peana

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Numerical Calculus Exam, 5

1) **(3 points)** Solve the system
$$\begin{cases} -x_1 + 3x_2 + 2x_3 = -1 \\ 2x_1 + 4x_2 = 8 \\ x_1 + 2x_2 - 3x_3 = 7 \end{cases}$$
using LUP decomposition with partial pivoting.

2) **(2 points)** Find the values of $c \in \mathbb{R}$ for which the iterative method $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converges to $\alpha = 1$ for a suitable x_0. Discuss the convergence of the method, the order of convergence and give a bound for the error.

3) Let $f \in C^2[-1, 1]$.

a) **(1.5 points)** Find $(Pf)(x)$, the polynomial interpolating the function f, given $f(-1)$ and $f'(1)$.

b) **(1 point)** Integrate $\int_{-1}^{1} (Pf)(x)dx$ to obtain a quadrature formula for $\int_{-1}^{1} f(x)\, dx$. Find its degree of precision.

c) **(1.5 points)** Find the remainder $R(f)$ of the quadrature formula in **b)**, using Peano's theorem.

$$Lf(x) = \int_a^b K_n(x,t) f^{(n+1)}(t)dt,$$

$$K_n(x,t) = \frac{1}{n!} L\left((x-t)_+^n\right)$$

$$Lf = \frac{1}{(n+1)!} f^{(n+1)}(\xi) L e_{n+1},$$

$$\int_a^b w(x)f(x)dx = \sum_{j=0}^{m} A_j F_j(f) + R(f),$$

$f(-1), f'(1)$

$$B_2 f(x) = b_{00}(x) f(\underset{x_0}{-1}) + b_{11}(x) f'(\underset{x_1}{1})$$

$$\boxed{\sum_{i,k} b_{ki} f^{(i)}(x_k)}$$

$b_{00}(x) = a_1 x + b_1$

$b_{11}(x) = a_2 x + b_2$

$$\begin{cases} b_{00}(x_0) = 1 \Rightarrow b_1 = 1 \\ b_{00}'(x_1) = 0 \Rightarrow a_1 = 0 \end{cases} \Rightarrow b_{00}(x) = 1$$

$$\begin{cases} b_{11}(x_0) = 0 \Rightarrow -1 + b_2 = 0 \Rightarrow b_2 = 1 \\ b_{11}'(x_1) = 1 \Rightarrow a_2 = 1 \end{cases} \Rightarrow b_{11}(x) = x+1$$

$$\Rightarrow \boxed{B_2 f(x) = f(-1) + (x+1) f'(1)}$$

b) $$\int_{-1}^{1} f(-1) + (x+1) f'(1) dx + R(f) = \int_{-1}^{1} f(x)dx$$

$$\Rightarrow 2f(-1) + f'(1) \frac{(x+1)^2}{2} \Big|_{-1}^{1} = 2f(-1) + 2f'(1)$$

$$R(f) = \int_{-1}^{1} f(x)dx - \int_{-1}^{1} (Pf)(x)dx$$

$$\Rightarrow R(f) = \int_{-1}^{1} f(x)dx - 2f'(1) - 2f(-1)$$

! $f(x) = e_i(x) = x^i$

$$R(e_0(x)) = \int_{-1}^{1} dx - 2e_0(-1) - 2e_0'(1) = x \Big|_{-1}^{1} - 2 - 0 = 0$$

$R(e_0(x)) = \int_{-1} dx - 2e_0(-1) - 2e_0'(\ )$

$R(e_1(x)) = \int_{-1}^{1} x\,dx - 2e_1(-1) - 2e_1'(1) = \frac{x^2}{2}\Big|_{-1}^{1} - 2\cdot(-1) - 2\cdot 1 =$

$= \frac{1}{2} - \frac{1}{2} + 2 - 2 = 0$

$R(e_2(x)) = \int_{-1}^{1} x^2 - 2e_2(-1) - 2e_2'(1) = \frac{x^3}{3}\Big|_{-1}^{1} - 2\cdot 1 - 2\cdot 2 =$

$= \frac{1}{3} + \frac{1}{3} - 2 - 4 = \frac{2}{3} - 6 = \frac{-16}{3} \neq 0$

$\Rightarrow$ degree $= 1$

c) $f(x) = B_2(f(x)) + R_1(f(x)) \Rightarrow R_1(f(x)) = f(x) - B_2(f(x))$

$R(f(x)) = \int_{-1}^{1} K_1(x,t) f''(t)\,dt$

$K_1(x,t) = \frac{1}{1!} R_1\left((x-t)_+^1\right)$

$R_1((x-t)_+) = (x-t)_+ - \underbrace{(-1-t)_+}_{=0} - (x+1)\frac{\partial}{\partial x}(x-t)_+\Big|_{x=1}$

$\cancel{B_2 f(x) = f(-1) + (x+1) f'(1)}$

$= (x-t)_+ - (x+1)\frac{\partial}{\partial x}(x-t)_+\Big|_{x=1}$

$x \le t \Rightarrow 0 - (x+1)\cdot 0 = 0$

$x \ge t \Rightarrow (x-t) - (x+1)\cdot 1 = -t-1 \le 0$ $\Big|\ \le 0$

$$\Rightarrow R(x) = \frac{1}{2!} f''(\xi) Re_2$$

$$\Rightarrow \int_{-1}^{1} R(x) = \int_{-1}^{1} \frac{1}{2!} f''(\xi) Re_2 = \frac{1}{2} f''(\xi) \int_{-1}^{1} Re_2 =$$

$$= \boxed{-\frac{8}{3} f''(\xi)}$$