Risk Parity: Convex, Non-convex, and Hierarchical Models

a talk by

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(in collab. with Prof. Daniel Palomar)

ELEC/IEDA3180 Data-driven Portfolio Optimization

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Contents

1. Markowitz and Portfolio Optimization
   Modern Portfolio Theory 101
2. Risk Parity Portfolio
   Historical events, convex, non-convex formulations, and bleeding-edge solvers in R and Python
3. Hierarchical Risk Parity
   Combining graph theory and portfolio design
4. Conclusions
   Some closing thoughts

Modern Portfolio Theory

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- problem: how to allocate B amount of money into N assets?

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- assume the log-returns $\boldsymbol{r}$ are i.i.d. Gaussian, then for a portfolio $\boldsymbol{w}$:
  - expected return: $\mathbb{E}\left[\boldsymbol{w}^\top \boldsymbol{r}\right] = \boldsymbol{w}^\top \boldsymbol{\mu}$
  - variance: $\mathbb{V}\left[\boldsymbol{w}^\top \boldsymbol{r}\right] = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$

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- criticisms:
  1. sensitivity to estimation errors in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
  2. does not consider risk diversification

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Risk Parity

From Capital to Risk Allocation

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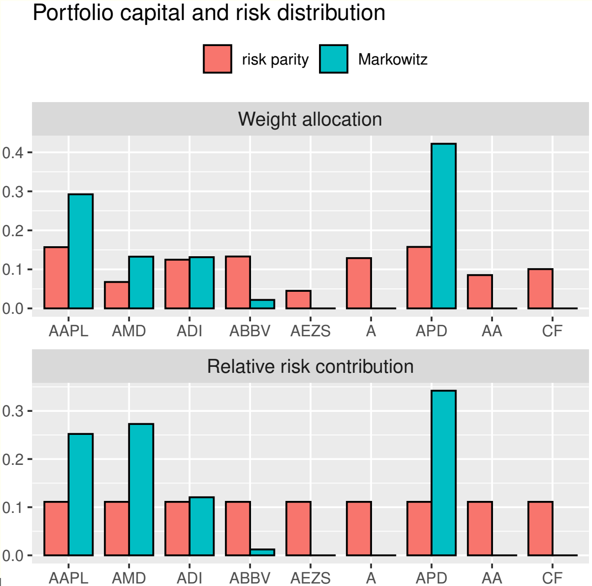
- first risk parity fund (1996): Bridgewater’s All Weather
- Bridgewater publishes "Engineering Targeted Returns and Risks" (2004)
- 2011-onwards: risk parity gains broad adoption
- **basic idea**: design a portfolio such that the risk is equally distributed among the asset classes (stocks, bonds, real state, etc.)

From Capital to Risk Allocation



Problem Formulation

- volatility: $\sigma(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}$
- from Euler's theorem:

$$\sigma(\boldsymbol{w}) = \sum_i w_i \frac{\partial \sigma(\boldsymbol{w})}{\partial w_i} = \sum_i \frac{w_i \left(\boldsymbol{\Sigma}\boldsymbol{w}\right)_i}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$$

- $\dfrac{\partial \sigma(\boldsymbol{w})}{\partial w_i}$: **marginal risk contribution**
- measures the sensitivity of the portfolio volatility to the i-th asset
- it can be defined for other risk measures like VaR and CVaR

Problem Formulation

- **risk contribution** of the i-th asset to the total risk $\sigma(\boldsymbol{w})$:

$$\mathsf{RC}_i = w_i \frac{\partial \sigma(\boldsymbol{w})}{\partial w_i}$$

- from Euler's theorem:

$$\sum_i \mathsf{RC}_i = \sigma(\boldsymbol{w})$$

- **relative risk contribution** of the i-th asset to the total risk $\sigma(\boldsymbol{w})$:

$$\mathsf{RCC}_i = \frac{\mathsf{RC}_i}{\sigma(\boldsymbol{w})} = \frac{w_i \left(\boldsymbol{\Sigma}\boldsymbol{w}\right)_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma}\boldsymbol{w}},$$

so that

$$\sum_i \mathsf{RCC}_i = 1$$

Risk Parity: Problem Formulation

- **risk parity portfolio**: $\mathsf{RCC}_i = b_i,\ i = 1, 2, \ldots, N$
- **feasibility** problem:

$$\begin{array}{ll} \underset{\boldsymbol{w} \succ \mathbf{0}}{\text{find}} & \boldsymbol{w} \\ \text{subject to} & \dfrac{w_i \left(\boldsymbol{\Sigma}\boldsymbol{w}\right)_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma}\boldsymbol{w}} = b_i,\ i = 1, 2, \ldots, N \end{array}$$

  doesn't look trivial
- **approximation**: Σ is diagonal

$$w_i \propto \frac{\sqrt{b_i}}{\sqrt{\boldsymbol{\Sigma}_{ii}}}$$

  i.e. **inverse volatility portfolio**

Solution to Risk Parity

- Spinu (2013):
  - change of variables $\boldsymbol{x} = \dfrac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$, then $\boldsymbol{w} = \dfrac{\boldsymbol{x}}{\mathbf{1}^\top \boldsymbol{x}}$
  - problem becomes: $\boldsymbol{\Sigma}\boldsymbol{x} = \dfrac{\boldsymbol{b}}{\boldsymbol{x}}$
  - $\underset{\boldsymbol{x} \succ \mathbf{0}}{\text{minimize}}\ f(\boldsymbol{x}) = \dfrac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} - \boldsymbol{b}^\top \log(\boldsymbol{x})$
  - **optimality condition:** $\nabla f(\boldsymbol{x}) = \boldsymbol{\Sigma}\boldsymbol{x} - \dfrac{\boldsymbol{b}}{\boldsymbol{x}} = \mathbf{0}$
  - then done? it's not a QP, LP, etc.

Cyclical Coordinate Descent (CCD)

Griveau-Billion (2013):

Algorithm 1: CCD to solve risk parity

$k \leftarrow 0$, initial $\boldsymbol{x}^{(0)}$
while *not converged* **do**
  for $i = 1$ *to* N **do**
    $x_i^{k+1} \leftarrow \arg\min_{x_i} f\left(x_1^{k+1}, \ldots, x_i^k, \ldots, x_N^k\right)$
  end
end

- closed-form update:

$$x_i^\star = \frac{-\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right) + \sqrt{\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right)^2 + 4\boldsymbol{\Sigma}_{ii} b_i}}{2\boldsymbol{\Sigma}_{ii}},$$

where $\boldsymbol{x}_{-i} = [x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N]^\top$

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- general purpose solvers: slow ☹

Non-convex Formulations

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$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \sum_{i=1}^{N}\left[g_i(\boldsymbol{w})\right]^2 + \lambda F(\boldsymbol{w}) \\ \text{subject to} & \mathbf{1}^\top \boldsymbol{w} = 1,\ \boldsymbol{w} \in \mathcal{W} \end{array}$$

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- "nice" approximation for $g_i(\boldsymbol{w})$: first order Taylor expansion (Feng & Palomar 2015)

Open Source Software Packages

- dppalomar/riskParityPortfolio (R version)
- dppalomar/riskparity.py (Python version)

Basic Usage

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```
> library(riskParityPortfolio)
> my_portfolio <- riskParityPortfolio(Sigma = Sigma, b = b)
> names(my_portfolio)
[1] "w"                 "risk_contributions"
```

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- Python version:

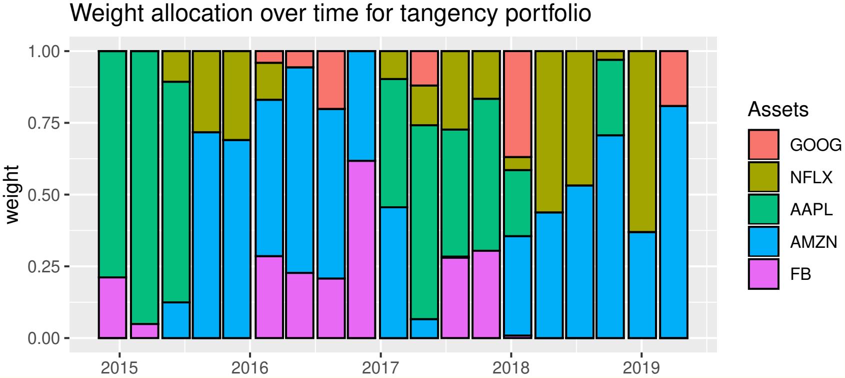
```
>>> import riskparityportfolio as rp
>>> my_portfolio = rp.RiskParityPortfolio(covariance=Sigma,
                                          budget=b)
>>> my_portfolio.design()
>>> my_portfolio.weights.numpy()
>>> my_portfolio.risk_contributions.numpy()
```

Practical Example

```r
library(portfolioBacktest)
library(riskParityPortfolio)
# download price data
faang_data <- stockDataDownload(
                    c("GOOG", "NFLX", "AAPL", "AMZN", "FB"),
                     from = "2014-01-01", to = "2019-06-25")
risk_parity <- function(dataset) {
  prices <- dataset$adjusted
  log_returns <- diff(log(prices))[-1]
  return(riskParityPortfolio(cov(log_returns))$w)
}
bt <- portfolioBacktest(
        list("risk parity" = risk_parity,
             "tangency"    = max_sharpe_ratio),
        list(faang_data), T_rolling_window = 12*20,
        optimize_every = 3*20, rebalance_every = 3*20)
```

Practical Example

```r
backtestChartStackedBar(bt, portfolio = "tangency",
                        legend = TRUE)
```



Practical Example

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backtestChartStackedBar(bt, portfolio = "risk parity",
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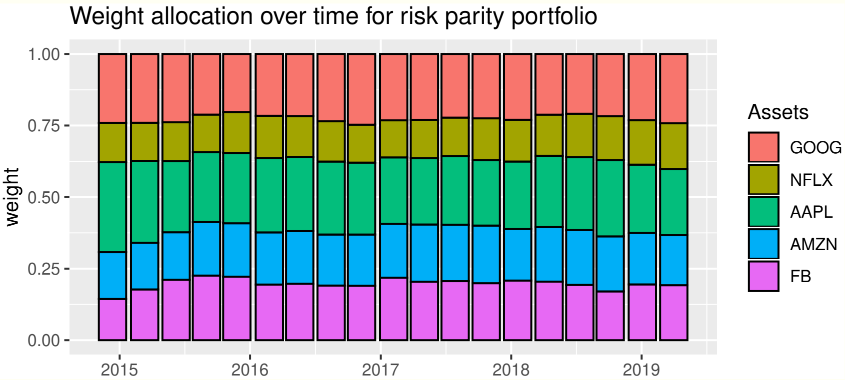
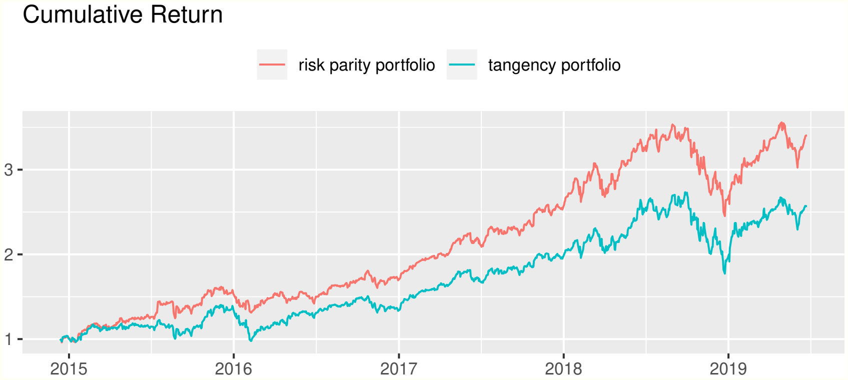


Practical Example

```
backtestChartCumReturns(bt) + theme(legend.position="top")
```



Testimonials

"I can easily run optimizations across new covariance matrices in seconds, which has helped streamline portfolio allocation testing"

Jonathan Dane, CFA. Director of Portfolio Strategy & Market Research, The Coury Firm.

"**riskParityPortfolio** provides a state-of-the-art implementation of rpps, otherwise only available at the top quantitative hedge funds"

Tharsis Souza, PhD. VP at TwoSigma. Founder, OpenQuants.com.

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Hierachical Risk Parity

Graph Theory 101

- A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$

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  - $W_{ii} = 0, W_{ij} > 0$ iff $(i, j) \in \mathcal{E}, W_{ij} = 0$ otherwise

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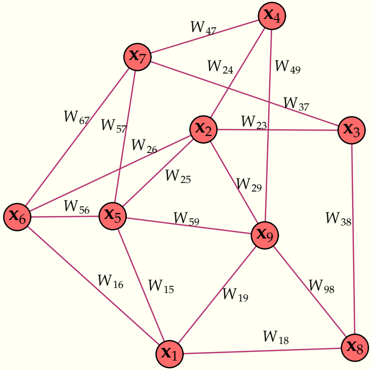

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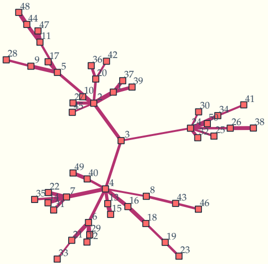
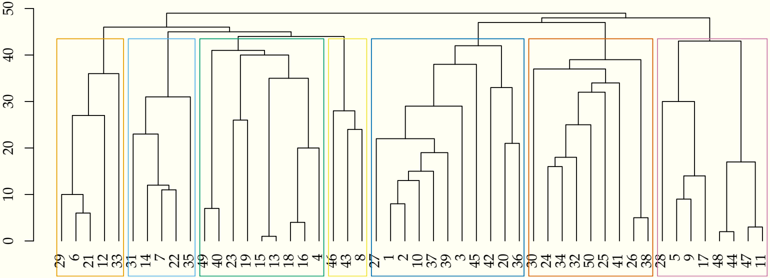
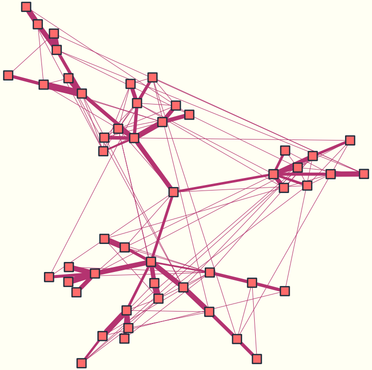




Graph Theory 101: Minimum Spanning Tree

- **def**: an MST of a graph $\mathcal{G}$ is a tree graph spans $\mathcal{G}$ with the minimum possible sum of edge weights
- An MST is **unique** as long as the edge weights are distinct



Graph Theory 101: Minimum Spanning Tree

Algorithm 2: BuildMST (Kruskal's algorithm)

Data: Distances between nodes $\{d_{ij}\}_{1\leq i,j\leq p}$

Result: $\mathcal{G}_{\mathsf{MST}}$

```
▷ start with an empty graph G_MST ← (V_MST, E_MST)
E_MST ← ∅
V_MST ← {i}_{1≤i≤p}
for {i, j} ∈ V²_MST ordered by increasing d_ij do
    ▷ Verify that i and j are not already connected by a path
    if not connected(i, j) then

    end
    E_MST ← E_MST ∪ {{i, j}}
end
```

Hierarchical Risk Parity

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- Alternatives:
  - Robustness
  - Bayesian priors, e.g., Black and Litterman
  - Improving inverse covariance matrix estimation, e.g., Ledoit and Wolf
- HRP idea: exploit the intrinsic hierarchy of the correlation matrix
- weights are allocated in a top-down fashion, just like how asset managers build their portfolios, e.g., from asset class, to sectors, to individual securities

Hierarchical Risk Parity

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1. Build an MST with input matrix: $d_{ij} = \sqrt{\frac{1}{2}(1 - \rho_{ij})}$, where ρ_{ij} is the correlation between stock i and j
2. Use **Recursive Bisection** to allocate the portfolio weights

Algorithm 3: Recursive Bisection

Data: Distances between nodes $\{d_{ij}\}_{1 \leq i,j \leq p}$, estimated covariance matrix $\boldsymbol{\Sigma}$

Result: Portfolio allocation $\boldsymbol{w}$

1 $\triangleright$ Initialization

2 • set the list of items $L = \{L_0\}$, with $L_0 = \{i\}_{i=1,\dots,p}$ according to the order obtained by the dendogram

3 • assign a unit weight to all assets: $w_i = 1,\ \forall\, i = 1, \dots, p$

4 $\triangleright$ if $|L_i| = 1,\ \forall\, L_i \in L$ then stop.

5 $\triangleright$ for each $L_i \in L$ such that $|L_i|$:

6 • bisect L_i into two subsets, $L_i^{(1)} \cup L_i^{(2)} = L_i$, where $|L_i^{(1)}| = \text{round}(\frac{1}{2}|L_i|)$, and the order is preserved

7 • define the variance of $L_i^{(j)}$ as $\tilde{\sigma}_i^{(j)} = (\theta_i^{(j)})^\top \boldsymbol{\Sigma}_i^{(j)} \theta_i^{(j)}$, where $V_i^{(j)}$ is the covariance matrix between the constituents of the $L_i^{(j)}$ bisection and $\theta_i^{(j)}$ are the inverse variance weights

8 • compute the split factor $\alpha_i = 1 - \dfrac{\tilde{\sigma}_i^{(1)}}{\tilde{\sigma}_i^{(1)} + \tilde{\sigma}_i^{(2)}}$

9 • re-scale allocations $\boldsymbol{w}_l$ by a factor of $\alpha_i,\ \forall\, l \in L_i^{(1)}$

10 • re-scale allocations $\boldsymbol{w}_l$ by a factor of $1 - \alpha_i,\ \forall\, l \in L_i^{(2)}$

11 go to **4**

HRP in Practice

- pip install **mfinlab**

```python
from mlfinlab.portfolio_optimization.hrp import HierarchicalRiskParity

prices = pdr.get_data_yahoo(stocks_, start="2018-12-31",
                            end="2019-12-31")[['Adj Close']]
hrp = HierarchicalRiskParity()
hrp.allocate(asset_prices=prices)
hrp_weights = hrp.weights
```

HRP in Practice

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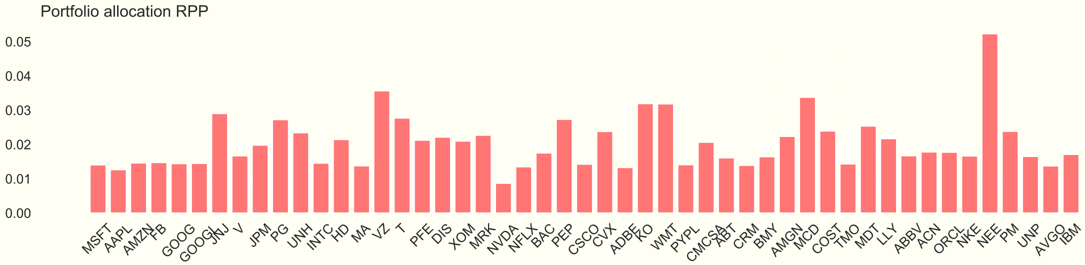

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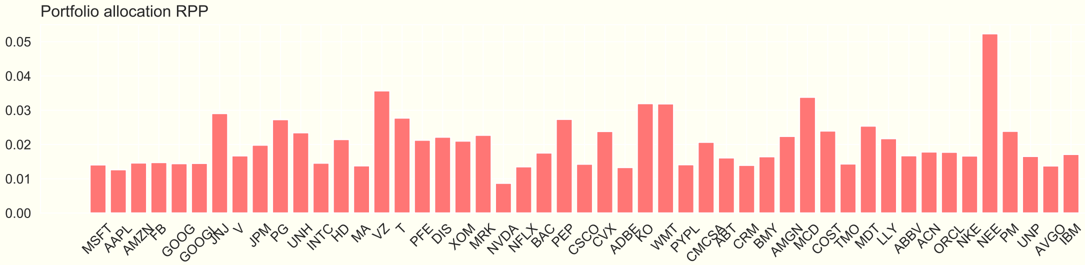

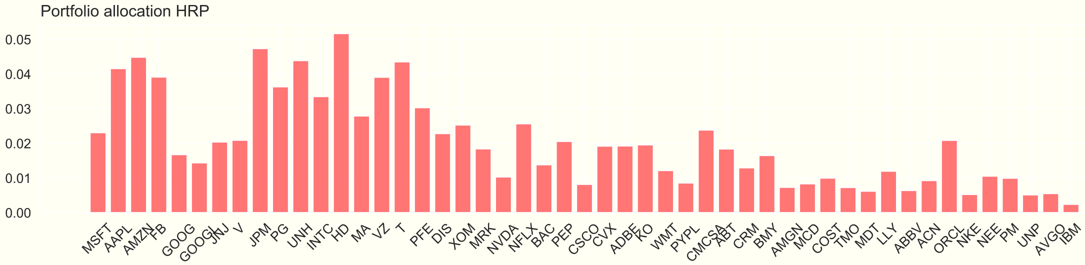

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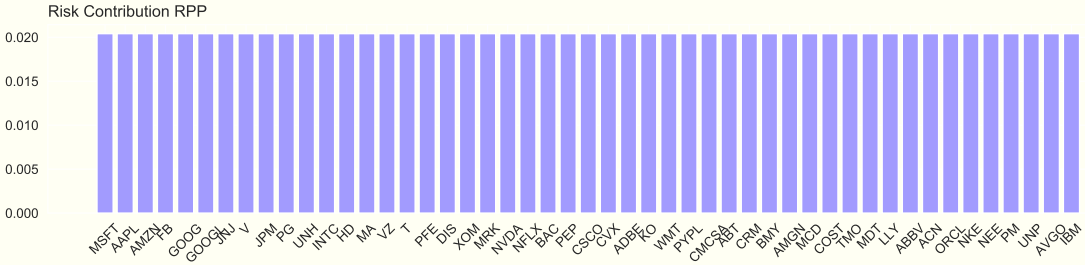
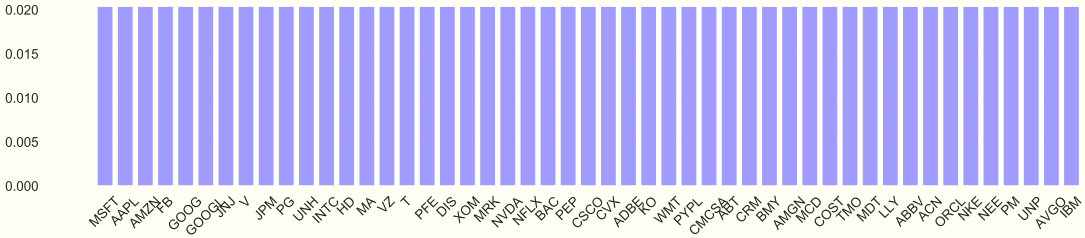
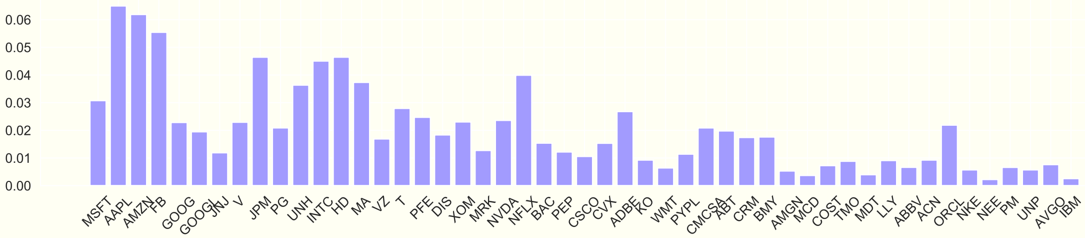


HRP in Practice

Risk Contribution RPP



Risk Contribution HRP



HRP in Practice

Now, let's evaluate those portfolios during the COVID-19 2020 crisis:

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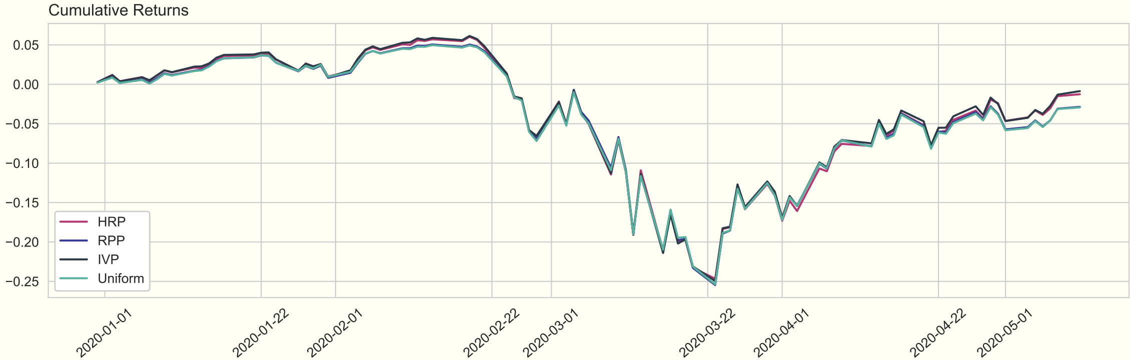

Takeaways

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2. non-convex? no problem!
3. hierarchical risk parity provides a natural way to apply graph theory in portfolio allocation
4. GitHub repositories to stay tuned:
   - dppalomar/riskParityPortfolio
   - dppalomar/riskparity.py
   - dppalomar/portfolioBacktest

Thank you! Questions?

References

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