Summary of *A tutorial on the free-energy framework for modelling perception and learning* by Rafal Bogacz

Marco Casari

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Complex system in neuroscience, 12 December 2023

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- In this presentation the main topics of the paper are presented in order of appearance.
- Source code of proposed exercises is available at https://github.com/mirasac/sistneur/tree/main/code.

Introduction

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2. Weight stimuli by their noise, learn features using their covariance, implement attentional modulation changing the variance of attended features.
3. Synaptic strenght is changed proportionally to activities of pre-synaptic and post-synaptic neurons.
4. Minimization of free energy can be seen as the base of many theories of perception.

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2. Synaptic plasticity depends only on the activities of pre-synaptic and post-synaptic neurons.
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4. Information gained and constantly updated from previous experience.

Exact solution to the inference problem

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2. In general, marginal likelihood is difficult to evaluate.
3. Complex calculations and infinite nodes are needed to represent each value of the posterior.

Approximate solution to the inference problem

- Most likely value of the feature is a scalar variable $\phi \in \Omega_v$.

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1. Evaluating the mode of the posterior instead of the whole function is more biologically plausible.
2. The most likely feature value is the fixed point of the gradient descent method applied to the negative free energy.
3. Prediction errors are introduced as new variables to extend the dynamical system and satisfy Hebbian plasticity.

Neural implementation

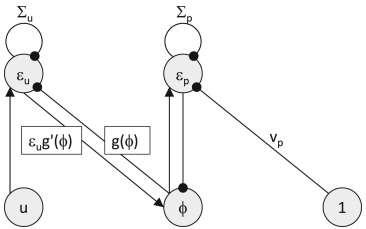
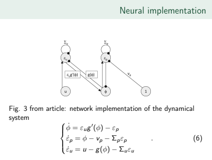


Fig. 3 from article: network implementation of the dynamical system

$$\begin{cases} \dot{\phi} = \varepsilon_u g'(\phi) - \varepsilon_p \\ \dot{\varepsilon}_p = \phi - v_p - \Sigma_p \varepsilon_p \\ \dot{\varepsilon}_u = u - g(\phi) - \Sigma_u \varepsilon_u \end{cases} . \qquad (6)$$



- Note that hypotheses and Hebbian plasticity are satisfied.

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1. Model parameters are mean and variance of variables.
2. The fixed point of this dynamical system exists only as sample mean over the occured events of perception, where most likely feature value and stimulus are known.
3. Without prediction errors, the computation is still local.

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$$g(v, \theta) = \theta v \quad . \tag{10}$$

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3. Same consideration of model parameters apply to the relation parameter.

Free energy framework

- Minimization of Kullback-Leibler divergence:

$$KL(q(v)||p(v|u)) = \int_{\Omega_v} q(v) \ln \left(\frac{q(v)}{p(v|u)} \right) \mathrm{d}v \quad . \qquad (13)$$

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2. Minimize KL divergence or maximize negative free energy to learn most likely model value, maximize marginal likelihood or maximize negative free energy to learn model parameters.
3. Equation (3) is recovered using delta function centered in the most likely feature value as probability distribution.

Multiple variables model

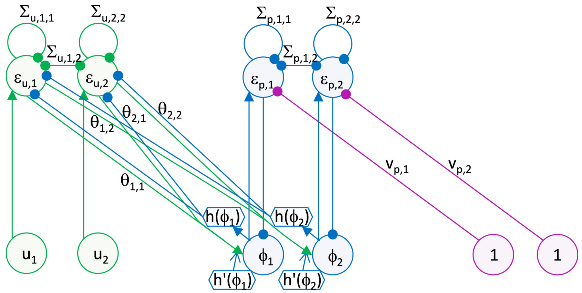

Fig. 5 from article: example of a model with 2 features and 2 stimuli. Equations are rewritten using matrix notation, but local plasticity is not satisfied.

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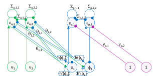

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- Calculus rules are extended to work elementwise on vectors and matrices, multivariate gaussian distribution and nonlinear relation between variables and stimuli are used.
- The inverse of covariance matrix depends on non-adjacent neurons, Hebbian plasticity is again partially satisfied.

Hierarchical structure implementation

- Parallel to structure of cortical areas.

1. Information is used and travels in different layers of the cortex.

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- Generalized equations for the inference task:

$$\dot{\vec{\phi}}_i = -\vec{\varepsilon}_i + h'(\vec{\phi}_i) \times \mathbf{\Theta}_{i-1}^{\mathsf{T}} \vec{\varepsilon}_{i-1} \quad , \tag{15}$$

$$\dot{\vec{\varepsilon}}_i = \vec{\phi}_i - \mathbf{\Theta}_i h(\vec{\phi}_{i+1}) - \mathbf{\Sigma}_i \vec{\varepsilon}_i \quad . \tag{16}$$

- Generalized equations for the learning task:

$$\frac{\partial F}{\partial \mathbf{\Sigma}_i} = \frac{1}{2}(\vec{\varepsilon}_i \vec{\varepsilon}_i^{\mathsf{T}} - \mathbf{\Sigma}_i^{-1}) \quad , \tag{17}$$

$$\frac{\partial F}{\partial \mathbf{\Theta}_i} = \vec{\varepsilon}_i h(\vec{\phi}_{i+1})^{\mathsf{T}} \quad . \tag{18}$$

Hierarchical structure implementation

- Parallel to structure of cortical areas.
- Generalized equations for the inference task:
- Generalized equations for the learning task:

1. Information is used and travels in different layers of the cortex.
2. Note the elementwise product and matrices of model and relation parameters.

Recover local plasticity

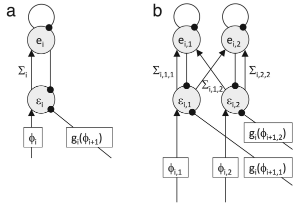

Fig. 7 from article: networks satisfying local plasticity for (a) single variable model and (b) multiple variables model. They implement the generalized dynamical system

$$\begin{cases} \dot{\vec{\varepsilon}}_i = \vec{\phi}_i - g_i(\vec{\phi}_{i+1}) - \vec{e}_i \\ \dot{\vec{e}}_i = \mathbf{\Sigma}_i \vec{\varepsilon}_i - \vec{e}_i \end{cases} . \qquad (19)$$

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- The update rule for the model parameters is Hebbian and contains the learning rate as hyperparameter of the model.
- Convergence of prediction errors to the sample variances is guaranteed if the most likely feature values change at slower time scales.

Conclusion

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- Learn covariance of stimuli.
- Attentional modulation.

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