Unraveling the Connections: Learning Undirected Graphs in Financial Markets

PhD Defense by

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Based on

Primary Publications

- **NeurIPS'22 Cardoso, J. V. M.**, Ying, J., and Palomar, D. P. "Learning Bipartite Graphs: Heavy Tails and Multiple Components", *Advances in Neural Information Processing Systems*, 14044–14057 (35), 2022
- **NeurIPS'21 Cardoso, J. V. M.**, Ying, J., and Palomar, D. P. "Graphical Models in Heavy-Tailed Markets", *Advances in Neural Information Processing Systems*, 19989–20001 (34), 2021

Relevant Publications

- **NeurIPS'20** Ying, J., **Cardoso, J. V. M.**, and Palomar, D. P. "Nonconvex Sparse Graph Learning under Laplacian Constrained Graphical Model", *Advances in Neural Information Processing Systems*, 7101–7113 (33), 2020
- **NeurIPS'19** Kumar, S., Ying, J., **Cardoso, J. V. M.**, and Palomar, D. P. "Structured Graph Learning Via Laplacian Spectral Constraints" *Advances in Neural Information Processing Systems*, (32), 2019

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2. Learning Graphs in Heavy Tailed Markets
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3. Learning Bipartite Graphs
   Bipartite structure, Stock Classification
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   Final Remarks

Introduction & Background

Motivation

what?

- how to go from a (financial) dataset $\boldsymbol{X}$ to a graph $\mathcal{G}$?

why?

- **estimating** relationships among (financial) entities $\Rightarrow$ enhance our **understanding** about their behavior

how?

- statistical estimation theory, optimization theory, numerical optimization frameworks

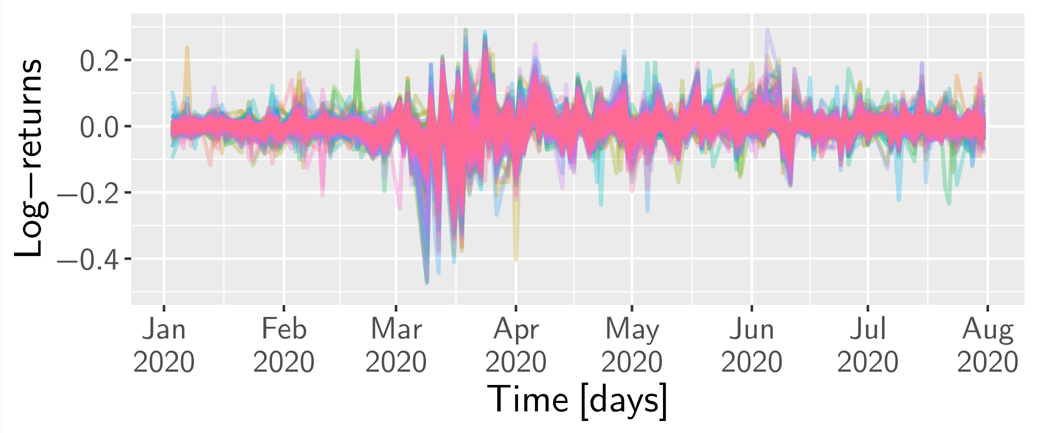
Financial Data

Mathematically:

$$\boldsymbol{X} \in \mathbb{R}^{n \times p}$$

- n is the number of observations
- p is the number of financial instruments

In real-life:

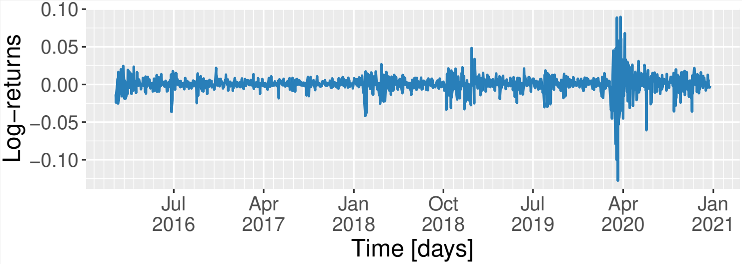


Log−returns
0.2
0.0
−0.2
−0.4
Jan 2020
Feb 2020
Mar 2020
Apr 2020
May 2020
Jun 2020
Jul 2020
Aug 2020
Time [days]

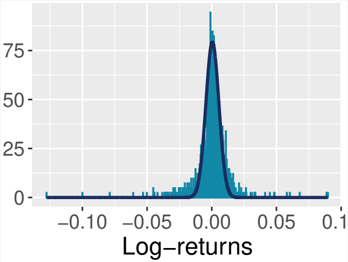
Stylized facts about finance data

Fact #1: Financial data is heavy-tailed

cf. S. I. Resnick. Heavy-Tail Phenomena: Probabilistic and Statistical Modeling. Springer-Verlag New York, 2007



(a) S&P500 log-returns.



(b) Histogram of S&P500 log-returns.

S. I. Resnick. Heavy-Tail Phenomena: Probabilistic and Statistical Modeling. Springer, 2007.

Fact #2: Stock markets are modular (stocks are more correlated within their sector)

cf. M. L. de Prado. Machine Learning for Asset Managers (Elements in Quantitative Finance). Cambridge University Press, 2020.

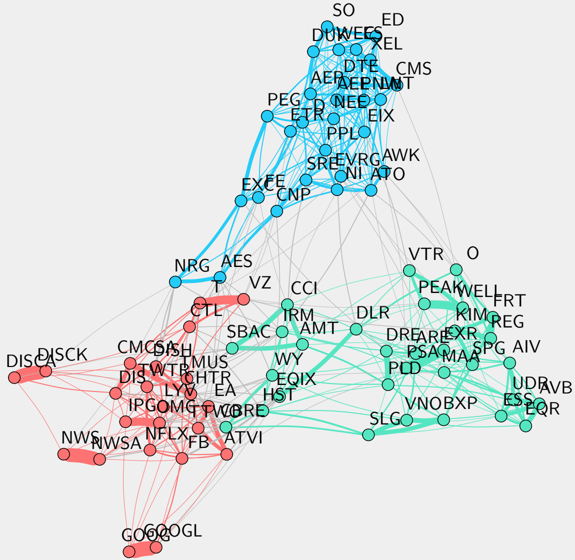
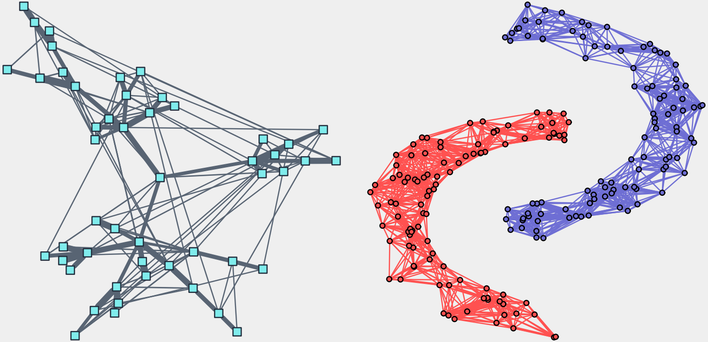


Figure: Graph showing three stock sectors of the US Stock Market, namely: Communication Services, Utilities, and Real Estate.

Graphs

Graphs

- a **set of nodes**
- a **set of edges** connecting these nodes
- many different flavours: directed (undirected), weighted (unweighted), single or multiple components
- in this thesis: **undirected**, **weighted**



Undirected Weighted Graphs

mathematically:

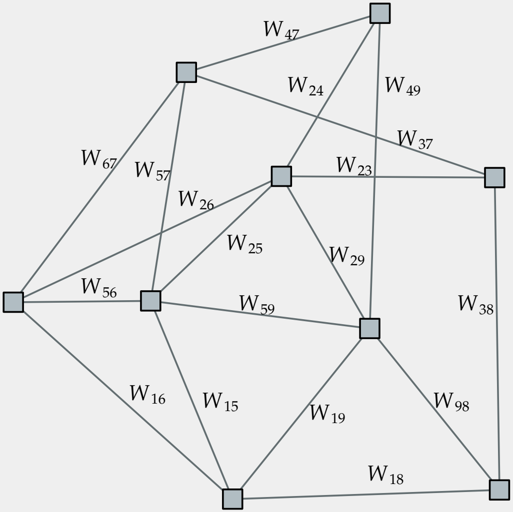
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$$

- $\mathcal{V} = \{1, 2, \dots, p\}$
- $\mathcal{E} \subseteq \{\{u, v\} : u, v \in \mathcal{V}, u \neq v\}$
- Adjacency Matrix: $\boldsymbol{W} \in \mathbb{R}_+^{p \times p}, \boldsymbol{W} = \boldsymbol{W}^\top, \mathrm{diag}(\boldsymbol{W}) = \mathbf{0}$
- Laplacian Matrix: $\boldsymbol{L} = \mathrm{Diag}(\boldsymbol{W}\mathbb{1}) - \boldsymbol{W}$
- Degree Matrix: $\boldsymbol{D} = \mathrm{Diag}(\boldsymbol{L}) = \mathrm{Diag}(\boldsymbol{W}\mathbb{1})$
- Number of components k: $\mathrm{rank}(\boldsymbol{L}) = p - k$

How to go from data to graphs?

$$\boldsymbol{X} \in \mathbb{R}^{n \times p}$$

- columns of $\boldsymbol{X}$ are **signals** generated at each node
- naive construction: pairwise correlation, norm difference, etc
- pro: **interpretability**
- con: **ignores joint dependencies** among nodes in the whole graph
- there must be a better way...



Laplacian Matrix as Precision Matrix

- **Gaussian Markov Random Field (GMRF) assumption**

$$\boldsymbol{X} \sim \mathcal{N}\left(0, \boldsymbol{L}^{\dagger}\right)$$

(P1) $\boldsymbol{L}\mathbf{1} = \mathbf{0}$

(P2) $L_{ij} = L_{ji} \leq 0 \; \forall i \neq j$

conditional correlation: $-\frac{L_{ij}}{\sqrt{L_{ii}L_{jj}}} \geq 0$

Laplacian Matrix as a Precision Matrix

- Penalized Maximum Likelihood Estimator (Lake and Tenenbaum, 2010; Egilmez et al., 2017; Zhao et al., 2019)

$$\begin{array}{ll} \underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{minimize}} & \underbrace{\operatorname{tr}\left(\boldsymbol{L}\boldsymbol{S}\right) - \log\det{}^{*}\left(\boldsymbol{L}\right)}_{\text{negative log-likelihood}} + \underbrace{\alpha h(\boldsymbol{L})}_{\text{regularizer}}, \\ \text{subject to} & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0},\ L_{ij} = L_{ji} \leq 0, \end{array}$$

- where $\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}^{\top}\boldsymbol{X}$ is the sample covariance matrix
- $\det^{*}$: pseudo-det, product of positive eigenvalues
- $h(\cdot)$ is a regularization function to impose properties on the estimated graph, *e.g.*, sparsity

Maximum Likelihood Estimator

- For connected graphs: $\det^*(\boldsymbol{L}) = \det(\boldsymbol{L} + \boldsymbol{J})$, $\boldsymbol{J} \triangleq \frac{1}{p}\mathbf{1}\mathbf{1}^\top$ (Egilmez et al., 2017)

$$\begin{array}{ll} \underset{\boldsymbol{L} \succeq \mathbf{0}}{\text{minimize}} & \operatorname{tr}(\boldsymbol{L}\boldsymbol{S}) - \log\det(\boldsymbol{L} + \boldsymbol{J}) + \alpha h(\boldsymbol{L}), \\ \text{subject to} & \boldsymbol{L}\mathbf{1} = \mathbf{0},\ L_{ij} = L_{ji} \leq 0, \end{array}$$

- pro: convex problem provided that $h(\cdot)$ is convex
- con: not scalable for disciplined convex programming languages ($p > 100$)
- con: may not be adequate in case $\boldsymbol{X}$ is heavy-tailed distributed
- **challenge**: develop scalable numerical optimization routines
- For $h(\boldsymbol{L}) = \|\boldsymbol{L}\|_1$: Block Coordinate Descent (BCD) (Egilmez et al., 2017), Alternating Direction Method of Multipliers (ADMM) (Zhao et al., 2019)
- For $h(\boldsymbol{L}) =$ concave_regularizer$(\boldsymbol{L})$: Majorization-Minimization (MM) (Sun et al., 2017) + Projected Gradient Descent (PGD) (Ying et al., 2020)

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2. **Learning Graphs in Heavy Tailed Markets**

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3. Learning Bipartite Graphs

   Bipartite structure, stock classification

4. Conclusion

   Final remarks and future works

Learning Graphs in Heavy Tailed Markets

State-of-the-art: k-component Graphs

- **Constrained Laplacian Rank** (Nie et al., 2016)
- **key property**: $\text{rank}(\boldsymbol{L}) = p - k$, k is the number of components
- **Two-stage approach:**

1. Obtain an initial adjacency matrix $\boldsymbol{W}_0$ (correlation graph, convex GMRF)
2. Find a projection of $\boldsymbol{W}_0$ that contains k-components:

$$\begin{array}{ll} \underset{\boldsymbol{W}, \boldsymbol{L} \succeq \boldsymbol{0}}{\text{minimize}} & \|\boldsymbol{W} - \boldsymbol{W}_0\|_{\mathsf{F}}^2, \\ \text{subject to} & \boldsymbol{W}\boldsymbol{1} = \boldsymbol{1}, \ \text{rank}(\boldsymbol{L}) = p - k, \\ & \boldsymbol{L} = \text{Diag}\left(\frac{\boldsymbol{W}^\top + \boldsymbol{W}}{2}\right) - \frac{\boldsymbol{W}^\top + \boldsymbol{W}}{2} \end{array}$$

- pro: simple approach with a fast alternating optimization algorithm
- con: graph estimation and k-component identification not done jointly
- con: no statistical foundation

State-of-the-art: k-component Graphs

- **Spectral Regularization** (Kumar et al., 2019)
- **key idea**: $\boldsymbol{L} = \boldsymbol{U}\mathsf{Diag}(\boldsymbol{\lambda})\boldsymbol{U}^\top$

$$\begin{array}{ll}
\underset{\boldsymbol{L},\boldsymbol{U},\boldsymbol{\lambda}}{\text{minimize}} & \underbrace{\operatorname{tr}(\boldsymbol{L}\boldsymbol{S}) - \sum_{i=1}^{p-k}\log(\lambda_i)}_{\text{neg log-likelihood}} + \underbrace{\frac{\eta}{2}\left\|\boldsymbol{L} - \boldsymbol{U}\mathsf{Diag}(\boldsymbol{\lambda})\boldsymbol{U}^\top\right\|_{\mathrm{F}}^2}_{k\text{-component structure}}, \\
\text{subject to} & \boldsymbol{L} \succeq \boldsymbol{0},\ \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0},\ L_{ij} = L_{ji} \le 0, \\
& \boldsymbol{U}^\top\boldsymbol{U} = \boldsymbol{I},\ \boldsymbol{U} \in \mathbb{R}^{p\times(p-k)}, \\
& \boldsymbol{\lambda} \in \mathbb{R}_+^{p-k},\ c_1 < \lambda_1 < \cdots < \lambda_{p-k} < c_2,\ c_1 > 0
\end{array}$$

- pros: statistically motivated, fast BCD optimization algorithm
- cons: **allows isolated nodes**, tunning η is not easy in practice

Graph Operators

- **Laplacian operator** (Kumar et al., 2020) $\mathcal{L} : \mathbb{R}_+^{p(p-1)/2} \to \mathbb{S}_+^p$, which takes a nonnegative vector $\boldsymbol{w}$ and outputs a Laplacian matrix $\boldsymbol{L}$.

Example

For $\boldsymbol{w} = [w_1, w_2, w_3]^\top$, $\mathcal{L}\boldsymbol{w} = \begin{bmatrix} w_1 + w_2 & -w_1 & -w_2 \\ -w_1 & w_1 + w_3 & -w_3 \\ -w_2 & -w_3 & w_2 + w_3 \end{bmatrix}$

- **Degree operator** (Cardoso et al., 2021): $\mathfrak{d}\boldsymbol{w} \triangleq \text{diag}\left(\mathcal{L}\boldsymbol{w}\right)$

Proposed Formulation: Connected Graphs

- **goal**: address the **heavy-tail nature** of financial returns and **leverage** that fact into the problem of **graph learning**
- assuming a **Student-t** data generating process

$$p(\boldsymbol{x}) \propto \sqrt{\det^{*}(\boldsymbol{\Theta})}\left(1+\frac{\boldsymbol{x}^{\top}\boldsymbol{\Theta}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+p}{2}}, \; \nu > 2,$$

- where $\boldsymbol{\Theta}$ is the so-called Inverse Scatter Matrix modeled as a Laplacian matrix
- robustified version of the MLE for **connected** graphs, *i.e.*

$$\begin{array}{ll} \underset{\boldsymbol{w}\geq\boldsymbol{0},\boldsymbol{\Theta}\succeq\boldsymbol{0}}{\text{minimize}} & \frac{p+\nu}{n}\sum_{i=1}^{n}\log\left(1+\frac{\boldsymbol{x}_i^{\top}\mathcal{L}\boldsymbol{w}\boldsymbol{x}_i}{\nu}\right) - \log\det\left(\boldsymbol{\Theta}+\boldsymbol{J}\right), \\ \text{subject to} & \boldsymbol{\Theta} = \mathcal{L}\boldsymbol{w}, \; \mathfrak{d}\boldsymbol{w} = \boldsymbol{d}, \end{array}$$

Proposed Formulation: k-component Graphs

- rank$(\mathcal{L}\boldsymbol{w}) = p - k$
- Fan's theorem (Fan, 1949):

$$\textstyle\sum_{i=1}^{k} \lambda_i\left(\mathcal{L}\boldsymbol{w}\right) = \underset{\boldsymbol{V}\in\mathbb{R}^{p\times k},\boldsymbol{V}^\top\boldsymbol{V}=\boldsymbol{I}}{\text{minimize}} \ \operatorname{tr}\left(\boldsymbol{V}^\top\mathcal{L}\boldsymbol{w}\boldsymbol{V}\right)$$

- k-component heavy-tailed graph learning:

$$\begin{array}{ll}\underset{\boldsymbol{w}\geq\boldsymbol{0},\boldsymbol{\Theta}\succeq 0,\boldsymbol{V}}{\text{minimize}} & \dfrac{p+\nu}{n}\textstyle\sum_{i=1}^{n}\log\left(1+\dfrac{\boldsymbol{x}_i^\top\mathcal{L}\boldsymbol{w}\boldsymbol{x}_i}{\nu}\right) - \log\det{}^*(\boldsymbol{\Theta}) + \underbrace{\eta\operatorname{tr}(\boldsymbol{V}^\top\mathcal{L}\boldsymbol{w}\boldsymbol{V})}_{\text{k-component regularizer}},\\ \text{subject to} & \boldsymbol{\Theta} = \mathcal{L}\boldsymbol{w},\ \text{rank}(\boldsymbol{\Theta}) = p-k,\ \underbrace{\mathfrak{d}\boldsymbol{w} = \boldsymbol{d}}_{\text{avoids isolated nodes}},\ \boldsymbol{V}^\top\boldsymbol{V} = \boldsymbol{I},\ \boldsymbol{V}\in\mathbb{R}^{p\times k}.\end{array}$$

- algorithms are derived from optimization frameworks: ADMM and MM

ADMM + MM Solution

- ADMM **key steps**: 1) divide, 2) relax, and 3) optimize alternatively
  - **build** the Augmented Lagrangian, e.g., connected graph Student-t case:

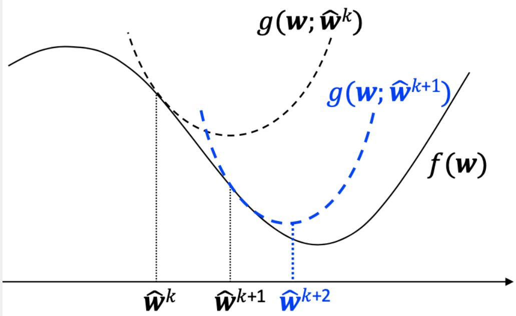
$$L_\rho(\boldsymbol{\Theta}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{y}) = \underbrace{\frac{p+\nu}{n}\sum_{i=1}^{n}\log\left(1 + \frac{\boldsymbol{x}_i^\top \mathcal{L}\boldsymbol{w}\boldsymbol{x}_i}{\nu}\right) - \log\det\left(\boldsymbol{\Theta} + \boldsymbol{J}\right)}_{\text{objective function}}$$

$$+ \underbrace{\langle \boldsymbol{y}, \mathfrak{d}\boldsymbol{w} - \boldsymbol{d}\rangle + \frac{\rho}{2}\left\|\mathfrak{d}\boldsymbol{w} - \boldsymbol{d}\right\|_2^2 + \langle \boldsymbol{Y}, \boldsymbol{\Theta} - \mathcal{L}\boldsymbol{w}\rangle + \frac{\rho}{2}\left\|\boldsymbol{\Theta} - \mathcal{L}\boldsymbol{w}\right\|_{\mathrm{F}}^2}_{\text{relaxed constraints}},$$

  - **optimize** over $\boldsymbol{w}, \boldsymbol{\Theta}, \boldsymbol{Y}, \boldsymbol{y}$ in an alternate fashion
  - observation: **not all** constraints have to be relaxed

ADMM + MM Solution

- MM key idea: approximate and solve
  - **approximate** nonconvex terms
  - **solve** the approximated problem
  - **iterate** until convergence
- heavy-tailed formulation:
  - approximate the log function by its 1st-order Taylor expansion
  - $\log\left(1+\frac{t}{b}\right) \leq \frac{t-a}{a+b} + \log\left(1+\frac{a}{b}\right)$
  - results in a sequences of "Gaussianized" problems with weighted sample covariance matrix



Experimental Results

Reproducibility

Open Source Software Package

https://github.com/convexfi/fingraph

Datasets and Benchmark Algorithms

Datasets (Log-returns)

- US Stock Market ($p = 82$ S&P500 stocks, from three sectors, $n = 1006$ daily observations)
- Foregin Exchange ($p = 34$ currencies, $n = 522$ daily observations)
- Cryptocurrencies ($p = 41$ most traded cryptos, $n = 1218$ daily observations)

- data matrix $\boldsymbol{X}$ constructed as:

$$X_{ij} = \log P_{i,j} - \log P_{i-1,j},$$

- $P_{i,j}$: is the closing price of the j-th instrument at the i-th day
- sector information on stocks are given by the Global Industry Classification System (GICS) (Morgan Stanley Capital International and S&P Dow Jones, 2018)

Datasets and Benchmark Algorithms

Benchmark Models (Connected Graphs)

- Gaussian formulation with ℓ_1-norm for sparsity (GLE) (Zhao et al., 2019; Egilmez et al., 2017)
- Gaussian formulation with concave regularizer for sparsity (NGL) (Ying et al., 2020)

Benchmark Models (k-component graphs)

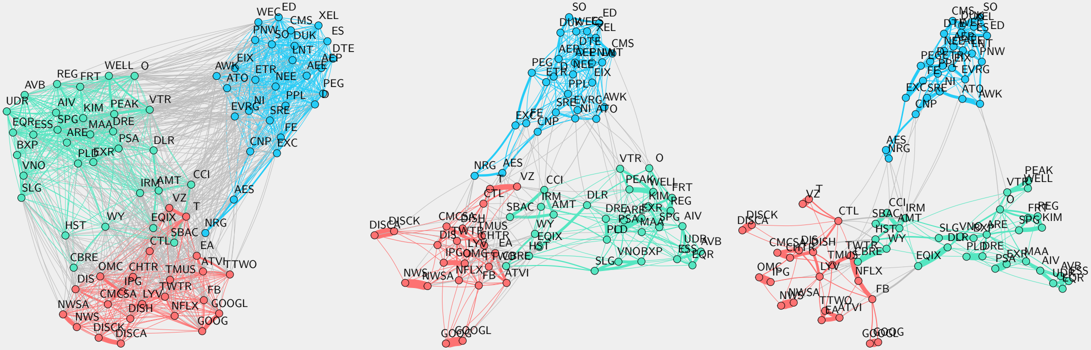
- Constrained Laplacian Rank (CLR) (Nie et al., 2016)
- Gaussian formulation with Spectral Constraints (SGL) (Kumar et al., 2019)

Performance Criteria

- Graph **modularity:** $Q(\mathcal{G}) \triangleq \frac{1}{2|\mathcal{E}|} \sum_{i,j \in \mathcal{V}} \left(\boldsymbol{W}_{ij} - \frac{d_i d_j}{2|\mathcal{E}|} \right) \delta(t_i = t_j)$, where d_i is the degree of the i-th node, t_i is the type (or label) of the i-th node, and $\delta(\cdot)$ is the indicator function

(a) modularity $= 0.1$ **(b)** modularity $= 0.37$ **(c)** modularity $= 0.7$

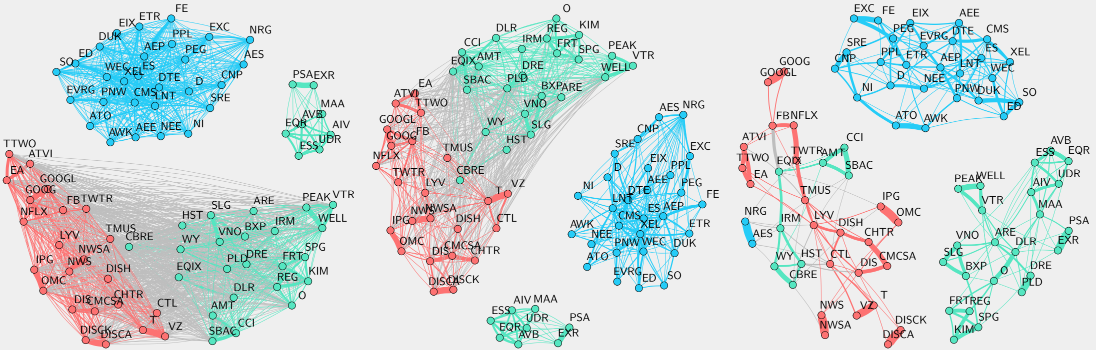
US Stock Market - Connected Graphs



(a) GLE, modularity $= 0.31$ **(b)** NGL, modularity $= 0.49$ **(c)** proposed, modularity $= 0.54$

- our method: **sparser** than Gaussian-based methods and shows **higher** modularity

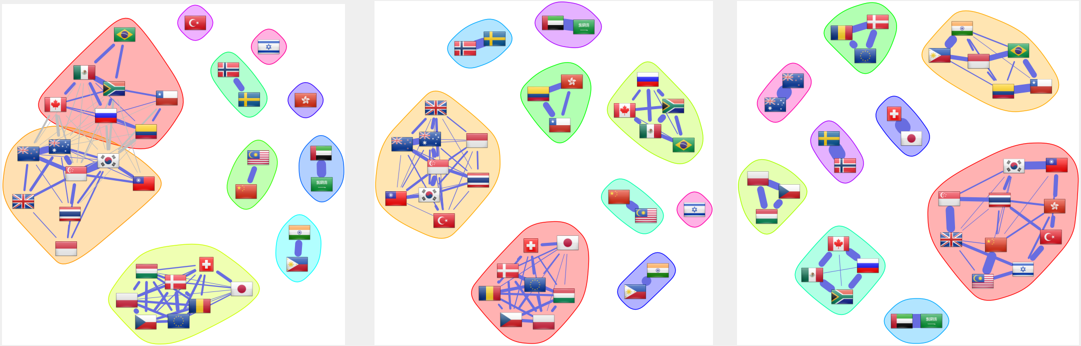
US Stock Market - k-component Graphs



(a) SGL, modularity $= 0.29$ **(b)** CLR, modularity $= 0.33$ **(c)** proposed, modularity $= 0.56$

- our method: clusters are more aligned with industry classification

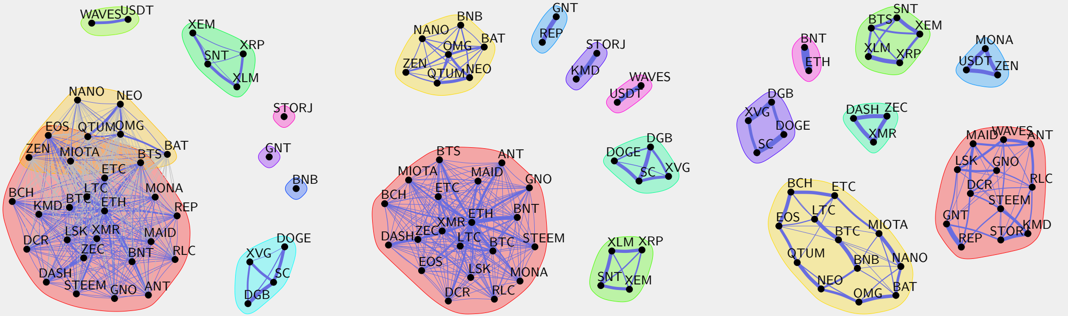
Foreign Exchange - k-component Graphs



(a) SGL, modularity $= 0.62$ **(b)** CLR, modularity $= 0.79$ **(c)** proposed, modularity $= 0.84$

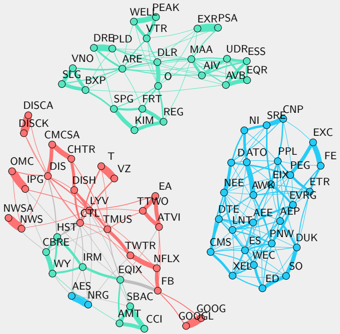
- our method: no isolated nodes, more reasonable clusters ({Australia & New Zealand}, {Hungary, Czech Republic, & Poland})

Cryptocurrencies - k-component Graphs

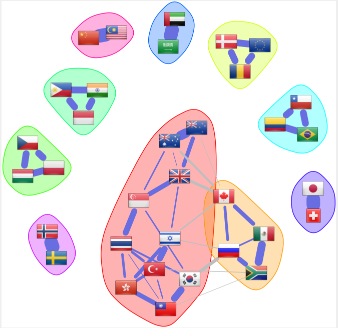


(a) SGL, modularity $= 0.36$ **(b)** CLR, modularity $= 0.66$ **(c)** proposed, modularity $= 0.79$

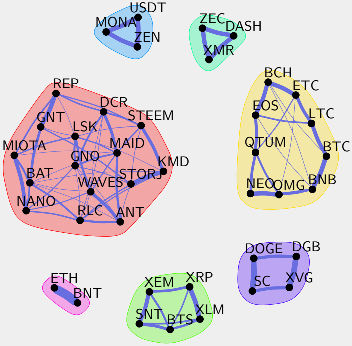
Effect of Initialization



(a) modularity $= 0.56$

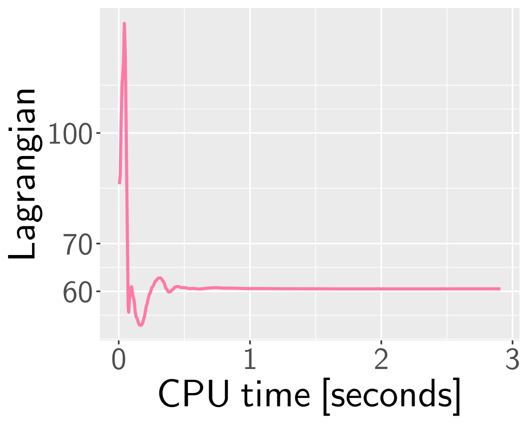


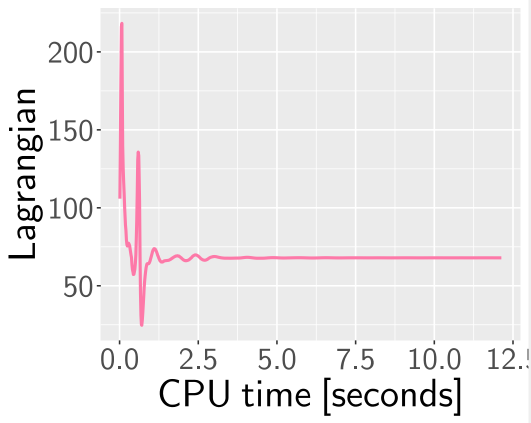
(b) modularity $= 0.80$



(c) modularity $= 0.78$

Empirical Convergence





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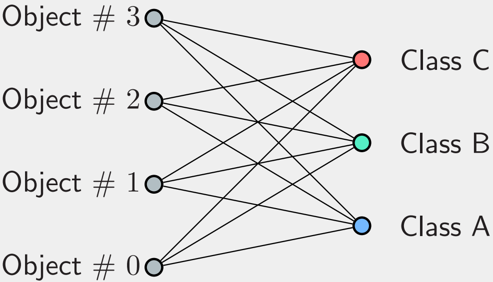
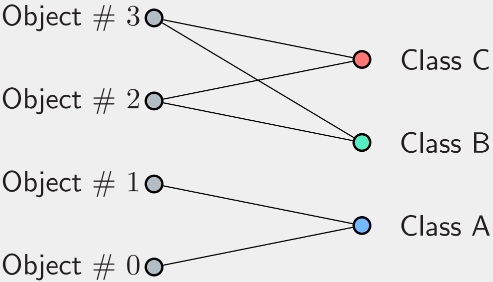
Learning Bipartite Graphs

Bipartite Graphs

- a **single component** bipartite graph:



- a 2-**component** bipartite graph:



Undirected Weighted Bipartite Graphs

$$\mathcal{G} = (\mathcal{V}_r, \mathcal{V}_q, \mathcal{E}, \boldsymbol{L})$$

- $\mathcal{V}_r = \{1, 2, \ldots, r\}$: **objects**
- $\mathcal{V}_q = \{r+1, r+2, \ldots, r+q\}$: **classes**
- $\mathcal{E} \subseteq \{\{u, v\} : u \in \mathcal{V}_r, v \in \mathcal{V}_q\}$
- **Laplacian Matrix:** $\boldsymbol{L} = \begin{bmatrix} \mathrm{Diag}\left(\boldsymbol{B}\mathbf{1}_q\right) & -\boldsymbol{B} \\ -\boldsymbol{B}^\top & \mathrm{Diag}\left(\boldsymbol{B}^\top\mathbf{1}_r\right) \end{bmatrix}, \boldsymbol{B} \in \mathbb{R}_+^{r \times q}$
- B_{ij}: edge weight between object i and class j

State-of-the-art Methods

- **Bipartite Structure** (Nie et al., 2017)

$$
\begin{aligned}
\underset{\boldsymbol{B},\boldsymbol{V}\in\mathbb{R}^{p\times k}}{\text{minimize}} \quad & \|\boldsymbol{B}-\boldsymbol{A}\|_{\mathrm{F}}^2 + \eta\,\mathrm{tr}\left(\boldsymbol{V}^\top \begin{bmatrix} \boldsymbol{I}_r & -\boldsymbol{B} \\ -\boldsymbol{B}^\top & \mathsf{Diag}\left(\boldsymbol{B}^\top \mathbf{1}_r\right) \end{bmatrix} \boldsymbol{V}\right), \\
\text{subject to} \quad & \boldsymbol{B} \geq \mathbf{0},\ \boldsymbol{B}\mathbf{1}_q = \mathbf{1}_r,\ \boldsymbol{V}^\top\boldsymbol{V} = \boldsymbol{I}_k
\end{aligned}
$$

- alternating optimization algorithm
- pros: simple optimization that works well in practice
- cons: lacks statistical foundations

State-of-the-art Methods

- **Spectral Regularization** (Kumar et al., 2020)

$$
\begin{aligned}
\underset{\boldsymbol{w}\geq\boldsymbol{0},\boldsymbol{V},\boldsymbol{U},\boldsymbol{\psi},\boldsymbol{\lambda}}{\text{minimize}} \quad & \operatorname{tr}\left(\mathcal{L}\boldsymbol{w}\boldsymbol{S}\right) - \log\det{}^{*}\left(\mathcal{L}\boldsymbol{w}\right) + \underbrace{\frac{\gamma}{2}\left\|\mathcal{A}\boldsymbol{w} - \boldsymbol{U}\mathsf{Diag}(\boldsymbol{\psi})\boldsymbol{U}^{\top}\right\|_{\mathrm{F}}^{2}}_{\text{bipartite structure}} \\
& + \underbrace{\frac{\beta}{2}\left\|\mathcal{L}\boldsymbol{w} - \boldsymbol{V}\mathsf{Diag}(\boldsymbol{\lambda})\boldsymbol{V}^{\top}\right\|_{\mathrm{F}}^{2}}_{k\text{-component structure}}, \\
\text{subject to} \quad & \boldsymbol{U}^{\top}\boldsymbol{U} = \boldsymbol{I}, \boldsymbol{V}^{\top}\boldsymbol{V} = \boldsymbol{I},\ \boldsymbol{\lambda} \in C_{\boldsymbol{\lambda}}, \boldsymbol{\psi} \in C_{\boldsymbol{\psi}} \\
& C_{\boldsymbol{\lambda}} = \left\{\boldsymbol{\lambda} \in \mathbb{R}_{+}^{(p-k)} c_1 \leq \lambda_1, \ldots, \leq \lambda_{p-k} \leq c_2\right\}, \\
C_{\boldsymbol{\psi}} = \{\boldsymbol{\psi} \in \mathbb{R}^{p} : \psi_i = -\psi_{p+1-i},\ & i = 1, 2, \ldots, p, \psi_1 \geq \psi_2 \geq \cdots \geq \psi_p\}.
\end{aligned}
$$

- BCD-like optimization algorithm
- pros: clever idea with statistical foundations!
- cons: tuning γ, β, c_1, and c_2 is difficult, postprocessing often needed!

Proposed Formulations

Connected Bipartite Graphs: Gaussian Case

- Gaussian:

$$\begin{array}{ll} \underset{\boldsymbol{L},\boldsymbol{B}}{\text{minimize}} & \operatorname{tr}\left(\boldsymbol{L}\boldsymbol{S}\right) - \log\det\left(\boldsymbol{L}+\boldsymbol{J}\right), \\ \text{subject to} & \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_r & -\boldsymbol{B} \\ -\boldsymbol{B}^\top & \text{Diag}\left(\boldsymbol{B}^\top\mathbf{1}_r\right) \end{bmatrix}, \boldsymbol{B} \geq \mathbf{0}, \boldsymbol{B}\mathbf{1}_q = \mathbf{1}_r, \end{array}$$

- $\boldsymbol{B}\mathbf{1}_q = \mathbf{1}_r$: normalizes the degrees of the set of objects
- **key idea**: simpler formulation in practice by plugging in the equality constraints and using the classical matrix determinant Lemma (Zhang, 2005):

$$\begin{array}{ll} \underset{\boldsymbol{B}\geq\mathbf{0},\boldsymbol{B}\mathbf{1}_q=\mathbf{1}_r}{\text{minimize}} & -\log\det\left(\text{Diag}(\boldsymbol{B}^\top\mathbf{1}_r) + \boldsymbol{J}_{qq} - (\boldsymbol{B}-\boldsymbol{J}_{rq})^\top(\boldsymbol{I}_r+\boldsymbol{J}_{rr})^{-1}(\boldsymbol{B}-\boldsymbol{J}_{rq})\right) \\ & +\operatorname{tr}\left(\boldsymbol{B}\boldsymbol{C}\right) \end{array}$$

- massive **reduction** in computational complexity!
- algorithm: projected gradient descent (Bertsekas, 1999)

Connected Bipartite Graphs: Student-t Case

- Student-t:

$$\begin{aligned} \underset{\boldsymbol{L},\boldsymbol{B}}{\text{minimize}} \quad & -\log\det\left(\boldsymbol{L}+\boldsymbol{J}\right)+\frac{p+\nu}{n}\sum_{i=1}^{n}\log\left(1+\tfrac{1}{\nu}\boldsymbol{x}_i^\top\boldsymbol{L}\boldsymbol{x}_i\right), \\ \text{subject to} \quad & \boldsymbol{L}=\begin{bmatrix}\boldsymbol{I}_r & -\boldsymbol{B} \\ -\boldsymbol{B}^\top & \mathsf{Diag}\left(\boldsymbol{B}^\top\mathbf{1}_r\right)\end{bmatrix}, \ \boldsymbol{B}\geq\mathbf{0}, \ \boldsymbol{B}\mathbf{1}_q=\mathbf{1}_r. \end{aligned}$$

- like in the Gaussian case, a formulation as a function of $\boldsymbol{B}$ can be obtained:

$$\begin{aligned} \underset{\boldsymbol{B}\geq\mathbf{0},\ \boldsymbol{B}\mathbf{1}_q=\mathbf{1}_r}{\text{minimize}} \quad & -\log\det\left(\mathsf{Diag}(\boldsymbol{B}^\top\mathbf{1}_r)+\boldsymbol{J}_{qq}-(\boldsymbol{B}-\boldsymbol{J}_{rq})^\top(\boldsymbol{I}_r+\boldsymbol{J}_{rr})^{-1}(\boldsymbol{B}-\boldsymbol{J}_{rq})\right) \\ & +\frac{p+\nu}{n}\sum_{i=1}^{n}\log\left(1+\frac{h_i+\mathrm{tr}\left(\boldsymbol{B}\boldsymbol{G}_i\right)}{\nu}\right) \end{aligned}$$

- algorithm: MM to deal with the concave terms

k-component Bipartite Graphs

- Student-t, k-component, bipartite graph

$$\begin{array}{ll}\underset{\boldsymbol{L}\succeq\mathbf{0},\boldsymbol{B}}{\text{minimize}} & \dfrac{p+\nu}{n}\sum_{i=1}^{n}\log\left(1+\dfrac{h_i+\operatorname{tr}\left(\boldsymbol{B}\boldsymbol{G}_i\right)}{\nu}\right)-\log\det^{*}\left(\boldsymbol{L}\right),\\ \text{subject to} & \boldsymbol{L}=\begin{bmatrix}\boldsymbol{I}_r & -\boldsymbol{B}\\ -\boldsymbol{B}^{\top} & \text{Diag}\left(\boldsymbol{B}^{\top}\mathbf{1}_r\right)\end{bmatrix},\ \text{rank}(\boldsymbol{L})=p-k,\ \boldsymbol{B}\geq\mathbf{0},\boldsymbol{B}\mathbf{1}_q=\mathbf{1}_r,\end{array}$$

- algorithmic solution: ADMM + MM

Experimental Results

Reproducibility

Open Source Software Packages

https://github.com/convexfi/bipartite

Experiments

Datasets (Log-returns)

- US Stock Market ($r = 333$ S&P500 stocks $q = 8$ S&P Sector Indices, from Jan. 5th 2016 to Jan. 5th 2021, $n = 1291$ daily observations)
- data matrix $\boldsymbol{X}$ constructed as:

$$X_{ij} = \log P_{i,j} - \log P_{i-1,j},$$

- $P_{i,j}$: is the closing price of the j-th instrument at the i-th day.

Benchmark Models

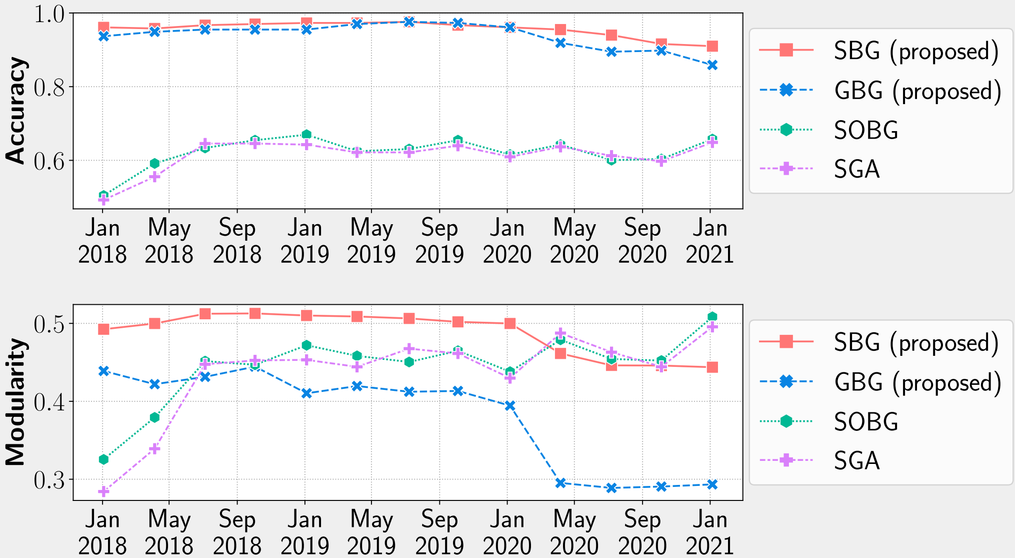
- Bipartite structure: SOBG, connected ($k = 1$) and k-components ($k > 1$) (Nie et al., 2017)
- Spectral regularization methods: SGA (connected), SGLA (k-components) (Kumar et al., 2020)

Experiments

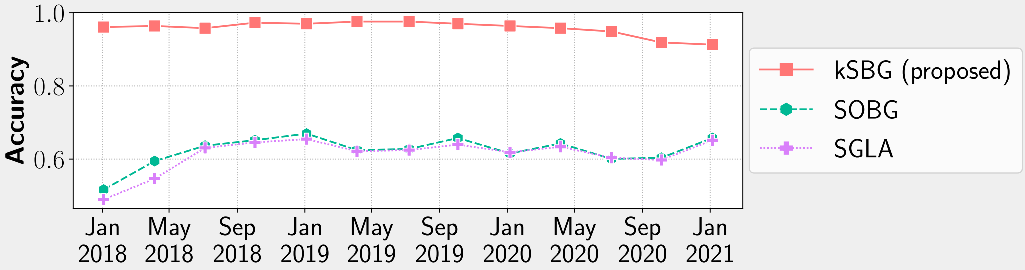
Performance Criteria

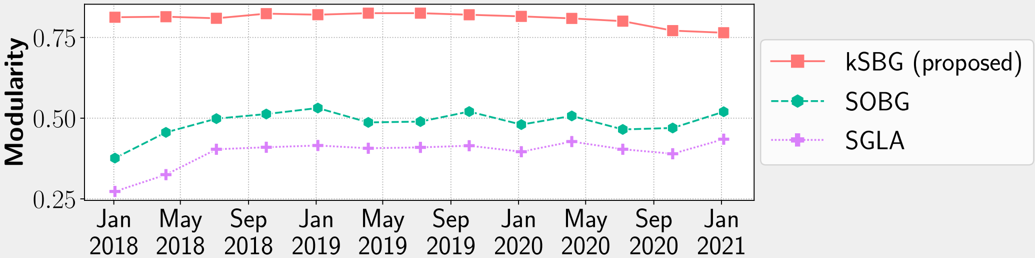
- Graph **modularity**
- Node **accuracy**: fraction of nodes whose sectors agree with those from the Global Industry Classification Standard (GICS) (Morgan Stanley Capital International and S&P Dow Jones, 2018)

Rolling Window Results: Connected Graphs

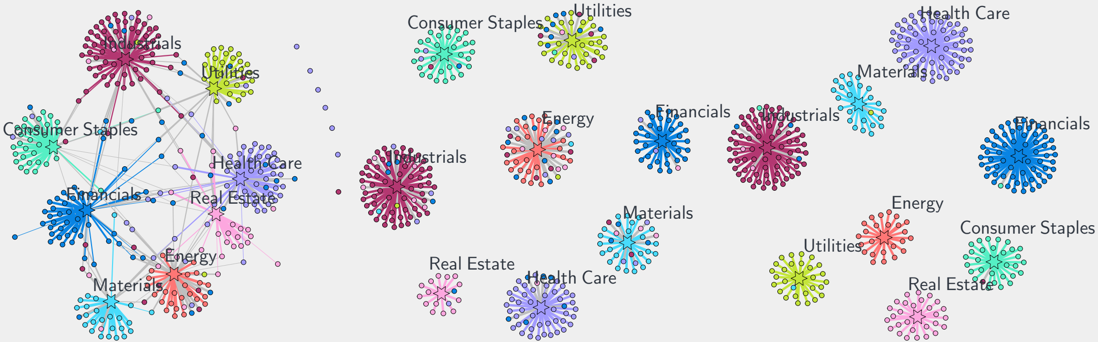


Rolling Window Results: $k = 8$-components



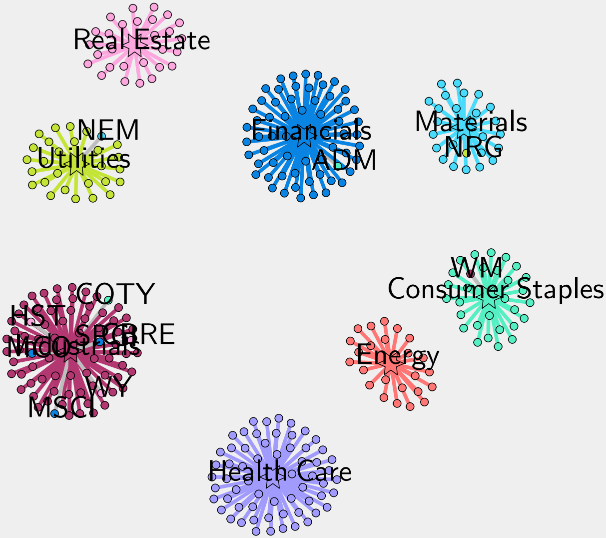


k-component Bipartite Graphs

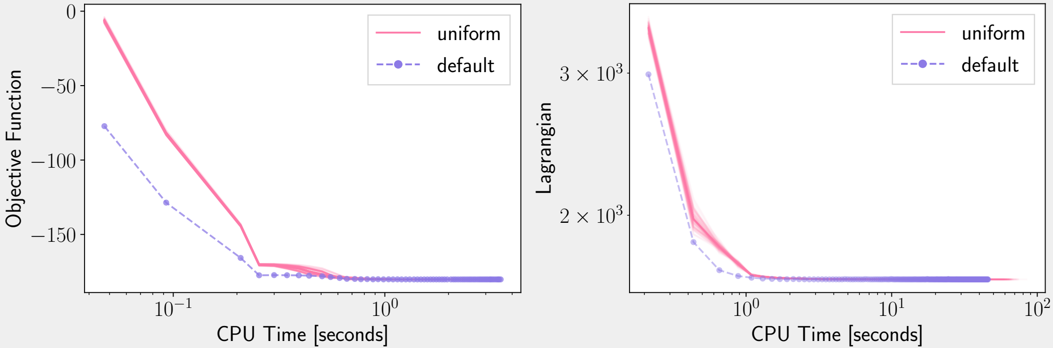


(a) SGLA, accuracy $= 0.77$, modularity $= 0.56$ **(b)** SOBG, accuracy $= 0.75$, modularity $= 0.61$ **(c)** proposed, accuracy $= 0.97$, modularity $= 0.82$

k-component Bipartite Graphs



Robustness to the choice of initial point



Conclusions

Conclusions

- graph learning formulations have received substantial attention from the scientific community in recent years
- **modeled** a financial networks as undirected graphs
- **developed** formulations as well as efficient algorithms to estimate the Laplacian matrix
  - k-component
  - bipartite
  - joint k-component & bipartite
- **worked** on heavy-tailed scenarios envisioning practical applications in finance
- **applied** the estimated graphs into clustering tasks of financial stocks and evaluate their performance via modularity and accuracy
- open source software for research **reproducibility** is made available on GitHub

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Thank you very much!