`yacoub`: a Python package for Simulating Generalized Fading Channels

José V. de M. Cardoso, Paulo R. Lins Júnior,
Wamberto J. L. Queiroz, and Marcelo S. Alencar, *IEEE Senior Member*
Universidade Federal de Campina Grande
Instituto Federal de Educação, Ciencia, e Tecnologia da Paraíba
Campina Grande, Paraíba, Brasil
{josevinicius,paulo,wamberto,malencar}@iecom.org.br

***Abstract*—We present a well tested Python-based library for simulating and computing generalized fading channels, named `yacoub`. We describe the applicability of `yacoub` using examples in recent communications systems challenges, namely: spectrum sensing, bit error rate computation, and parameter estimation in generalized fading channels. The development of `yacoub` open source and its code is avaliable at `http://github.com/mirca/yacoub`.**

I. Introduction

A. Note on notation

Scalars and random variables are denoted as *italic* small-case letters *e.g.* x; vectors and random vectors are denoted as *italic*, boldface, small-case letters *e.g.* $\boldsymbol{x}$. The $n-$th component of a vector $\boldsymbol{x}$ is denoted as x_n. A complex vector of length n is defined as $\boldsymbol{x} \in \mathbb{C}^{n\times 1}$. All vectors are column vectors. Matrices are denoted as *italic*, boldface, capital letters as in $\boldsymbol{X}$; the identity matrix of order n is denoted as $\boldsymbol{I}_n$. We define a discrete-time circularly symmetric Gaussian process $\boldsymbol{z}$ as any collection of random varibles $\boldsymbol{z} = \boldsymbol{x} + j\boldsymbol{y}$, $j \triangleq \sqrt{-1}$, such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are iid jointly Gaussian with zero mean vector and covariance matrix given by $\mathbb{E}\left[\boldsymbol{z}\boldsymbol{z}^{\dagger}\right]$, in which $\boldsymbol{z}^{\dagger}$ means the conjugate transpose of $\boldsymbol{z}$. The expectation value wrt a random variable x is denoted as $\mathbb{E}_x$. The probability of an event A is denoted as $\mathbb{P}(A)$. The indicator function is denoted as $\mathbb{I}(\cdot)$ and it evaluates to 1 if its argument is true and 0 otherwise.

II. The Acceptance-Rejection sampler in log-space

III. Majorization-Minimization Algorithms

IV. Examples

A. Spectrum Sensing in Complex Generalized Fading Channels

The spectrum sensing problem consists in deciding whether or not a given channel frequency band is being occupied by a licensed (primary) user and, in case that such frequency band is available, how to opportuniscally allocate secondary users such that the interference on the primary user is negligible.

From a probabilistic point of view, the spectrum sensing problem may be framed as a decision theory problem, as follows

$$H_0 : \ \boldsymbol{y} = \boldsymbol{w}, \tag{1}$$
$$H_1 : \ \boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w}, \tag{2}$$

in which $\boldsymbol{y} \in \mathbb{C}^{n\times 1}$ is the decoded received vector signal, $\boldsymbol{w} \in \mathbb{C}^{n\times 1}$ is complex Gaussian noise process with zero mean vector and covariance matrix given as $\sigma^2\boldsymbol{I}_n$, and h is the channel gain.

In [**?**], the authors have shown that the probability distribution of the energy statistic $\tilde{y} \triangleq \boldsymbol{y}^{\dagger}\boldsymbol{y}$ conditioned on the knowledge of h, in case that $\boldsymbol{s}$ is an M-PSK signal such that every symbol has the same probability of occurrence, $\mathbb{P}(s_n = s) = \frac{1}{M}$, is given as

$$p(\tilde{y}|h, H_1) = 1 - Q_n\left(\sqrt{\frac{2n|h|^2E_s}{\sigma^2}}, \sqrt{\frac{2\tilde{y}}{\sigma^2}}\right), \tag{3}$$

in which Q_n is the Marcum-Q function and E_s is the energy per symbol.

The pdf of $\tilde{y}$ can be written using the Law of Total Expectation

$$p(\tilde{y}|H_1) = \mathbb{E}_h\left[p(\tilde{y}|h, H_1)\right] = \int_{-\infty}^{+\infty} p(\tilde{y}|h, H_1)p(h)\,\mathrm{d}h. \tag{4}$$

Recall that the energy detection rule can be expressed as

$$d_\delta(\tilde{y}) = \mathbb{I}(\tilde{y} > \delta) \tag{5}$$

in which δ is a strictly positive real number known as energy threshold, and $d_\delta(\tilde{y}) = j,\ j \in \{0, 1\}$, means that the detector has decided in favor of the hypothesis H_j.

As a result, the probabilities of false alarm and miss detection can be written as

$$p_f \triangleq \mathbb{P}\left(d_\delta(\tilde{y}) = 1|H_0\right) = 1 - p(\delta|H_0), \tag{6}$$
$$p_d \triangleq \mathbb{P}\left(d_\delta(\tilde{y}) = 0|H_1\right) = \mathbb{E}_h\left[p(\delta, h|H_1)\right], \tag{7}$$

B. Parameter Estimation in Nakagami-m fading

$$p(\boldsymbol{h}) = \prod_{i=1}^{n} \frac{2m^m}{\Gamma(m)\Omega^m} h_i^{2m-1} \exp\left(-\frac{mh_i^2}{\Omega}\right) \tag{8}$$
$$= \left(\frac{2m^m}{\Gamma(m)\Omega^m}\right)^n \exp\left(-\frac{m\sum_{i=1}^{n} h_i^2}{\Omega}\right) \prod_{i=1}^{n} h_i^{2m-1} \tag{9}$$

$$\log p(\boldsymbol{h}) \cong n\left(m\left(\log m-\log \Omega\right)-\log \Gamma(m)\right) \\ -m\left(\frac{\sum_{i=1}^{n} h_i^2}{\Omega}-2\sum_{i=1}^{n}\log h_i\right) \tag{10}$$

A direct maximum likelihood estimator for (10) has been investigated to be infeasible [?].

Therefore, we use a Majorization-Minimization algorithm to find smooth and easy to optimize upper bounds for $\log p(\boldsymbol{h})$.

Note that,

$$-\log \Gamma(m) \geq -\log \Gamma(m_t)+(m-m_t) \\ +\left(m_t-\frac{1}{2}\right)\log(m_t-1) \\ -\left(m-\frac{1}{2}\right)\log(m-1), m \geq m_t > 1 \tag{11}$$

and

$$m\log m \geq m\log m_t + m - m_t,\ m>0 \tag{12}$$

$$-m\log(m-1) \geq -\log(m-1)-(m-1)\left(\log(m_t-1)+\frac{m-1}{m_t-1}-1\right),\ m>1 \tag{13}$$

(in all inequalities, equality is achieved iff $m=m_t$).

Therefore,

C. Parameter Estimation in α-μ fading

The α-μ fading probability density is given as

$$p(h)=\frac{\alpha\mu^{\mu}h^{\alpha\mu-1}}{\Gamma(\mu)}\exp\left(-\mu h^{\alpha}\right), \tag{19}$$

$h \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+$, and $\mu \in \mathbb{R}_+$.

Given a vector $\boldsymbol{h} \in \mathbb{R}^{n\times 1}$ of iid samples from the $\alpha-\mu$ distribution, we would like to estimate the value of... The pdf of $\boldsymbol{h}$ is given as

$$p(\boldsymbol{h})=\prod_{i=1}^{n}p(h_i)=\prod_{i=1}^{n}\frac{\alpha\mu^{\mu}h_i^{\alpha\mu-1}}{\Gamma(\mu)}\exp\left(-\mu h_i^{\alpha}\right) \tag{20}$$

$$=\left(\frac{\alpha\mu^{\mu}}{\Gamma(\mu)}\right)^{n}\exp\left(-\mu\sum_{i=1}^{n}h_i^{\alpha}\right)\prod_{i=1}^{n}h_i^{\alpha\mu-1} \tag{21}$$

and the loglikelihood is given as

$$\log p(\boldsymbol{h}) \cong n\left(\log\alpha+\mu\log\mu-\log\Gamma(\mu)\right) \\ -\mu\sum_{i=1}^{n}h_i^{\alpha}+\alpha\mu\sum_{i=1}^{n}\log h_i \tag{22}$$

Therefore, we want to solve

$$\text{minimize } -\log p(\boldsymbol{h}) \\ \text{s.t. } \alpha>0, \boldsymbol{h}>0 \tag{23}$$

D. BER in Complex $\alpha-\mu$ Fading

V. Conclusions

Acknowledgement

The authors would like to thank the Federal University of Campina Grande (UFCG) and the Institute for Advanced Studies in Communications (Iecom) for supporting this research.

$$
\begin{aligned}
-\log p(\boldsymbol{h}) \leq -n \Bigg(& m \log m_t + 2(m - m_t) - m \log \Omega - \log \Gamma(m_t) + \left(m_t - \frac{1}{2}\right) \log(m_t - 1) - \frac{1}{2} \log(m - 1) \\
& -(m-1)\left(\log(m_t - 1) + \frac{m-1}{m_t - 1} - 1\right)\Bigg) + m \left(\frac{\sum_{i=1}^{n} h_i^2}{\Omega} - 2 \sum_{i=1}^{n} \log h_i \right), m \geq m_t > 1.
\end{aligned} \tag{14}
$$

$$
\frac{\partial g(m|m_t)}{\partial m} = -n \left(\log m_t + 2 - \log \Omega - \frac{1}{2(m-1)} - (\log(m_t - 1) - 1) + \frac{-2m+1}{m_t - 1} \right) + \frac{\sum_{i=1}^{n} h_i^2}{\Omega} - 2 \sum_{i=1}^{n} \log h_i = 0 \tag{15}
$$

$$
2(m-1)\left(-n\left(\log m_t + 2 - \log \Omega - (\log(m_t - 1) - 1)\right) + \frac{\sum_{i=1}^{n} h_i^2}{\Omega} - 2 \sum_{i=1}^{n} \log h_i \right) + n - \frac{2n}{m_t - 1}(-2m^2 + 3m - 1) = 0 \tag{16}
$$

$$
2(m-1)a + n - \frac{2n}{m_t - 1}(-2m^2 + 3m - 1) = 0 \tag{17}
$$

$$
m^2 \left(\frac{4n}{m_t - 1} \right) + m \left(\frac{6n}{m_t - 1} + 2a \right) - 2a + n \left(\frac{m_t + 1}{m_t - 1} \right) = 0 \tag{18}
$$