Lab 12

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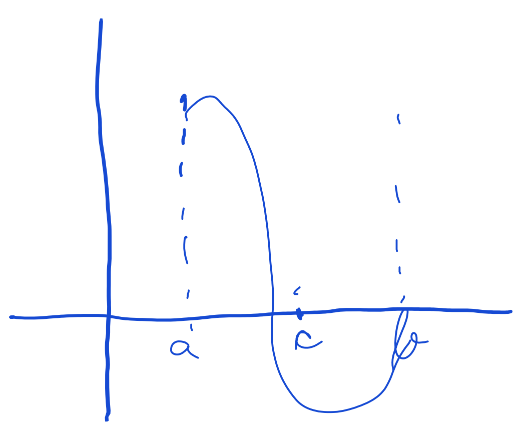
$f(x) = 0$

1) Bisection
2) Secant
3) Newton

$x_0, x_1, \dots x_n \to \alpha$ → solution for $f(x)=0$ i.e. $f(\alpha)=0$

1) **Bisection** $f(a) \cdot f(b) < 0$

$c = \frac{a+b}{2}$



2) Secant

$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$!

Stop condition: $|x_{n+1} - x_n| < \varepsilon$

3) Tangent / **Newton**

$x_{n+1} = x_n - f(x_n) \frac{1}{f'(x_n)}$

$x_n \to \alpha$ → solution

↳ sequence of approximation

Order of convergence

$|x_{n+1} - \alpha| \le c|x_n - \alpha|^p, \forall n \in \mathbb{N}$

$p \ge 1$ order

$c > 0$ const

α solution

$p = 1$ linear

$p = 2$ quadratic

$p = 3$ cubic

4) Fixed point

$f(x) = 0 \; | + x \Rightarrow f(x) + x = x$

$g(x) = x$

T1 (Banach)

1) $g \in C[a,b]$

2) $g([a,b]) \subseteq [a,b]$

3) $\exists \alpha \in (0,1)$ s.t. $|g(x) - g(y)| \le \alpha|x - y|, \forall x, y \in [a,b]$ (contraction)

$\Rightarrow \exists! \; \alpha \in [a,b]$ s.t. $g(\alpha) = \alpha$

Moreover

$x_{n+1} = g(x_n) \xrightarrow{n \to \infty} \alpha$

$\forall x_0 \in [a,b]$

and $|x_n - \alpha| \le \alpha|x_{n-1} - \alpha|, \forall n \ge 1$

T2 $g \in C^1[a,b]$

$g([a,b]) \subseteq [a,b]$

$\lambda = \max_{x \in [a,b]} |g'(x)| < 1$

$\Rightarrow \exists! \alpha \in [a,b]$ s.t. $g(x) = \alpha$ and $x_{n+1} = g(x_n) \to \alpha$

successive approximation

$f(x) = x - e^{-x}, x \in \mathbb{R}$

$f(0) = 0 - e^{-0} = 1$

$x = 1: f(1) = 1 - e^{-1} = 1 - \frac{1}{e} > 0$

$\Rightarrow f(0) \cdot f(1) < 0 \Rightarrow \exists$ at least one solution

$f'(x) = 1 + e^{-x} > 0 \Rightarrow f$ increasing on $\mathbb{R}$

$\Downarrow$

$\exists! \alpha \in (0,1)$ s.t. $f(\alpha) = 0$

Ex 3 $f(x) = x + e^{-\beta x^2} \cos x, \beta > 0, x \in (-1, 0)$

1) $\beta = 1$

$f(x) = x + e^{-x^2} \cos x$

$f(-1) = -1 + e^{-1} \cos(-1) \le -1$

$f(0) = 0 + e^{-0} \cos 0 = 1 > 0$

$f(-1) \cdot f(0) < 0 \Rightarrow \exists \alpha \in [-1, 0]$

$f'(x) = 1 - 2\beta x e^{-\beta x^2} \cos x - e^{-\beta x^2} \sin x =$

$= 1 - e^{-\beta x^2}(2\beta x \cos x + \sin x)$

$f'(x) = 1 - e^{-x^2}(2x \cos x + \sin x)$

Newton: $x_0 = 0$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{1}{1} = -1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{-1 + e^{-1}\cos(-1)}{1 - \frac{1}{e}(-2\cos(-1) + \sin(-1))} \approx -0,53$

" Real value: $-0,0518$

Ex 4: $g(x) = \frac{1}{5}(x^2 - 4), x \in [-2, 2]$

1) $g \in C[-2,2]$

2) $g([-2,2]) \subseteq [-2,2]$

$g'(x) = \frac{2}{5}x$

x	-2		0		2
$g(x)$	0	↘	$-\frac{4}{5}$	↗	0
$g'(x)$	$- - - - -$		0		$+ + + + +$

$\forall x \in [-2,2]: g(x) \in [-\frac{4}{5}, 0] \subseteq [-2,2]$

iii) $|g'(x)| = \left|\frac{2}{5}x\right| = \frac{2}{5}|x| \le \frac{4}{5} < 1$

$x \in [-2,2]$

$\exists! \alpha \in [-2,2]$ s.t. $g(\alpha) = \alpha$

$x^2 - 5x - 4 = 0$

$x_{1,2} = \frac{5 \pm \sqrt{41}}{2}$