



# Introduction

2023-10-03

## Radiative-convective equilibrium in a grey atmosphere

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1. The analysed quantity is the atmospheric temperature profile averaged over all latitudes and longitudes.

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- Radiative Transfer Equation (RTE).

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2. RTE describes radiative processes.

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- Radiative Transfer Equation (RTE).
- Fluid dynamics equations.

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2. RTE describes radiative processes.
3. Fluid dynamics equations describe convective processes.

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- Radiative Transfer Equation (RTE).
- Fluid dynamics equations.

## Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho c_P} \frac{\partial q}{\partial z} \quad . \quad (1)$$

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- Radiative-convective equilibrium.

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## └ Hypotheses

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- Radiative-convective equilibrium.
- Grey atmosphere.

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## Radiative-convective equilibrium in a grey atmosphere

## └ Introduction

## └ Hypotheses

1. Thermodynamic energy equation describes average vertical temperature profile.
2. The study is conducted on an atmosphere in radiative-convective equilibrium.
3. Quantities do not depend on the frequency of electromagnetic radiation.

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho c_P} \frac{\partial q}{\partial z} \quad (1)$$

- Radiative-convective equilibrium.
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# Additional hypotheses

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## Radiative-convective equilibrium in a grey atmosphere

### └ Introduction

### └ Additional hypotheses

- Hypotheses on the planet.

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1. Diurnal cycle, constant irradiance, constant Bond albedo, surface emits blackbody radiation, constant gravitational acceleration.



# Additional hypotheses

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## Radiative-convective equilibrium in a grey atmosphere

### └ Introduction

### └ Additional hypotheses

- Hypotheses on the planet.
- Hypotheses on the composition of atmosphere.

1. Diurnal cycle, constant irradiance, constant Bond albedo, surface emits blackbody radiation, constant gravitational acceleration.
2. Hydrostatic equilibrium, constant specific heat at constant pressure, scattering is neglected, absorption coefficient depends only on altitude, constant mass attenuation coefficient, ideal gas.

- Hypothesis on the planet.
- Hypothesis on the composition of atmosphere.

# Additional hypotheses

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## Radiative-convective equilibrium in a grey atmosphere

### └ Introduction

### └ Additional hypotheses

- Hypotheses on the planet.
- Hypotheses on the composition of atmosphere.
- Hypotheses on total heat flux.
- Resulting thermodynamic energy equation:

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho c_p} \frac{\partial}{\partial z} (E_U - E_D) \quad . \quad (2)$$

1. Diurnal cycle, constant irradiance, constant Bond albedo, surface emits blackbody radiation, constant gravitational acceleration.
2. Hydrostatic equilibrium, constant specific heat at constant pressure, scattering is neglected, absorption coefficient depends only on altitude, constant mass attenuation coefficient, ideal gas.
3. Heat flux determined only by radiative and convective processes, two-stream approximation, numerical correction for convection.

- Hypothesis on the planet.
- Hypothesis on the composition of atmosphere.
- Hypothesis on total heat flux.
- Resulting thermodynamic energy equation:

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## Vertical coordinates

- Relation between pressure and altitude:

$$P(z) = P_g \exp \left( - \frac{z - z_g}{z_0} \right) . \quad (3)$$

- Relation between optical depth and pressure:

$$\delta(P) = \frac{\mu_m}{g} (P - P_{\text{TOA}}) . \quad (4)$$

- Relation between optical depth and altitude:

$$\delta(z) = \frac{\mu_m}{g} \left( P_g \exp \left( - \frac{z - z_g}{z_0} \right) - P_{\text{TOA}} \right) . \quad (5)$$

## Radiative-convective equilibrium in a grey atmosphere

## └ Introduction

## └ Vertical coordinates

1. Obtained from hydrostatic equilibrium and ideal gas law.
2. Obtained from definition of optical depth, hydrostatic equilibrium and hypotheses on attenuation coefficient.
3. Obtained by combining relations (3) and (4).

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# Equations in radiative equilibrium

- RTE for non-scattering medium in LTE:

$$\frac{1}{\mu} \frac{\partial L}{\partial z} = B_\nu - L \quad . \quad (6)$$

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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative equilibrium

└ Equations in radiative equilibrium

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- Diffusion approximation:  $\delta' = D\delta$ .

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## Radiative-convective equilibrium in a grey atmosphere

### └ Radiative equilibrium

### └ Equations in radiative equilibrium

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- To describe irradiances, RTE is integrated over the whole spectrum of radiation and over the solid angle of a hemisphere.

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# Equations in radiative equilibrium

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- Integration over frequency and solid angle.
- Diffusion approximation:  $\delta' = D\delta$ .
- Equations for irradiances:

$$-\frac{\partial}{\partial \delta'} E_U(t, \delta') = \sigma T(t, \delta')^4 - E_U(t, \delta') \quad , \quad (7)$$

$$\frac{\partial}{\partial \delta'} E_D(t, \delta') = \sigma T(t, \delta')^4 - E_D(t, \delta') \quad . \quad (8)$$

## Radiative-convective equilibrium in a grey atmosphere

### └ Radiative equilibrium

### └ Equations in radiative equilibrium

1. RTE describes radiance in a medium.
2. To describe irradiances, RTE is integrated over the whole spectrum of radiation and over the solid angle of a hemisphere.
3. Equations for irradiances and temperature are written in terms of  $\delta'$ .

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- Steady state.

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Radiative-convective equilibrium in a grey atmosphere

- Radiative equilibrium

└ Initial Value Problem

1. Analytical solutions are found for temperature at the steady state.





## Initial Value Problem

- Steady state.
- Initial conditions on irradiances:

$$E_D(0) = 0 \quad , \quad (9)$$

$$E_U(0) = S_t \quad . \quad (10)$$

- Additional relations:

$$\frac{d}{d\delta'} (E_U(\delta') - E_D(\delta')) = 0 \quad , \quad (11)$$

$$E_U(\delta') - E_D(\delta') = S_t \quad . \quad (12)$$

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## Radiative-convective equilibrium in a grey atmosphere

## └ Radiative equilibrium

## └ Initial Value Problem

- Analytical solutions are found for temperature at the steady state.
- At TOA downward irradiance does not deposit energy and radiative equilibrium fixes upward irradiance, i.e. outgoing longwave radiation.
- Hypotheses of atmosphere in radiative equilibrium at all altitudes and atmosphere transparent to radiation coming from outside the planet are equivalent to hypotheses of atmosphere in radiative equilibrium at TOA and steady state.

• Steady state.

• Initial conditions on irradiances:

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- Initial condition on temperature:

$$T(0) = \left( \frac{S_t}{2\sigma} \right)^{\frac{1}{4}} \quad . \quad (13)$$

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## Radiative-convective equilibrium in a grey atmosphere

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- Analytical solutions are found for temperature at the steady state.
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- Hypotheses of atmosphere in radiative equilibrium at all altitudes and atmosphere transparent to radiation coming from outside the planet are equivalent to hypotheses of atmosphere in radiative equilibrium at TOA and steady state.
- Initial condition on temperature is obtained by combining all conditions on irradiances.

Initial Value Problem

- Steady state.
- Initial conditions on irradiances:
 
$$E_D(0) = 0 \quad , \quad (9)$$

$$E_U(0) = S_t \quad . \quad (10)$$
- Additional relations:
 
$$\frac{d}{d\delta'} (E_U(\delta') - E_D(\delta')) = 0 \quad , \quad (11)$$

$$E_U(\delta') - E_D(\delta') = S_t \quad . \quad (12)$$
- Initial condition on temperature:
 
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- Temperature:

$$T(\delta) = \left( \frac{S_t}{2\sigma} (1 + D\delta) \right)^{\frac{1}{4}}. \quad (14)$$

- Irradiances:

$$E_U(\delta) = \frac{S_t}{2}(2 + D\delta) \quad , \quad (15)$$

$$E_D(\delta) = \frac{S_t}{2} D\delta \quad . \quad (16)$$

## Numerical solution

- Normalisation:

$$Y_0 = \frac{T^4}{T_0^4} \quad , \quad Y_1 = \frac{E_U}{S_t} \quad , \quad Y_2 = \frac{E_U}{S_t} \quad (17)$$

- Resulting system of ODEs:

$$\begin{cases} \frac{dY_0}{d\delta} = \frac{D}{2} \\ \frac{dY_1}{d\delta} = D(Y_1 - Y_0) \\ \frac{dY_2}{d\delta} = D(Y_0 - Y_2) \end{cases} \quad (18)$$

- Runge-Kutta method of order 4.
- Non-uniform step size.

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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative equilibrium

└ Numerical solution

- Normalisation:

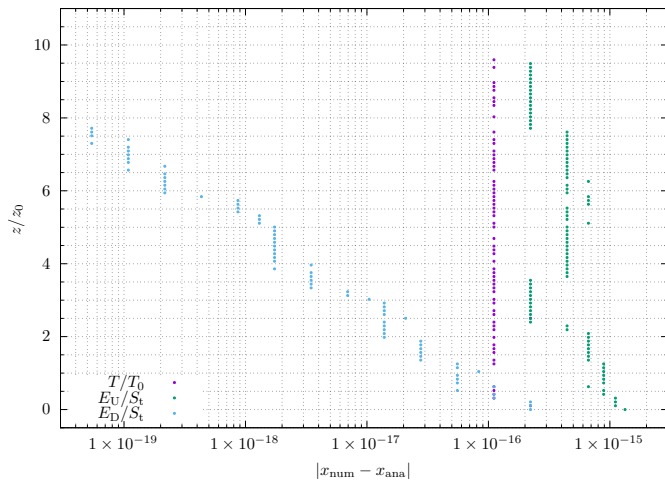
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# Errors of numerical solutions

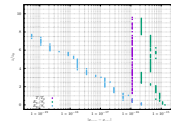


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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative equilibrium

└ Errors of numerical solutions



- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium, assuming steady state. Points at some altitudes are not shown because their value is exactly 0.
- Errors are compatible with 0 based on precision of double-precision floating-point numbers.



- Iterative procedure.

## Time integration

- Iterative procedure.
- Additional normalisation:

$$Y_3 = \frac{T}{T_0} \quad . \quad (19)$$

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- Resulting system of PDEs:

$$\begin{cases} \frac{\partial}{\partial t} Y_3(t, \delta) = \frac{\mu_m S_t}{c_P T_0} \frac{\partial}{\partial \delta} (Y_1(t, \delta) - Y_2(t, \delta)) \\ \frac{\partial}{\partial \delta} Y_1(t, \delta) = D(Y_1(t, \delta) - Y_3(t, \delta)^4) \\ \frac{\partial}{\partial \delta} Y_2(t, \delta) = D(Y_3(t, \delta)^4 - Y_2(t, \delta)) \end{cases} \quad . \quad (20)$$

- Euler method for temporal integration.
- Constant time step.

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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative equilibrium

└ Time integration

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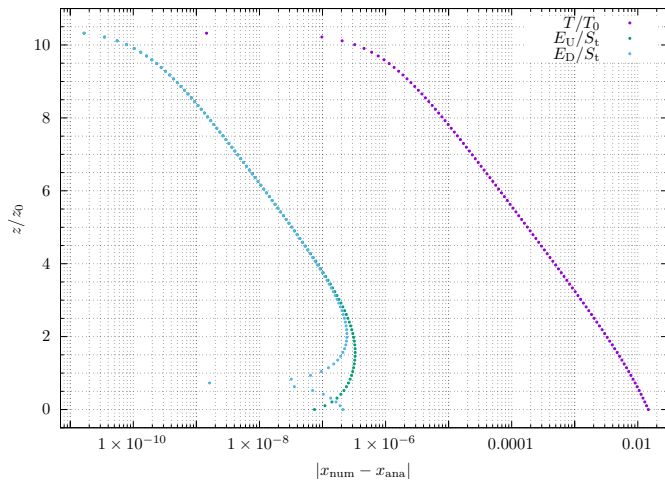
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- Resulting system of PDEs:

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- Euler method for temporal integration.
- Constant time step.

# Errors of time integration at steady state

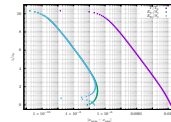


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## Radiative-convective equilibrium in a grey atmosphere

### └ Radiative equilibrium

### └ Errors of time integration at steady state



- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium. Steady state is reached through iterative temporal and spatial integrations. Precision is reduced by propagation of errors during successive iterations. Points at TOA are omitted being compatible with 0 within precision of double-precision floating-point numbers.
- Time step is chosen arbitrarily to reduce the errors of irradiances below the precision of numeric values outputs.

## Radiative-convective equilibrium

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## Radiative-convective equilibrium in a grey atmosphere

- Radiative-convective equilibrium

- Radiative-convective equilibrium

- Lapse rate at steady state.

1. Convective equilibrium correspond to steady state of lapse rate.

- Lapse rate at steady state.

# Radiative-convective equilibrium

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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative-convective equilibrium

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- Lapse rate at steady state.
- Convection faster than radiative processes.

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1. Convective equilibrium correspond to steady state of lapse rate.
2. Different time scales between processes.

## Radiative-convective equilibrium

- Lapse rate at steady state.
- Convection faster than radiative processes.
- Convective adjustment if  $-\frac{\partial T}{\partial z} > \Gamma_0$ .

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## Radiative-convective equilibrium in a grey atmosphere

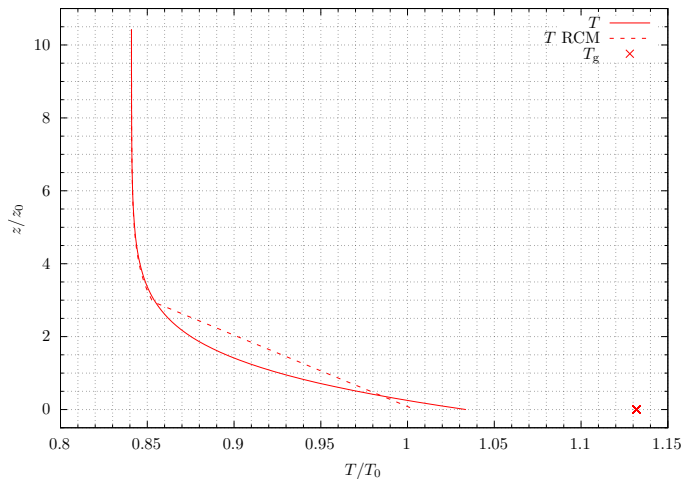
- └ Radiative-convective equilibrium

- Radiative-convective equilibrium

1. Convective equilibrium correspond to steady state of lapse rate.
2. Different time scales between processes.
3. If convective adjustment is applied, temperature is set to obtain the value of critical lapse rate.

- Lapse rate at steady state.
- Convection faster than radiative processes.
- Convective adjustment if  $-\frac{\partial T}{\partial z} > \Gamma_0$ .

# Temperature plot

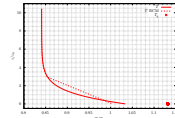


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Radiative-convective equilibrium in a grey atmosphere

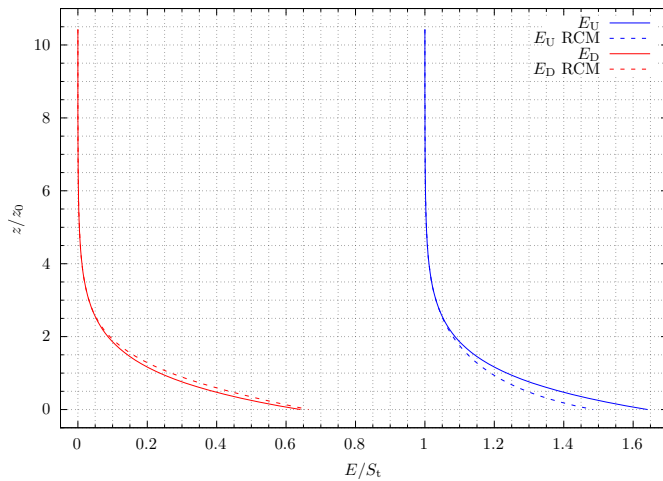
└ Radiative-convective equilibrium

└ Temperature plot



Vertical temperature profile of a grey atmosphere. Continuous line is the analytical solution in radiative equilibrium, dashed line is the numerical solution in radiative-convective equilibrium. Convective adjustment is visible at lower altitudes, but values are affected by the low precision of the numerical procedure. A discontinuity is present at ground level due to the lack of heat exchange between surface and the lowest atmospheric layer.

## Irradiances plot



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## Radiative-convective equilibrium in a grey atmosphere

└ Radiative-convective equilibrium

└ Irradiances plot

Upward and downward irradiances in a grey atmosphere. Continuous line is the solution in radiative equilibrium, dashed line is the solution in radiative-convective equilibrium. Greater values than TOA at lower altitudes are indicators of greenhouse effect. Irradiances of the RCM are not directly subject to convective adjustment, but they are affected due to the dependence on temperature.

