Radiative-convective equilibrium in a grey atmosphere

Marco Casari

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Complex systems in climate physics, 3 October 2023

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- A radiative-convective model is used to study a grey atmosphere.
- Comparison between numerical and analytical solutions is possible in radiative equilibrium.

Introduction

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2. RTE describes radiative processes.
3. Fluid dynamics equations describe convective processes.

Hypotheses

- Thermodynamic energy equation in Local Thermodynamic Equilibrium (LTE):

$$\frac{\partial T}{\partial t} = -\frac{1}{\varrho c_P}\frac{\partial q}{\partial z} \quad . \tag{1}$$

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1. Thermodynamic energy equation describes average vertical temperature profile.
2. The study is conducted on an atmosphere in radiative-convective equilibrium.
3. Quantities do not depend on the frequency of electromagnetic radiation.

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- Hypotheses on total heat flux.
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2. Hydrostatic equilibrium, constant specific heat at constant pressure, scattering is neglected, absorption coefficient depends only on altitude, constant mass attenutation coefficient, ideal gas.
3. Heat flux determined only by radiative and convective processes, two-stream approximation, numerical correction for convection.

Vertical coordinates

- Relation between pressure and altitude:

$$P(z) = P_{\mathrm{g}} \exp\left(- \frac{z - z_{\mathrm{g}}}{z_0} \right) \quad . \tag{3}$$

- Relation between optical depth and pressure:

$$\delta(P) = \frac{\mu_{\mathrm{m}}}{g}(P - P_{\mathrm{TOA}}) \quad . \tag{4}$$

- Relation between optical depth and altitude:

$$\delta(z) = \frac{\mu_{\mathrm{m}}}{g}\left(P_{\mathrm{g}} \exp\left(- \frac{z - z_{\mathrm{g}}}{z_0} \right) - P_{\mathrm{TOA}} \right) \quad . \tag{5}$$

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1. Obtained from hydrostatic equilibrium and ideal gas law.
2. Obtained from definition of optical depth, hydrostatic equilibrium and hypotheses on attenutation coefficient.
3. Obtained by combining relations (3) and (4).

Equations in radiative equilibrium

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- Equations for irradiances:

$$-\frac{\partial}{\partial \delta'} E_\mathrm{U}(t, \delta') = \sigma T(t, \delta')^4 - E_\mathrm{U}(t, \delta') \quad , \tag{7}$$

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2. To describe irradiances, RTE is integrated over the whole spectrum of radiation and over the solid angle of a hemisphere.
3. Equations for irradiances and temperature are written in terms of δ'.

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- Steady state.

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2. At TOA downward irradiance does not deposit energy and radiative equilibrium fixes upward irradiance, i.e. outgoing longwave radiation.

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2. At TOA downward irradiance does not deposit energy and radiative equilibrium fixes upward irradiance, i.e. outgoing longwave radiation.
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3. Hypotheses of atmosphere in radiative equilibrium at all altitudes and atmopshere transparent to radiation coming from outside the planet are equivalent to hypotheses of atmopshere in radiative equilibrium at TOA and steady state.
4. Initial condition on temperature is obtained by combining all conditions on irradiances.

Analytical solution

- Temperature:

$$T(\delta) = \left(\frac{S_\mathrm{t}}{2\sigma}(1 + D\delta)\right)^{\frac{1}{4}} \quad . \tag{14}$$

- Irradiances:

$$E_\mathrm{U}(\delta) = \frac{S_\mathrm{t}}{2}(2 + D\delta) \quad , \tag{15}$$

$$E_\mathrm{D}(\delta) = \frac{S_\mathrm{t}}{2}D\delta \quad . \tag{16}$$

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Numerical solution

- Normalisation:

$$Y_0 = \frac{T^4}{T_0^4} \quad , \quad Y_1 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \quad , \quad Y_2 = \frac{E_\mathrm{U}}{S_\mathrm{t}} \tag{17}$$

- Resulting system of ODEs:

$$\begin{cases} \frac{\mathrm{d}Y_0}{\mathrm{d}\delta} = \frac{D}{2} \\ \frac{\mathrm{d}Y_1}{\mathrm{d}\delta} = D(Y_1 - Y_0) \\ \frac{\mathrm{d}Y_2}{\mathrm{d}\delta} = D(Y_0 - Y_2) \end{cases} \quad . \tag{18}$$

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- Non-uniform step size.

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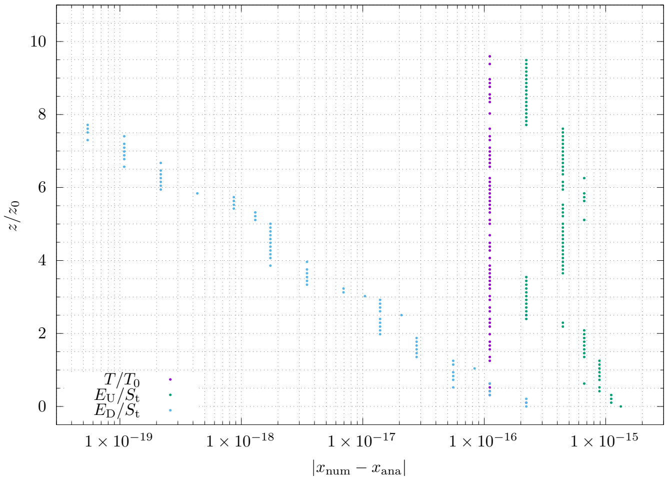
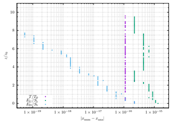
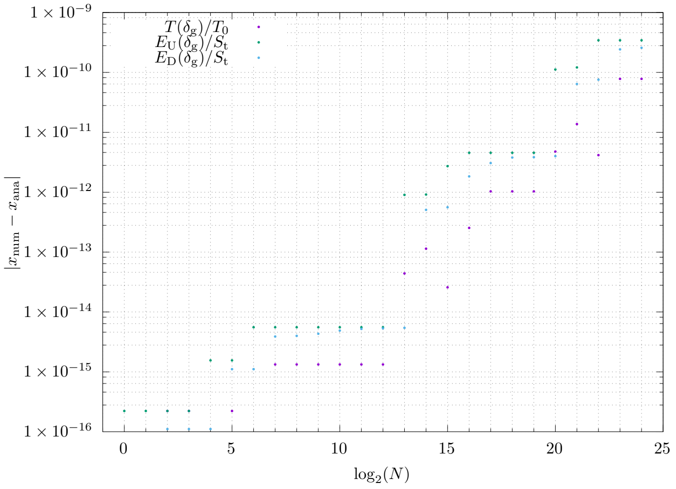
Errors of numerical solutions

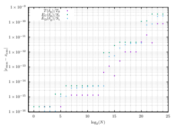




- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium, assuming steady state. Points at some altitudes are not shown because their value is exactly 0.
- Errors are compatible with 0 based on precision of double-precision floating-point numbers.

Stability analysis





- Stability of numerical solution for ODEs system in radiative equilibrium with respect to spatial grid size. Errors are negligible up to 4096 layers, then error propagation dominates reducing the precision of the method. Missing points have value 0.
- Errors are evaluated as absolute differences between numerical and analytical values at ground level of each normalised function.

Time integration

- Iterative procedure.

1. The iterations continue until a steady state for temperature is reached:
   1.1 an arbitrary temperature profile is chosen;
   1.2 equations (7) and (8) are solved with respect to δ';
   1.3 the resulting E_{U} and E_{D} are used to step forward T with respect to t for each layer;
   1.4 the obtained temperature profile is used to restart the loop from point 2.

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2. Same initial conditions for analytical solution are used, point $Y_3(0, 0)$ is not used during the integration.

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- Resulting system of PDEs:

$$\begin{cases} \frac{\partial}{\partial t} Y_3(t,\delta) = \frac{\mu_\mathrm{m} S_\mathrm{t}}{c_P T_0} \frac{\partial}{\partial \delta}(Y_1(t,\delta) - Y_2(t,\delta)) \\ \frac{\partial}{\partial \delta} Y_1(t,\delta) = D(Y_1(t,\delta) - Y_3(t,\delta)^4) \\ \frac{\partial}{\partial \delta} Y_2(t,\delta) = D(Y_3(t,\delta)^4 - Y_2(t,\delta)) \end{cases} \quad . \tag{20}$$

- Euler method for temporal integration.
- Constant time step.

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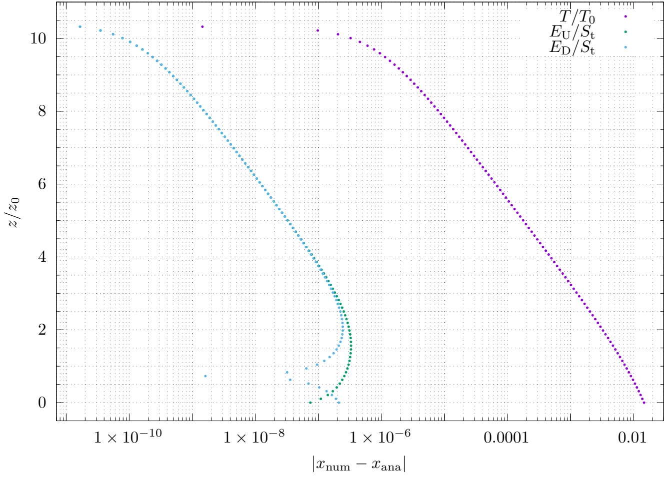
2. Same initial conditions for analytical solution are used, point $Y_3(0,0)$ is not used during the integration.
3. Integration over vertical coordinate remains unchanged.

Errors of time integration at steady state



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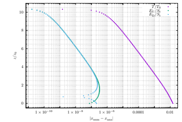

- Errors between numerical and analytical solutions of a grey atmosphere in radiative equilibrium. Steady state is reached throught iterative temporal and spatial integrations. Precision is reduced by propagation of errors during successive iterations. Points at TOA are omitted being compatible with 0 within precision of double-precision floating-point numbers.
- Time step is chosen arbitrarily to reduce the errors of irradiances below the precision of numeric values outputs.

Radiative-convective equilibrium

- Lapse rate at steady state.

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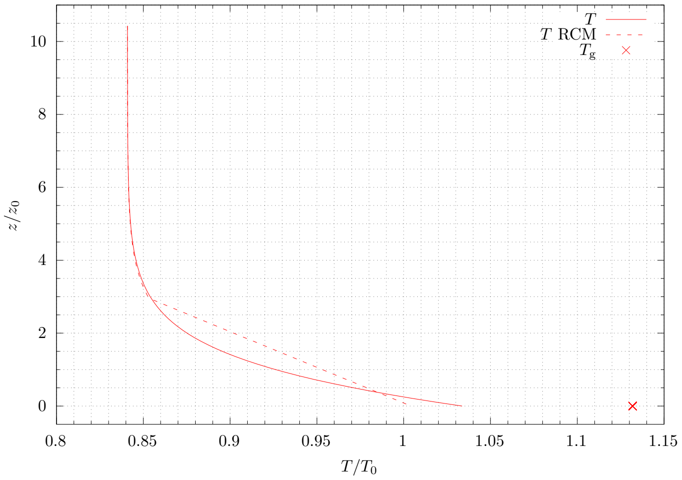
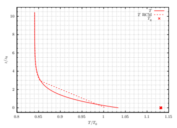
3. If convective adjustment is applied, temperature is set to obtain the value of critical lapse rate.

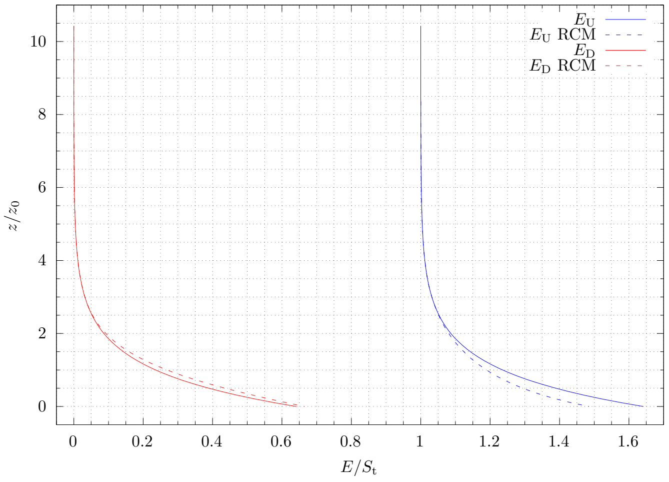
Temperature plot

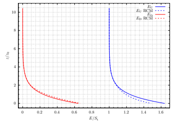




Vertical temperature profile of a grey atmosphere. Continuous line is the analytical solution in radiative equilibrium, dashed line is the numerical solution in radiative-convective equilibrium. Convective adjustment is visible at lower altitudes, but values are affected by the low precision of the numerical procedure. A discontinuity is present at ground level due to the lack of heat exchange between surface and the lowest atmospheric layer.

Irradiances plot





Upward and downward irradiances in a grey atmosphere. Continuous line is the solution in radiative equilibrium, dashed line is the solution in radiative-convective equilibrium. Greater values than TOA at lower altitudes are indicators of greenhouse effect. Irradiances of the RCM are not directly subject to convective adjustment, but they are affected due to the dependence on temperature.

Conclusion

- Vertical temperature profile of a grey atmopshere described in general by a system of PDEs and at the steady state by a system of ODEs.
- Negligible errors for ODEs numerical solutions.
- Possible improvements on description of physical phenomena.
- Possible improvements on precision of PDEs numerical solutions.

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