Seminar 5

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ℓ line, $A(x_0,y_0)\in \ell$

$\vec{\ell} = \{\vec{w}\in V^2 \mid \ell \parallel \vec{w}\}$

$\vec{v}(a,b)\in\vec{\ell}$

Parametric equation: $\begin{cases} x = x_0 + a t\\ y = y_0 + b t\end{cases}$

Cartesian equation

a) Symmetric form: $\frac{x-x_0}{a} = \frac{y-y_0}{b}$

$x = x_0$, if $a=0$

$y=y_0$, if $b=0$

if $a,b\neq 0$, implicit form: $b(x-x_0)-a(y-y_0)=0$

explicit form: $y = \frac{b}{a}x + y_0 - \frac{b}{a}x_0$, $\frac{b}{a}$ slope

The direction vectors of ℓ are $\vec{\ell} = \{\vec{w}\in V^2 \mid \vec{w}\parallel \ell\}$

normal vectors of ℓ are $\vec{\ell}^{\perp} = \{\vec{w}\in V^2\mid \vec{w}\perp\ell\} = \langle(a,b)\rangle^{\perp} = \langle(b,-a)\rangle$

1. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2. For the lines ℓ in the previous exercise

a) give a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

3. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ has slope −5 and contains the point $A(1,-2)$,

b) ℓ has slope 1 and is at distance 2 from the origin,

c) ℓ contains the point $A(-2,3)$ and has an angle of 60° with the Ox-axis,

d) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

4. For the lines ℓ in the previous exercise

a) give parametric equations for ℓ,

b) describe all normal vectors for ℓ.

(1,2) a) $b(x-x_0)-a(y-y_0)=0$

$-1(x-1)-3(y-2)=0$

$-x+1-3y+6=0$

$x+3y=7$

$\vec{\ell} = \langle(3,-1)\rangle$

$\vec{\ell}^{\perp} = \langle(-1,-3)\rangle$

Parametric equations $\begin{cases}x=1+3\lambda\\ y=2-\lambda\end{cases}$

b) $O(0,0)\in\ell$, $\ell\parallel\vec{b}(4,5)$

Parametric equations

$\begin{cases}x=x_0+a\cdot t\\y=y_0+b\cdot t\end{cases} \Rightarrow \begin{cases}x=4t\\y=5t\end{cases}$

Cartesian form: $\frac{x-x_0}{a}=\frac{y-y_0}{b}$ → Symmetric form

$\frac{x}{4}=\frac{y}{5}$

Implicit form: $b(x-x_0)-a(y-y_0)=0$

$5x-4y=0$

Explicit form: $y=\frac{b}{a}x+y_0-\frac{b}{a}x_0$

$y=\frac{5}{4}x$

Normal vectors of ℓ are $\vec{b}^{\perp}=\langle(5,-4)\rangle$

Dir vectors of ℓ are $\vec{\ell}=\langle(4,5)\rangle$

c) $M(1,7)\in\ell$

$\ell\parallel Oy \Rightarrow a=0$

Parametric eq: $\begin{cases}x=1\\y=7+bt,\ b\in\mathbb{R}\end{cases}$

Cartesian eq: $x=1$

Normal vectors: $\vec{\ell}^{\perp}=\langle(b,0)\rangle=\langle(1,0)\rangle$

Dir vectors: $\vec{\ell}=\langle(0,1)\rangle$

d) $M(2,4)$, $N(2,-5)\in\ell$

$\vec{MN}=(0,-9)$

Parametric eq: $\begin{cases}x=2\\y=4-9t\end{cases}$

$\vec{\ell}=\langle(0,1)\rangle$

$\vec{\ell}^{\perp}=\langle(-1,0)\rangle$

(3,4) a) $\frac{b}{a}=-5=m_\ell=\frac{y-y_0}{x-x_0}$

$y-y_0=-5(x-x_0)$

$y+2=-5(x-1)$

$y+2=-5x+5$

$5x+y-3=0$

Parametric: $\begin{cases}x=t\\y=3-5t\end{cases}$

$\vec{\ell}=\langle(1,-5)\rangle$

$\vec{\ell}^{\perp}=\langle(5,1)\rangle$

b)

c) $A(-2,3)$

$\sphericalangle(\ell,Ox)=60^\circ \Rightarrow m(\ell)=tg\left(\frac{\pi}{3}\right)=\sqrt{3}$

$\frac{b}{a}=\sqrt{3}\Rightarrow b=a\sqrt{3}\Rightarrow\vec{\ell}=\langle(1,\sqrt{3})\rangle$

$\vec{\ell}^{\perp}=\langle(\sqrt{3},-1)\rangle$

$\frac{y-y_0}{b}=\frac{x-x_0}{a}\Leftrightarrow\frac{y-3}{a\sqrt{3}}=\frac{x+2}{a}\Leftrightarrow\frac{y-3}{\sqrt{3}}=x+2$

$y-3=(x+2)\sqrt{3}$

$y-x\sqrt{3}=2\sqrt{3}+3$

$\begin{cases}x=t\\y=2\sqrt{3}+3+t\sqrt{3}\end{cases}$

d) $v\perp n\Rightarrow v\cdot n=0$

$4a=-3b$

$\frac{b}{a}=-\frac{4}{3}\Rightarrow\vec{\ell}=\langle(-\frac{3}{4},1)\rangle$

$\vec{\ell}^{\perp}=\langle(1,\frac{3}{4})\rangle$

$\frac{y-7}{x-1}=\frac{4}{3}$

$3y-21=4x-4$

$4x-3y+17=0$

$\begin{cases}x=t\\y=\frac{4t+17}{3}\end{cases}$

5. Consider a line ℓ. Show that

c) if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,

d) if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

a) $V(v_1,v_2)$

$V\perp n\Leftrightarrow v\cdot n=0$

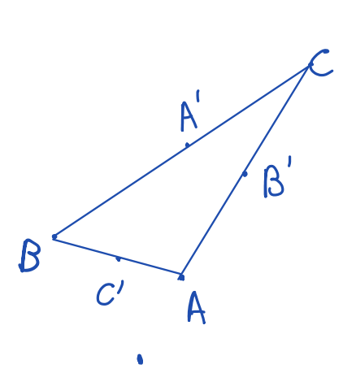
$v_1v_2-v_1v_2=0$

b) $n(n_1,n_2)$

$n\perp v\Leftrightarrow n\cdot v=0$

$n_1n_2-n_1n_2=0$

6. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.



$A'(x_{A'},y_{A'})$

$x_{A'}=\frac{x_B+x_C}{2}=1$

$y_{A'}=\frac{y_B+y_C}{2}=5$

Parametric eq

$x=x_A$

$a'(0,3)$

$B'(x_{B'},y_{B'})$

$x_{B'}=\frac{x_A+x_C}{2}=\frac{5}{2}$

$y_{B'}=\frac{y_A+y_C}{2}=\frac{9}{2}$

$\vec{b}'=\left(\frac{9}{2},\frac{3}{2}\right)$

$\vec{b}=\langle(3,1)\rangle$

$\vec{\ell}^{\perp}=\langle(1,-3)\rangle$

Parametric eq

$\begin{cases}x=\frac{5}{2}+\frac{9}{2}t\\y=\frac{9}{2}+\frac{3}{2}t\end{cases}$

$C'(x_{C'},y_{C'})$

$x_{C'}=\frac{x_A+x_B}{2}=-\frac{1}{2}$

$y_{C'}=\frac{y_A+y_B}{2}=\frac{5}{2}$

$\vec{c}'=\left(-\frac{9}{2},-\frac{9}{2}\right)$

$\vec{c}'=\langle(1,1)\rangle$

$\vec{c}^{\perp}=\langle(1,-1)\rangle$

Parametric eq

$\begin{cases}x=-\frac{1}{2}-\frac{9}{2}t\\y=\frac{5}{2}-\frac{9}{2}t\end{cases}$

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

$A_{\Delta ABC}=\frac{1}{2}\left|\begin{vmatrix}1&3&1\\-4&3&1\\2&9&1\end{vmatrix}\right|=\frac{1}{2}\cdot30=15$

$A_{\Delta ABC}=\frac{d(A,BC)\cdot BC}{2}=15\Leftrightarrow d(A,BC)=\frac{5\sqrt{2}}{2}$

$BC=\sqrt{6^2+6^2}=6\sqrt{2}$

b) $m_{BC}=\frac{y_C-y_B}{x_C-x_B}=\frac{9-3}{2+4}=\frac{6}{6}=1\Rightarrow m_{AA'}=-1$

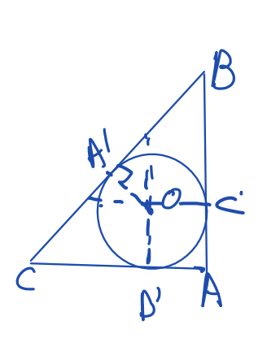
$y-y_A=m_{AA'}(x-x_A)$

$y-3=-1(x-1)$

$x+y-4=0$

$y=4-x$

9. Determine the circumcenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.



$m_{BC}=\frac{x_C-x_B}{y_C-y_B}=\frac{2}{8}=\frac{1}{4}$

$OA'\perp BC\Rightarrow m_{OA'}=-4$

$m_{AC}=\frac{x_C-x_A}{y_C-y_A}=\frac{4}{4}=1$

$m_{OC'}=-1$

$A'\left(\frac{x_B+x_C}{2},\frac{y_B+y_C}{2}\right)$

$A'(4,2)$

$C'\left(\frac{x_A+x_B}{2},\frac{y_A+y_B}{2}\right)$

$C'(2,0)$

$OA': y=-4x+y_0+4x_0$

$y=-4x+2+16$

$4x+y=18$

$OC': y=-x+y_0+x_0$

$y=-x+2$

$x+y=2$

$\begin{cases}4x+y=18\\x+y=2\end{cases}\Leftrightarrow\begin{cases}x_0=\frac{16}{3}\\y_0=-\frac{10}{3}\end{cases}$

10. Determine the angle between the lines $\ell_1: y=2x+1$ and $\ell_2: y=-x+2$.

$\left.\begin{matrix}\ell_1: y=2x+1\\ y=\frac{b_1}{a_1}x+y_0-\frac{b_1}{a_1}x_0\end{matrix}\right\}\Rightarrow\frac{b_1}{a_1}=2\Rightarrow b_1=2a_1\Rightarrow\vec{\ell}_1\langle(1,2)\rangle$

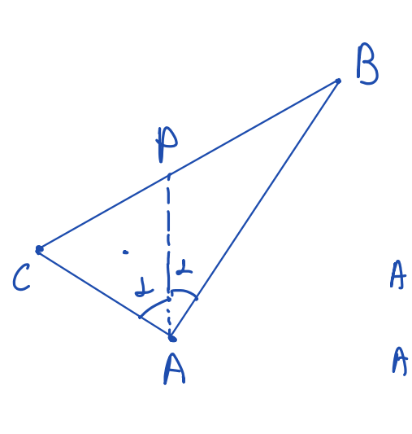
$\left.\begin{matrix}\ell_2: y=-x+2\\ y=\frac{b_2}{a_2}x+y_0-\frac{b_2}{a_2}x_0\end{matrix}\right\}\Rightarrow\frac{b_2}{a_2}=-1\Rightarrow\vec{\ell}_2\langle(1,-1)\rangle$

$\vec{v}_1\cdot\vec{v}_2=\|\vec{v}_1\|\cdot\|\vec{v}_2\|\cdot\cos(\sphericalangle(v_1,v_2))$

$\left.\begin{matrix}\vec{v}_1\cdot\vec{v}_2=-1\\ \|v_1\|=\sqrt{5}\\ \|v_2\|=\sqrt{2}\end{matrix}\right\}\Rightarrow\cos(\sphericalangle(v_1,v_2))=\frac{-1}{\sqrt{10}}=-\frac{\sqrt{10}}{10}$

$\sphericalangle v_1v_2=\arccos -\frac{\sqrt{10}}{10}$

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle ∡A.



$\frac{AB}{AC}=\frac{BP}{PC}$

$\left.\begin{matrix}AB=\sqrt{52}=2\sqrt{13}\\AC=\sqrt{13}\end{matrix}\right\}\Rightarrow\frac{BP}{PC}=2\Rightarrow\frac{PC}{BC}=\frac{1}{3}$

$\Rightarrow P\left(\frac{(x_B+x_C)}{3},\frac{(y_B+y_C)}{3}\right)$

$P\left(2,\frac{4}{3}\right)$

$\vec{AP}\left(1,\frac{10}{3}\right)$

$A(1,-2)$

$\begin{cases}x=1+\lambda\\y=-2+\frac{10}{3}\lambda\end{cases}$

$x-1=\frac{(y+2)\cdot3}{10}$

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell: 3x+4y-20=0$. Determine the coordinates of A'.

$\ell: 3x+4y-20=0$

$4y=-3x-20 \quad |:4$

$y=\boxed{-\frac{3}{4}}x-5$

$\frac{b}{a}$

↓ slope

$m_{AA'}=\frac{4}{3}\Rightarrow\vec{AA'}\langle(3,4)\rangle$

$\frac{x_0-x_A}{3}=\frac{y_0-y_A}{4}$

$4x_0-4x_A=3y_0-3y_A$

$\begin{cases}4x_0-3y_0=10\\3x_0+4y_0=20\end{cases}\Leftrightarrow\begin{cases}12x_0-9y_0=30\\12x_0+16y_0=80\end{cases}\Leftrightarrow\begin{cases}x_0=4\\y_0=2\end{cases}$

$\left.\begin{matrix}x_O=\left(\frac{x_A+x_{A'}}{2},\frac{y_A+y_{A'}}{2}\right)\\x_O=(4,2)\end{matrix}\right\}\Rightarrow\begin{cases}x_{A'}=-2\\y_{A'}=-6\end{cases}$

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

Let a be the distance

I $M(0,a)$, $N(a,0)$

$\vec{AM}(-2,a-5)$

$\frac{x_N+2}{-2}=\frac{y_N-5}{a-5}$

$\frac{a+2}{-2}=\frac{-5}{a-5}$

$(a+2)(a-5)=10$

$a^2-3a-10=10$

$a^2-3a-20=0$

$\Delta=9+80=89$

$a_1=\frac{3+\sqrt{89}}{2}$

$a_2=\frac{3-\sqrt{89}}{2}$

II $M(0,a)$, $N(-a,0)$

$\vec{AM}(-2,a-5)$

$\frac{x_N+2}{-2}=\frac{y_N-5}{a-5}$

$\frac{-a+2}{-2}=\frac{-5}{a-5}$

$(a-2)(a-5)=-10$

$a^2-7a+20=0$

$\Delta=49-80<0$

↳ $a\notin\mathbb{R}$

nate axes in congruent segments.

14. Determine Cartesian equations for the lines situated at distance 4 from the line $12x-5y-15=0$.

Since the distance is 4, $\ell_1\parallel\ell_2$

$d(\ell_1,\ell_2)=d(P,\ell_1)$

$P(x_p,y_p)$

$d(P,\ell_1)=\frac{|ax_p+bx_p+c|}{\sqrt{a^2+b^2}}\Rightarrow\frac{|12x_p-5y_p-15|}{13}=4$

$|12x_p-5y_p-15|=52$

$\ell_2: 12x_p-5y_p-67=0$

$\ell_3: 12x_p-5y_p+37=0$

15. Determine the values k for which the distance from the point $(2,3)$ to the line $8x+15y+k=0$ equals 5.

$d(P,\ell)=\frac{|ax_p+by_p+c|}{\sqrt{a^2+b^2}}$

$\frac{|8\cdot2+15\cdot3+k|}{\sqrt{289}}=5$

$|51+k|=81\Rightarrow k\in\{30,-132\}$

16. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

Let $\ell_1: a_1x+b_1y+c_1=0$, $d(A,\ell_1)=d(B,\ell_1)$

$\ell_2: a_2x+b_2y+c_2=0$, $d(A,\ell_2)=d(C,\ell_2)$

$\ell_3: a_3x+b_3y+c_3=0$, $d(B,\ell_3)=d(C,\ell_3)$

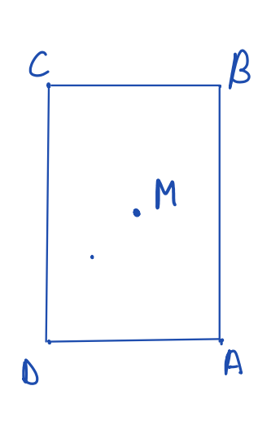
$d(A,\ell_1)=\frac{|a_1x_A+b_1y_A+c_1|}{\sqrt{a_1^2+b_1^2}}=\frac{|3a_1-b_1+c_1|}{\sqrt{a_1^2+b_1^2}}$

$d(B,\ell_1)=\frac{|a_1x_B+b_1y_B+c_1|}{\sqrt{a_1^2+b_1^2}}=\frac{|9a_1+b_1+c_1|}{\sqrt{a_1^2+b_1^2}}$

⇒ ? ? ?

$\Rightarrow|3a_1-b_1+c_1|=|9a_1+b_1+c_1|$

17. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.



$\left.\begin{matrix}x_C=2x_M-x_A=-1\\y_C=2y_M-y_A=4\end{matrix}\right\}\Rightarrow\left.\begin{matrix}C(-1,4)\\A(3,-2)\end{matrix}\right\}\Rightarrow\begin{matrix}B(3,4)\\D(-1,-2)\end{matrix}$

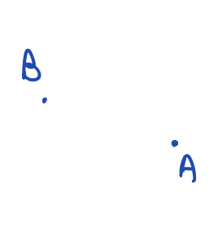
$AB: x=3$

$DA: y=-2$

$DC: x=-1$

$CB: y=4$

18. Determine a point on the line $5x-4y-4=0$ which is equidistant to the points $A(1,0)$ and $B(-2,1)$.



$P\in\ell: 5x_p-4y_p-4=0$

$AP=BP$

$\left.\begin{matrix}AP=\sqrt{(x_p-1)^2+y_p^2}\\BP=\sqrt{(x_p+2)^2+(y_p-1)^2}\end{matrix}\right\}\Rightarrow(x_p-1)^2+y_p^2=(x_p+2)^2+(y_p-1)^2$

$2y_p-1=3(2x_p+1)$

$2y_p-1=6x_p+3$

$\left.\begin{matrix}y_p=3x_p+2\\4y_p=5x_p-4\end{matrix}\right\}\Rightarrow12x_p+8=5x_p-4$

$7x_p=-12$

$x_p=-\frac{12}{7}$

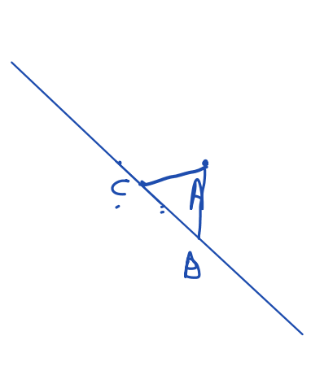
$y_p=2-\frac{36}{7}=-\frac{22}{7}$

19. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x+y-1=0$. Determine Cartesian equations for the lines containing the other two sides.

(M1) Direct

$AB=AC=BC$

$\begin{cases}(x_B-2)^2+y_B^2=(x_C-2)^2+y_C^2=(x_B-x_C)^2+(y_B-y_C)^2\\x_B+y_B-1=0\\x_C+y_C-1=0\end{cases}$



$AH\perp BC$

$AH=\frac{|ax_A+by_A+c|}{\sqrt{a^2+b^2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

$AH=\frac{\ell\sqrt{3}}{2}$

$\ell=\frac{\sqrt{6}}{3}$

$\ell^2=\frac{2}{3}$