maoud: a Python package for Simulating Generalized Fading Channels

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***Abstract*—We present a well tested Python-based library for simulating and computing generalized fading channels, named `maoud`. We describe the applicability of `maoud` using examples in scenarios of communications channels impaired by generalized fading, namely: spectrum sensing, bit error rate computation, and fading estimation. For the latter, we develop an iterative algorithm using the Majorization-Minimization framework, which allows reliable estimation of the fading parameter. The development of `maoud` is open source and its code, along with the code for the examples presented in this paper, are avaliable at `http://github.com/mirca/maoud`.**

I. Introduction

The study of modern wireless communications systems heavily relies on fading channel simulation. By fading channel simulation, we refer to the generation of samples from a probability distribution that resembles the effects and impairments caused by real communications fading channels on the transmitted signal.

Although accurate and precise distributions for generalized fading have been estabilished in the literature, such as α-μ [1], κ-μ [2] and η-μ [2], the generation of samples following these distributions is usually a time-consuming task. In [3], the authors built an efficient algorithm, based on the rejection method, for generation of samples from those distributions.

however, there are neither open nor closed source implementations available to the scientific community.

In this paper, we present an open source Python package, named `maoud`, for generation of samples following the α-μ, κ-μ, and η-μ distributions. The usefullness of `maoud` is illustrated through examples involving spectrum sensing, bit error rate (BER) computation, and fading estimation.

Notation

Scalars and random variables are denoted as italic, small-case letters *e.g.* x; sets and events are denoted as italic, capital letters *e.g.*, A; vectors and random vectors are denoted as italic, boldface, small-case letters *e.g.* $\boldsymbol{x}$. The $n-$th component of a vector $\boldsymbol{x}$ is denoted as x_n. A complex vector of length n is defined as $\boldsymbol{x} \in \mathbb{C}^{n\times 1}$. All vectors are column vectors. Matrices are denoted as italic, boldface, capital letters as in $\boldsymbol{X}$; the identity matrix of order n is denoted as $\boldsymbol{I}_n$. We define a discrete-time circularly symmetric Gaussian process $\boldsymbol{z}$ as any collection of random variables $\boldsymbol{z} = \boldsymbol{x} + j\boldsymbol{y}$, $j \triangleq \sqrt{-1}$, such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are i.i.d. jointly Gaussian, with zero mean vector and covariance matrix given by $\mathbb{E}\left(\boldsymbol{z}\boldsymbol{z}^\dagger\right)$, in which $\boldsymbol{z}^\dagger$ means the conjugate transpose of $\boldsymbol{z}$. The expectation value with respect to the probability distribution of a random variable x is denoted as $\mathbb{E}_x$. The probability of an event A is denoted as $\mathbb{P}(A)$. The indicator function is denoted as $\mathbb{I}(\cdot)$, it evaluates to one if its argument is true and zero otherwise. For any given two real functions f and g defined on the same domain D, $f \cong g$ means that there exist a constant c such that $f(\boldsymbol{x}) = g(\boldsymbol{x}) + c$, $\forall\ \boldsymbol{x} \in D$. The natural logarithm of a scalar $x > 0$ is denoted as $\log x$.

II. Rejection sampling

III. Examples

A. Spectrum Sensing in Complex Generalized Fading Channels

The spectrum sensing problem consists in deciding whether or not a given channel frequency band is being occupied by a licensed (primary) user and, in case that such frequency band is available, how to opportuniscally allocate secondary users such that the interference on the primary user is negligible.

From a probabilistic point of view, the spectrum sensing problem may be framed as a decision theory problem, as follows

$$H_0: \ \boldsymbol{y} = \boldsymbol{w}, \tag{1}$$
$$H_1: \ \boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w}, \tag{2}$$

in which $\boldsymbol{y} \in \mathbb{C}^{n\times 1}$ is the received vector signal, $\boldsymbol{w} \in \mathbb{C}^{n\times 1}$ is complex Gaussian noise process with zero mean vector and covariance matrix given as $\sigma^2\boldsymbol{I}_n$, and h is the channel gain.

In [4], the authors have shown that the cumulative distribution function (cdf) of the energy statistic $\tilde{y} \triangleq \boldsymbol{y}^\dagger\boldsymbol{y}$ conditioned on the knowledge of h, in case that $\boldsymbol{s}$ is an M-PSK signal such that every symbol has the same probability of occurrence, $\mathbb{P}(s_n = s) = \frac{1}{M}$, is given as

$$P(\tilde{y}|h, H_1) = 1 - \mathrm{Q}_n\left(\sqrt{\frac{2n|h|^2E_s}{\sigma^2}}, \sqrt{\frac{2\tilde{y}}{\sigma^2}}\right), \tag{3}$$

in which Q_n is the Marcum-Q function and E_s is the energy per symbol.

Recall that the energy detection rule can be expressed as

$$d_\delta(\tilde{y}) = \mathbb{I}(\tilde{y} > \delta) \tag{4}$$

in which δ is a strictly positive real number known as energy threshold, and $d_\delta(\tilde{y}) = j,\ j \in \{0, 1\}$, means that the detector has decided in favor of the hypothesis H_j.

As a result, the probabilities of false alarm and miss detection can be written as

$$\mathbb{P}\left(d_\delta(\tilde{y}) = 1|H_0\right) = 1 - P(\delta|H_0) = 1 - \gamma\left(n, \frac{\delta}{\sigma^2}\right), \tag{5}$$

$$\begin{aligned}\mathbb{P}\left(d_\delta(\tilde{y}) = 0|H_1\right) &= \mathbb{E}_h\left(P(\delta|h, H_1)\right) \\ &= \int_{-\infty}^{+\infty} P(\delta|h, H_1)p(h)\ \mathrm{d}h,\end{aligned} \tag{6}$$

in which γ is the regularized lower incomplete Gamma function, $p(h)$ is the pdf of the fading, and δ is the energy detection threshold.

The performance of detection schemes can be measured by computing the Receiver Operating Characteristic (ROC), which consists in varying δ and computing the pairs of probability of false alarm and miss detection, as illustrated in Fig. 1.

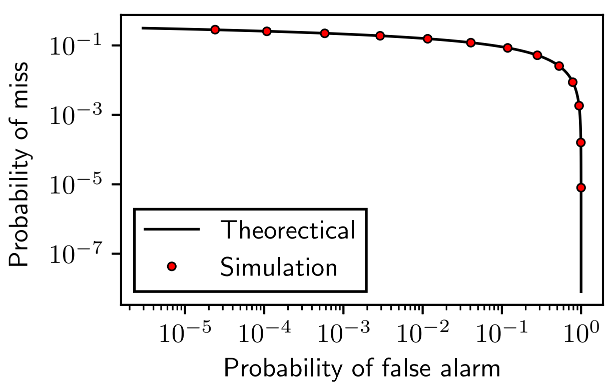

Fig. 1. Receiver Operating Characteristic for the energy detector in α-μ fading channel with $\alpha = 2$ and $\mu = 1$, i.e., Nakagami-m, $m = 1$. The solid curve represents the theorectical probabilities as stated on (5) and (6) for different values of δ, whereas bullets represent Monte Carlo simulations acquired with 10^6 realizations. The input signal $\boldsymbol{s}$ consists in a vector of length $n = 25$ in which each entry represents a symbol of a 64-PSK constellation. Symbols are assumed to be equiprobable. The signal-to-noise ratio is set to 5 dB.

B. Parameter Estimation in Nakagami-m fading

The Nakagami-m density is given as

$$\begin{aligned}p(\boldsymbol{h}|m) &= \prod_{i=1}^{n} \frac{2m^m}{\Gamma(m)\Omega^m} h_i^{2m-1} \exp\left(-\frac{m h_i^2}{\Omega}\right) \\ &= \left(\frac{2m^m}{\Gamma(m)\Omega^m}\right)^n \exp\left(-\frac{m\sum_{i=1}^{n} h_i^2}{\Omega}\right) \prod_{i=1}^{n} h_i^{2m-1},\end{aligned} \tag{7}$$

in which m is the fading parameter and $\Omega = \mathbb{E}(h_i^2)$ is the scale parameter.

The negative log-likelihood function (up to an additive constant) is given as

$$\begin{aligned}-\log p(\boldsymbol{h}|m) \cong &- n\left(m\left(\log m - \log \Omega\right) - \log\Gamma(m)\right) \\ &+ m\sum_{i=1}^{n}\left(\frac{h_i^2}{\Omega} - 2\log h_i\right).\end{aligned} \tag{8}$$

A direct maximum likelihood estimator (MLE) for (8) has been shown to be infeasible [5]. Thus, we use the Majorization-Minimization technique [6] to find smooth and easy to optimize upper bounds for $-\log p(\boldsymbol{h}|m)$.

Basically, the Majorization-Minimization (MM) algorithm consists in constructing a sequence of functions $g(\cdot|m_t)$, initiliazed at m_0, such that

$$g(m|m_t) \geq -\log p(\boldsymbol{h}|m) + c_t, \tag{9}$$

$$c_t = g(m_t|m_t) + \log p(\boldsymbol{h}|m), \tag{10}$$

$$m_{t+1} \in \underset{m > \frac{1}{2}}{\arg\min}\, g(m|m_t). \tag{11}$$

Note that (9) is the majorization step and (11) is the minimization step, whereas (10) is a condition to ensure that the difference of $g(\cdot|m_t)$ and $-\log p(\boldsymbol{h}|\cdot)$ is minimized at m_t.

Therefore, one possible way to construct $g(\cdot|m_t)$ is to find upper bounds for $-m\log m$ and $\log\Gamma(m)$ for $m \geq \frac{1}{2}$. The former function is concave for $m \geq 0$, hence it can be upper bounded by its first order Taylor series as

$$-m\log m \leq -m(1 + \log m_t) + m_t, \tag{12}$$

The function $\log\Gamma(m)$ is convex for $m > 0$, therefore, it can be upper bounded by its second order Taylor series expansion as

$$\begin{aligned}\log\Gamma(m) \leq &\log\Gamma(m_t) + \psi(m_t)(m - m_t) \\ &+ \frac{\psi'\left(\frac{1}{2}\right)}{2}(m - m_t)^2,\end{aligned} \tag{13}$$

in which $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is known as the digamma function. For both inequalities presented above, equality is achieved at $m = m_t$.

Applying inequalities (12) and (13) into (8), an upper bound for $\log p(\boldsymbol{h})$ is obtained as follows

$$\begin{aligned}g(m|m_t) = -n\Bigg[&- m\log\Omega + m(1 + \log m_t) - m_t \\ &- \log\Gamma(m_t) - \psi(m_t)(m - m_t) - \frac{\psi'\left(\frac{1}{2}\right)}{2}(m - m_t)^2\Bigg] \\ &+ m\left(\frac{\sum_{i=1}^{n} h_i^2}{\Omega} - 2\sum_{i=1}^{n}\log h_i\right)\end{aligned} \tag{14}$$

Due to the simple form of $g(m|m_t)$, its minimizer can be found in closed form, and an updating rule for the MLE can be written as

$$m_{t+1} = m_t + \frac{1}{\psi'(\frac{1}{2})}\left(1 + \log\frac{m_t}{\Omega} - \psi(m_t) + \frac{1}{n}\sum_{i=1}^{n}\left(2\log h_i - \frac{h_i^2}{\Omega}\right)\right). \tag{15}$$

Additionally, note that $\{m_{t+1}\}_{t\in\mathbb{N}}$ is a sequence of estimators that converges to the MLE of the parameter value m. This fact is a consequence of the convergence properties of the MM algorithm [6]. Therefore, the proposed estimator is asymptotically consistent and efficient.

We experimentally note, and it is also illustrated in Fig 2, that the estimator takes on average 30 iterations to converge within an absolute tolerance of 10^{-4}.

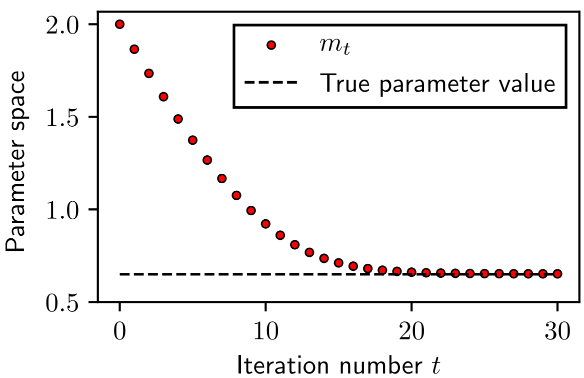

Fig. 2. Typical convergence behavior of the proposed estimator.

Moreover, we perform an experimental analysis of the bias and variance of the proposed estimator. First, we recall that the Cramér-Rao Lower Bound for any unbiased estimator of m, say $\hat{m}$, is given as [5]

$$\mathrm{var}(\hat{m}) \geq \frac{1}{n\left(\psi'(m) - \frac{1}{m}\right)}. \tag{16}$$

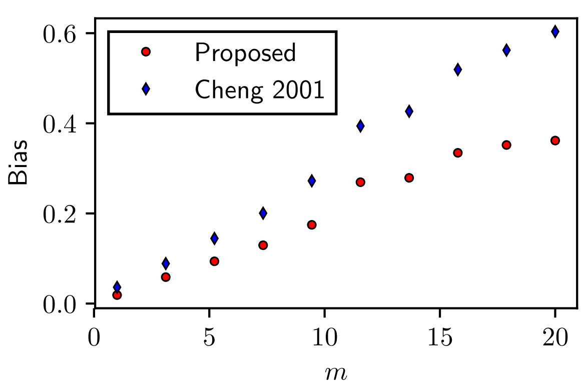

Fig. 3. Bias of the proposed estimator and the estimator presented in [5]. The bias was computed via Monte Carlo simulation with 10^4 realizations.

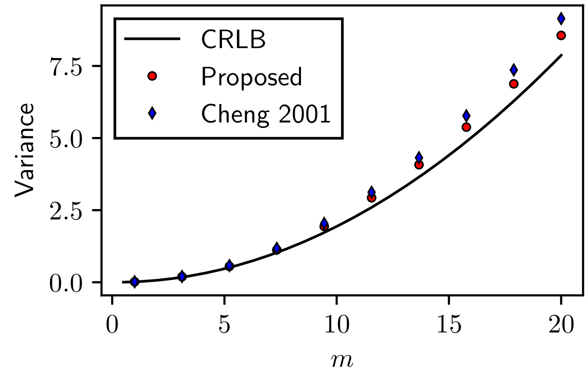

Fig. 4. Variance of the proposed estimator and the estimator presented in [5] compared against the Cramér-Rao Lower Bound. The variance was computed via Monte Carlo simulation with 10^4 realizations.

C. BER in α-μ Fading

Consider the system

$$y = hs + w \tag{17}$$

in which $s \in \{0, a\}$, $a \in \mathbb{R}_+$, is a transmitted signal, h is an α-μ random variable and $\boldsymbol{w}$ is a Gaussian random variable with zero mean vector and variance σ^2, and y is the received signal.

Assume that the binary symbols are equiprobable, then the probability of one bit error is given as

$$p_e = \frac{1}{2}\left(\mathbb{P}\left(\hat{y} = 0|s = a\right) + \mathbb{P}\left(\hat{y} = 1|s = 0\right)\right). \tag{18}$$

Further, assume that the decoded bit $\hat{y}$ is estimated using the minimum distance decoding rule, i.e.,

$$\hat{y} = \mathbb{I}(|y - a|^2 < |y|^2) \tag{19}$$

therefore

$$\begin{aligned}\mathbb{P}\left(\hat{y} = 1|s = 0\right) &= \mathbb{P}\left(|w - a|^2 - |w|^2 < 0\right)\\ &= \mathbb{P}\left(w > \frac{a}{2}\right) = 1 - \Phi\left(\frac{a}{2\sigma}\right)\end{aligned} \tag{20}$$

and likewise

$$\begin{aligned}\mathbb{P}\left(\hat{y} = 0|s = a\right) &= \mathbb{P}\left(ha + w < \frac{a}{2}\right)\\ &= \mathbb{E}_h\left(\Phi\left(\frac{a(1 - 2h)}{2\sigma}\right)\right).\end{aligned} \tag{21}$$

IV. Conclusions

References

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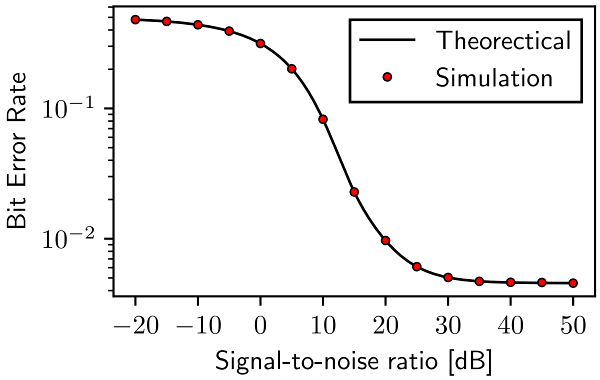

Fig. 5. BER

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