Seminar 7

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4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

$rank(v_1, v_2, \dots, v_n)$ = maximum nr. of linear independent vectors among them

If $v_1, v_2, \dots, v_n \in K^N$

$rank(v_1, v_2, \dots, v_n) = rank(v_1 | v_2 | \dots | v_n)$

$Ker\, f = \{v \in \mathbb{R}^3 \mid f(v) = (0,0)\} = \{(x,y,z) \in \mathbb{R}^3 \mid (y,-x) = (0,0)\} = \{(0,0,z) \in \mathbb{R}^3\} = \{z(0,0,1) \mid z \in \mathbb{R}^3\} = <0,0,1>$

$(0,0,1) \neq (0,0,0) \to (0,0,1)$ is a basis $\Rightarrow \dim Ker\, f = 1$

$Im\, f = \{f(v) \mid v \in \mathbb{R}^3\}$

$Im\, f = \{(y,-x) \mid (y,-x) \in \mathbb{R}^2\}$

$(y,-x) = (y,0) + (0,-x) = y(1,0) - x(0,1)$

$Im\, f = <(1,0),(0,1)>$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow rank \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2 \Rightarrow (1,0),(0,-1)$ linearly independent $\Rightarrow \dim Im\, f = 2$

Def: V K-v.s.

$S_1, S_2 \leq_{\mathbb{R}} V$

$V = S_1 \oplus S_2 \Leftrightarrow \begin{cases} V = S_1 + S_2 \\ S_1 \cap S_2 = 0 \end{cases}$

$\Rightarrow \forall v \in V, \exists!\, s_1 \in S_1, s_2 \in S_2$

s.t. $v = s_1 + s_2$

$s_1 + s_2 = < s_1, s_2 >$

Now if $S = V$, then we can always find another subspace $T \leq V$ s.t. $V = S \oplus T$

Here's how:

① Find a basis of S, $v_1, v_2, \dots, v_t$

② Complete this basis to a basis of V

$(v_1, v_2, \dots, v_k, w_{k+1}, w_{k+2}, \dots, w_n)$

③ $(w_{k+1}, \dots, w_n)$ will be a basis of T, so $T = < w_k, w_{k+1}, \dots, w_n >$

7. Determine a complement for the following subspaces:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(ii) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

$(x,y,z) = x<1,0,0> + y<0,1,0> + z<0,0,1>$

$A = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$

$x = -2y - 3z$

$A = \{(-2y-3z, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R}^3\}$

$(-2y-3z, y, z) = y(-2,1,0) + z(-3,0,1)$

$A = <(-2,1,0),(-3,0,1)>$

$M = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$, $rank\, M \leq 2$

$rank\, M = ?$

$\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3 \neq 0 \Rightarrow rank\, M \geq 2$

$\Rightarrow rank\, M = 2 \Rightarrow (\underbrace{(-2,1,0)}_{v_1}, \underbrace{(-3,0,1)}_{v_2})$ basis for A

We will complete this basis to a basis of $\mathbb{R}^3$

It suffices to choose $v_3 \in \mathbb{R}^3 \setminus <v_1, v_2> = \mathbb{R}^3 \setminus A$

Let $v_3 = (1,2,3)$

The complement is $<(1,2,3)>$

ii) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$

$B = <X, X^3>$

$\alpha X + \beta X^3 = 0 \Rightarrow \alpha = \beta = 0 \Rightarrow X, X^3$ are lin. indep

$\mathbb{R}_3[X] = <1, X, X^2, X^3>$

The compliment of B in $\mathbb{R}_3[X]$ is $<1, X^2>$

Theorem: V, V' - K-v.s.

$f: V \to V'$ linear map $\Rightarrow \dim V = \dim Ker\, f + \underbrace{\dim Im\, f}_{\text{rank of } f}$

$\left(\begin{array}{l} V / Ker \cong Im\, f \\ \text{1st isom. th} \\ \dim V - \dim Ker\, f = \dim Im\, f \end{array} \right)$

V, K-v.s., $S, T \leq_K V$

$\dim(S + T) = \dim(S) + \dim(T) - \dim(S \cap T)$

10. Determine the dimensions of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

$\alpha_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_1 \\ \alpha_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\left. \begin{array}{l} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 = 0 \\ \alpha_2 = 0 \end{array} \right\} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are linearly independent $\Rightarrow \dim S = 2$

$\alpha_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 + \alpha_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\left. \begin{array}{l} \alpha_1 = 0 \\ \alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{array} \right\} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are linearly independent $\Rightarrow \dim T = 2$

$S + T = < \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} > \Rightarrow \dim S + T = 4$

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

$S \cap T = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0,\ (x,y,z) = \alpha(0,1,1) + \beta(1,1,0)\}$

$(0,y,z) = (\beta, \alpha + \beta, \alpha)$

$(0,y,z) = (0, \alpha, \alpha)$

$\{(0, \alpha, \alpha) \mid \alpha \in \mathbb{R}\} = <(0,1,1)>$

$\dim S+T = \dim S + \dim T - \dim S \cap T$

$S = <(0,1,0),(0,0,1)>$

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

$A = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$

$A = \{(x,y,0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

$A: (x,y,0) = (x,0,0) + (0,y,0) = x(1,0,0) + y(0,1,0) = <(1,0,0),(0,1,0)>$

$\dim A = 2$

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$B = \{(x, y, -x-y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

$B = \{x(1,0,-1) + y(0,1,-1) \mid x, y \in \mathbb{R}\} = <(1,0,-1),(0,1,-1)>$

$(1,0,-1)$ and $(0,1,-1)$ are lin. indep

$\dim B = 2$

$C = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$

$C = \{x(1,1,1) \mid x \in \mathbb{R}\} = <(1,1,1)>$

$\dim C = 1$

2. Let K be a field and $S = \{(x_1, \dots, x_n) \in K^n \mid x_1 + \dots + x_n = 0\}$.
(i) Prove that S is a subspace of the canonical vector space K^n over K.
(ii) Determine a basis and the dimension of S.

S subspace of $K \Leftrightarrow \begin{cases} S \neq \emptyset \ (0,0,\dots,0) \in S \\ \forall a, b \in K, \forall x, y \in S,\ ax + by \in S \end{cases}$

$ax + by = a(x_1, \dots, x_n) + b(y_1, y_2, \dots, y_n) = (ax_1, ax_2, \dots, ax_n) + (by_1, by_2, by_3, \dots, by_n) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3, \dots, ax_n + by_n)$

$ax + by \in S \Leftrightarrow ax_1 + by_1 + ax_2 + by_2 + \dots + ax_n + by_n = 0 \Leftrightarrow$

$\Leftrightarrow a\underbrace{(x_1 + x_2 + \dots + x_n)}_{=0} + b\underbrace{(y_1 + y_2 + \dots + y_n)}_{=0} = 0 \Leftrightarrow$

$\Leftrightarrow 0 = 0$, "T"

b) $S = \{(x_1, x_2, \dots, x_n) \in K^n \mid x_1 + x_2 + \dots + x_n = 0\}$

$S = \{(x_1, x_2, \dots, x_{n-1}, -x_1 - x_2 - \dots - x_{n-1})\} = \{x_1(1,0,0,\dots,-1) + x_2(0,1,0,\dots,-1) + x_3(0,0,1,\dots,-1) + \dots + x_{n-1}(0,0,\dots,1,-1)\} =$

$= <\underbrace{(1,0,0,\dots,-1)}_{v_1}, \underbrace{(0,1,0,\dots,-1)}_{v_2}, \underbrace{(0,0,1,\dots,-1)}_{v_3}, \dots, \underbrace{(0,0,0,\dots,1,-1)}_{v_{n-1}}>$

$v_1, v_2, \dots, v_{n-1}$ are linearly independent

$\Rightarrow \dim S = n-1$

3. Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

$(\mathbb{C}, +)$ abelian group

$\mathbb{C}$ v. space

1) $\forall k_1, k_2 \in \mathbb{C}$, $z \in \mathbb{C}$

$(k_1 + k_2) \cdot z_1 = k_1 z_1 + k_2 z_1$ (True)

2) $\forall k \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C}$

$k(z_1 + z_2) = kz_1 + kz_2$ (True)

3) $\forall k_1, k_2 \in \mathbb{C}$, $z_1 \in \mathbb{C}$

$k_1(k_2 z) = (k_1 k_2)(z)$ (True)

4) $\forall z \in \mathbb{C}$, $1 \cdot z = z$ (True)

$\forall z \in \mathbb{C}, \exists a, b \in \mathbb{R}$, $z = a \cdot 1 + b \cdot i \in \mathbb{C} \Rightarrow \mathbb{C} = <1, i>$ is a basis

$\dim(\mathbb{C}) = 2$

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

$f \in End_{\mathbb{R}}(\mathbb{R}^3)$

$f(x,y,z) = (-y + 5z, x, y - 5z)$

$Ker\, f = \{(x,y,z) \mid f(x,y,z) = (0,0,0)\}$

$\begin{cases} 5z - y = 0 \\ y - 5z = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 5z \\ x = 0 \\ y = 5z \end{cases}$

$Ker\, f = \{(0, 5z, z) \mid z \in \mathbb{R}\} = \{z(0,5,1)\} = <(0,5,1)>$

$\dim(Ker(f)) = 1$

$Im\, f = \{f(x,y,z) \mid x, y, z \in \mathbb{R}\}$

$Im\, f = \{(-y + 5z, x, y - 5z)\} = \{x(0,1,0) + (y - 5z)(-1,0,1)\} = <(0,1,0),(-1,0,1)>$

$\dim Im\, f = 2$

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

$A = <(1,0,0),(0,1,0)>$

Let $v_3 = (x,y,z)$ be part of the basis

v_1, v_2, v_3 lin indep

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$\alpha(1,0,0) + \beta(0,1,0) + \gamma(x,y,z) = 0$

$(\alpha + \gamma x, \beta + \gamma y, \gamma z) = (0,0,0) \Rightarrow \left. \begin{cases} \alpha + \gamma x = 0 \\ \beta + \gamma y = 0 \\ \gamma z = 0 \end{cases} \right\}$ $x, y, z \in \mathbb{R}$, $(x,y,z) = (0,0,1)$

$B = <\underbrace{(1,0,-1)}_{v_1}, \underbrace{(0,1,-1)}_{v_2}>$

$av_1 + bv_2 + cv_3 = 0 \Leftrightarrow a, b, c = 0$

$(a,0,-a) + (0,b,-b) + (cx, cy, cz) = 0$

$(a + cx, b + cy, -a - b + cz) = (0,0,0)$

$\begin{cases} a + cx = 0 \\ b + cy = 0 \\ -a - b + cz = 0 \end{cases}$

$c(x + y + z) = 0 \Rightarrow x + y + z \neq 0 \Rightarrow (x,y,z) = (1,1,0)$

$C = <(1,1,1)>$

We have to find 2 vectors that are linearly independent with $(1,1,1)$

$v_2 = (1,0,1)$

$v_3 = (1,1,0)$

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

V K v.space

$S \leq V$

$T \leq V$

$U \leq V$

$\left. \begin{array}{l} \dim(S + U) = \dim(S) + \dim(U) - \dim(S \cap U) \\ \dim(T + U) = \dim(T) + \dim(U) - \dim(T \cap U) \\ \dim(S + U) = \dim(T + U) \end{array} \right\} \Rightarrow \left. \begin{array}{l} \dim(S) = \dim(T) \\ T - K.v.s \\ S \subseteq T \\ \Rightarrow \dim S \leq \dim T \end{array} \right\} \Rightarrow S = T$