Breaking Down Risk Parity Portfolios: A Practical Open Source Implementation

a talk by

Zé Vinícius
(in collab. with Daniel Palomar)

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whoami

- made in Brazil
- electrical engineer by training, computer scientist by passion/experience
- guest researcher at NIST[1]
- software developer for Google summer of code with OpenAstronomy
- software engineer intern NASA[2]
- **since Sept. 2019**: phd student at HKUST[3]

[1] National Institute of Standards and Technology, Washington DC, US
[2] National Aeronautics and Space Administration, Ames Research Center, Silicon Valley, US
[3] The Hong Kong University of Science and Technology

Portfolio Optimization

Stock Data


CMHK
4:11 PM
7%
Stocks
November 10
Edit
Search
AAPL
Apple Inc.
260.14
+0.27%
MSFT
Microsoft Corporat...
145.96
+1.18%
AMZN
Amazon.com, Inc.
1,785.88
-0.13%
NFLX
Netflix, Inc.
291.57
+0.69%
GOOG
Alphabet Inc.
1,311.37
+0.19%
Business News
From Yahoo
YAHOO!

Portfolio Optimization

- problem: how to allocate B amount of money into N assets?

[4] 1990 Nobel Memorial Prize in Economic Sciences

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- assume the log-returns $\boldsymbol{r}$ are i.i.d. Gaussian, then for a portfolio $\boldsymbol{w}$:
  - expected return: $\mathbb{E}\left[\boldsymbol{w}^\top \boldsymbol{r}\right] = \boldsymbol{w}^\top \boldsymbol{\mu}$
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- Dr. Harry Markowitz[4] (1952):

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\top \boldsymbol{\mu} - \lambda \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \succeq \boldsymbol{0},\ \boldsymbol{1}^\top \boldsymbol{w} = 1 \end{array}$$

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- criticisms:
  1. sensitivity to estimation errors in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
  2. does not consider risk diversification

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Risk Parity

From Capital to Risk Allocation

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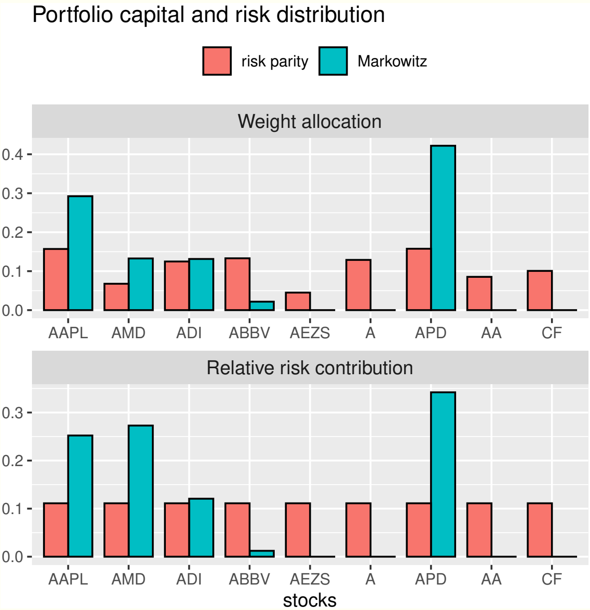
- allocate **risk** rather than **capital**
- first risk parity fund (1996): Bridgewater's All Weather
- Bridgewater publishes "Engineering Targeted Returns and Risks" (2004)
- 2011-onwards: risk parity gains broad adoption
- **basic idea**: design a portfolio such that the risk is equally distributed among the asset classes (stocks, bonds, real state, etc.)

From Capital to Risk Allocation



Risk Parity: Problem Formulation

- **risk parity portfolio**: $\mathsf{RCC}_i = b_i, \; i = 1, 2, \ldots, N$
- **feasibility** problem:

$$\begin{array}{ll} \underset{\boldsymbol{w} \succ \boldsymbol{0}}{\text{find}} & \boldsymbol{w} \\ \text{subject to} & \dfrac{w_i \left(\boldsymbol{\Sigma} \boldsymbol{w}\right)_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}} = b_i, \; i = 1, 2, \ldots, N \end{array}$$

  doesn't look trivial
- **approximation**: Σ is diagonal

$$w_i \propto \frac{\sqrt{b_i}}{\sqrt{\boldsymbol{\Sigma}_{ii}}}$$

  i.e. **inverse volatility portfolio**

Solution to Risk Parity

- Spinu (2013):
  - change of variables $\boldsymbol{x} = \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$, then $\boldsymbol{w} = \frac{\boldsymbol{x}}{\mathbf{1}^\top \boldsymbol{x}}$
  - problem becomes: $\boldsymbol{\Sigma} \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}$
  - $\underset{\boldsymbol{x} \succ \mathbf{0}}{\text{minimize}} \; f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} - \boldsymbol{b}^\top \log(\boldsymbol{x})$
  - **optimality condition:** $\nabla f(\boldsymbol{x}) = \boldsymbol{\Sigma} \boldsymbol{x} - \frac{\boldsymbol{b}}{\boldsymbol{x}} = \mathbf{0}$
  - then done? it's not a QP, LP, etc.

Cyclical Coordinate Descent (CCD)

Griveau-Billion (2013):

Algorithm 1: CCD to solve risk parity

$k \leftarrow 0$, initial $\boldsymbol{x}^{(0)}$

while *not converged* **do**

  for $i = 1$ *to* N **do**

    $x_i^{k+1} \leftarrow \arg\min_{x_i} f\left(x_1^{k+1}, \ldots, x_i^k, \ldots, x_N^k\right)$

  end

end

- closed-form update:

$$x_i^\star = \frac{-\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right) + \sqrt{\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right)^2 + 4\boldsymbol{\Sigma}_{ii} b_i}}{2\boldsymbol{\Sigma}_{ii}},$$

where $\boldsymbol{x}_{-i} = \left[x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N\right]^\top$

Non-convex Formulations

- limitations of the previous formulation: $\boldsymbol{w} \succ \mathbf{0}, \mathbf{1}^\top \boldsymbol{w} = 1$

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- Bruder & Roncalli (2012):

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\underset{\boldsymbol{w}}{\text{minimize}} & \sum_{i=1}^{N} \left(\dfrac{w_i (\boldsymbol{\Sigma} \boldsymbol{w})_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}} - b_i \right)^2 \\
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- how to "solve" the non-convex formulation?
- general purpose solvers: slow ☹

Non-convex Formulations

- Feng & Palomar (2015):

$$
\begin{array}{ll}
\underset{\boldsymbol{w}}{\text{minimize}} & \sum_{i=1}^{N} \left[g_i(\boldsymbol{w})\right]^2 + \lambda F(\boldsymbol{w}) \\
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- "nice" approximation for $g_i(\boldsymbol{w})$: first order Taylor expansion (Feng & Palomar 2015)

Open Source Software Packages

- dppalomar/riskParityPortfolio (R version)
- dppalomar/riskparity.py (Python version)

Basic Usage

- R version:

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```
> library(riskParityPortfolio)
> my_portfolio <- riskParityPortfolio(Sigma = Sigma, b = b)
> names(my_portfolio)
[1] "w"                  "risk_contributions"
```

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- R version:

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- Python version:

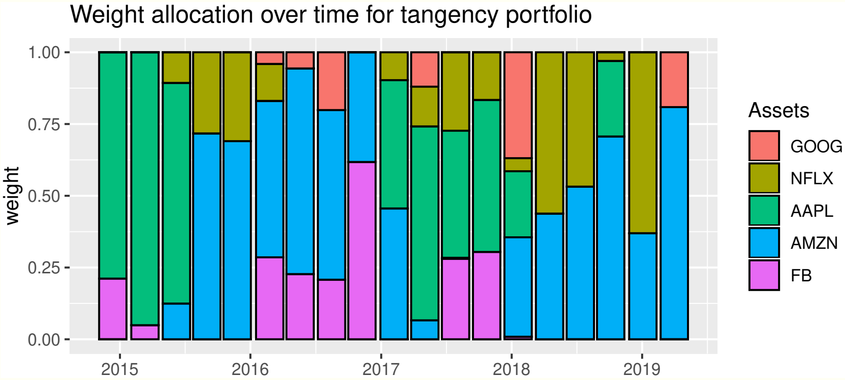
```
>>> import riskparityportfolio as rp
>>> my_portfolio = rp.RiskParityPortfolio(covariance=Sigma,
                                          budget=b)
>>> my_portfolio.design()
>>> my_portfolio.weights.numpy()
>>> my_portfolio.risk_contributions.numpy()
```

Practical Example

```r
library(portfolioBacktest)
library(riskParityPortfolio)
# download price data
faang_data <- stockDataDownload(
                    c("GOOG", "NFLX", "AAPL", "AMZN", "FB"),
                     from = "2014-01-01", to = "2019-06-25")
risk_parity <- function(dataset) {
  prices <- dataset$adjusted
  log_returns <- diff(log(prices))[-1]
  return(riskParityPortfolio(cov(log_returns))$w)
}
bt <- portfolioBacktest(
        list("risk parity" = risk_parity,
             "tangency"    = max_sharpe_ratio),
        list(faang_data), T_rolling_window = 12*20,
        optimize_every = 3*20, rebalance_every = 3*20)
```

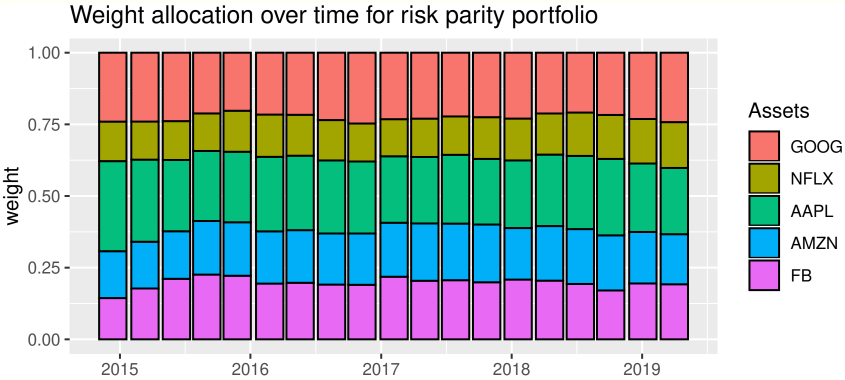
Practical Example

```r
backtestChartStackedBar(bt, portfolio = "tangency",
                        legend = TRUE)
```



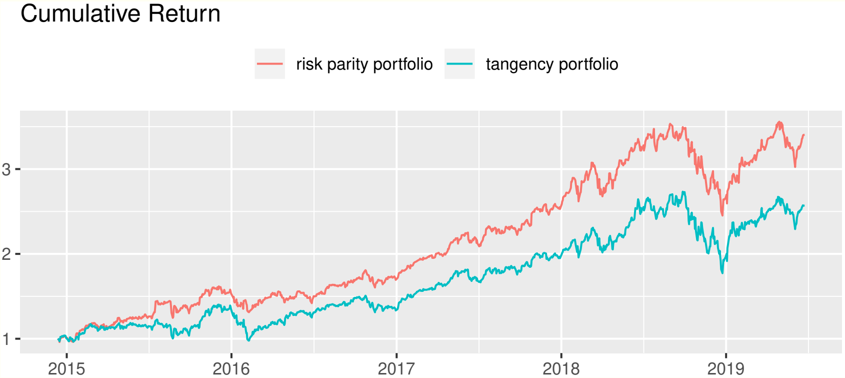
Practical Example

```r
backtestChartStackedBar(bt, portfolio = "risk parity",
                        legend = TRUE)
```



Practical Example

```
backtestChartCumReturns(bt) + theme(legend.position="top")
```



Testimonials

"I can easily run optimizations across new covariance matrices in seconds, which has helped streamline portfolio allocation testing"

Jonathan Dane, CFA. Director of Portfolio Strategy & Market Research, The Coury Firm.

"**riskParityPortfolio** provides a state-of-the-art implementation of rpps, otherwise only available at the top quantitative hedge funds"

Tharsis Souza, PhD. Director of Strategic Innovation, Axioma Inc. Founder, OpenQuants.com.

Disclaimer: views, thoughts, and opinions expressed in the text belong solely to the author, and not necessarily to the author's employer, organization, committee or other group or individual.

Takeaways

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1. risk parity represents a shift from **capital** allocation to **risk** allocation
2. risk parity portfolios have been praised for their robustness in different market "weathers"
3. non-convex? no problem!
4. GitHub repositories to stay tuned:
    - dppalomar/riskParityPortfolio
    - dppalomar/riskparity.py
    - dppalomar/portfolioBacktest

Thank you! Questions?

References

- H .M. Markowitz. "Portfolio selection". The Journal of Finance. 7 (1): 77–91, 1952.
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- T. Souza, "DIY Ray Dalio ETF: How to build your own Hedge Fund strategy with risk parity portfolios". `https://www.openquants.com`