7. Lagrange interpolation in Newton's divided differences form

$\boxed{1}$ $f : [0,1] \to \mathbb{R}$, $f(x) = \cos(\pi x)$

x	0	$1/3$	$1/2$	1
$f(x)$	1	$1/2$	0	-1

$m = 3$

$\Downarrow$

$\boxed{n = 3}$

a) Lagrange polynomial

$$L_3 f(x) = N_3 f(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

x	$f(x)$	$D^1 f$	$D^2 f$	$D^3 f$
0	1	$-3/2$	-3	$9/2$
$1/3$	$1/2$	-3	$3/2$	
$1/2$	0	-2		
1	-1			

$$D^i f(x_0) = f[x_0, \dots, x_i]$$

Hence, $L_3 f(x) = 1 - \frac{3}{2}(x - 0) - 3(x - 0)(x - \frac{1}{3}) + \frac{9}{2}(x - 0)(x - \frac{1}{3})(x - \frac{1}{2})$

$$= 1 - \frac{3}{2}x - \frac{3x(3x-1)}{3} + \frac{9x(3x-1)(2x-1)}{12}$$

$$\Rightarrow L_3 f(x) = 1 - \frac{3}{2}x - x(3x-1) + \frac{3x(3x-1)(2x-1)}{4}$$

$$= \frac{9}{2}x^3 - \frac{27}{4}x^2 + \frac{1}{4}x + 1$$

b). Approx. for $\cos\left(\frac{\pi}{5}\right)$

$$\cos\left(\frac{\pi}{5}\right) = f\left(\frac{1}{5}\right) \approx L_3 f\left(\frac{1}{5}\right),$$

where $L_3 f\left(\frac{1}{5}\right) = \frac{9}{2}\left(\frac{1}{5}\right)^3 - \frac{27}{4}\left(\frac{1}{5}\right)^2 + \frac{1}{4}\cdot\frac{1}{5} + 1 = \ldots = 0{,}816$

c). The error $R_3 f$

$$R_3 f(x) = \frac{u(x)}{(3+1)!} f^{IV}(\xi), \text{ where } \xi \in (0,1)$$

$$f(x) = \cos(\pi x) \Rightarrow f'(x) = -\pi \sin(\pi x)$$

$$f''(x) = -\pi^2 \cos(\pi x)$$

$$f'''(x) = \pi^3 \sin(\pi x)$$

$$\boxed{f^{IV}(x) = \pi^4 \cos(\pi x)}$$

and $u(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$

$$= (x-0)\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right)(x-1)$$

Then $R_3 f(x) = \frac{x(3x-1)(2x-1)(x-1)}{6 \cdot 4!} \pi^4 \cdot \cos(\pi \cdot \xi)$,

$\xi \in (0,1)$.

d) A bound for the error

$$0 \leq |R_3 f(x)| = \left| \frac{x(3x-1)(2x-1)(x-1)}{6 \cdot 4!} \pi^4 \cdot \cos(\pi \cdot \xi) \right|$$

$$\leq \frac{|x(3x-1)(2x-1)(x-1)|}{6 \cdot 24} \cdot \pi^4 \cdot \underbrace{|\cos(\pi \cdot \xi)|}_{\leq 1}$$

$$\leq \frac{\pi^4}{144} \cdot \underbrace{|\overbrace{x(3x-1)(2x-1)(x-1)}^{v(x)}|}_{\leq \max\limits_{x \in [0,1]} v(x)}, \quad x \in [0,1]$$

- In this case, v is of degree 4 and is difficult to determine by hand the maximum value (see Lecture 5 - pag. 7 for an example)

e) A bound for the error of approx. $f(\frac{1}{5})$

$$0 \leq \left|R_3 f\left(\tfrac{1}{5}\right)\right| \leq \frac{\pi^4}{6 \cdot \cancel{24}} \cdot \frac{\cancel{1}}{5} \cdot \frac{\cancel{2}}{5} \cdot \frac{\cancel{3}}{5} \cdot \frac{\cancel{4}}{5} = \frac{\pi^4}{5^4 \cdot 6} = \frac{1}{6}\left(\frac{\pi}{5}\right)^4$$

$$\Rightarrow 0 \leq \left|R_3 f\left(\tfrac{1}{5}\right)\right| \leq \frac{1}{6} \cdot \left(\frac{\pi}{5}\right)^4 \approx 0.0259$$

Indeed, $\left|L_3 f\left(\frac{1}{5}\right) - f\left(\frac{1}{5}\right)\right| = 0.00698 < 0.0259$

$\square$