Seminar 1

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1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

1. Addition: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
2. Substraction: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
3. Multiplication: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
4. Substraction: $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

2. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

i) 3 elements in 3 spaces, so we have 3^9

	a_1	a_2	a_3
a_1			
a_2			
a_3			

$\varphi : A \times A \longrightarrow A$

ii)

	a_1	a_2	a_3
a_1	\	a_3	a_2
a_2	a_3	\	a_1
a_3	a_2	a_1	\

3 elements in 3 free spaces $\to 3^3$
3 commutative elements in 3 spaces 3^3 $\Big\} \Rightarrow 3^6$

iii)

	e	b	c
e	e	b	c
b	b		
c	c		

3 elements in 4 free spaces $\Rightarrow 3^4$
3 elements which can be e $\Rightarrow 3$ $\Big\} \Rightarrow 3^5$

Generalization

i) n^{n^2}

ii) $n^n \cdot n^{\frac{n(n-1)}{2}}$

iii) $n^{(n-1)^2+1}$

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{Q}^*, \cdot), (\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

i) Associativity

$x + y + xy = x(y+1) + (y+1) - 1 = (x+1)(y+1) - 1$

$(x * y) * z = [(x+1)(y+1) - 1] * z = [(x+1)(y+1) - 1 + 1](z+1) - 1 = (x+1)(y+1)(z+1) - 1$

Commutativity

$x * y = (x+1)(y+1) - 1$

$y * x = (y+1)(x+1) - 1$

$x * y = y * x$

Identity element

$\exists\, e \in \mathbb{R}, \forall x \in \mathbb{R}, \ x * e = e * x = x$

$x * e = (x+1)(e+1) - 1 = x$

$(x+1)e + x + \cancel{1} - \cancel{1} = x$

$(x+1)e = 0, \forall x \in \mathbb{R} \Rightarrow e = 0$

$(\mathbb{R}, *)$ commutative monoid

b) $\forall x, y \in [-1, \infty), \ x * y \in [-1, \infty)$

$x \geq -1 \Leftrightarrow (x+1) \geq 0$
$y \geq -1 \Leftrightarrow (y+1) \geq 0$ $\Big\} \Rightarrow (x+1)(y+1) \geq 0 \ \Big| -1$

$(x+1)(y+1) - 1 \geq -1$

$x * y \geq -1 \Leftrightarrow x * y \in [-1, \infty) \Rightarrow [-1, \infty)$ stable part

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

i) Associativity

$(x * y) * z = x * (y * z)$

$(x * y) * z = \gcd(x, y) * z = \gcd(\gcd(x, y), z) = d \Rightarrow d \mid \gcd(x, y)$, $d \mid z$

$\gcd(x, y) = d \Rightarrow x = d x_1$, $y = d y_1$, $d \mid d$ $\Big\} \Rightarrow d \mid x$, $d \mid y$, $d \mid z$ $\Big\} \Rightarrow d \mid \gcd(y, z) \Rightarrow d \mid \gcd(x, \gcd(y, z))$

ii) $\forall x, y \in D_n \Rightarrow x \mid n$, $y \mid n$ $\Big\} \Rightarrow n = x d_1$, $n = y d_2$

$x * y = \gcd(x, y) = d \Rightarrow x = d x_1$, $y = d y_1$ $\Rightarrow$ $n = d x_1 d_1$, $n = d y_1 d_2$ $\Big\} \Rightarrow d \mid n \Rightarrow \gcd(x, y) \mid n \Rightarrow x * y \in D_n$

iii)

$*$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2 y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

1) $(G, \cdot)$ group

a) Commutativity: $\forall x, y \in G, \ xy = yx$

$\Rightarrow (xy)^2 = xy\,xy = xx\,yy = x^2 y^2$

$\Leftarrow (xy)^2 = x^2 y^2$

$xy\,xy = xx\,yy \ \big| \cdot y^{-1}$

$(G, \cdot)$ group $\Rightarrow \exists\, y^{-1} \in G$, $x^{-1} \in G$

$x^{-1} \big| \ xyx = xxy$

$yx = xy$

b) $x^2 = 1, \forall x \in G$, G abelian

$\forall x \in G, \ x^2 = 1 \Rightarrow x = x^{-1}$

$y^2 = 1 \Rightarrow y = y^{-1}$

$xy = x^{-1} y^{-1}$

$(xy)^2 = 1 \Rightarrow xy = (xy)^{-1} = y^{-1} x^{-1}$

$(yx)^2 = 1 \Rightarrow yx = (yx)^{-1} = x^{-1} y^{-1}$

$x^{-1} y^{-1} = y^{-1} x^{-1} \Leftrightarrow xy = yx, \forall x, y \in G$

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.

a) $(A, \cdot)$ monoid $\Rightarrow$ $\cdot$ associative

$\exists\, e \in A, \ a \cdot e = e \cdot a = e$

$X, Y, Z \in P(A)$: $(X * Y) * Z = \{(xy)z \mid x \in X, y \in Y, z \in Z\} = \{x(yz) \mid x \in X, y \in Y, z \in Z\} \Rightarrow *$ is associative

$X \in P(A)$: $X * \{e\} = \{xe \mid x \in X\} = \{ex \mid x \in X\} = \{x \mid x \in X\} \Rightarrow$ "$*$" has an identity element

b) $(A, \cdot)$ group $\Rightarrow (P(A), *)$ is not a group

Counterexample: $A = \emptyset$, $P(A)$ is a group

$A \neq \emptyset$, $A = \{e\} \Rightarrow P(A) = \{\emptyset, e\}$ (not a group)