maoud: a Python package for Simulating Generalized Fading Channels

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***Abstract*—We present a well tested Python-based library for simulating and computing generalized fading channels, named `maoud`. We describe the applicability of `maoud` using examples in scenarios of communications channels impaired by generalized fading, namely: spectrum sensing, bit error rate computation, and fading estimation. For the latter, we develop an iterative algorithm using the Majorization-Minimization framework, which allows reliable estimation of the fading parameter. The development of `maoud` is open source and its code along with examples are avaliable at `http://github.com/mirca/maoud`.**

I. Introduction

Notation

Scalars and random variables are denoted as *italic*, small-case letters *e.g.* x; sets and events are denoted as *italic*, capital letters *e.g.*, A; vectors and random vectors are denoted as *italic*, boldface, small-case letters *e.g.* $\boldsymbol{x}$. The $n-$th component of a vector $\boldsymbol{x}$ is denoted as x_n. A complex vector of length n is defined as $\boldsymbol{x} \in \mathbb{C}^{n\times 1}$. All vectors are column vectors. Matrices are denoted as *italic*, boldface, capital letters as in $\boldsymbol{X}$; the identity matrix of order n is denoted as $\boldsymbol{I}_n$. We define a discrete-time circularly symmetric Gaussian process $\boldsymbol{z}$ as any collection of random variables $\boldsymbol{z} = \boldsymbol{x} + j\boldsymbol{y}$, $j \triangleq \sqrt{-1}$, such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are i.i.d. jointly Gaussian, with zero mean vector and covariance matrix given by $\mathbb{E}\left(\boldsymbol{z}\boldsymbol{z}^{\dagger}\right)$, in which $\boldsymbol{z}^{\dagger}$ means the conjugate transpose of $\boldsymbol{z}$. The expectation value with respect to the probability distribution of a random variable x is denoted as $\mathbb{E}_x$. The probability of an event A is denoted as $\mathbb{P}(A)$. The indicator function is denoted as $\mathbb{I}(\cdot)$, it evaluates to one if its argument is true and zero otherwise. For any given two real functions f and g defined on the same domain D, $f \cong g$ means that there exist a constant c such that $f(\boldsymbol{x}) = g(\boldsymbol{x}) + c$, $\forall\ \boldsymbol{x} \in D$. The natural logarithm of a scalar $x > 0$ is denoted as $\log x$.

II. The Acceptance-Rejection sampler in log-space

III. Examples

A. Spectrum Sensing in Complex Generalized Fading Channels

The spectrum sensing problem consists in deciding whether or not a given channel frequency band is being occupied by a licensed (primary) user and, in case that such frequency band is available, how to opportuniscally allocate secondary users such that the interference on the primary user is negligible.

From a probabilistic point of view, the spectrum sensing problem may be framed as a decision theory problem, as follows

$$H_0 :\ \boldsymbol{y} = \boldsymbol{w}, \tag{1}$$
$$H_1 :\ \boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w}, \tag{2}$$

in which $\boldsymbol{y} \in \mathbb{C}^{n\times 1}$ is the decoded received vector signal, $\boldsymbol{w} \in \mathbb{C}^{n\times 1}$ is complex Gaussian noise process with zero mean vector and covariance matrix given as $\sigma^2 \boldsymbol{I}_n$, and h is the channel gain.

In [1], the authors have shown that the probability distribution of the energy statistic $\tilde{y} \triangleq \boldsymbol{y}^{\dagger}\boldsymbol{y}$ conditioned on the knowledge of h, in case that $\boldsymbol{s}$ is an M-PSK signal such that every symbol has the same probability of occurrence, $\mathbb{P}(s_n = s) = \frac{1}{M}$, is given as

$$p(\tilde{y}|h, H_1) = 1 - \mathrm{Q}_n\left(\sqrt{\frac{2n|h|^2 E_s}{\sigma^2}}, \sqrt{\frac{2\tilde{y}}{\sigma^2}}\right), \tag{3}$$

in which Q_n is the Marcum-Q function and E_s is the energy per symbol.

The pdf of $\tilde{y}$ can be written using the Law of Total Expectation

$$p(\tilde{y}|H_1) = \mathbb{E}_h\left[p(\tilde{y}|h, H_1)\right] = \int_{-\infty}^{+\infty} p(\tilde{y}|h, H_1)p(h)\ \mathrm{d}h. \tag{4}$$

Recall that the energy detection rule can be expressed as

$$d_{\delta}(\tilde{y}) = \mathbb{I}(\tilde{y} > \delta) \tag{5}$$

in which δ is a strictly positive real number known as energy threshold, and $d_{\delta}(\tilde{y}) = j,\ j \in \{0, 1\}$, means that the detector has decided in favor of the hypothesis H_j.

As a result, the probabilities of false alarm and miss detection can be written as

$$p_f \triangleq \mathbb{P}\left(d_{\delta}(\tilde{y}) = 1|H_0\right) = 1 - p(\delta|H_0), \tag{6}$$
$$p_d \triangleq \mathbb{P}\left(d_{\delta}(\tilde{y}) = 0|H_1\right) = \mathbb{E}_h\left[p(\delta, h|H_1)\right], \tag{7}$$

B. Parameter Estimation in Nakagami-m fading

The Nakagami-m density is given as

$$p(\boldsymbol{h}|m,\Omega) = \prod_{i=1}^{n} \frac{2m^m}{\Gamma(m)\Omega^m} h_i^{2m-1} \exp\left(-\frac{m h_i^2}{\Omega}\right)$$
$$= \left(\frac{2m^m}{\Gamma(m)\Omega^m}\right)^n \exp\left(-\frac{m\sum_{i=1}^{n} h_i^2}{\Omega}\right) \prod_{i=1}^{n} h_i^{2m-1} \quad (8)$$

And the log-likelihood function (up to an additive constant) is given as

$$\log p(\boldsymbol{h}|m,\Omega) \cong n\left(m\left(\log m - \log\Omega\right) - \log\Gamma(m)\right)$$
$$- m\sum_{i=1}^{n}\left(\frac{h_i^2}{\Omega} - 2\log h_i\right) \quad (9)$$

A direct maximum likelihood estimator (MLE) for (9) has been investigated to be infeasible [2].

Therefore, we use a Majorization-Minimization algorithm to find smooth and easy to optimize lower bounds for $\log p(\boldsymbol{h}|m,\Omega)$. More precisely, we need to find lower bounds for $m\log m$ and $-\log\Gamma(m)$ for $m \geq \frac{1}{2}$. The former function is convex for $m \geq 0$, hence it can be lower bounded by its first order Taylor series as

$$m\log m \geq m(1+\log m_t) - m_t, \quad (10)$$

The function $-\log\Gamma(m)$ is concave, therefore, it can be lower bounded by its second order Taylor series expansion as

$$-\log\Gamma(m) \geq -\log\Gamma(m_t) - \psi(m_t)(m-m_t)$$
$$-\frac{\psi'\left(\frac{1}{2}\right)}{2}(m-m_t)^2,\ m \geq \frac{1}{2},\ m_t \geq \frac{1}{2}, \quad (11)$$

in which $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is known as the digamma function. For both inequalities presented above, equality is achieved at $m = m_t$.

Substituting (10) and (11) into (9), a lower bound for $\log p(\boldsymbol{h})$ is obtained in (18). Due to the simple form of $g(m|m_t)$, its maximizer can be found in closed form, and an updating rule for the MLE can be written as

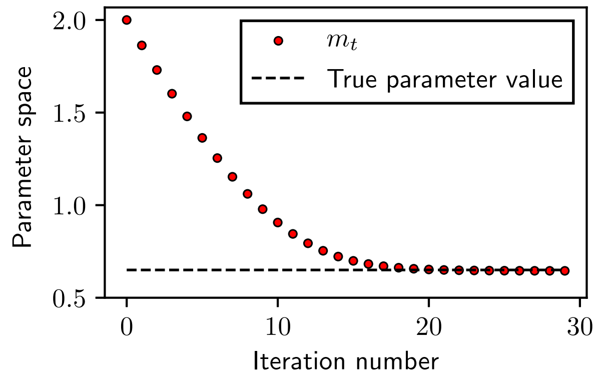
$$m_{t+1} = m_t + \frac{1}{\psi'(\frac{1}{2})}\left(1 + \log\frac{m_t}{\Omega} - \psi(m_t)\right.$$
$$\left. + \frac{1}{n}\sum_{i=1}^{n}\left(2\log h_i - \frac{h_i^2}{\Omega}\right)\right) \quad (12)$$

Additionally, note that $m_{t+1},\ t \in \mathbb{N}$, is a sequence of estimators that converges to the maximum likelihood estimator of the parameter value m. The expected value of m_{t+1}, conditioned on the knowledge of m_t, is given as

$$\mathbb{E}(m_{t+1}|m_t) = m_t + \frac{1}{\psi'(\frac{1}{2})}\left(\log\frac{m_t}{m} + \psi(m) - \psi(m_t)\right), \quad (13)$$

in which we used the fact that [3]

$$\mathbb{E}\left(\log h_i\right) = \frac{1}{2}\left(\psi(m) - \log\left(\frac{m}{\Omega}\right)\right). \quad (14)$$



Most importantly, as $m_t \to m$, then $\mathbb{E}(m_{t+1}|m_t) \to m$. The variance of m_{t+1}, given m_t, can be expressed as

$$\operatorname{var}(m_{t+1}|m_t) = \frac{1}{n\left(\psi'(\frac{1}{2})\right)^2}\operatorname{var}\left(2\log h_1 - \frac{h_1^2}{\Omega}\right). \quad (15)$$

$$\mathbb{E}\left(\log^2 h_i\right) = \frac{1}{4}\left\{\left[\psi(m) - \log\frac{m}{\Omega}\right]^2 + \zeta(2,m)\right\} \quad (16)$$

The Cramér-Rao Lower Bound for any unbiased estimator of m, say $\hat{m}$, is given as [2]

$$\operatorname{var}(\hat{m}) \geq \frac{1}{n\left(\psi'(m) - \frac{1}{m}\right)} \quad (17)$$

C. BER in Complex $\alpha - \mu$ Fading

Consider the system

$$\boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w} \quad (19)$$

in which $\boldsymbol{s} \in \mathbb{C}^{n\times 1}$ is a complex On-Off Keying (OOK) signal, h is a complex $\alpha - \mu$ random variable and $\boldsymbol{w}$ is a complex Gaussian process with zero mean vector and covariance matrix equals $\sigma^2\boldsymbol{I}_n$, and $\boldsymbol{y}$ is the received complex vector signal.

Assume that the OOK symbols are equiprobable and that there exist no interference between the in-phase and quadrature components, then the probability of one bit error is given as

$$p_e = \frac{1}{2}\left(\mathbb{P}\left(\hat{y}_i = 0|s_i = 1\right) + \mathbb{P}\left(\hat{y}_i = 1|s_i = 0\right)\right). \quad (20)$$

Assume that the decoded vector $\hat{\boldsymbol{y}}$ is estimated using the minimum distance decoding rule, i.e.,

IV. Conclusions

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$$g(m|m_t) = n\left(-m\log\Omega + m(1+\log m_t) - m_t - \log\Gamma(m_t) - \psi(m_t)(m-m_t) - \frac{\psi'\left(\frac{1}{2}\right)}{2}(m-m_t)^2\right) - m\left(\frac{\sum_{i=1}^{n} h_i^2}{\Omega} - 2\sum_{i=1}^{n}\log h_i\right) \tag{18}$$