Summary of *A tutorial on the free-energy framework for modelling perception and learning* by Rafal Bogacz

Marco Casari

University of Turin

Complex system in neuroscience, 12 December 2023

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- In this presentation the main topics of the paper are presented, in the same order they appear in the paper.

Introduction

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2. Weight stimuli by their noise, learn features using their covariance, implement attentional modulation changing the variance of attended features.
3. Synaptic strenght is changed proportionally to activities of pre-synaptic and post-synaptic neurons.
4. Minimization of free energy can be seen as the base of many theories of perception.

Working hypotheses

- Local computation.

1. The state of a neuron is determined only by the synaptic weight and the state of its input neurons.

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3. The state of a neuron is the result of the application of a monotonic function to the linear combination of states and synaptic weights of input neurons.

Single variable model

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2. In general inferred variable and sensory input are related by some smooth function.
3. Sensory input and stimulus are drafted from the same space.
4. Information gained and constantly updated from previous experience.

Exact solution to the inference problem

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2. In general, marginal likelihood is difficult to evaluate.
3. Complex calculations and infinite nodes are needed to represent each value of the posterior.

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- Most likely value of the feature is a scalar variable $\phi \in \Omega_v$.

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1. Evaluating the mode of the posterior instead of the whole function is more biologically plausible.
2. The most likely feature value is the fixed point of the gradient descent method applied to the negative free energy.
3. Prediction errors are introduced as new variables to extend the dynamical system and satisfy Hebbian plasticity.

Neural implementation

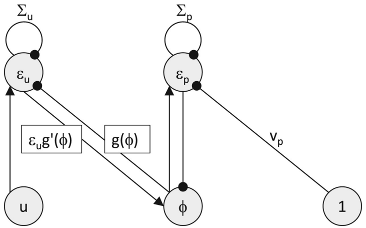
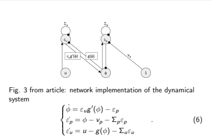


Fig. 3 from article: network implementation of the dynamical system

$$\begin{cases} \dot{\phi} = \varepsilon_u g'(\phi) - \varepsilon_p \\ \dot{\varepsilon_p} = \phi - v_p - \Sigma_p \varepsilon_p \\ \dot{\varepsilon_u} = u - g(\phi) - \Sigma_u \varepsilon_u \end{cases} . \tag{6}$$



- Note that all three hypotheses are satisfied.

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1. Model parameters are mean and variance of variables.
2. The fixed point of this dynamical system exists only as sample mean over the occured events of perception.
3. Without prediction errors, the computation is still local.

Learning relation parameter

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3. Same consideration of model parameters apply to the relation parameter.

Free energy framework

- Minimization of Kullback-Leibler divergence:

$$KL(q(v)||p(u|v)) = \int_{\Omega_v} q(v) \ln \left(\frac{q(v)}{p(v|u)} \right) \mathrm{d}v \quad . \qquad (13)$$

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- For the models discussed in the paper: $q(v) = \delta(v - \phi)$.

1. In general, the posterior is approximated by a simpler probability distribution and the divergence between the two is minimized.
2. Minimize KL divergence or maximize negative free energy to learn most likely model value, maximize marginal likelihood or maximize negative free energy to learn model parameters.
3. Equation (3) is recovered using delta function centered in the most likely feature value as probability distribution.

Multiple variables model

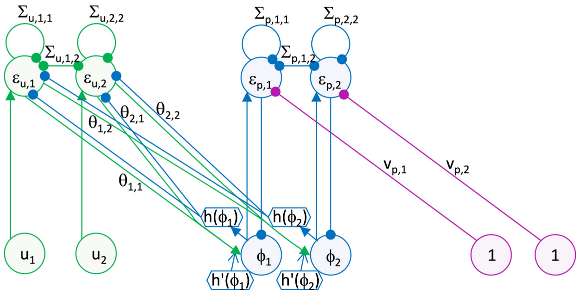

Fig. 5 from article: example of a model with 2 features and 2 stimuli. Equations are rewritten using matrix notation, but local plasticity is not satisfied.

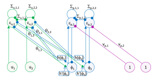

Fig. 5 from article: example of a model with 2 features and 2 stimuli. Equations are rewritten using matrix notation, but local plasticity is not satisfied.

- Calculus rules are extended to work elementwise on vectors and matrices, multivariate gaussian distribution and nonlinear relation between variables and stimuli are used.
- The inverse of covariance matrix depends on non-adjacent neurons, Hebbian plasticity is partially satisfied due to the nonlinear relation.

Hierarchical structure implementation

- Parallel to structure of cortical areas.

1. Information is used and travels in different layers of the cortex.

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- Generalized equations for the inference task:

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1. Information is used and travels in different layers of the cortex.
2. Note the elementwise product and matrices of model and relation parameters.

Recover local plasticity

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Conclusion

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