Seminar 2

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1) $M = \{2,3,4,5,6\}$

$r = \{R, S, T, V\}$

a) $x R y \Leftrightarrow x < y$

$R = \{(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

b) $x S y \Leftrightarrow x \mid y$

$S = \{(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(5,5),(6,6)\}$

c) $x T y \Leftrightarrow \gcd(x,y) = 1$

$T = \{(2,3),(2,5),(3,2),(3,4),(3,5),(4,3),(4,5),(5,2),(5,3),(5,4),(5,6),(6,5)\}$

d) $x V y \Leftrightarrow x \equiv y \pmod 3$

$V = \{(2,2),(2,5),(3,3),(3,6),(4,4),(5,2),(5,5),(6,3),(6,6)\}$

2. Let A and B be sets with n and m elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:

(i) relations having the domain A and the codomain B;

(ii) homogeneous relations on A.

a) $\rho : A \to B \Rightarrow$ Number of $\rho = 2^{|A \times B|} = 2^{mn}$

b) $\rho : A \to A \Rightarrow$ Number of $\rho = 2^{|A \times A|} = 2^{n^2}$

(ii) homogeneous relations on A.

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

a) Reflexivity

$A = \{1,2,3\}$

$R = \{(1,1),(2,2),(3,3)\}$, $x - y = 0$

b) Transitivity

$A = \{1,2,3\}$ $x < y$

$T = \{(1,2),(1,3),(2,3)\}$

c) Symmetry $x \not\equiv y \pmod 3$

$A = \{2,4,5\}$

$S = \{(2,4),(4,5),(4,2),(5,4)\}$

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

a) $x r y \Leftrightarrow x \neq y$

r is
- symmetric because, if $x \neq y$, then $y \neq x$, $\forall x, y \in \mathbb{R}$
- not reflexive, because $x = x$, $\forall x \in \mathbb{R}$
- transitive: $\forall x, y, z \in \mathbb{R}$, if $x \neq y$ and $y \neq z$ then $x \neq z$

b) $x r y \Leftrightarrow x \mid y$

r is
- reflexive: $\forall x \in \mathbb{N}$, $x \mid x$, $\forall x, y \in \mathbb{N}$
- !symmetric: if $x \mid y$, then $y \nmid x$, $\forall x, y \in \mathbb{N}$
- transitive: if $x \mid y$ and $y \mid z$, then $x \mid z$, $\forall x, y, z \in \mathbb{N}$

c) Same for $\mathbb{Z}$

d) $(V^3, \perp)$

$d_1 r d_2 : d_1 \perp d_2$

r is
- not reflexive : $d \not\perp d$
- not transitive : $\left.\begin{array}{l} d_1 \perp d_2 \\ d_2 \perp d_3 \end{array}\right\} \Rightarrow d_1 \not\perp d_3$
- symmetric : $d_1 \perp d_2 \Leftrightarrow d_2 \perp d_1$

e) $(V^3, \parallel)$

$d_1 r d_2 \Leftrightarrow d_1 \parallel d_2$

r is
- not reflexive: $d \nparallel d$
- symmetric: $d_1 \parallel d_2 \Rightarrow d_2 \parallel d_1$
- transitive: $\left.\begin{array}{l} d_1 \parallel d_2 \\ d_2 \parallel d_3 \end{array}\right\} \Rightarrow d_1 \parallel d_3$

f) $(V^2, \equiv)$

$\Delta_1 r \Delta_2 \Rightarrow \Delta_1 \equiv \Delta_2$

r is
- reflexive : $x \equiv x$, $\forall x \in V_2$
- symmetric : $x \equiv y \Rightarrow y \equiv x$, $\forall x, y \in V_2$
- transitive : $\left.\begin{array}{l} x \equiv y \\ y \equiv z \end{array}\right\} \Rightarrow x \equiv z$, $\forall x, y, z$

g) Same for Associativity

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

$R_1 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(4,4)\}$

R_1 is reflexive $\{(1,1),(2,2),(3,3),(4,4)\}$

R_1 is symmetric $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$

R_1 is transitive $\{(1,2),(2,3),(1,3)\}$

r_1 is an equivalence relation $\pi = \{1,2,3,4\}$

$R_2 = r \cup \{(1,2),(1,3)\}$

R_2 is not symmetric

π_1 is a partition

$R = \{(1,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$

π_2 is not a partition

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1 \, r \, z_2 \Longleftrightarrow |z_1| = |z_2| ; \qquad z_1 \, s \, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0 .$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

$z_1 r z_2 \Leftrightarrow |z_1| = |z_2|$

r is
- reflexive, $\forall z \in \mathbb{C}$, $|z| = |z|$
- symmetric, $\forall z_1, z_2 \in \mathbb{C}$, $|z_1| = |z_2|$ and $|z_2| = |z_1|$
- transitive, $\forall z_1, z_2, z_3 \in \mathbb{C}$ $\left.\begin{array}{l} |z_1| = |z_2| \\ |z_2| = |z_3| \end{array}\right\} \Rightarrow |z_1| = |z_3|$

$\mathbb{C}/r = \{r\langle x \rangle \mid x \in \mathbb{C}\} = \{x \, r \, x \mid x \in \mathbb{C}\} = \{r(z) \mid |z| = |\bar{z}|, z \in \mathbb{C}\} = \{0\} \cup \{C(0, |z|)\}$

s is
- reflexive, $\forall z \in \mathbb{C}$, $\arg z = \arg z$
- symmetric, $\forall z_1, z_2 \in \mathbb{C}$, $\arg z_1 = \arg z_2$, then $\arg z_2 = \arg z_1$
- transitive, $\forall z_1, z_2, z_3 \in \mathbb{C}$, $\left.\begin{array}{l} \arg z_1 = \arg z_2 \\ \arg z_2 = \arg z_3 \end{array}\right\} \Rightarrow \arg z_1 = \arg z_3$

$\mathbb{C}/s = \{s\langle x \rangle \mid x \in \mathbb{C}\} = \{x \, s \, x \mid x \in \mathbb{C}\} = \{s(x) \mid \arg x = \arg \bar{x}, x \in \mathbb{C}\} \Rightarrow$ the line starting from 0 which has the angle $\arg(z)$ with $O_x\} \cup \{0\}$

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo n*, defined by:

$$x \, \rho_n \, y \Longleftrightarrow n \mid (x - y) .$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

ρ_n is reflexive : $\forall x \in \mathbb{Z}$, $n \mid (x - x)$

ρ_n is symmetric, : $\forall x, y \in \mathbb{Z}$, $n \mid (x - y) \Rightarrow n \mid (y - x)$

ρ_n is transitive $\forall x, y, z \in \mathbb{Z}$, $\left.\begin{array}{l} n \mid (x - y) \\ n \mid (y - z) \end{array}\right\} \Rightarrow n \mid (x - z)$

$\mathbb{Z}/\rho_0 = \{\{x\} \mid x \in \mathbb{Z}\} \Leftrightarrow 0 \mid x - y \Leftrightarrow x = y$

$\mathbb{Z}/\rho_1 = \{\mathbb{Z}\} \Leftrightarrow 1 \mid x - y$

$\mathbb{Z}/\rho_n = \{\hat{0}, \hat{1}, \dots, \widehat{n-1}\}$

ρ_n. Discuss the cases $n = 0$ and $n = 1$.

8. Determine all equivalence relations and all partitions on the set $M = \{1,2,3\}$

$M = \{1,2,3\}$

$\pi_1 = \{\{1\},\{2\},\{3\}\}$, $R_1 = \{(1,1),(2,2),(3,3)\}$

$\pi_2 = \{\{1,2\},\{3\}\}$, $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$

$\pi_3 = \{\{1\},\{2,3\}\}$, $R_3 = \{(1,1),(1,3),(3,2),(2,2),(3,3)\}$

$\pi_4 = \{\{1,3\},\{2\}\}$, $R_4 = \{(1,1),(1,3),(3,1),(3,3),(2,2)\}$

$\pi_5 = \{\{1,2,3\}\}$, $R_5 = \{(1,1),(1,2),(1,3),(2,2),(2,1),(2,3),(3,1),(3,2),(3,3)\}$

9. Let $M = \{0,1,2,3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\} .$$

Is h a function?

h function $\Leftrightarrow \forall x \in \mathbb{Z}, \exists y$ unique

$\exists z \in \mathbb{Z}$, $x = 4z + y$

$|h\langle x \rangle| = 1$

$h\langle x \rangle = \{y \in M \mid (x,y) \in \mathbb{Z} \times M\} = \{y \in M \mid \exists z \in \mathbb{Z} : x = 4z + y\}$

y is the remainder of the division of x to 4 $\Rightarrow y \in \{0,1,2,3\}$

it is unique

$\Rightarrow |h\langle x \rangle| = 1 \Rightarrow h$ is a function

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m \, r \, n \Longleftrightarrow \exists a \in \mathbb{N} : m = 2^a n ,$$

$$m \, s \, n \Longleftrightarrow (m = n \text{ or } m = n^2 \text{ or } n = m^2) .$$

Are r and s equivalence relations?

r is not symmetric

$m = 2^{\alpha} n \mid \cdot 2^{\beta}$ $\Rightarrow$ $2^{\beta} m = 2^{\alpha+\beta} \cdot n$

$n = 2^{\beta} m$

$2^{\alpha+\beta} = 2^0$

$\alpha + \beta = 0$

$\alpha = -\beta$

$\alpha, \beta \in \mathbb{N} \Rightarrow \alpha = \beta = 0$

s is an equivalence relation

s is reflexive ($n = m$)

s is transitive ($m s n$ and $n s q$)

$\left.\begin{array}{l} m = n^2 q \\ n^2 = q \end{array}\right\} \Rightarrow m = q$

s is symmetric ($m s n$ and $n s m$)