`maoud`: a Python package for Simulating Generalized Fading Channels

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***Abstract*—We present a well tested Python-based library for simulating and computing generalized fading channels, named `maoud`. We describe the applicability of `maoud` using examples in scenarios of communications channels impaired by generalized fading, namely: spectrum sensing, bit error rate computation, and fading estimation. For the latter, we develop an iterative algorithm using the Majorization-Minimization framework, which allows reliable estimation of the fading parameter. The development of `maoud` is open source and its code along with examples are avaliable at `http://github.com/mirca/maoud`.**

I. Introduction

The study of wireless communications systems heavily depends on channel simulation. By channel simulation, we mean the generation of samples from a random variable which resembles the effects of real communications channels on the transmitted signal.

Although accurate and precise distributions for generalized fading have been estabilished in the literature, such as $\alpha - \mu$, $\kappa - \mu$, and $\eta - \mu$, the generation of samples following these distributions is usually a time-consuming task. In [], an efficient algorithm for generation of samples from those distributions was proposed, however, there are neither open nor closed source implementations available to the scientific community.

In this paper, we present an open source Python package, named `maoud`, for generation of samples following the $\alpha - \mu$, $\kappa - \mu$, and $\eta - \mu$ distributions. The usefullness of `maoud` is illustrated through examples involving spectrum sensing, bit error rate computation, and fading estimation.

Notation

Scalars and random variables are denoted as italic, small-case letters *e.g.* x; sets and events are denoted as italic, capital letters *e.g.*, A; vectors and random vectors are denoted as italic, boldface, small-case letters *e.g.* $\boldsymbol{x}$. The $n-$th component of a vector $\boldsymbol{x}$ is denoted as x_n. A complex vector of length n is defined as $\boldsymbol{x} \in \mathbb{C}^{n\times 1}$. All vectors are column vectors. Matrices are denoted as italic, boldface, capital letters as in $\boldsymbol{X}$; the identity matrix of order n is denoted as $\boldsymbol{I}_n$. We define a discrete-time circularly symmetric Gaussian process $\boldsymbol{z}$ as any collection of random variables $\boldsymbol{z} = \boldsymbol{x} + j\boldsymbol{y}$, $j \triangleq \sqrt{-1}$, such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are i.i.d. jointly Gaussian, with zero mean vector and covariance matrix given by $\mathbb{E}\left(\boldsymbol{z}\boldsymbol{z}^\dagger\right)$, in which $\boldsymbol{z}^\dagger$ means the conjugate transpose of $\boldsymbol{z}$. The expectation value with respect to the probability distribution of a random variable x is denoted as $\mathbb{E}_x$. The probability of an event A is denoted as $\mathbb{P}(A)$. The indicator function is denoted as $\mathbb{I}(\cdot)$, it evaluates to one if its argument is true and zero otherwise. For any given two real functions f and g defined on the same domain D, $f \cong g$ means that there exist a constant c such that $f(\boldsymbol{x}) = g(\boldsymbol{x}) + c$, $\forall\ \boldsymbol{x} \in D$. The natural logarithm of a scalar $x > 0$ is denoted as $\log x$.

II. Rejection sampling

III. Examples

A. Spectrum Sensing in Complex Generalized Fading Channels

The spectrum sensing problem consists in deciding whether or not a given channel frequency band is being occupied by a licensed (primary) user and, in case that such frequency band is available, how to opportuniscally allocate secondary users such that the interference on the primary user is negligible.

From a probabilistic point of view, the spectrum sensing problem may be framed as a decision theory problem, as follows

$$H_0 :\ \boldsymbol{y} = \boldsymbol{w}, \tag{1}$$
$$H_1 :\ \boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w}, \tag{2}$$

in which $\boldsymbol{y} \in \mathbb{C}^{n\times 1}$ is the decoded received vector signal, $\boldsymbol{w} \in \mathbb{C}^{n\times 1}$ is complex Gaussian noise process with zero mean vector and covariance matrix given as $\sigma^2 \boldsymbol{I}_n$, and h is the channel gain.

In [1], the authors have shown that the probability distribution of the energy statistic $\tilde{y} \triangleq \boldsymbol{y}^\dagger\boldsymbol{y}$ conditioned on the knowledge of h, in case that $\boldsymbol{s}$ is an M-PSK signal such that every symbol has the same probability of occurrence, $\mathbb{P}(s_n = s) = \frac{1}{M}$, is given as

$$p(\tilde{y}|h, H_1) = 1 - \mathrm{Q}_n\left(\sqrt{\frac{2n|h|^2 E_s}{\sigma^2}}, \sqrt{\frac{2\tilde{y}}{\sigma^2}}\right), \tag{3}$$

in which Q_n is the Marcum-Q function and E_s is the energy per symbol.

The pdf of $\tilde{y}$ can be written using the Law of Total Expectation

$$p(\tilde{y}|H_1) = \mathbb{E}_h \left[p(\tilde{y}|h, H_1)\right] = \int_{-\infty}^{+\infty} p(\tilde{y}|h, H_1)p(h) \, \mathrm{d}h. \quad (4)$$

Recall that the energy detection rule can be expressed as

$$d_\delta(\tilde{y}) = \mathbb{I}(\tilde{y} > \delta) \quad (5)$$

in which δ is a strictly positive real number known as energy threshold, and $d_\delta(\tilde{y}) = j, \ j \in \{0, 1\}$, means that the detector has decided in favor of the hypothesis H_j.

As a result, the probabilities of false alarm and miss detection can be written as

$$p_f \triangleq \mathbb{P}\left(d_\delta(\tilde{y}) = 1|H_0\right) = 1 - p(\delta|H_0), \quad (6)$$
$$p_d \triangleq \mathbb{P}\left(d_\delta(\tilde{y}) = 0|H_1\right) = \mathbb{E}_h\left[p(\delta, h|H_1)\right], \quad (7)$$

B. Parameter Estimation in Nakagami-m fading

The Nakagami-m density is given as

$$\begin{aligned} p(\boldsymbol{h}|m, \Omega) &= \prod_{i=1}^{n} \frac{2m^m}{\Gamma(m)\Omega^m} h_i^{2m-1} \exp\left(-\frac{m h_i^2}{\Omega}\right) \\ &= \left(\frac{2m^m}{\Gamma(m)\Omega^m}\right)^n \exp\left(-\frac{m\sum_{i=1}^n h_i^2}{\Omega}\right) \prod_{i=1}^{n} h_i^{2m-1} \quad (8) \end{aligned}$$

And the log-likelihood function (up to an additive constant) is given as

$$\begin{aligned} \log p(\boldsymbol{h}|m, \Omega) &\cong n\left(m\left(\log m - \log \Omega\right) - \log\Gamma(m)\right) \\ &\quad - m\sum_{i=1}^{n}\left(\frac{h_i^2}{\Omega} - 2\log h_i\right) \quad (9) \end{aligned}$$

A direct maximum likelihood estimator (MLE) for (9) has been investigated to be infeasible [2].

Therefore, we use a Majorization-Minimization algorithm to find smooth and easy to optimize lower bounds for $\log p(\boldsymbol{h}|m, \Omega)$. More precisely, we need to find lower bounds for $m \log m$ and $-\log\Gamma(m)$ for $m \geq \frac{1}{2}$. The former function is convex for $m \geq 0$, hence it can be lower bounded by its first order Taylor series as

$$m \log m \geq m(1 + \log m_t) - m_t, \quad (10)$$

The function $-\log\Gamma(m)$ is concave for $m > 0$, therefore, it can be lower bounded by its second order Taylor series expansion as

$$\begin{aligned} -\log\Gamma(m) &\geq -\log\Gamma(m_t) - \psi(m_t)(m - m_t) \\ &\quad - \frac{\psi'\left(\frac{1}{2}\right)}{2}(m - m_t)^2, \quad (11) \end{aligned}$$

in which $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is known as the digamma function. For both inequalities presented above, equality is achieved at $m = m_t$.

Substituting (10) and (11) into (9), a lower bound for $\log p(\boldsymbol{h})$ is obtained as follows

$$\begin{aligned} g(m|m_t) = n \Bigg[&- m\log\Omega + m(1 + \log m_t) - m_t \\ &- \log\Gamma(m_t) - \psi(m_t)(m - m_t) - \frac{\psi'\left(\frac{1}{2}\right)}{2}(m - m_t)^2 \Bigg] \\ &- m\left(\frac{\sum_{i=1}^n h_i^2}{\Omega} - 2\sum_{i=1}^{n}\log h_i\right) \quad (12) \end{aligned}$$

Due to the simple form of $g(m|m_t)$, its maximizer can be found in closed form, and an updating rule for the MLE can be written as

$$\begin{aligned} m_{t+1} = m_t + \frac{1}{\psi'(\frac{1}{2})} \Bigg(&1 + \log\frac{m_t}{\Omega} - \psi(m_t) \\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(2\log h_i - \frac{h_i^2}{\Omega}\right)\Bigg) \quad (13) \end{aligned}$$

Additionally, note that m_{t+1}, $t \in \mathbb{N}$, is a sequence of estimators that converges to the maximum likelihood estimator of the parameter value m. The expected value of m_{t+1}, conditioned on the knowledge of m_t, is given as

$$\mathbb{E}(m_{t+1}|m_t) = m_t + \frac{1}{\psi'(\frac{1}{2})}\left(\log\frac{m_t}{m} + \psi(m) - \psi(m_t)\right), \quad (14)$$

in which we used the fact that [3]

$$\mathbb{E}\left(\log h_i\right) = \frac{1}{2}\left(\psi(m) - \log\left(\frac{m}{\Omega}\right)\right). \quad (15)$$

Most importantly, as $m_t \to m$, then $\mathbb{E}(m_{t+1}|m_t) \to m$. The variance of m_{t+1}, given m_t, can be expressed as

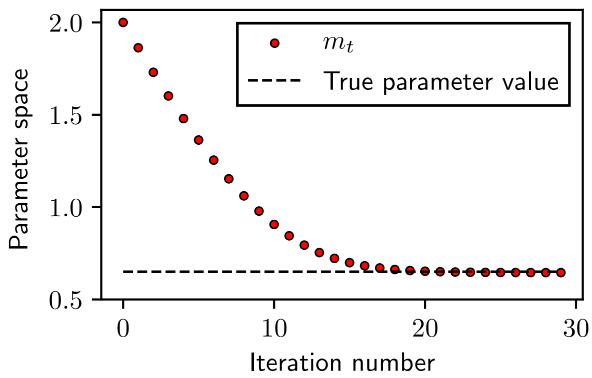
$$\mathrm{var}(m_{t+1}|m_t) = \frac{1}{n\left(\psi'(\frac{1}{2})\right)^2}\mathrm{var}\left(2\log h_1 - \frac{h_1^2}{\Omega}\right). \quad (16)$$

$$\mathbb{E}\left(\log^2 h_1\right) = \frac{1}{4}\left\{\left[\psi(m) - \log\frac{m}{\Omega}\right]^2 + \zeta(2, m)\right\} \quad (17)$$
$$\mathbb{E}\left(h_1^2 \log h_1\right) = \frac{\Omega}{2}\left(\psi(m+1) - \log\frac{m}{\Omega}\right) \quad (18)$$
$$\mathbb{E}\left(h_1^2\right) = \Omega \quad (19)$$
$$\mathbb{E}\left(h_1^4\right) = \Omega^2\frac{m+1}{m} \quad (20)$$

$$\mathrm{var}(m_{t+1}|m_t) = \frac{\zeta(2, m) + \frac{1}{m} + 2(\psi(m) - \psi(m+1))}{n\left(\psi'(\frac{1}{2})\right)^2} \quad (21)$$

The Cramér-Rao Lower Bound for any unbiased estimator of m, say $\hat{m}$, is given as [2]

$$\mathrm{var}(\hat{m}) \geq \frac{1}{n\left(\psi'(m) - \frac{1}{m}\right)} \quad (22)$$



C. BER in Complex $\alpha - \mu$ Fading

Consider the system

$$\boldsymbol{y} = h\boldsymbol{s} + \boldsymbol{w} \tag{23}$$

in which $\boldsymbol{s} \in \mathbb{C}^{n\times 1}$ is a complex On-Off Keying (OOK) signal, h is a complex $\alpha - \mu$ random variable and $\boldsymbol{w}$ is a complex Gaussian process with zero mean vector and covariance matrix equals $\sigma^2 \boldsymbol{I}_n$, and $\boldsymbol{y}$ is the received complex vector signal.

Assume that the OOK symbols are equiprobable and that there exist no interference between the in-phase and quadrature components, then the probability of one bit error is given as

$$p_e = \frac{1}{2}\left(\mathbb{P}\left(\hat{y}_i = 0 | s_i = 1\right) + \mathbb{P}\left(\hat{y}_i = 1 | s_i = 0\right)\right). \tag{24}$$

Assume that the decoded vector $\hat{\boldsymbol{y}}$ is estimated using the minimum distance decoding rule, i.e.,

IV. Conclusions

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