

yacoub: a Python package for Simulating Generalized Fading Channels

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Abstract—We present a well tested Python-based library for simulating and computing generalized fading channels, named **yacoub**. We describe the applicability of **yacoub** using examples in scenarios of communications channels impaired by generalized fading, namely: spectrum sensing, bit error rate computation, and fading estimation. For the latter, we develop an iterative algorithm using the Majorization-Minimization framework, which allows reliable estimation of the fading parameter. The development of **yacoub** is open source and its code along with examples are available at <http://github.com/mirca/yacoub>.

I. INTRODUCTION

Notation

Scalars and random variables are denoted as *italic*, small-case letters *e.g.* x ; sets and events are denoted as *italic*, capital letters *e.g.* A ; vectors and random vectors are denoted as *italic*, boldface, small-case letters *e.g.* \mathbf{x} . The n -th component of a vector \mathbf{x} is denoted as x_n . A complex vector of length n is defined as $\mathbf{x} \in \mathbb{C}^{n \times 1}$. All vectors are column vectors. Matrices are denoted as *italic*, boldface, capital letters as in \mathbf{X} ; the identity matrix of order n is denoted as \mathbf{I}_n . We define a discrete-time circularly symmetric Gaussian process \mathbf{z} as any collection of random variables $\mathbf{z} = \mathbf{x} + j\mathbf{y}$, $j \triangleq \sqrt{-1}$, such that \mathbf{x} and \mathbf{y} are i.i.d. jointly Gaussian, with zero mean vector and covariance matrix given by $\mathbb{E}[\mathbf{z}\mathbf{z}^\dagger]$, in which \mathbf{z}^\dagger means the conjugate transpose of \mathbf{z} . The expectation value w.r.t. to the probability distribution of a random variable x is denoted as \mathbb{E}_x . The probability of an event A is denoted as $\mathbb{P}(A)$. The indicator function is denoted as $\mathbb{I}(\cdot)$, it evaluates to one if its argument is true and zero otherwise. For any given two real functions f and g defined on the same domain D , $f \cong g$ means that there exist a constant c such that $f(\mathbf{x}) = g(\mathbf{x}) + c$, $\forall \mathbf{x} \in D$.

II. THE ACCEPTANCE-REJECTION SAMPLER IN LOG-SPACE

III. MAJORIZATION-MINIMIZATION ALGORITHMS

IV. EXAMPLES

A. Spectrum Sensing in Complex Generalized Fading Channels

The spectrum sensing problem consists in deciding whether or not a given channel frequency band is being occupied by a licensed (primary) user and, in case that such frequency band

is available, how to opportunistically allocate secondary users such that the interference on the primary user is negligible.

From a probabilistic point of view, the spectrum sensing problem may be framed as a decision theory problem, as follows

$$H_0 : \mathbf{y} = \mathbf{w}, \quad (1)$$

$$H_1 : \mathbf{y} = h\mathbf{s} + \mathbf{w}, \quad (2)$$

in which $\mathbf{y} \in \mathbb{C}^{n \times 1}$ is the decoded received vector signal, $\mathbf{w} \in \mathbb{C}^{n \times 1}$ is complex Gaussian noise process with zero mean vector and covariance matrix given as $\sigma^2 \mathbf{I}_n$, and h is the channel gain.

In [?], the authors have shown that the probability distribution of the energy statistic $\tilde{y} \triangleq \mathbf{y}^\dagger \mathbf{y}$ conditioned on the knowledge of h , in case that \mathbf{s} is an M -PSK signal such that every symbol has the same probability of occurrence, $\mathbb{P}(s_n = s) = \frac{1}{M}$, is given as

$$p(\tilde{y}|h, H_1) = 1 - Q_n \left(\sqrt{\frac{2n|h|^2 E_s}{\sigma^2}}, \sqrt{\frac{2\tilde{y}}{\sigma^2}} \right), \quad (3)$$

in which Q_n is the Marcum- Q function and E_s is the energy per symbol.

The pdf of \tilde{y} can be written using the Law of Total Expectation

$$p(\tilde{y}|H_1) = \mathbb{E}_h [p(\tilde{y}|h, H_1)] = \int_{-\infty}^{+\infty} p(\tilde{y}|h, H_1) p(h) dh. \quad (4)$$

Recall that the energy detection rule can be expressed as

$$d_\delta(\tilde{y}) = \mathbb{I}(\tilde{y} > \delta) \quad (5)$$

in which δ is a strictly positive real number known as energy threshold, and $d_\delta(\tilde{y}) = j$, $j \in \{0, 1\}$, means that the detector has decided in favor of the hypothesis H_j .

As a result, the probabilities of false alarm and miss detection can be written as

$$p_f \triangleq \mathbb{P}(d_\delta(\tilde{y}) = 1|H_0) = 1 - p(\delta|H_0), \quad (6)$$

$$p_d \triangleq \mathbb{P}(d_\delta(\tilde{y}) = 0|H_1) = \mathbb{E}_h [p(\delta, h|H_1)], \quad (7)$$

B. Parameter Estimation in Nakagami- m fading

The Nakagami- m density is given as

$$p(\mathbf{h}) = \prod_{i=1}^n \frac{2m^m}{\Gamma(m)\Omega^m} h_i^{2m-1} \exp\left(-\frac{mh_i^2}{\Omega}\right) \\ = \left(\frac{2m^m}{\Gamma(m)\Omega^m}\right)^n \exp\left(-\frac{m\sum_{i=1}^n h_i^2}{\Omega}\right) \prod_{i=1}^n h_i^{2m-1} \quad (8)$$

And the log likelihood function (up to an additive constant) is given as

$$\log p(\mathbf{h}) \cong n(m(\log m - \log \Omega) - \log \Gamma(m)) \\ - m \left(\frac{\sum_{i=1}^n h_i^2}{\Omega} - 2 \sum_{i=1}^n \log h_i \right) \quad (9)$$

A direct maximum likelihood estimator (MLE) for (9) has been investigated to be infeasible [?].

Therefore, we use a Majorization-Minimization algorithm to find smooth and easy to optimize lower bounds for $\log p(\mathbf{h})$. More precisely, we need to find lower bounds for $m \log m$ and $-\log \Gamma(m)$ for $m \geq \frac{1}{2}$. The former function is convex for $m \geq 0$, hence it can be lower bounded by its first order Taylor series as

$$m \log m \geq m(1 + \log m_t) - m_t, \quad (10)$$

The function $-\log \Gamma(m)$ is concave, therefore, it can be lower bounded by its second order Taylor series expansion as

$$-\log \Gamma(m) \geq -\log \Gamma(m_t) - \psi(m_t)(m - m_t) \\ - \frac{\psi'(\frac{1}{2})}{2}(m - m_t)^2, \quad m \geq \frac{1}{2}, \quad m_t \geq \frac{1}{2}, \quad (11)$$

in which $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is known as the digamma function. For both inequalities presented above, equality is achieved at $m = m_t$.

Substituting (10) and (11) into (9), a lower bound for $\log p(\mathbf{h})$ is obtained in (14). Due to the simple form of $g(m|m_t)$, its maximizer can be found in closed form, and an updating rule for the MLE can be written as

$$m_{t+1} = 1 + \frac{1}{\psi'(\frac{1}{2})} \left(1 + \log \frac{m_t}{\Omega} - \psi(m_t) + \right. \\ \left. + \frac{1}{n} \left(2 \sum_{i=1}^n \log h_i - \frac{\sum_{i=1}^n h_i^2}{\Omega} \right) \right) \quad (12)$$

Additionally, note that m_{t+1} is an estimator of the true parameter value m . Its expected value, conditioned on the knowledge of m_t , is given as

$$\mathbb{E}(m_{t+1}|m_t) = 1 + \frac{1}{\psi'(\frac{1}{2})} \left(1 + \log \frac{m_t}{\Omega} - \psi(m_t) + \right. \\ \left. + \frac{1}{n} \left(2 \sum_{i=1}^n \mathbb{E}(\log h_i) - \frac{\sum_{i=1}^n \mathbb{E}(h_i^2)}{\Omega} \right) \right) \quad (13)$$

C. BER in Complex $\alpha - \mu$ Fading

Consider the system

$$\mathbf{y} = h\mathbf{s} + \mathbf{w} \quad (15)$$

in which $\mathbf{s} \in \mathbb{C}^{n \times 1}$ is a complex On-Off Keying (OOK) signal, h is a complex $\alpha - \mu$ random variable and \mathbf{w} is a complex Gaussian process with zero mean vector and covariance matrix equals $\sigma^2 \mathbf{I}_n$, and \mathbf{y} is the received complex vector signal.

Assume that the OOK symbols are equiprobable and that there exist no interference between the in-phase and quadrature components, then the probability of one bit error is given as

$$p_e = \frac{1}{2} (\mathbb{P}(\hat{y}_i = 0 | s_i = 1) + \mathbb{P}(\hat{y}_i = 1 | s_i = 0)). \quad (16)$$

Assume that the decoded vector $\hat{\mathbf{y}}$ is estimated using the minimum distance decoding rule, i.e.,

V. CONCLUSIONS

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$$g(m|m_t) = n \left(-m \log \Omega + m(1 + \log m_t) - m_t - \log \Gamma(m_t) - \psi(m_t)(m - m_t) - \frac{\psi'(\frac{1}{2})}{2}(m - m_t)^2 \right) - m \left(\frac{\sum_{i=1}^n h_i^2}{\Omega} - 2 \sum_{i=1}^n \log h_i \right)$$

(14)