs:

$$\begin{cases} \frac{\mathrm{d}Y_0}{\mathrm{d}\delta} = \frac{D}{2} \\ \frac{\mathrm{d}Y_1}{\mathrm{d}\delta} = D(Y_1 - Y_0) \\ \frac{\mathrm{d}Y_2}{\mathrm{d}\delta} = D(Y_0 - Y_2) \end{cases} \quad . \tag{18}$$

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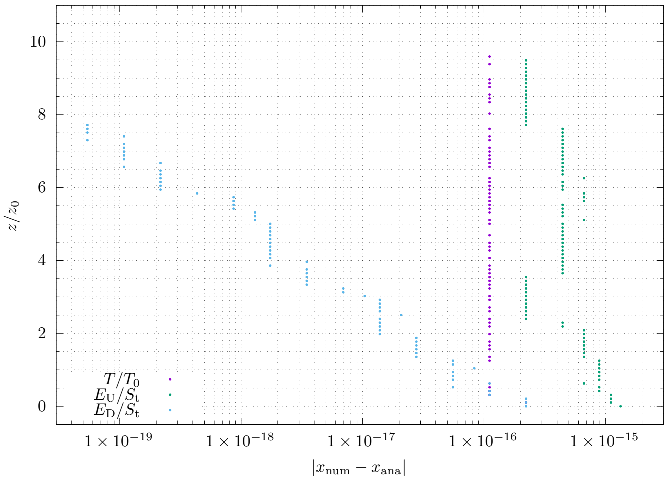
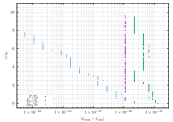
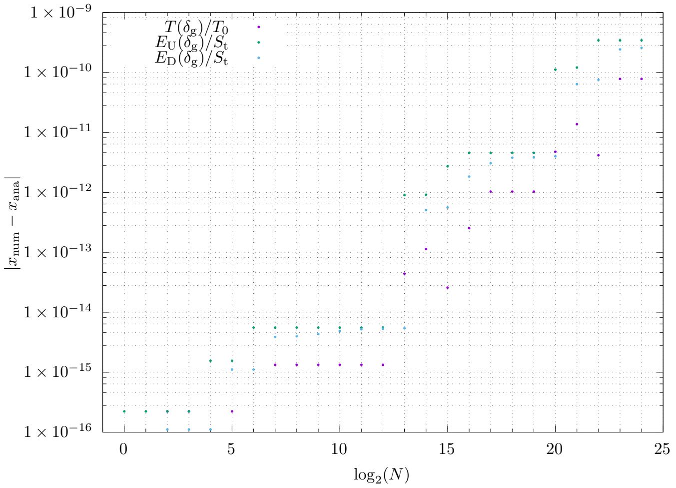
Errors of numerical solutions

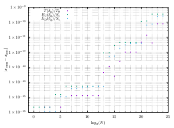




- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium, assuming steady state. Points at some altitudes are not shown because their value is exactly 0.
- Errors are compatible with 0 based on precision of double-precision floating-point numbers.

Stability analysis





- Stability of numerical solution for ODEs system in radiative equilibrium with respect to spatial grid size. Errors are negligible up to 4096 layers, then error propagation dominates reducing the precision of the method. Missing points have value 0.
- Errors are evaluated as absolute differences between numerical and analytical values at ground level of each normalised function.

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- Euler method for temporal integration.
- Constant time step.

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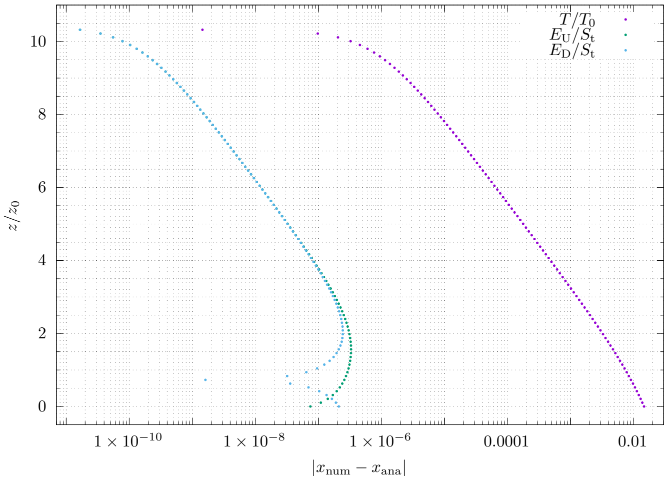
Errors of time integration at steady state



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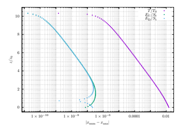

- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium. Steady state is reached throught iterative temporal and spatial integrations. Precision is reduced by propagation of errors during successive iterations. Points at TOA are omitted being compatible with 0 within precision of double-precision floating-point numbers.
- Time step is chosen arbitrarily to reduce the errors of irradiances below the precision of numeric values outputs.

Radiative-convective equilibrium

Conclusion