Seminar 9

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Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

① $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \underset{\substack{L_2 \to L_2 - 2L_3 \\ L_4 \to L_4 - 2L_3}}{\sim} \begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_1 \to L_1 - L_2}{\sim} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_1 \to L_1 - L_4}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank } A = 3$

② $B = \begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{\substack{L_2 \to L_2 + 2L_1 \\ L_3 \leftarrow L_3 + L_1}}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix} \overset{L_3 \leftarrow 2L_3 + L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 0 & 15 & 5 \end{pmatrix} \Rightarrow \text{rank } B = 3$

③ $C = \begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{\substack{L_2 \to \beta L_2 - L_1 \\ L_3 \to \beta L_3 - 2L_1}}{\sim} \begin{pmatrix} \beta & 1 & 3 & 4 \\ 0 & \alpha\beta - 1 & 3\beta - 3 & 3\beta - 4 \\ 0 & 3\alpha\beta - 2 & 4\beta - 6 & 7\beta - 8 \end{pmatrix} \overset{L_3 \to L_3 - 3L_2}{\sim} \begin{pmatrix} \beta & 1 & 3 & 4 \\ 0 & \alpha\beta - 1 & 3\beta - 3 & 3\beta - 4 \\ 0 & 1 & 3 - 5\beta & -2\beta + 4 \end{pmatrix}$

If $\beta = 0$, rank $C = 1$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

④ $D = \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 2L_1}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \overset{L_2 \to 3L_2 + 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -9 & 0 & -2 & -1 & 2 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \overset{L_1 \leftarrow 9L_1 + 2L_2 - 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 9 & 0 & 0 & 1 & 2 & 2 \\ 0 & -9 & 0 & -2 & -1 & 2 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right) \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

⑤ $E = \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \to 5L_3 - 12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{\substack{L_1 \to L_1 - 2L_3 \\ L_2 \leftarrow L_2 + 3L_3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & -5 & 0 & 25 & -35 & 15 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_2 \leftarrow -\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_1 \leftarrow L_1 - 4L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & 3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \Rightarrow E^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & 3 \\ 9 & -12 & 5 \end{pmatrix}$

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

$[X]_B = \begin{pmatrix} [v_1]_B^t \\ [v_2]_B^t \\ [v_3]_B^t \end{pmatrix}$

$X = \begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \underset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - L_1}}{\sim} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank } X = 2 \Rightarrow v_1, v_2, v_3$ linearly dependent

For the following exercises, for a list X of vectors in a canonical vector space $\mathbb{R}^n$, use that dim $< X >$ is equal to the rank of an echelon form C of the matrix consisting of the components of the vectors of X, and a basis of $< X >$ is given by the non-zero rows of C.

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine *dim* $< X >$ and a basis of $< X >$.

$X = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \overset{L_4 \leftarrow L_4 - 2L_3}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 0 & 0 & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 + 9L_1 - 5L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim \langle X \rangle = 2$

$\langle X \rangle = \langle (1,0,4), (0,1,-8) \rangle$

8. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine *dim* $< X >$ and a basis of $< X >$.

$X = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim \langle X \rangle = 2$, $\langle X \rangle = \langle (1,0,4,3), (0,2,3,1) \rangle$

9. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S =< (1, 0, 4), (2, 1, 0), (1, 1, -4) >,$$

$$T =< (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > .$$

$M_S = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & -4 \\ 2 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 + L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(S) = 2$

$S = \langle (1,0,4), (2,1,0) \rangle$

$M_T = \begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_2 \leftarrow L_2 + L_1 + L_3}{\sim} \begin{pmatrix} -3 & -2 & 4 \\ 0 & 0 & 0 \\ -2 & 0 & -8 \end{pmatrix} \underset{\substack{L_2 \leftarrow -1L_2 \\ L_2 \leftrightarrow L_3}}{\sim} \begin{pmatrix} -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(T) = 2$

$T = \langle (-3,-2,4), (-2,0,-8) \rangle$

$\dim(S+T) = \dim \langle S \cup T \rangle$

$\dim(S+T) = \dim(S) + \dim(T) - \dim(T \cap S)$

$M_{S+T} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{L_4 \leftarrow L_4 - 2L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ -3 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(T+S) = 3$

$\dim S \cap T = \dim S + \dim T - \dim S+T = 2 + 2 - 3 = 1$