Seminar 4

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Def: $(V,+)$ abelian group
$(K,+,\cdot)$ field

Let $\cdot : K\times V \to V$
$(k,v)\mapsto k\cdot v$

V is a k vector space if:

① $\forall \alpha,\beta\in K, \forall v\in V$
$(\alpha+\beta)\cdot v=\alpha v+\beta v$

② $\forall \alpha\in K, \forall v_1,v_2\in V$
$\alpha(v_1+v_2)=\alpha v_1+\alpha v_2$

③ $\forall \alpha,\beta\in K, v\in V$
$(\alpha\cdot\beta)v=\alpha\cdot(\beta v)$

④ $\forall v\in V$
$1_k\cdot v=v$

Ex: ${}_K K[x]$, ${}_K K^{m\,n}(K)$, ${}_K K^A$ where $K^A=\{f: A\to K\}$, ${}_K K^n$

4. Let $V=\{x\in\mathbb{R}\mid x>0\}$ and define the operations: $x\perp y=xy$ and $k\top x=x^k$, $\forall k\in\mathbb{R}$ and $\forall x,y\in V$. Prove that V is a vector space over $\mathbb{R}$.

$V=\{x\in\mathbb{R}\mid x>0\}$, $x\perp y=xy$
$k\top x=x^k$
$\forall k\in\mathbb{R}, \forall x,y\in V$

Show that ${}_{\mathbb{R}}V$

$x,y\in V\Rightarrow x>0, y>0$
$\forall x,y\in V$, $x\perp y>0\Rightarrow x\perp y\in V$

① Associativity: $(x\perp y)\perp z=(xy)\perp z=xyz$
$x\perp(y\perp z)=x\perp(yz)=xyz$ $\Big\}\Rightarrow (x\perp y)\perp z=x\perp(y\perp z)$

② Commutativity: $x\perp y=xy$
$y\perp x=yx$ $\Big\}\Rightarrow xy=yx$ (T), $\forall x,y\in V$

③ Identity element: $\exists! e\in V, x\perp e=x$
$xe=x$
$x>0$ $\Big\}\Rightarrow e=1$

④ Inverse low, $\forall x\in V, \exists x^{-1}\in V$, $x\perp x^{-1}=e$
$x\cdot x^{-1}=1$
$x^{-1}=\frac{1}{x}>0\Rightarrow x^{-1}=\frac{1}{x}, \forall x\in V$

$\Rightarrow (V,\perp)$ is an abelian group

Let $k\in\mathbb{R}, x\in V$, $k\top x=x^k$
$x>0$
$x^k>0, \forall x\in V$ $\Big\}\Rightarrow x^k\in V, \forall x\in V$, $\forall k\in\mathbb{R}$

Axioms

① $\forall \alpha\in\mathbb{R}, \forall x,y\in V$
$k\top(x\perp y)=k\top x\perp k\top y$
$k\top(x\perp y)=k\top(xy)=(xy)^k$
$k\top x\perp k\top y=x^k\cdot y^k=(xy)^k$ $\Big\}\Rightarrow k\top(x\perp y)=(k\top x)\perp(k\top y)$

② $\forall \alpha,\beta\in\mathbb{R}, \forall x\in V$
$(\alpha+\beta)\top x=\alpha\top x\perp\beta\top x$
$(\alpha+\beta)\top x=x^{\alpha+\beta}$
$(\alpha\top x)\perp(\beta\top x)=x^\alpha\perp x^\beta=x^\alpha\cdot x^\beta=x^{\alpha+\beta}$ $\Big\}\Rightarrow (\alpha+\beta)\top x=(\alpha\top x)\perp(\beta\top x)$

③ $\forall \alpha,\beta\in\mathbb{R}, \forall x\in V$
$(\alpha\cdot\beta)\top x=\alpha\top(\beta\top x)$
$(\alpha\cdot\beta)\top x=x^{\alpha\cdot\beta}$
$\alpha\top(\beta\top x)=\alpha\top(x^\beta)=(x^\beta)^\alpha=x^{\alpha\beta}$ $\Big\}\Rightarrow(\alpha\cdot\beta)\top x=\alpha\top(\beta\top x)$

④ $\forall x\in V$
$1\top x=x^1=x$

$\Rightarrow V$ is a vector space over $\mathbb{R}$ (${}_{\mathbb{R}}V$)

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\, k\in K$, $\forall f=a_0+a_1X+\cdots+a_nX^n\in K[X]$,

$$k\cdot f=(ka_0)+(ka_1)X+\cdots+(ka_n)X^n\,.$$

① $\forall k_1,k_2\in K$
$\forall f\in K[X]$
$(k_1+k_2)\cdot f=k_1f+k_2f$
$(k_1+k_2)\cdot f=(k_1+k_2)a_0+(k_1+k_2)a_1X+(k_1+k_2)a_2X^2+\ldots+(k_1+k_2)a_nX^n$
$k_1f+k_2f=k_1a_0+k_1a_1X+k_1a_2X^2+\ldots+k_1a_nX^n+k_2a_0+k_2a_1X+k_2a_2X^2+\ldots+k_2a_nX^n$
$k_1f+k_2f=(k_1+k_2)a_0+(k_1+k_2)a_1X+(k_1+k_2)a_2X^2+\ldots+(k_1+k_2)a_nX^n$ $\Big\}\Rightarrow(k_1+k_2)f=k_1f+k_2f$

② $\forall k\in K$
$\forall f_1,f_2\in K[X]$
$k(f_1+f_2)=kf_1+kf_2$
$k(f_1+f_2)=k(a_0+a_1X+a_2X^2+\ldots+a_nX^n+b_0+b_1X+b_2X^2+\ldots+b_nX^n)=[(a_0+b_0)+(a_1+b_1)X+(a_2+b_2)X^2+\ldots+(a_n+b_n)X^n]\cdot k=k(a_0+b_0)+k(a_1+b_1)X+k(a_2+b_2)X+\ldots+k(a_n+b_n)X$
$kf_1+kf_2=ka_0+ka_1X+ka_2X^2+\ldots+ka_nX^n+kb_0+kb_1X+kb_2X^2+\ldots+kb_nX^n=k(a_0+b_0)+k(a_1+b_1)X+\ldots+k(a_n+b_n)X^n$ $\Big\}\Rightarrow k(f_1+f_2)=kf_1+kf_2$

③ $\forall k_1,k_2\in K, \forall f\in V$
$(k_1k_2)f=k_1(k_2f)\Leftrightarrow(k_1k_2)(a_0+a_1X+a_2X^2+\ldots+a_nX^n)=k_1(k_2a_0+k_2a_1X+k_2a_2X^2+\ldots+k_2a_nX^n)\Leftrightarrow k_1k_2a_0+k_1k_2a_1X+k_1k_2a_2X^2+\ldots+k_1k_2a_nX^n=k_1k_2a_0+k_1k_2a_1X+k_1k_2a_2X^2+\ldots+k_1k_2a_nX^n$ „T"

④ $\forall f\in K[X]$, $1\cdot f=f$ (T)

$\Rightarrow {}_K K[X]$

Vector subspaces

Def: $(V,+)$ is a k-v.s.

$S\subset V$, then S is a K-subspace of V (denoted by $S\le_K V$) if

i) $S\neq\emptyset$
ii) $(S,+)\le(V,+)$ ($\forall x,y\in S$, $x-y\in S$)
iii) $\cdot: K\times S\Rightarrow S$ well defined ($\forall k\in K$, $\forall v\in S$) $\Rightarrow kv\in S$

$\Big\}$ $\forall\alpha,\beta\in K$, $\forall v_1,v_2\in S$, $\alpha v_1+\beta v_2\in S$ ($\alpha v_1+\beta v_2$ is a linear combination of v_1 and v_2)

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A=\{(x,y,z)\in\mathbb{R}^3\mid x=0\}$;
(ii) $B=\{(x,y,z)\in\mathbb{R}^3\mid x=0 \text{ or } z=0\}$;
(iii) $C=\{(x,y,z)\in\mathbb{R}^3\mid x\in\mathbb{Z}\}$;
(iv) $D=\{(x,y,z)\in\mathbb{R}^3\mid x+y+z=0\}$;
(v) $E=\{(x,y,z)\in\mathbb{R}^3\mid x+y+z=1\}$;
(vi) $F=\{(x,y,z)\in\mathbb{R}^3\mid x=y=z\}$?

i) $A\neq\emptyset$
$(0,1,2)\in A$
Let $\alpha,\beta\in\mathbb{R}$, $(0,y_1,z_1)\in\mathbb{R}^3$, $(0,y_2,z_2)\in\mathbb{R}^3$
$\alpha(0,y_1,z_1)+\beta(0,y_2,z_2)=(0,\alpha y_1+\beta y_2,\alpha z_1+\beta z_2)\in\mathbb{R}^3\Rightarrow A\le_{\mathbb{R}}\mathbb{R}^3$

ii) For $\alpha=1,\beta=2, v_1=(1,2,0), v_2=(0,2,1)$
$\alpha v_1+\beta v_2=(1,2,0)+(0,4,2)=(1,6,2)\notin B\Rightarrow B\not\le_{\mathbb{R}}\mathbb{R}^3$

iii) For $\alpha=\frac12,\beta=\frac14, v_1=(1,2,4), v_2=(1,4,8)$
$\alpha v_1+\beta v_2=(\frac12,1,2)+(\frac14,1,2)=(\frac34,2,4)\notin C\Rightarrow C\not\le_{\mathbb{R}}\mathbb{R}^3$

iv) $D=\{(x,y,z)\in\mathbb{R}^3\mid x+y+z=0\}$
$D\neq\emptyset$ ($(0,0,0)\in D$)
$\forall v_1,v_2\in D$
$\forall\alpha,\beta\in\mathbb{R}$
$\alpha v_1+\beta v_2=(\alpha x_1,\alpha y_1,\alpha z_1)+(\beta x_2,\beta y_2,\beta z_2)=(\alpha x_1+\beta x_2,\alpha y_1+\beta y_2,\alpha z_1+\beta y_1)$
$\alpha x_1+\beta x_2+\alpha y_1+\beta y_2+\alpha z_1+\beta z_2=\alpha(x_1+y_1+z_1)+\beta(x_2+y_2+z_2)=\alpha\cdot0+\beta\cdot0=0\Rightarrow\alpha v_1+\beta v_2\in D, \forall\alpha,\beta\in\mathbb{R}, v_1,v_2\in D$

v) For $\alpha=1,\beta=2, v_1=(0,0,1), v_2=(1,0,0)$
$\alpha v_1+\beta v_2=(0,0,1)+(2,0,0)=(2,0,1)\notin E\Rightarrow E\not\le_{\mathbb{R}}\mathbb{R}^3$

vi) $v_1=(x,x,x)$, $v_2=(y,y,y)$
$\alpha v_1+\beta v_2=(\alpha x+\beta y,\alpha x+\beta y,\alpha x+\beta y)\in F\Rightarrow F\le_{\mathbb{R}}\mathbb{R}^3$

Theorem 2.1.4 *Let V be a vector space over K. Then $\forall k,k'\in K$ and $\forall v,v'\in V$ we have:*
(i) $k\cdot0=0\cdot v=0$.
(ii) $k(-v)=(-k)v=-kv$.
(iii) $k(v-v')=kv-kv'$.
(iv) $(k-k')v=kv-k'v$.

Proof

i) $k\cdot0+k\cdot v=k(0+v)=kv\Rightarrow k\cdot0=0$
$0\cdot v+k\cdot v=(0+k)\cdot v=kv\Rightarrow0\cdot v=0$

ii) $k\cdot v+k\cdot(-v)=k(v-v)=k\cdot0=0\Rightarrow k(-v)=-kv$
$k\cdot v+(-k)\cdot v=(k-k)\cdot v=0\cdot v=0\Rightarrow(-k)v=-kv$

iii) $k(v-v')+kv'=k(v-v'+v')=kv\Rightarrow k(v-v')=kv-kv'$

iv) $(k-k')\cdot v+k'v=(k-k'+k')v=kv\Rightarrow(k-k')v=kv-k'v$

Hence all the above properties are true. □

Theorem 2.1.5 *Let V be a vector space over K and let $k\in K$ and $v\in V$. Then*

$$kv=0\Longleftrightarrow k=0 \text{ or } v=0\,.$$

Proof. $\boxed{\Longrightarrow}$ Assume that $kv=0$. Suppose that $k\neq0$. Then k is invertible in the field K and we have

$$kv=0\Longrightarrow kv=k\cdot0\Longrightarrow k^{-1}(kv)=k^{-1}(k\cdot0)\Longrightarrow(k^{-1}k)v=(k^{-1}k)\cdot0\Longrightarrow v=0\,.$$

$\boxed{\Longleftarrow}$ This is Theorem 2.1.4 (i). □

2. Let K be a field and $m,n\in\mathbb{N}$, $m,n\ge2$. Show that $M_{m,n}(K)$ is a K-vector space, with the usual addition and scalar multiplication of matrices.

① $\forall k_1,k_2\in K, \forall A\in M_{m,n}(K)$
$(k_1+k_2)\cdot A=k_1A+k_2A$

$$A=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{2,1}&a_{22}&\cdots&a_{2n}\\\vdots&&&\\a_{m,1}&a_{m,2}&\cdots&a_{m,n}\end{pmatrix}$$

② $\forall k\in K, \forall A\in M_{m,n}(K)$
$\forall B\in M_{m,n}(K)$
$k(A+B)=kA+kB$

$$B=\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{2,1}&b_{22}&\cdots&b_{2n}\\\vdots&&&\\b_{m,1}&b_{m,2}&\cdots&b_{m,n}\end{pmatrix}$$

③ $\forall k_1,k_2\in K$, $A\in M_{m,n}(K)$
$(k_1k_2)\cdot A=k_1(k_2A)$ (T)

④ $1\cdot A=A, \forall A\in M_{m,n}(K)$

3. Let K be a field, $A\neq\emptyset$ and denote $K^A=\{f\mid f:A\to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f,g\in K^A$, $\forall k\in K$, $f+g\in K^A$, $kf\in K^A$,

$$(f+g)(x)=f(x)+g(x)\,,\quad(k\cdot f)(x)=k\cdot f(x)\,,\forall x\in A\,.$$

① $\forall k_1,k_2\in K, f\in K^A$
$(k_1+k_2)f(x)=k_1f(x)+k_2f(x)$

② $\forall k\in K, f,g\in K^A$
$[k(f+g)](x)=kf(x)+kg(x)\Leftrightarrow$
$\Leftrightarrow k(f(x)+g(x))=kf(x)+kg(x)\Leftrightarrow$
$\Leftrightarrow kf(x)+kg(x)=kf(x)+kg(x)$ (T)

③ $\forall k_1,k_2\in K, f\in K^A$
$[(k_1\cdot k_2)f](x)=k_1[(k_2f)(x)]\Leftrightarrow[k_1k_2f](x)=k_1(k_2f(x))\Leftrightarrow k_1k_2f(x)=k_1k_2f(x)$ „T"

④ $(1\cdot f)(x)=f(x)$ (T)

$\forall A\neq0$, K-field $\Rightarrow K^A$ is a vector space
$A=\mathbb{R}$, $\mathbb{R}$-field $\Rightarrow\mathbb{R}^{\mathbb{R}}$ is a vector space

5. Let K be a field and let $V=K\times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:
(i) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+2y_2)$ and $k\cdot(x_1,y_1)=(kx_1,ky_1)$, $\forall(x_1,y_1),(x_2,y_2)\in V$ and $\forall k\in K$.
(ii) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $k\cdot(x_1,y_1)=(kx_1,y_1)$, $\forall(x_1,y_1),(x_2,y_2)\in V$ and $\forall k\in K$.

i) ① $\forall\alpha,\beta\in K$, $v\in K\times K$
$(\alpha+\beta)v=\alpha v+\beta v$
$(\alpha+\beta)(x,y)=\alpha(x,y)+\beta(x,y)$
$((\alpha+\beta)x,(\alpha+\beta)y)=(\alpha x,\alpha y)+(\beta x,\beta y)$
$((\alpha+\beta)x,(\alpha+\beta)y)=((\alpha+\beta)x,(\alpha+2\beta)y)$ „F" $\Rightarrow V$ is not a K vector space

ii) $\forall\alpha,\beta\in K$, $v\in V$
$(\alpha+\beta)\cdot v=\alpha v+\beta v$
$(\alpha+\beta)\cdot v=(\alpha+\beta)(x,y)=((\alpha+\beta)x,y)$
$\alpha v+\beta v=\alpha(x,y)+\beta(x,y)=(\alpha x,y)+(\beta x,y)=((\alpha+\beta)x,2y)$ $\Big\}\Rightarrow(\alpha+\beta)\cdot v\neq\alpha v+\beta v\Rightarrow V$ is not a K-vector space

6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.
(i) Prove that $\underbrace{x+\cdots+x}_{p\text{ times}}=0$, $\forall x\in V$.

ii) Ask !!!

$\hat1x=x$
$\hat1x+\hat1x=(\hat1+\hat1)x=\hat2x$
$\hat1x+\hat2x=(\hat1+\hat2)x=\hat3x$
$\vdots$
$\hat1x+\widehat{p-1}\,x=(\hat1+\widehat{p-1})x=\hat0\cdot x=0$

8. Which ones of the following sets are subspaces:
(i) $[-1,1]$ of the real vector space $\mathbb{R}$;
(ii) $\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\le1\}$ of the real vector space $\mathbb{R}^2$;
(iii) $M=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix}\middle|\, a,b,c\in\mathbb{Q}\right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(iv) $\{f:\mathbb{R}\to\mathbb{R}\mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

i) For $x_1=\frac12$, $x_2=\frac14$
$\alpha=2$, $\beta=4$
$\alpha x_1+\beta x_2=1+1=2\notin[-1,1]\Rightarrow[-1,1]$ is not a subspace

ii) $B=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\le1\}$
For $\alpha=4,\beta=2$, $v_1=(1,0)$, $v_2=(0,1)$
$\alpha v_1+\beta v_2=4(1,0)+2(0,1)=(4,2)$
$4^2+2^2\le1$ „F" $\Rightarrow B\not\le\mathbb{R}$

iii) a) $\forall\alpha,\beta\in\mathbb{Q}$, $A_1\in M_2(\mathbb{Q})$
$A_2\in M_2(\mathbb{Q})$
$\alpha A_1+\beta A_2=\alpha\begin{pmatrix}a_1&b_1\\0&c_1\end{pmatrix}+\beta\begin{pmatrix}a_2&b_2\\0&c_2\end{pmatrix}=\begin{pmatrix}\alpha a_1+\beta a_2&\alpha b_1+\beta b_2\\0&\alpha c_1+\beta c_2\end{pmatrix}\in M_2(\mathbb{Q})\Rightarrow M\le_{\mathbb{Q}}M_2(\mathbb{Q})$

b) For $\alpha=\sqrt2$, $\beta=\sqrt3$, $A_1=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $A_2=\begin{pmatrix}2&2\\0&2\end{pmatrix}$
$\alpha A_1+\beta A_2=\begin{pmatrix}\sqrt2&\sqrt2\\0&\sqrt2\end{pmatrix}+\begin{pmatrix}2\sqrt3&2\sqrt3\\0&2\sqrt3\end{pmatrix}=\begin{pmatrix}\sqrt2+2\sqrt3&\sqrt2+2\sqrt3\\0&\sqrt2+2\sqrt3\end{pmatrix}\in M_2(\mathbb{R})$
But $\sqrt2+2\sqrt3\notin\mathbb{Q}$ $\Big\}\Rightarrow M\not\le_{\mathbb{R}}M_2(\mathbb{R})$

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(i) $K_n[X]=\{f\in K[X]\mid\text{degree}(f)\le n\}$ $(n\in\mathbb{N})$;
(ii) $K'_n[X]=\{f\in K[X]\mid\text{degree}(f)=n\}$ $(n\in\mathbb{N})$.

i) $k,l\le n$, $k\le l$
$f=a_0+a_1X+\ldots+a_kX^k$
$g=b_0+b_1X+\ldots+b_lX^l$
$\alpha,\beta\in K$
$\alpha f+\beta g=\alpha a_0+\alpha a_1X+\alpha a_2X^2+\ldots+\alpha a_kX^k+\beta b_0+\beta b_1X+\beta b_2X^2+\ldots+\beta b_lX^l=a_0\alpha+b_0\beta+(a_1\alpha+b_1\beta)X+(a_2\alpha+b_2\beta)X^2+\ldots+(a_k\alpha+b_k\beta)X^k+b_{k+1}\beta X^{k+1}+\ldots+b_l\beta X^l\Rightarrow\text{degree}(\alpha f+\beta g)=l\le n\Rightarrow K_n[X]$ is a subspace of K

ii) $f=a_0+a_1X+\ldots+a_nX^n$
$g=b_0+b_1X+\ldots+b_nX^n$
$\alpha,\beta\in K$
$\alpha f+\beta g=\alpha a_0+\beta b_0+(\alpha a_1+\beta b_1)X+(\alpha a_2+\beta b_2)X^2+\ldots+(\alpha a_n+\beta b_n)X^n$
For $\alpha=-4$, $a_n=1$, $\beta=2$, $b_n=2$
$\alpha a_n+\beta b_n=-4+4=0$
$\Big\}\Rightarrow\text{degree}(\alpha f+g\beta)\le n\Rightarrow K'_n[X]$ is not a subspace of K

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

$S=\{(x_1,x_2)\mid(x_1,x_2)\text{ - system solutions}\}$
$S\neq\emptyset$ $(0,0)$ is a solution of the system
Let $v_1=(x_1,x_2)$, $v_2\in(y_1,y_2)\in S$
$\alpha,\beta\in K$ $\Big\}\Rightarrow\alpha(x_1,x_2)+\beta(y_1,y_2)\in S$

$\alpha(x_1,x_2)+\beta(y_1,y_2)=(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2)$ should be a solution $\Rightarrow a(\alpha x_1+\beta y_1)+b(\alpha x_2+\beta y_2)=\alpha(ax_1+bx_2)+\beta(ay_1+by_2)=0$
$ax_1+bx_2=0$
$ay_1+by_2=0$ $\Big\}\Rightarrow S$ is a subspace of $\mathbb{R}^2$