Radiative-convective equilibrium in a grey atmosphere

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Introduction

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- Fluid dynamics equations.

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- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

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- Hypotheses on the composition of atmosphere.
- Hypotheses on total heat flux.
- Resulting thermodynamic energy equation:

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial}{\partial z}\left(E_\mathrm{U} - E_\mathrm{D}\right) \quad . \tag{2}$$

Vertical coordinates

- Relation between pressure and altitude:

$$P(z) = P_\mathrm{g} \exp\left(- \frac{z - z_\mathrm{g}}{z_0} \right) \quad . \tag{3}$$

- Relation between optical depth and pressure:

$$\delta(P) = \frac{\mu_\mathrm{m}}{g} (P - P_\mathrm{TOA}) \quad . \tag{4}$$

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Equations in radiative equilibrium

- RTE for non-scattering medium in LTE:

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- Equations for irradiances:

$$-\frac{\partial}{\partial \delta'} E_{\mathrm{U}}(t,\delta') = \sigma T(t,\delta')^4 - E_{\mathrm{U}}(t,\delta') \quad , \tag{7}$$

$$\frac{\partial}{\partial \delta'} E_{\mathrm{D}}(t,\delta') = \sigma T(t,\delta')^4 - E_{\mathrm{D}}(t,\delta') \quad . \tag{8}$$

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- Initial conditions on irradiances:

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- Additional relations:

$$\frac{\mathrm{d}}{\mathrm{d}\delta'}\left(E_{\mathrm{U}}(\delta') - E_{\mathrm{D}}(\delta')\right) = 0 \quad , \tag{11}$$

$$E_{\mathrm{U}}(\delta') - E_{\mathrm{D}}(\delta') = S_{\mathrm{t}} \quad . \tag{12}$$

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- Initial condition on temperature:

$$T(0) = \left(\frac{S_{\mathrm{t}}}{2\sigma}\right)^{\frac{1}{4}} \quad . \tag{13}$$

Analytical solution

- Temperature:

$$T(\delta) = \left(\frac{S_t}{2\sigma}(1 + D\delta)\right)^{\frac{1}{4}} \quad . \tag{14}$$

- Irradiances:

$$E_U(\delta) = \frac{S_t}{2}(2 + D\delta) \quad , \tag{15}$$

$$E_D(\delta) = \frac{S_t}{2}D\delta \quad . \tag{16}$$

Numerical solution

- Normalisation:

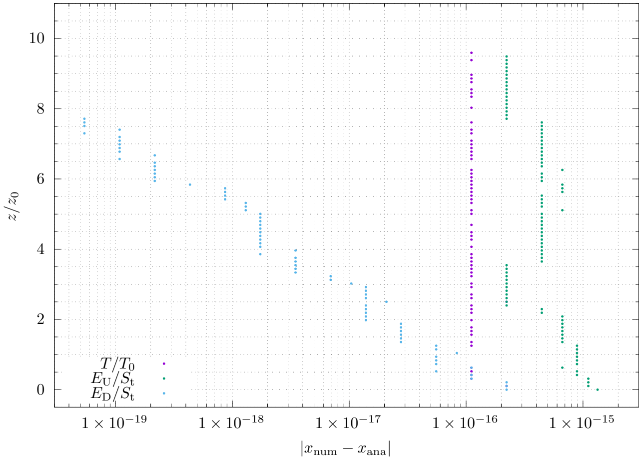
$$Y_0 = \frac{T^4}{T_0^4} \quad , \quad Y_1 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \quad , \quad Y_2 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \tag{17}$$

- Resulting system of ODEs:

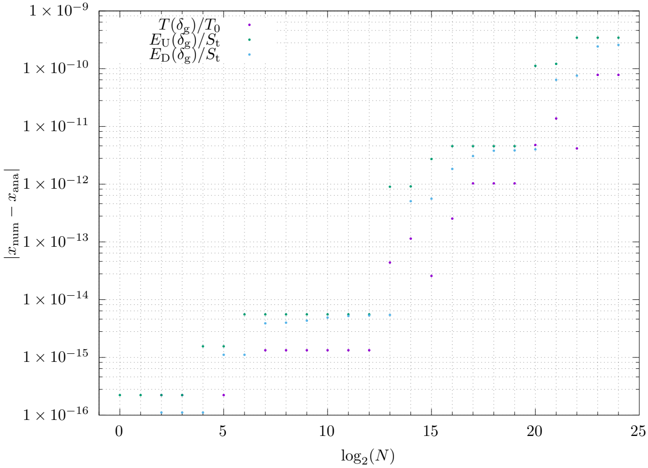
$$\begin{cases} \frac{\mathrm{d}Y_0}{\mathrm{d}\delta} = \frac{D}{2} \\ \frac{\mathrm{d}Y_1}{\mathrm{d}\delta} = D(Y_1 - Y_0) \\ \frac{\mathrm{d}Y_2}{\mathrm{d}\delta} = D(Y_0 - Y_2) \end{cases} \quad . \tag{18}$$

- Runge-Kutta method of order 4.
- Non-uniform step size.

Errors of numerical solutions



Stability analysis



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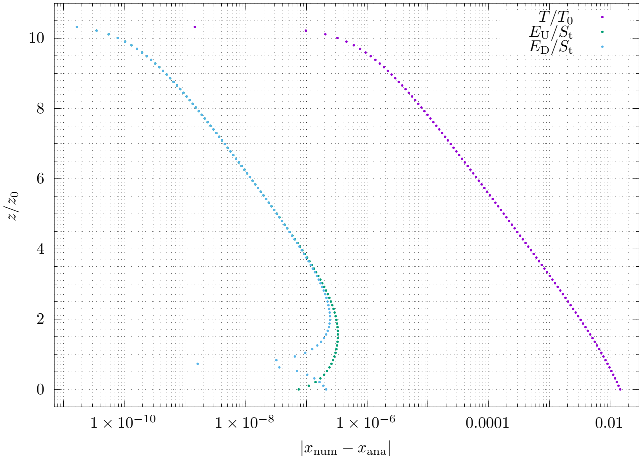
- Resulting system of PDEs:

$$\begin{cases} \frac{\partial}{\partial t} Y_3(t,\delta) = \frac{\mu_\mathrm{m} S_\mathrm{t}}{c_P T_0} \frac{\partial}{\partial \delta} \left(Y_1(t,\delta) - Y_2(t,\delta) \right) \\ \frac{\partial}{\partial \delta} Y_1(t,\delta) = D\left(Y_1(t,\delta) - Y_3(t,\delta)^4 \right) \\ \frac{\partial}{\partial \delta} Y_2(t,\delta) = D\left(Y_3(t,\delta)^4 - Y_2(t,\delta) \right) \end{cases} \quad . \tag{20}$$

- Euler method for temporal integration.
- Constant time step.

Errors of time integration at steady state



Radiative-convective equilibrium

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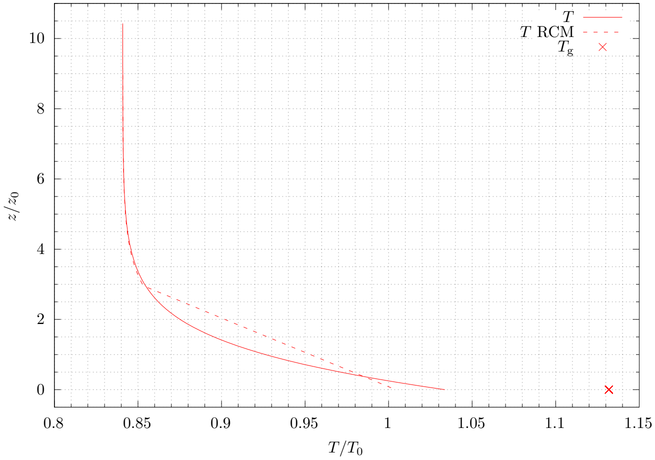
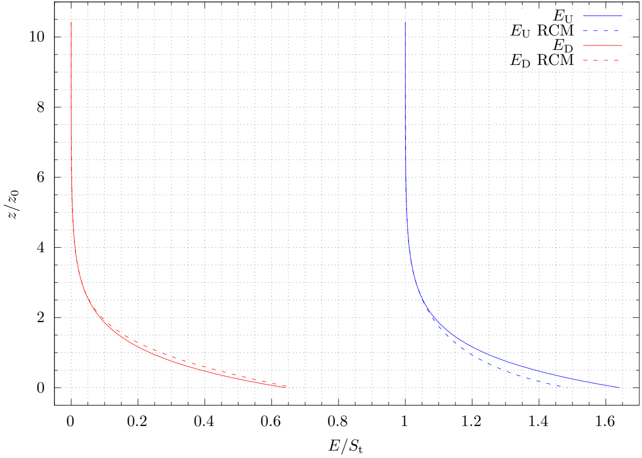
- Convective adjustment if $-\frac{\partial T}{\partial z} > \Gamma_0$.

Temperature plot



Irradiances plot



Conclusion

- Vertical temperature profile of a grey atmosphere described in general by a system of PDEs and at the steady state by a system of ODEs.
- Negligible errors for ODEs numerical solutions.
- Possible improvements on description of physical phenomena.
- Possible improvements on precision of PDEs numerical solutions.