Risk Parity Portfolio: Theory and Practice

a talk by

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(in collab. with Prof. Daniel Palomar)

ELEC/IEDA3180 Data-driven Portfolio Optimization

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Modern Portfolio Theory

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Modern Portfolio Theory

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- assume the log-returns $\boldsymbol{r}$ are i.i.d. Gaussian, then for a portfolio $\boldsymbol{w}$:
  - expected return: $\mathbb{E}\left[\boldsymbol{w}^\top \boldsymbol{r}\right] = \boldsymbol{w}^\top \boldsymbol{\mu}$
  - variance: $\mathbb{V}\left[\boldsymbol{w}^\top \boldsymbol{r}\right] = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$

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- Dr. Harry Markowitz[1] (1952):

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- criticisms:
  1. sensitivity to estimation errors in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
  2. does not consider risk diversification

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Risk Parity

From Capital to Risk Allocation

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- first risk parity fund (1996): Bridgewater's All Weather
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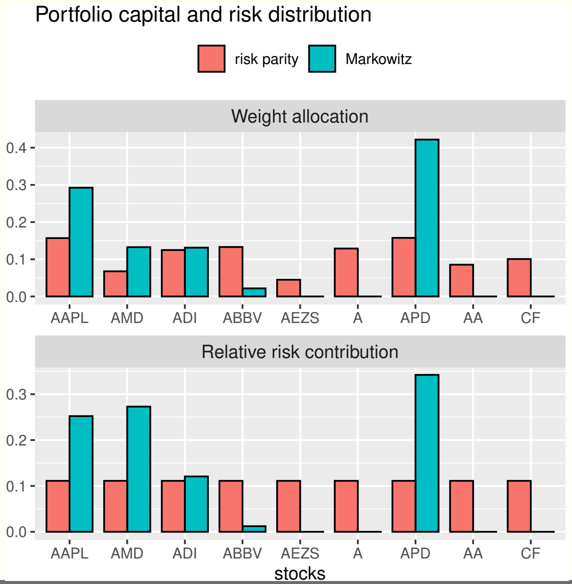
- 2011-onwards: risk parity gains broad adoption
- **basic idea**: design a portfolio such that the risk is equally distributed among the asset classes (stocks, bonds, real state, etc.)

From Capital to Risk Allocation



Problem Formulation

- volatility: $\sigma(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}$
- from Euler's theorem:

$$\sigma(\boldsymbol{w}) = \sum_i w_i \frac{\partial \sigma(\boldsymbol{w})}{\partial w_i} = \sum_i \frac{w_i \left(\boldsymbol{\Sigma} \boldsymbol{w}\right)_i}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$$

- $\frac{\partial \sigma(\boldsymbol{w})}{\partial w_i}$: **marginal risk contribution**
- measures the sensitivity of the portfolio volatility to the i-th asset
- it can be defined for other risk measures like VaR and CVaR

Problem Formulation

- **risk contribution** of the i-th asset to the total risk $\sigma(\boldsymbol{w})$:

$$\mathsf{RC}_i = w_i \frac{\partial \sigma(\boldsymbol{w})}{\partial w_i}$$

- from Euler's theorem:

$$\sum_i \mathsf{RC}_i = \sigma(\boldsymbol{w})$$

- **relative risk contribution** of the i-th asset to the total risk $\sigma(\boldsymbol{w})$:

$$\mathsf{RCC}_i = \frac{\mathsf{RC}_i}{\sigma(\boldsymbol{w})} = \frac{w_i \left(\boldsymbol{\Sigma w}\right)_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma w}},$$

so that

$$\sum_i \mathsf{RCC}_i = 1$$

Risk Parity: Problem Formulation

- **risk parity portfolio**: $\mathsf{RCC}_i = b_i,\ i = 1, 2, \ldots, N$
- **feasibility** problem:

$$\begin{array}{ll} \underset{\boldsymbol{w} \succ \boldsymbol{0}}{\text{find}} & \boldsymbol{w} \\ \text{subject to} & \dfrac{w_i \left(\boldsymbol{\Sigma} \boldsymbol{w}\right)_i}{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}} = b_i,\ i = 1, 2, \ldots, N \end{array}$$

  doesn't look trivial
- **approximation**: Σ is diagonal

$$w_i \propto \frac{\sqrt{b_i}}{\sqrt{\boldsymbol{\Sigma}_{ii}}}$$

  i.e. **inverse volatility portfolio**

Solution to Risk Parity

- Spinu (2013):
  - change of variables $\boldsymbol{x} = \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$, then $\boldsymbol{w} = \frac{\boldsymbol{x}}{\mathbf{1}^\top \boldsymbol{x}}$
  - problem becomes: $\boldsymbol{\Sigma} \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}$
  - $\underset{\boldsymbol{x} \succ \mathbf{0}}{\text{minimize}}\ f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} - \boldsymbol{b}^\top \log(\boldsymbol{x})$
  - **optimality condition:** $\nabla f(\boldsymbol{x}) = \boldsymbol{\Sigma} \boldsymbol{x} - \frac{\boldsymbol{b}}{\boldsymbol{x}} = \mathbf{0}$
  - then done? it's not a QP, LP, etc.

Cyclical Coordinate Descent (CCD)

Griveau-Billion (2013):

Algorithm 1: CCD to solve risk parity

$k \leftarrow 0$, initial $\boldsymbol{x}^{(0)}$

while *not converged* **do**

  for $i = 1$ *to* N **do**

    $x_i^{k+1} \leftarrow \arg\min_{x_i} f\left(x_1^{k+1}, \ldots, x_i^k, \ldots, x_N^k\right)$

  end

end

- closed-form update:

$$x_i^\star = \frac{-\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right) + \sqrt{\left(\boldsymbol{x}_{-i}^\top \boldsymbol{\Sigma}_{:,i}\right)^2 + 4\boldsymbol{\Sigma}_{ii} b_i}}{2\boldsymbol{\Sigma}_{ii}},$$

where $\boldsymbol{x}_{-i} = [x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N]^\top$

Non-convex Formulations

- limitations of the previous formulation: $\boldsymbol{w} \succ \mathbf{0}, \mathbf{1}^\top \boldsymbol{w} = 1$

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- how to "solve" the non-convex formulation?
- general purpose solvers: slow ☹

Non-convex Formulations

- Feng & Palomar (2015):

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \sum_{i=1}^{N} \left[g_i(\boldsymbol{w})\right]^2 + \lambda F(\boldsymbol{w}) \\ \text{subject to} & \mathbf{1}^\top \boldsymbol{w} = 1,\ \boldsymbol{w} \in \mathcal{W} \end{array}$$

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- "convexify" the objective function: fast 😄
- **basic idea**: find a "nice" approximation to the non-convex term, solve it, repeat (Scutari et al. 2014)
- "nice" approximation for $g_i(\boldsymbol{w})$: first order Taylor expansion (Feng & Palomar 2015)

Open Source Software Packages

- dppalomar/riskParityPortfolio (R version)
- dppalomar/riskparity.py (Python version)

Basic Usage

- R version:

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```
> library(riskParityPortfolio)
> my_portfolio <- riskParityPortfolio(Sigma = Sigma, b = b)
> names(my_portfolio)
[1] "w"                 "risk_contributions"
```

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- R version:

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- Python version:

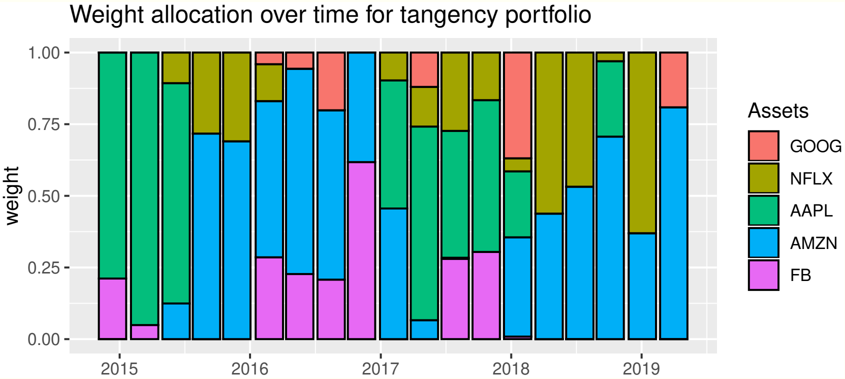
```
>>> import riskparityportfolio as rp
>>> my_portfolio = rp.RiskParityPortfolio(covariance=Sigma,
                                          budget=b)
>>> my_portfolio.design()
>>> my_portfolio.weights.numpy()
>>> my_portfolio.risk_contributions.numpy()
```

Practical Example

```r
library(portfolioBacktest)
library(riskParityPortfolio)
# download price data
faang_data <- stockDataDownload(
                   c("GOOG", "NFLX", "AAPL", "AMZN", "FB"),
                   from = "2014-01-01", to = "2019-06-25")
risk_parity <- function(dataset) {
  prices <- dataset$adjusted
  log_returns <- diff(log(prices))[-1]
  return(riskParityPortfolio(cov(log_returns))$w)
}
bt <- portfolioBacktest(
        portfolio_funs = list("risk parity" = risk_parity,
                              "tangency"    = max_sharpe_ratio),
        dataset_list = list(faang_data), T_rolling_window = 12*20,
        optimize_every = 3*20, rebalance_every = 3*20)
```

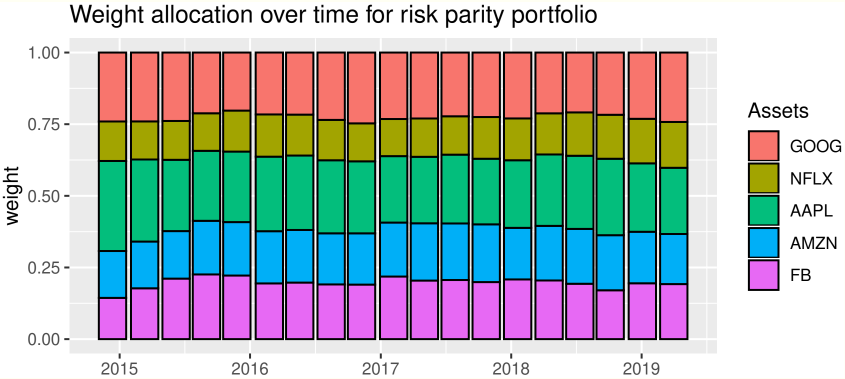
Practical Example

```
backtestChartStackedBar(bt, portfolio = "tangency",
                        legend = TRUE)
```



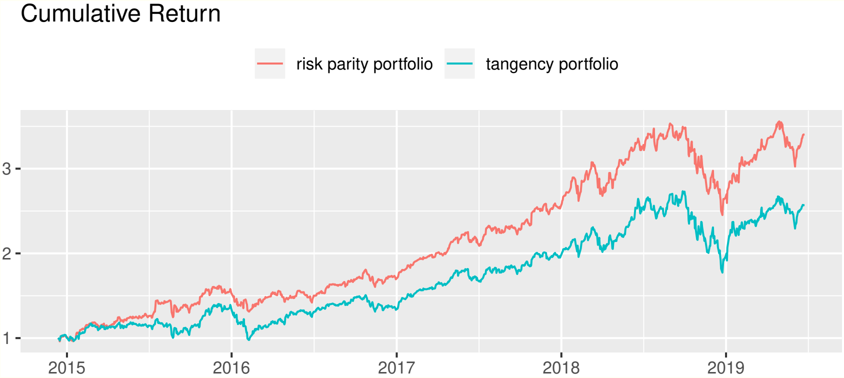
Practical Example

```
backtestChartStackedBar(bt, portfolio = "risk parity",
                        legend = TRUE)
```



Practical Example

```
backtestChartCumReturns(bt) + theme(legend.position="top")
```



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