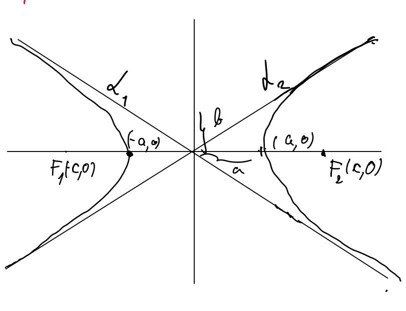
Seminar 12

05.06.2023 22:16

Hyperbolas and parabolas



Foci: F_1, F_2

H = locus of points M in the plane for which $|MF_1 - MF_2| = 2a$

$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a, b > 0 \Leftrightarrow y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$

$c = \sqrt{a^2 + b^2}$

H has 2 asymptotes at $\pm\infty$: $d_1: y = -\frac{b}{a}x$

$d_2: y = \frac{b}{a}x$

$T_{H,(x_0,y_0)}: \frac{x x_0}{a} - \frac{y y_0}{b} = 1$

$\frac{\partial}{\partial x} F(x - x_0) + \frac{\partial}{\partial y} F(y - y_0) = 1$

The tangent to H of a line with slope m: $y = mx \pm \sqrt{m^2a^2 - b^2}$

1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

$$\begin{cases} 2x - y - 10 = 0 \\ \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \mid \cdot 20 \Leftrightarrow x^2 - 4y^2 - 20 = 0 \end{cases}$$

$y = 2(5 - x)$

$x^2 - 4 \cdot 4(5 - x)^2 - 20 = 0$

$x^2 - 16(25 - 10x + x^2) - 20 = 0$

$x^2 - 400 + 160x - 16x^2 - 20 = 0$

$-15x^2 + 160x - 420 = 0 \mid :5$

$-3x^2 + 32x - 84 = 0 \mid \cdot(-3)$

$9x^2 - 2 \cdot 16 \cdot (3x) + 256 - 508 = 0$

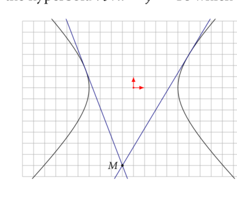
$(3x - 16)^2 = 508$

$|3x - 16| = 2\sqrt{127}$

$x = \pm \frac{2\sqrt{127}}{3} + 16$

$y = 42 \pm 4\frac{\sqrt{127}}{3}$

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell : 4x + 2y - 5 = 0$.

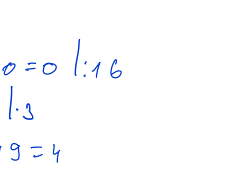


$\ell: 4x + 2y - 5 = 0 \Rightarrow m = -2$

$\frac{x^2}{16} - \frac{y^2}{8} = 1 \Rightarrow a^2 = 16$, $b^2 = 8$

$y = -2x \pm \sqrt{4 \cdot 16 - 8} = -2x \pm 2\sqrt{14}$

3. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1, 7)$.



$H: x^2 - y^2 = 16 \Leftrightarrow \frac{x^2}{16} - \frac{y^2}{16} = 0$

Let ℓ_1, ℓ_2 be the tangents $\Rightarrow T_{(x_0,y_0)}: \frac{x x_0}{16} - \frac{y y_0}{16} = 1$, $P(x_0, y_0) \in H$, $\forall A \in \ell_1$

$$\begin{cases} -x_0 - 7y_0 = 16 \Leftrightarrow x_0 = -7y_0 - 16 \\ x_0^2 - y_0^2 = 16 \end{cases}$$

$(-7y_0 - 16)^2 - y_0^2 = 16$

$48y_0^2 + 224y_0 + 240 = 0 \mid :16$

$3y_0^2 + 14y_0 + 15 = 0 \mid \cdot 3$

$(3y_0)^2 + 2 \cdot 3y_0 \cdot 7 + 49 = 4$

$(3y_0 + 7)^2 = 4$

$y_0 = -\frac{5}{3}$ or $y_0 = -3$

$x = -16 - 7 \cdot \left(-\frac{5}{3}\right) = -\frac{13}{3}$ or $x = -16 - 7 \cdot (-3) = 5$

4. Determine the relations between the coordinates (x_P, y_P) of the point P such that P does not belong to any tangent line to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

$$\begin{cases} \frac{x^2}{4} - \frac{y^2}{9} = 1 \Leftrightarrow (3x)^2 - (2y)^2 = 36 \\ 9x_P \cdot x - 4y_P y = 36 \Rightarrow x = \frac{36 - 4y_P y}{9x_P} \end{cases}$$

$\left(\frac{36 - 4y_P y}{3x_P}\right)^2 - (2y)^2 = 36 \mid \cdot 9x_P^2$

$(36 - 4y_P y)^2 - 9 \cdot 4x_P^2 y^2 = 36 \cdot 9x_P^2$

$4^2(9 - y_P y)^2 - 9 \cdot 4x_P^2 y^2 = 4 \cdot 9^2 x_P^2$

$4(9 - y_P y)^2 - 9x_P^2 y^2 = 9^2 x_P^2$

$4(81 - 18y_P y + y_P^2 y^2) - 9x_P^2 y^2 - 9x_P^2 = 0$

$324 - 64y_P y + 4y_P^2 y^2 - 9x_P^2 y^2 - 9x_P^2 = 0$

$(4y_P^2 - 9x_P^2)y^2 - 64y_P y + 324 - 9x_P^2 = 0$

5. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

$\ell: 9x + 2y - 24 = 0 \Leftrightarrow y = -\frac{9}{2}x + 12$

$H: 9x^2 - 4y^2 - 36 = 0$

$d_1: y = \frac{3}{2}x$

$y = -\frac{3}{2}x$

$$d_1 \cap \ell \begin{cases} y = -\frac{3}{2}x \\ y = -\frac{9}{2}x + 12 \end{cases} \Rightarrow \begin{cases} -\frac{3}{2}x = -\frac{9}{2}x + 12 \\ y = -\frac{3}{2}x \end{cases} \Leftrightarrow \begin{cases} x = 4 \\ y = -6 \end{cases}$$

$$d_1 \cap d_2 \begin{cases} y = -\frac{3}{2}x \\ y = \frac{3}{2}x \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$d_2 \cap \ell \begin{cases} y = \frac{3}{2}x \\ y = -\frac{9}{2}x + 12 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = 3 \end{cases}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 4 & -6 & 1 \end{vmatrix} = \frac{1}{2}(12 + 12) = 12$$

6. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell : 4x + 3y - 7 = 0$;

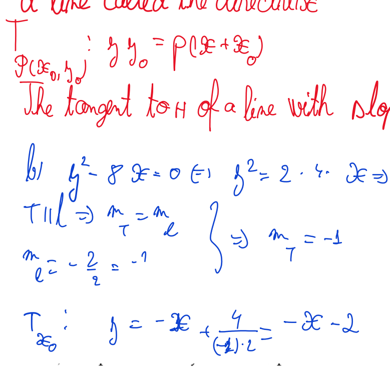
b) the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $\ell : 2x + 2y - 3 = 0$.

a) $\ell: 4x + 3y - 7 \Rightarrow m = -\frac{4}{3}$

$\ell_1 \perp \ell \Rightarrow m_{\ell_1} \cdot m_\ell = -1 \Rightarrow m_{\ell_1} = \frac{3}{4}$

$y = mx \pm \sqrt{a^2m^2 - b^2} = \frac{3}{4}x \pm \sqrt{20 \cdot \frac{9}{16} - 5} = \frac{3}{4}x \pm \sqrt{\frac{25}{4}} = \frac{3}{4}x \pm \frac{5}{2}$

b) Parabolas



$P: y^2 = 2px \Leftrightarrow x = \frac{y^2}{2p}$

P: locus of points M in the plane for which $MF = dist(M, d)$, F focus, d line called the directrice

$T_{P(x_0,y_0)}: y y_0 = p(x + x_0)$

The tangent to H of a line with slope m: $y = mx + \frac{p}{2m}$

b) $y^2 - 8x = 0 \Leftrightarrow y^2 = 2 \cdot 4 \cdot x \Rightarrow p = 4$

$$\left.\begin{matrix} T \parallel \ell \Rightarrow m_T = m_\ell \\ m_\ell = -\frac{2}{2} = -1 \end{matrix}\right\} \Rightarrow m_T = -1$$

$T_{x_0}: y = -x + \frac{4}{(-1) \cdot 2} = -x - 2$

7. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1, -5)$;

b) the paraola $\mathcal{P} : y^2 - 36x = 0$, passing through $P(2, 9)$.

$$\text{a)} \begin{cases} \frac{x_0^2}{3} - \frac{y_0^2}{5} - 1 = 0 \\ \frac{x_P x_0}{3} - \frac{y_P y_0}{5} = 1 \end{cases} \Leftrightarrow \begin{cases} 5x_0^2 - 3y_0^2 = 15 \\ \frac{x_0}{3} + y_0 = 1 \end{cases} \Leftrightarrow \begin{cases} 5x_0^2 - 3y_0^2 = 15 \\ x_0 + 3y_0 = 3 \end{cases} \Leftrightarrow \begin{cases} 5x_0^2 - 3y_0^2 = 15 \\ x_0 = 3(1 - y_0) \end{cases}$$

$5 \cdot 9(1 - y_0)^2 - 3y_0^2 = 15 \mid :3$

$15(1 - 2y_0 + y_0^2) - y_0^2 = 5$

$15 - 30y_0 + 15y_0^2 - y_0^2 = 15$

$14y_0^2 - 30y_0 = 0 \mid :2$

$7y_0^2 - 15y_0 = 0$

$(7y_0 - 15)y_0 = 0 \Rightarrow y_0 = 0 \Rightarrow x_0 = 3$; $y_0 = \frac{15}{7} \Rightarrow x_0 = -\frac{24}{7}$

$$\text{b)} \begin{cases} y_0^2 - 36x_0 = 0 \Rightarrow y_0^2 = 2 \cdot 18x \\ y_P y_0 = 18(x_P + x_0) \end{cases}$$

$$\begin{cases} y_0 = 2 \cdot 18x_0 \mid \cdot 5 \\ 9y_0 = 18(x_0 + 2) \end{cases}$$

$18^2 x_0 = 18(x_0 + 2) \mid :18$

$18x_0 = x_0 + 2$

$x_0 = \frac{2}{17}$

$y_0 = \frac{72}{17}$

$\frac{72}{17}y = 18\left(x + \frac{2}{17}\right)$

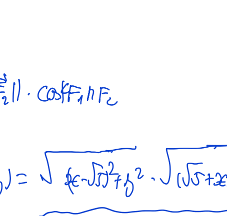
$72y = 18(17x + 2)$

8. Consider the hyperbola $\mathcal{H} : x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

a) The angle $\measuredangle F_1MF_2$ is right;

b) The angle $\measuredangle F_1MF_2$ is 60°;

c) The angle $\measuredangle F_1MF_2$ is θ.



$\frac{x^2}{1} - \frac{y^2}{4} - 1 = 0 \Rightarrow a = 1$, $b = 2$, $c = \sqrt{1^2 + 2^2} = \sqrt{5}$

$F_1(-\sqrt{5}, 0)$, $F_2(\sqrt{5}, 0)$

$M(x, y)$

$\overrightarrow{MF_1}(-\sqrt{5} - x, -y)$

$\overrightarrow{MF_2}(\sqrt{5} - x, -y)$

$\overrightarrow{MF_1} \cdot \overrightarrow{MF_2} = \|\overrightarrow{MF_1}\| \cdot \|\overrightarrow{MF_2}\| \cdot \cos F_1MF_2$

$\measuredangle F_1MF_2 = \theta$

$(-\sqrt{5} - x, -y) \cdot (\sqrt{5} - x, -y) = \sqrt{(x - \sqrt{5})^2 + y^2} \cdot \sqrt{(\sqrt{5} + x)^2 + y^2} \cdot \cos\theta$

$-(\sqrt{5} - x)(\sqrt{5} + x) + y^2 = \sqrt{((x - \sqrt{5})^2 + y^2)((x + \sqrt{5})^2 + y^2)} \cdot \cos\theta$

$\frac{x^2 + y^2 - 5}{\sqrt{((x - \sqrt{5})^2 + y^2)((x + \sqrt{5})^2 + y^2)}} = \cos\theta$

$y^2 = 4x^2 - 1$

$x^2 + y^2 - 5 = x^2 + 4x^2 - 1 - 5 = 5x^2 - 6$

$(x - \sqrt{5})^2 + y^2 = x^2 - 2\sqrt{5}x + 5 + 4x^2 - 1 = 5x^2 - 2\sqrt{5}x + 5$

$(x + \sqrt{5})^2 + y^2 = 5x^2 + 2\sqrt{5}x + 5$

$((x - \sqrt{5})^2 + y^2)((x + \sqrt{5})^2 + y^2) = 25(x^2 + 1)^2 - 20x^2 = 25(x^4 + 2x^2 + 1) - 20x^2 =$

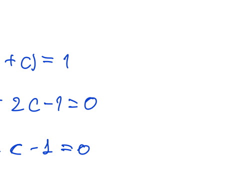
$= 25x^4 + 30x^2 + 25$

$\cos\theta = \frac{5x^2 - 6}{\sqrt{25x^4 + 30x^2 + 25}}$

$t = x^2$

$\cos\theta = \frac{5t - 6}{\sqrt{25t^2 + 30t + 25}}$

10. Consider the hyperbola $\mathcal{H} : x^2 - 2y^2 = 1$. Determine the geometric locus described by the midpoints of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.



$x^2 - 2y^2 = 1$

$\frac{x^2}{1} - \frac{y^2}{\frac{1}{2}} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \frac{\sqrt{5}}{2}$

$F_1\left(-\frac{\sqrt{5}}{2}, 0\right)$, $F_2\left(-\frac{\sqrt{5}}{2}, 0\right)$

$$\begin{cases} x^2 - 2y^2 = 1 \\ y = 2x + c \end{cases}$$

$x^2 - 2(4x^2 + 4xc + c) = 1$

$x^2 - 8x^2 - 8xc - 2c - 1 = 0$

$-7x^2 - 8xc - 2c - 1 = 0$

$x_1 + x_2 = -\frac{8}{7}c$

$x_M = \frac{x_1 + x_2}{2} = -\frac{4}{7}c$

$y_M = -\frac{8c}{7} + c = -\frac{1}{7}c$

$M\left(-\frac{4}{7}c, -\frac{1}{7}c\right) \Rightarrow v = \langle (4, 1) \rangle$