

# Radiative-convective equilibrium in a grey atmosphere

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## Abstract

Radiative-convective models provide an intermediate complexity approach to the simulation of climate. These models evaluate the atmospheric temperature profile averaged over all latitudes and longitudes, which is function of time and altitude. Physical processes which determine the energy exchange in the model are absorption, transmission, reflection of electromagnetic radiation and convection of fluid. In this work a radiative-convective model is used to derive the temperature of an atmosphere where the optical depth is constant with respect to the frequency of radiation. The resulting temperature profile is compared with the analytical solution provided under the condition of radiative equilibrium.

## 1 Introduction

Climate dynamics can be studied with models of varying complexity. One of the quantities analysed is the temporal and spatial distribution of temperature in the atmosphere, which is the result of the heat exchange between different processes. Electromagnetic (EM) radiation is emitted, absorbed and scattered by the chemical species distributed in the atmosphere and by the planetary surface. Moreover, the atmosphere receives EM radiation by other celestial bodies, e.g. stars. Radiative processes are described by the Radiative Transfer Equation (RTE). Local temperature differences generate motion of fluid parcels, hence convection. A rough planetary surface can hinder horizontal heat transport. Fluid dynamics equations are needed to represent these processes.

Fluid parcels are treated as open systems and the temperature distribution is obtained by a suitable thermodynamic energy equation, coupled with

the equations of the processes occurring in the atmosphere. Even by limiting the analysis to radiative processes and convection, which are the main drivers of temperature variations, the equations involved are not solvable analytically and are not tractable numerically without simplifications.

A first approximation is to consider the average over all latitudes and longitudes for quantities which depend on spatial position. The resulting atmosphere is represented by plane-parallel layers, identified by their altitude  $z$  ranging from  $z_g$  at ground level to  $z_{\text{TOA}}$  at the Top Of the Atmosphere (TOA). With this hypothesis, the differential equation describing temperature variation with respect to time  $t$  is

$$\frac{\partial}{\partial t}T(t, z) = -\frac{1}{\varrho c_P} \frac{d}{dz}q(z) \quad , \quad (1)$$

where  $\varrho$  is the volumetric mass density of the atmosphere, in general a function of  $z$ ,  $c_P$  is the specific heat at constant pressure of the atmosphere, assumed constant, and  $q(z)$  is the total flux exchanged due to heat transfer, which depends on altitude  $z$  also through  $T(t, z)$ . Details on the derivation are in [1, p. 466], where the equation is written initially in terms of volumetric power densities.

A second hypothesis is the radiative-convective equilibrium of the atmosphere. This translates in the existence of a steady state  $\frac{\partial}{\partial t}T(t, z) = 0$  where the atmosphere is both in radiative equilibrium, i.e. the total radiant flux at TOA is null, and in convective equilibrium, i.e. fluid parcels are stable with respect to vertical motion.

A model with the previous assumptions is called Radiative-Convective Model (RCM). To simplify further the RTE, in this work the dependence of optical depth  $\delta$  on the frequency of EM radiation is neglected. An atmosphere with this property is called grey atmosphere.

Additional hypotheses are considered to simplify this study. Specific gas constant  $R_m$  is used in thermodynamic relations, which is defined as the gas constant  $R$  divided by the molar mass of the gas. Gravitational acceleration  $g$  is constant. The atmosphere is supposed to be in hydrostatic equilibrium described by

$$dP = -\rho g dz \quad (2)$$

and to obey the ideal gas law

$$P = \rho R_m T \quad (3)$$

A bijective relation exists between altitude  $z$  and pressure  $P$  (cfr. section B.1), hence they can be used interchangeably as vertical coordinates. Pressure decreases with altitude from standard value  $P_g$  at ground level to value  $P_{\text{TOA}}$  at TOA. To simplify radiative calculations, scattering is neglected and optical depth is used as vertical coordinate, starting from 0 at TOA and increasing downward up to value  $\delta_g$  at ground level. Optical depth is related to altitude by

$$\delta(z) = \mu_m \int_z^{z_{\text{TOA}}} \varrho(z') dz' \quad (4)$$

where  $\mu_m$  is the mass attenuation coefficient of the atmosphere, which is the sum of mass attenuation coefficients of the chemical species if no chemical reaction is supposed to occur in atmosphere. A simpler relation exists between  $\delta$  and  $P$  using equation (15) to evaluate the integral in equation (4):

$$\delta(P) = \frac{\mu_m}{g} (P - P_{\text{TOA}}) \quad (5)$$

For simplicity, the planet is supposed to have a diurnal cycle, to receive a constant irradiance  $S_0$  and to have a constant Bond albedo  $A$ . Atmosphere is supposed transparent to radiation from outside the planet.

The total heat flux  $q(z)$  is determined by radiative transfer and atmospheric convection. The RTE is not solved directly by the RCM, instead a two-stream approximation is adopted: radiation is split in upward and downward directions and they are treated separately. Neither the contribution to  $q(z)$  due to atmospheric convection is obtained by solving the proper fluid dynamics equations, in its place a numerical correction is adopted and is described

in section 3.6. With these considerations, equation (17) can be rewritten as

$$\frac{\partial}{\partial t} T(t, z) = -\frac{1}{\rho c_P} \frac{d}{dz} (E_U(z) - E_D(z)) \quad (6)$$

where  $E_U(z)$  and  $E_D(z)$  are the irradiances of upward and downward radiations, respectively.

## 2 Analytical solution of radiative equilibrium

The modelled atmosphere is in Local Thermal Equilibrium (LTE) and does not scatter radiation. With these hypotheses the RTE becomes Schwarzschild's equation

$$\frac{1}{\mu_m \varrho} \frac{d}{ds} L(\nu, s) = B_\nu(\nu, T) - L(\nu, s) \quad (7)$$

where  $L(\nu, s)$  is the spectral radiance at point  $s$  of the optical path and  $B_\nu(\nu, T)$  is Planck's function (cfr. section B.2).

To apply equation (7) to irradiances, first it is integrated over the whole EM spectrum, then it is integrated over the solid angle corresponding to a hemisphere. This second integration brings a factor  $D$  which is multiplied to  $\delta$  to consider the diffusion of radiation. Optical depth  $\delta' = D\delta$  is used as vertical coordinate, hence the resulting equations for the upward and downward irradiances are:

$$-\frac{d}{d\delta'} E_U(\delta') = \sigma T(\delta')^4 - E_U(\delta') \quad (8)$$

$$\frac{d}{d\delta'} E_D(\delta') = \sigma T(\delta')^4 - E_D(\delta') \quad (9)$$

Boundary conditions are  $E_D(0) = 0$  because energy released to atmosphere at TOA by the downward flux is negligible and  $E_U(0)$  is a constant called Outgoing Longwave Radiation (OLR).

For the condition of radiative equilibrium the OLR is equal to the irradiance arrived at TOA from outside the atmosphere:

$$E_U(0) = (1 - A) \frac{S_0}{4} \quad (10)$$

When temperature reaches the steady state, equations (8) and (9) are coupled by equation

$$\frac{d}{d\delta'} (E_U(\delta') - E_D(\delta')) = 0 \quad (11)$$

which has solution

$$E_U(\delta') - E_D(\delta') = (1 - A) \frac{S_0}{4} \quad (12)$$

in agreement with the hypothesis of atmosphere transparent to radiation from outside the atmosphere.

With these information and by adding and subtracting equations (8) and 9, the equation for the temperature of the steady state is obtained,

$$2\sigma \frac{d}{d\delta'} T(\delta')^4 = (1 - A) \frac{S_0}{4} \quad , \quad (13)$$

and the solution is:

$$T(\delta) = \left( (1 - A) \frac{S_0}{8\sigma} (1 + D\delta) \right)^{\frac{1}{4}} \quad . \quad (14)$$

### 3 Methods

The RCM developed in the present work approximates the atmosphere as a vertical column divided in plane-parallel layers. A bijective relation exists between altitude  $z$  and pressure  $P$  (cfr. section B.1), hence they can be used interchangeably as vertical coordinates. The origin of coordinate  $z$  is set at ground level and it is identified by value  $z_g$  to keep generality. On the contrary, the Top-Of-Atmosphere (TOA) altitude  $z_{TOA}$  is chosen arbitrarily. Pressure decreases with altitude from standard value  $P_g$  at ground level to value  $P_{TOA}$ , which is calculated using equation (31).

Each layer is identified by a value of  $z$  corresponding to the upper boundary of the layer and by its thickness. The corresponding values of pressure are evaluated at each time instant with equation (31).

#### 3.1 Hypotheses and conventions

To simplify some calculations of radiative fluxes, the definition of wavenumber  $\nu = \frac{1}{\lambda}$  is used in this work, where  $\lambda$  is the wavelength of the radiation. The electromagnetic (EM) spectrum is limited to interval  $[\nu_{min}, \nu_{max}]$  and the choice of these values is discussed in section 5.1.

Specific gas constant is used in thermodynamical relations, which is defined as the gas constant  $R$  divided by the gas molar mass. Henceforth, specific gas constant is represented with symbol  $R_m$ .

Some assumptions are made on state and composition of Earth's atmosphere. Gravitational acceleration  $g$  is constant. Atmospheric layers are supposed to be in hydrostatic equilibrium described by

$$dP = -\rho g dz \quad (15)$$

and gaseous components obey the ideal gas law

$$P = \rho R_m T \quad , \quad (16)$$

where  $\rho$  is the volumetric mass density of the gas. Each layer is homogenous, in the sense that quantities affecting radiative calculations are not dependent on spatial position inside the layer (e.g. gas densities, attenuation coefficients  $\mu$  and specific heats at constant pressure  $c_P$  are constant inside the layer), and emits in both upward and downward directions as a blackbody. No chemical reactions are supposed to occur in atmosphere. The average Earth's surface considered in the model is approximated as a planar blackbody emitting in the normal direction with temperature  $T_{earth}$ . Average Bond albedo for Earth is identified with constant  $A$ .

Solar radiation is made by parallel rays forming an angle of incidence  $\varepsilon$  with the normal of the plane-parallel atmospheric layers (i.e. zenith angle). Sun's surface is approximated as an emitting blackbody with temperature  $T_{sun}$ . Constant value  $S_0$  for total solar irradiance is used in calculations.

Solar spectral irradiance extends in the infrared (IR), visible (VIS) and ultraviolet (UV) bands, while spectral irradiance of Earth's surface is peaked in the IR band. The difference of their shapes allows the division of the spectrum at the intersection of the curves. The wavenumber of the intersection point is  $\nu_{div}$ . In this work, radiation with wavenumbers lesser than  $\nu_{div}$  is addressed as long-wave radiation, otherwise the denomination short-wave radiation is used. The two resulting spectra are treated independently and the overlap between curves is neglected. Accuracy of these approximations and value of  $\nu_{div}$  are discussed in section 5.1.

Data on constants used in this work are listed in table 1.

#### 3.2 Temperature equation

Intermediate layers of the atmosphere are treated as open systems where in general air parcels can

Table 1: Data on constants used in the present work. The middle rule separates standard values on top from arbitrary values chosen for the present work on bottom.

Symbol	Value	Unit	Notes
$R$	8.31446261815324	J/(K mol)	Gas constant
$g$	9.80665	m/s <sup>2</sup>	Standard gravitational acceleration of Earth
$S_0$	1361.0	W/m <sup>2</sup>	Nominal total solar irradiance, from [2]
$R_{\text{sun}}$	$6.957 \times 10^8$	m	Nominal solar radius, from [2]
$T_{\text{sun}}$	5772	K	Solar surface temperature, from [2]
$T_{\text{earth}}$	288.15	K	Earth's surface temperature based on [3]
$A$	0.3		Bond albedo value compatible with various observations, cfr. [4, p. 1281]
$A_g$	0.1		Average surface albedo of Earth, cfr. [5, p. 217]
$P_g$	$1.013250 \times 10^5$	Pa	Standard pressure at ground level, from [3, p. 2]
$c_{P,\text{air}}$	$1.004 \times 10^3$	J/(K kg)	Specific heat at constant pressure of air, from [6, p. 16]
$z_g$	0	m	Nominal ground level
$D$	1.66		Diffusivity factor, from [1, p. 469]
$z_{\text{TOA}}$	55 000	m	Arbitrary TOA altitude
$\nu_{\text{min}}$	100	1/cm	Arbitrary minimum boundary for EM spectrum
$\nu_{\text{max}}$	100 000	1/cm	Arbitrary maximum boundary for EM spectrum
$N$	20		Number of atmospheric layers

move vertically, therefore temperature is determined by horizontally averaging the thermodynamic energy equation. Details on the derivation are in [1, p. 466], where the equation is written in terms of volumetric power densities. The resulting differential equation describing temperature variation with time across layer  $z$  is

$$\frac{\partial}{\partial t} T(t, z) = -\frac{1}{\rho c_P} \frac{d}{dz} q(z) \quad (17)$$

where  $q(z)$  is the total flux exchanged due to heat transfer, which depends on altitude  $z$  also through  $T(t, z)$ .

The total heat flux  $q(z)$  is determined by radiative transfer and atmospheric convection. The first process is described by the radiative transfer equation (RTE), while the second process is governed by fluid dynamics. The RTE is an integro-differential equation in five dimensions and it is not solved directly by the RCM, instead longwave and shortwave radiations are treated separately. Neither the contribution to  $q(z)$  due to atmospheric convection is obtained by solving the proper fluid dynamics equations, in its place a numerical correction is adopted and is described in section 3.6.

With these considerations and using equation (15) to drop the dependence on  $\varrho$ , equation (17) can be rewritten as

$$\frac{\partial}{\partial t} T(t, z) = \frac{g}{c_P} \frac{d}{dP} (E_L(z) + E_S(z)) \quad , \quad (18)$$

where  $E_L(z)$  and  $E_S(z)$  are the irradiances of longwave and shortwave radiations, respectively.

### 3.3 Longwave radiation

In the region  $[\nu_{\text{min}}, \nu_{\text{div}}]$  the attenuation of radiation is dominated by absorption, scattering is neglected. Absorbing species in this bandwidth are H<sub>2</sub>O as water vapor, CO<sub>2</sub> and trace gases like N<sub>2</sub>O, chlorofluorocarbons (CFCs), CH<sub>4</sub>.

The absorption by the chemical species is quantified by the spectral internal transmittance

$$\tau_1(\nu, s, s_0) = e^{-\delta(\nu, s, s_0)} \quad , \quad (19)$$

where  $\delta(\nu, s, s_0)$  is the spectral optical depth, which depends only on the spectral attenuation coefficient  $\mu(\nu, s')$  of the medium traversed by the radiation from  $s_0$  to  $s$  on the optical path. In the RCM optical paths are straight and form an angle  $\theta$  with the

direction normal to the layers, hence the definition of the spectral optical depth becomes:

$$\delta(\nu, s, s_0) = \frac{1}{\cos \theta} \int_{s_0}^s \mu(\nu, s') ds' \quad (20)$$

The absorbing species do not interact, hence  $\mu(\nu, s')$  is simply the sum of the spectral attenuation coefficients of the individual components of the medium. Moreover, due to layers homogeneity it depends only on the concentration of the absorbing species in the layer, hence the spectral attenuation coefficient can be rewritten as

$$\mu(\nu, s') = \mu_m(\nu) \rho(s') \quad (21)$$

where  $\mu_m(\nu)$  is the spectral mass attenuation coefficient and  $\rho(s')$  is the volumetric mass density of the absorber.

The longwave irradiance at altitude  $z$  is given by

$$E_L(z) = \int_{\nu_{\min}}^{\nu_{\div}} E_L(\nu, z) d\nu \quad (22)$$

where  $E_L(\nu, z)$  is the longwave spectral irradiance

$$\begin{aligned} E_L(\nu, z) = & \pi B_\nu(\nu, T(t, 0)) \tau_i(\nu, z, 0) + \\ & + \int_{z_g}^z \pi B_\nu(\nu, T(t, z')) \frac{d\tau_i(\nu, z, z')}{dz'} dz' \\ & + \int_z^{z_{\text{TOA}}} \pi B_\nu(\nu, T(t, z')) \frac{d\tau_i(\nu, z', z)}{dz'} dz' \end{aligned} \quad (23)$$

with  $\tau_i(\nu, z, z')$  and  $\tau_i(\nu, z', z)$  spectral internal transmittances of the medium between altitudes  $z'$  and  $z$ . Time  $t$  appears only as a parameter. In equation (23) each addend corresponds to a spectral irradiance depositing energy at layer  $z$ : the first one is generated by Earth's surface, the second and the third ones are the total spectral irradiances of layers beneath and above  $z$ , respectively.

### 3.4 Shortwave radiation

Both absorption and scattering occur in bandwidth  $[\nu_{\div}, \nu_{\max}]$ . Chemical species absorbing in this bandwidth are  $\text{O}_3$ ,  $\text{H}_2\text{O}$  as water vapor,  $\text{CO}_2$  and  $\text{O}_2$ .

Explicit evaluation of scattering of solar radiation is not performed due to the complexity of equations involved. A simpler parametrisation is

adopted instead, where the division of the atmosphere in plane-parallel layers is used to assign constant reflectance, internal absorptance and internal transmittance to each layer. In particular, layer  $i$  has reflectance  $\rho(i)$  assigned a priori (e.g.  $A$  at the TOA,  $A_g$  at ground level), internal transmittance  $\tau_i(i)$  and internal absorptance  $\alpha_i(i) = 1 - \tau_i(i)$ . The internal transmittance is obtained by integrating equation 19 in the shortwave spectrum,

$$\tau_i(s, s_0) = \int_{\nu_{\div}}^{\nu_{\max}} \tau_i(\nu, s, s_0) d\nu \quad (24)$$

and assigning  $\tau_i(i) = \tau_i(z, z')$  with  $z$  and  $z'$  altitudes corresponding to layers  $i$  and  $i - 1$ , respectively.

Multiple reflections can occur but only one is considered in this model because successive reflections from atmospheric layers have negligible intensities compared to the first one. With these simplifications, shortwave irradiance at layer  $i$  is given by

$$E_S(i) = \alpha_S(i) \frac{S_0}{4} \quad (25)$$

where absorptance  $\alpha_S(i)$  measures the total absorption of shortwave radiation by layer  $i$ . A derivation for  $\alpha_S(i)$  is found in section B.3.

Specific intensity of solar radiation can be expressed by a differential equation whose resolution is complex even applying approximations and numerical methods

### 3.5 Numerical resolution

Euler's method is used to solve the Ordinary Differential Equation (ODE) for the time-dependant temperature function. A solution for each atmospheric layer is calculated, hence the resulting values are triplets of temperature, altitude (i.e. proxy for the atmospheric layer) and time (i.e. simulation time). Further information on storage and plotting of data are presented in Section C.1.

Spectral bands are identified by two arrays: one listing the lower bound of each band, the other containing the width of each band. This choice simplify the use of functions for numerical integration. Values related to spectral bands are stored as integers.

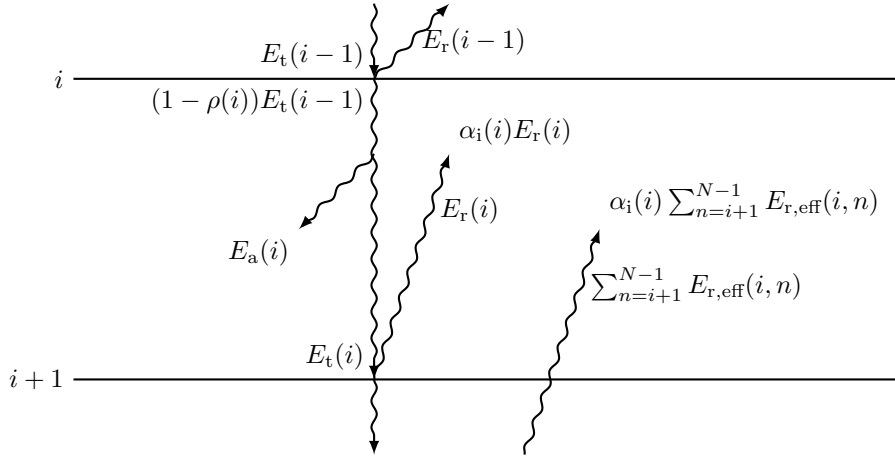


Figure 1:

Vertical spatial dimension is discretised in  $N$  layers, numbered downward starting from 0 at the TOA layer.

Equation (18) is solved in the time domain through Euler method.

### 3.6 Convective adjustment

## 4 Results

### 4.1 Stability analysis

## 5 Discussion

### 5.1 Division of the electromagnetic spectrum

Spectral irradiances of Sun's and Earth's surfaces are obtained from blackbody spectral radiance (cfr. section B.2). They are compared inside Earth's atmosphere, whose thickness is neglected with respect to the other spatial quantities involved in these calculations, hence the comparison is performed directly on Earth's surface. Nevertheless, the reflexion of solar radiation by atmospheric components is still considered and it is parametrised through Earth's Bond albedo. Moreover, attenuation of solar radiation due to distance is considered: given a generic coordinate system, spectral radiant intensity of Sun is obtained by multiplying the spectral radiance by the area of a spherical surface with radius  $R_{\text{sun}}$ , then the result is divided by the area

of a spherical surface with radius 1 au to measure solar radiance at Earth's surface.

The resulting solar spectral irradiance is

$$E_{\text{sun}}(\nu) = (1 - A) \left( \frac{R_{\text{sun}}}{1 \text{ au}} \right)^2 \pi B_{\nu}(\nu, T_{\text{sun}}) \quad , \quad (26)$$

while Earth's surface spectral irradiance is

$$E_{\text{earth}}(\nu) = \pi B_{\nu}(\nu, T_{\text{earth}}) \quad . \quad (27)$$

The boundaries of the EM spectrum are chosen arbitrarily in such a way that both equation (26) and equation (27) integrated in the resulting interval are over 99 % of the respective radiant exitances (cf. section B.2) given by the Stefan-Boltzmann law

$$M(T) = \sigma T^4 \quad (28)$$

where  $\sigma$  is the Stefan-Boltzmann constant and  $T$  is the temperature of the emitting body. The chosen values are shown in table 1.

Figure 2 shows the plots of equation (26), equation (27) and the division of the spectrum at the wavenumber  $\nu_{\text{div}}$  where the curves intersect. Value  $\nu_{\text{div}} \approx 2154/\text{cm}$  is obtained numerically (cfr. section A.1).

Considering only interval  $[\nu_{\text{min}}, \nu_{\text{div}}]$ , the largest contribution to the total irradiance is brought by Earth's surface and solar irradiance is neglected, being about 1.6 % of the total irradiance. The removed irradiance is 0.65 % of the total Sun's surface irradiance. Instead, in  $[\nu_{\text{div}}, \nu_{\text{max}}]$  Earth's surface

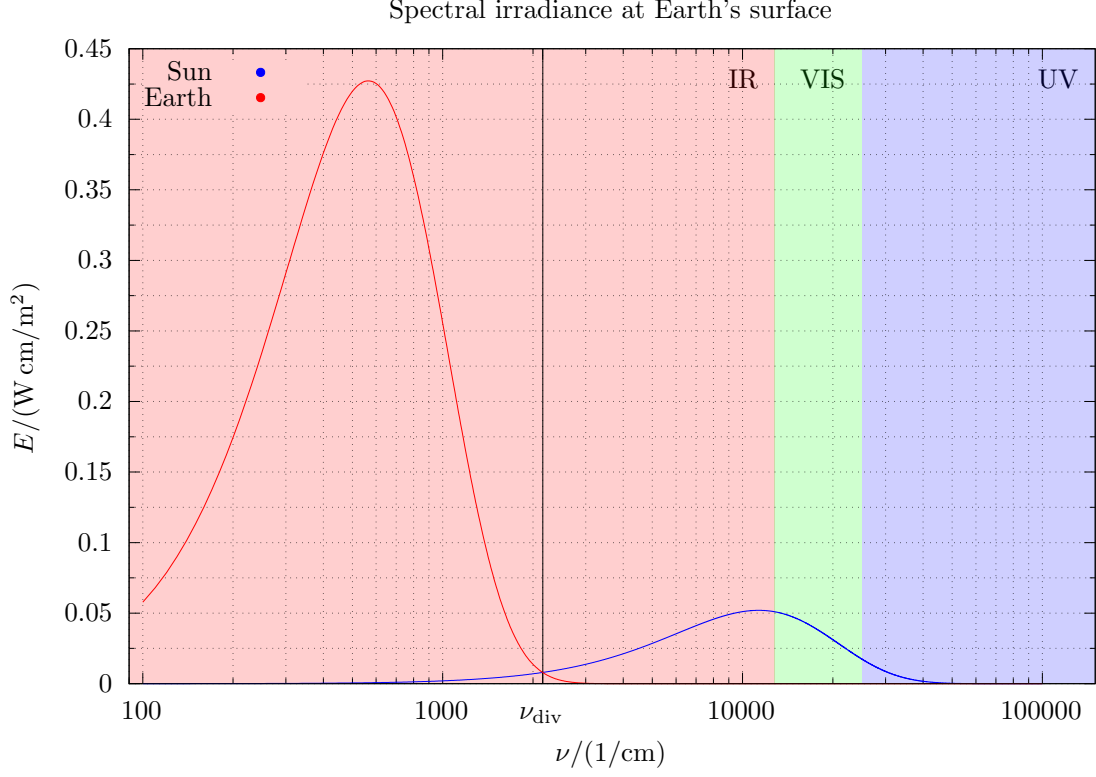


Figure 2: The vertical black line marks the division of the spectrum at  $\nu_{\text{div}}$ . In the longwave region  $[\nu_{\text{min}}, \nu_{\text{div}}]$  Earth’s surface brings the largest contribution to the total irradiance, in the shortwave region  $[\nu_{\text{div}}, \nu_{\text{max}}]$  solar radiation dominates instead. Infrared (IR), visible (VIS) and ultraviolet (UV) spectral bands as specified by [7] are shown in background.

irradiance is 0.22 % of the total irradiance in this interval and is neglected. The removed quantity is 0.55 % of the total Earth’s surface irradiance.

## A Source code

In this section the C++ code used to obtain the results presented in this work is shown and commented.

First, the parametrization of the vertical coordinate is chosen among three alternatives: altitude  $z$  in m, atmospheric pressure  $P$  in Pa, coordinate  $\sigma = \frac{P - P_{\text{TOA}}}{P_S - P_{\text{TOA}}}$  adimensional, with  $P_{\text{TOA}}$  pressure at the top of the modelised atmosphere and  $P_S$  pressure at the surface. One parametrisation can be written in term of another through a monotonic function (e.g. pressure decreases with altitude, cfr. section B.1). For the initial development the altitude  $z$  is chosen as vertical coordinate because it is more intuitive, moreover plots in TTAPS-I are expressed in terms of both  $z$  and  $P$ .

Second, the atmospheric layers are configured. In

TTAPS-I 20 layers are used (cfr. [8, p. 396]) and they are numbered from the top of the atmosphere down as it is common in RCMs. The vertical coordinate refers to the center of each layer, with the exception of the last layer which is in direct contact with the surface and needs to be treated separately. Therefore two arrays are needed: one for the point and one with the corresponding layer thicknesses. Values are then assigned as double precision numbers. The value corresponding to the top of the atmosphere is set in a proper variable and a uniform distribution of layer thicknesses is assumed for ease.

To reduce the computation load of radiative fluxes, only some spectral intervals are considered and they are specific to each atmospheric layer, since the available absorbing gases and aerosols differ among layers. First absorbers for each layer are stored in an array, then the spectral bandwidths needed for each layer are evaluated. Spectral bandwidths are expressed in terms of wavenumbers with unit 1/cm to manage integer values or double precision values close to unity. For each chemical species the absorption intervals are identified by their width and central wavenumber, the former are obtained from the latter and using the exponential wide band model (cfr. [9, p. 360]). These two values are stored in separate arrays.

## A.1 Hypotheses

## A.2 Classes

# B Mathematical derivations

In this appendix mathematical derivations of some ancillary results and formulae used in the main text are explicitly shown.

## B.1 Relation between pressure and altitude

A general result regarding planetary atmosphere is that atmospheric pressure decreases with increasing altitude. Theoretical relations which approximate this behaviour can be obtained. Hypotheses considered in Section 3.1 are valid.

If density is assumed constant, equation (15) can be solved easily resulting in a linear dependence of

pressure  $P$  on altitude  $z$ ,

$$P(z) = P_0 - \rho g(z - z_0) \quad , \quad (29)$$

where  $(z_0, P_0)$  is a reference point inside the atmosphere.

If density is not constant its expression is given by the ideal gas law (cfr. equation (16)) and, assuming constant temperature  $T$ , equation (15) becomes

$$\begin{aligned} dP &= -\frac{Pg}{R_m T} dz \iff \\ \iff \frac{dP}{P} &= -\frac{g}{R_m T} dz \end{aligned} \quad (30)$$

with solution

$$\begin{aligned} \ln(P') \Big|_{P_0}^{P(z)} &= -\frac{g}{R_m T} z' \Big|_{z_0}^z \iff \\ \iff P(z) &= P_0 \exp \left( -\frac{g}{R_m T} (z - z_0) \right) \quad . \end{aligned} \quad (31)$$

This relation is not meaningful, since the aim of the work is to derive the non-constant temperature profile of the atmosphere. However, it can be used inside atmospheric layers where the temperature is considered constant (e.g. stratosphere).

A better approximation assumes non-constant density and constant lapse rate  $\Gamma$ , hence temperature depends linearly on altitude,

$$\Gamma = -\frac{dT}{dz} \iff T(z) = T_0 - \Gamma(z - z_0) \quad , \quad (32)$$

with  $T_0$  temperature corresponding to reference altitude  $z_0$ . Using these assumptions and the density rewritten through the ideal gas law (16), equation (15) becomes

$$\begin{aligned} dP &= -\frac{Pg}{R_m T} \left( -\frac{dT}{\Gamma} \right) \iff \\ \iff \frac{dP}{P} &= \frac{g}{R_m \Gamma} \frac{dT}{T} \quad , \end{aligned} \quad (33)$$

which has solution

$$\begin{aligned} \ln(P') \Big|_{P_0}^{P(z)} &= \frac{g}{R_m \Gamma} \ln(T') \Big|_{T_0}^{T(z)} \iff \\ \iff P(z) &= P_0 \left( \frac{T_0 - \Gamma(z - z_0)}{T_0} \right)^{\frac{g}{R_m \Gamma}} \quad . \end{aligned} \quad (34)$$



Equation (34) can be used also with a piecewise constant lapse rate in altitude intervals where it is not null. Otherwise, in altitude intervals where lapse rate is null, equation (31) is valid with appropriate boundary conditions to ensure continuity between layers.

## B.2 Radiometric quantities

Refer to [7] for more details on quantities reviewed in this section.

Consider electromagnetic radiation emitted by a point source. The total emitted power is called *radiant flux*, with unit W. The density of radiant flux with respect to a solid angle in the direction of emission is called *radiant intensity*, expressed in W/sr. When radiation interacts with a surface, i.e. it gets absorbed, transmitted or reflected, its radiant intensity distributed over the surface is measured through *radiance* in W/(m<sup>2</sup> sr). If the area on which the radiation is incident is expressed through the solid angle it subtends, the integral of radiance over this solid angle is called *irradiance*, expressed in W/m<sup>2</sup>. Note that the coordinate system where the solid angles of radiant intensity and irradiance are defined may not be the same. Radiant flux emitted by a body normalised over the surface of emission is measured by *radiant exitance* in W/m<sup>2</sup>.

All previous quantities can be expressed as densities with respect to the wavelength or the wavenumber and the adjective *spectral* is prefixed to their names. Their units are divided by the respective spectral quantity (e.g. spectral radiance with wavenumber in 1/cm has units W cm/(m<sup>2</sup> sr)).

Spectral radiance of a blackbody is given by Planck's law

$$B_\nu(\nu, T) = 2hc^2\nu^3 \frac{1}{e^{\frac{hc\nu}{k_B T}} - 1} , \quad (35)$$

where  $\nu$  is the wavenumber in unit 1/cm (cfr. notation in section 3.1),  $T$  in unit K is the temperature of the emitting body and the other quantities are constants (cfr. table 1). Note that Planck's law has different form when it is expressed in terms of wavelength, due to its definition as density and the resulting change of variables:

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} . \quad (36)$$

If radiance is isotropic, i.e. it is not dependent on the direction of the radiation, the corresponding irradiance is proportional. For instance, if the radiation is absorbed by a hemispheric surface approximated by a blackbody, the spectral irradiance of the surface is

$$\begin{aligned} & \int B_\nu(\nu, T) d\phi \sin(\theta) d\theta \cos(\theta) = \\ & = B_\nu(\nu, T) \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin(\theta) \cos(\theta) d\theta = \\ & = 2\pi B_\nu(\nu, T) \int_0^1 \sin(\theta) d(\sin(\theta)) = \\ & = 2\pi B_\nu(\nu, T) \frac{1}{2} = \\ & = \pi B_\nu(\nu, T) , \end{aligned} \quad (37)$$

where spherical coordinates are used to describe the surface and the term  $\cos(\theta)$  considers the component of radiation along the normal of the infinitesimal solid angle.

Equation (23) is obtained by integrating the RTE under the hypotheses in section 3.1:

$$\begin{aligned} E_L(\nu, z) = & \pi B_\nu(\nu, T(t, 0)) \tau_1(\nu, z, 0) + \\ & + \int_{z_g}^z \pi B_\nu(\nu, T(t, z')) \frac{d\tau_1(\nu, z, z')}{dz'} dz' \\ & + \int_z^{z_{TOA}} \pi B_\nu(\nu, T(t, z')) \frac{d\tau_1(\nu, z', z)}{dz'} dz' \end{aligned} \quad (38)$$

## B.3 Radiation attenuation

Radiation crossing a medium loses energy due to absorption and scattering. This attenuation is measured by the *transmittance* of the medium as the spectral radiant fluxes ratio of transmitted to incident radiation.

Experimentally it is observed that the ratio between follows the Beer-Lambert-Bouguer law

$$E(s) = E(0) \int_0^s \dots \quad (39)$$

between radiation and of radiant flux along the direction of radiation is given by the *attenuation coefficient* (commonly called *extinction coefficient* in atmospheric sciences), which has different definitions based on the way it is derived (cfr. [6,

p. 44]). The attenuation coefficient is the sum of *absorption coefficient* and *scattering coefficient* which contain information on the attenuation due to the respective physical processes. In general these coefficients are functions of wavelength or wavenumber, in which case the prefix *spectral* is adopted. If the medium is a fluid, they depends on temperature and pressure of the medium.

Due to conservation of energy, the sum of *internal transmittance* and *internal absorptance* is 1, as well as the sum of *reflectance*, *absorptance* and *transmittance*.

Names of radiative properties ending with suffix *-ance* are generally used for rough surfaces, while suffix *-ivity* indicates smooth surfaces. In this work the former is adopted. Refer to [9, p. 20] for more information and to the definition of spectral absorptivity in [7] for an example of the difference.

To derive the reflection term in equation 25, it is useful to consider a generic index  $n$  with  $i \leq n \leq N-1$  and to write the irradiances in terms of the total solar irradiance as discussed in section 3.4. The irradiance transmitted by layer  $n$  is:

$$\begin{aligned} E_t(n) &= \\ &= \tau_i(n)(1 - \rho(n))E_t(n-1) = \\ &= \frac{S_0}{4} \prod_{j=0}^n \tau_i(j)(1 - \rho(j)) \quad . \end{aligned} \quad (40)$$

Multiple reflections are neglected, hence the irradiance reflected by layer  $n+1$ ,

$$\begin{aligned} E_r(n) &= \\ &= \rho(n+1)E_t(n) = \\ &= \rho(n+1)\frac{S_0}{4} \prod_{j=0}^n \tau_i(j)(1 - \rho(j)) \quad , \end{aligned} \quad (41)$$

is reduced only by the transmittances of the crossed layers. The effective irradiance reflected by layer  $n+1$  and arriving to layer  $i$  is

$$\begin{aligned} E_{r,\text{eff}}(i, n) &= \\ &= \tau_i(i+1) \cdots \tau_i(n)E_r(n) = \\ &= \rho(n+1)E_t(n) \prod_{k=i+1}^n \tau_i(k) = \\ &= \rho(n+1)\frac{S_0}{4} \prod_{j=0}^n \tau_i(j)(1 - \rho(j)) \prod_{k=i+1}^n \tau_i(k) \quad . \end{aligned} \quad (42)$$

Irradiance absorbed directly by layer  $n$  is

$$\begin{aligned} E_a(n) &= \alpha_i(n)(1 - \rho(n))E_t(n-1) = \\ &= \alpha_i(n)(1 - \rho(n))\frac{S_0}{4} \prod_{j=0}^{n-1} \tau_i(j)(1 - \rho(j)) \quad . \end{aligned} \quad (43)$$

In conclusion, the total irradiance absorbed by layer  $i$  due to shortwave radiation is given by:

$$\begin{aligned} E_S(i) &= \\ &= E_a(i) + \alpha_i(i) \sum_{n=i}^{N-1} E_{r,\text{eff}}(i, n) = \\ &= \alpha_i(i)(1 - \rho(i))\frac{S_0}{4} \prod_{l=0}^{i-1} \tau_i(l)(1 - \rho(l)) + \\ &+ \alpha_i(i) \sum_{n=i}^{N-1} \left( \rho(n+1)\frac{S_0}{4} \prod_{j=0}^n \tau_i(j)(1 - \rho(j)) \right. \\ &\quad \left. \prod_{k=i+1}^n \tau_i(k) \right) = \\ &= \alpha_i(i)\frac{S_0}{4} \prod_{j=0}^i \tau_i(j)(1 - \rho(j)) \left( \frac{1}{\tau_i(i)} + \right. \\ &+ \sum_{n=i}^{N-1} \left( \rho(n+1) \prod_{k=i+1}^n \tau_i(k)^2(1 - \rho(k)) \right) \Bigg) \\ &= (1 - \tau_i(i))\frac{S_0}{4} \prod_{j=0}^i \tau_i(j)(1 - \rho(j)) \left( \frac{1}{\tau_i(i)} + \right. \\ &+ \sum_{n=i}^{N-1} \left( \rho(n+1) \prod_{k=i+1}^n \tau_i(k)^2(1 - \rho(k)) \right) \Bigg) \end{aligned} \quad (44)$$

and the total absorptance of layer  $i$  due to short-wave radiation can be defined as:

$$\begin{aligned} \alpha_S(i) &= \\ &= \frac{E_S(i)}{\frac{S_0}{4}} = \\ &= (1 - \tau_i(i)) \prod_{j=0}^i \tau_i(j)(1 - \rho(j)) \left( \frac{1}{\tau_i(i)} + \right. \\ &+ \sum_{n=i}^{N-1} \left( \rho(n+1) \prod_{k=i+1}^n \tau_i(k)^2(1 - \rho(k)) \right) \Bigg) \quad . \end{aligned} \quad (45)$$

## B.4 Quantities commonly used in atmospheric sciences

Earth’s surface horizontal profile is not uniform, hence altitude and pressure near ground level could present sudden variations. In models where this is taken into consideration, the sigma coordinate system is commonly used instead, defined by

$$\sigma = \frac{P - P_{\text{TOA}}}{P_g - P_{\text{TOA}}} . \quad (46)$$

To avoid confusion, in this work symbol  $\sigma$  is used for the Stefan-Boltzmann constant (cfr. table 1), except for equation (46).

An alternative quantity evaluated in place of  $T(t, z)$  for a given parcel of fluid is the potential temperature

$$\theta(t, z) = T(t, z) \left( \frac{P_0}{P(z)} \right)^{\frac{R_m}{c_P}} , \quad (47)$$

where  $P_0$  is a reference pressure and quantities  $P(z)$ ,  $R_m$  and  $c_P$  refer to the fluid.

## C Supplementary information

### C.1 Plotting

Software gnuplot is used to generate plots shown in this work. Output values from the simulation are stored in a DAT file with the structure reproduced in listing 1.

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Listing 1: Structure of DAT file containing output values. The line spacing between data blocks is mandatory and ellipsis ... indicates a logical continuation of the sequence. Number  $N_t$  of temporal steps needed to reach convergence is not known a priori. Values for the ground level are shown as elements of the sequence with index  $N$ . Quantities  $P$ ,  $\sigma$  and  $\theta$  are added for additional plots or for comparison with other models. To maintain file portability, sigma coordinate and potential temperature are presented with their names  $\sigma$  and  $\theta$  and not by their symbols from equation (46) and equation (47), respectively.

```

t[0]      z[0]      T[0][0]      P[0]      sigma[0]      theta[0][0]
...      ...      ...      ...      ...      ...
t[0]      z[N - 1] T[0][N - 1] P[N - 1] sigma[N - 1] theta[0][N - 1]
t[0]      z[N]     T[0][N]     P[N]      sigma[N]      theta[0][N]

t[1]      z[0]      T[1][0]      P[0]      sigma[0]      theta[1][0]
...      ...      ...      ...      ...      ...
t[1]      z[N - 1] T[1][N - 1] P[N - 1] sigma[N - 1] theta[1][N - 1]
t[1]      z[N]     T[1][N]     P[N]      sigma[N]      theta[1][N]

...

t[N_t - 1] z[0]      T[N_t - 1][0] P[0]      sigma[0]      theta[N_t - 1][0]
...      ...      ...      ...      ...      ...
t[N_t - 1] z[N - 1] T[N_t - 1][N - 1] P[N - 1] sigma[N - 1] theta[N_t - 1][N - 1]
t[N_t - 1] z[N]     T[N_t - 1][N] P[N]      sigma[N]      theta[N_t - 1][N]

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