Lab 11

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$$\int_a^b w(x) f(x) = \sum_{k=1}^{m} A_k f(x_k) + R_m(f)$$

- m nodes (x_k)
- m coeff (A_k)

$\Big\}$ $2m$ unknowns

Degree of precision $d = 2m-1$

$$R_m(e_0) = R_m(e_1) = \dots = R(e_{2m-1}) = 0 \Leftrightarrow R_m(e_j) = 0,\ j = \overline{0, 2m-1}$$

$$e_i(x) = x^i,\ i \in \mathbb{N}$$

$$R_m(f) = \int_a^b w(x) f(x)\,dx - \sum_{k=1}^{m} A_k f(x_k)$$

$$R_m(e_j) = \int_a^b w(x) x^j\,dx - \sum_{k=1}^{m} A_k x_k^j,\ j = \overline{0, 2m-1}$$

$$\underbrace{\int_a^b w(x) x^j\,dx}_{\mu_j \text{ moments}} = \sum_{k=1}^{m} A_k x_k^j$$

Ex: $i = \int_{-1}^{1} f(x)\,dx \simeq A_1 f(x_1)$ x_1, A_1 unknowns

$$\begin{cases} j=0: & A_1 x_1^0 = \int_{-1}^{1} 1 \cdot x^0\,dx \Rightarrow A_1 = 2 \\ j=1: & A_1 x_1^1 = \int_{-1}^{1} 1 \cdot x^1\,dx \Rightarrow A_1 x_1 = 0 \Rightarrow x_1 = 0 \end{cases}$$

Legendre: $l_m(x) = [(x^2-1)^m]^{(m)}$

$l_1(x) = [(x^2-1)^1]' = 2x$

$x_1 = 0$

$m = 2 \Rightarrow d = 2m-1 = 3$

$A_1 + A_2 = \int_{-1}^{1} 1\,dx = 2$

$A_1 x_1 + A_2 x_2 = \int_{-1}^{1} 1\, x^1\,dx = 0$

$A_1 x_1^2 + A_2 x_2^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$A_1 x_1^3 + A_2 x_2^3 = \int_{-1}^{1} x^3\,dx = 0$

$l_2(x) = ((x^2-1)^2)'' = (x^4 - 2x^2 - 1)'' = 12x^2 - 4 = 4(3x^2-1) \Rightarrow x_1, x_2 = \pm\frac{\sqrt{3}}{3}$

$$\begin{cases} A_1 + A_2 = 2 \\ \frac{\sqrt{3}}{3} A_1 - \frac{\sqrt{3}}{3} A_2 = 0 \end{cases} \Rightarrow A_1 = A_2 = 1$$

$i = \int_{-1}^{1} f(x)\,dx = f(x_1) + f(x_2) = f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right)$

4) $i = \int_0^\infty e^{-x} \sin x\,dx = \frac{1}{2}$

Interval: $[0, \infty)$

Weight: $w(x) = x^\alpha e^{-x}$, $a > -1$

$f(x) = \sin(x)$

$m = 2$

Laguerre: $L_m^a(x) = x^{-a} e^x (x^{m+a} e^{-x})^{(a)}$

$L_2^0(x) = e^x (x^2 e^{-x})'' = e^x (2x e^{-x} - x^2 e^{-x})' =$

$= e^x (2e^{-x} - 2x e^{-x} - 2x e^{-x} + x^2 e^{-x}) =$

$= 2 - 4x + x^2 = 0 \Rightarrow x_{1,2} = 2 \pm \sqrt{2}$

$$\begin{cases} A_1 + A_2 = \int_0^\infty e^{-x}\,dx \\ A_1 x_1 + A_2 x_2 = \int_0^\infty e^{-x}\,dx \end{cases}$$

$\Gamma: [0, \infty) \to (0, \infty)$, $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\,dx$

$\int_0^\infty e^{-x}\,dx = \Gamma(1) = 0! = 1$

$\int_0^\infty e^{-x} x\,dx = \Gamma(2) = 1! = 1$

$A_1 + A_2 = 1$

$2A_1 + \sqrt{2} A_1 + 2A_2 - \sqrt{2} A_2 = 1$

$\sqrt{2}(A_1 - A_2) = -1$

$A_1 - A_2 = -\frac{\sqrt{2}}{2}$

$A_1 + A_2 = 1$

$\int_0^\infty e^{-x} \sin x\,dx \simeq \frac{2+\sqrt{2}}{4} \sin(2+\sqrt{2}) + \frac{2-\sqrt{2}}{4} \sin(2+\sqrt{2})$

b) $\int_{-\infty}^{\infty} e^{-x^2} \cos x\,dx$

Interval: $(-\infty, +\infty)$

Weight: $w(x) = e^{-x^2}$

Fct: $f(x) = \cos(x)$

$m = 2 \Rightarrow d = 3$

Hermite: $(-1)^m e^{x^2} (e^{-x^2})^{(m)}$

$(-1)^2 e^{x^2} (e^{-x^2})'' = e^{x^2} \cdot (e^{-x^2} \cdot (-2x))' = -2e^{x^2}(e^{-x^2} - x e^{-x^2} 2x) = -2 + 4x^2 \Rightarrow x = \pm\frac{\sqrt{2}}{2}$

$A_1 + A_2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$A_1 x_1 + A_2 x_2 = \int_{-\infty}^{\infty} e^{-x^2} x\,dx$

$\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$

$\int_{\mathbb{R}} x e^{-x^2}\,dx = -\frac{1}{2} \int_{\mathbb{R}} -2x e^{-x^2}\,dx = -\frac{1}{2} e^{-x^2} \Big|_{-\infty}^{+\infty} = -\frac{1}{2} \left(\lim_{x \to \infty} e^{-x^2} - \lim_{x \to -\infty} e^{-x^2}\right) = 0$

$A_1 + A_2 = \sqrt{\pi}$

$\frac{\sqrt{2}}{2} A_1 + \frac{\sqrt{2}}{2} A_2 = 0 \Rightarrow A_1 = A_2$

$I \approx \frac{\sqrt{\pi}}{2} \cos\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} \cos\left(\frac{\sqrt{2}}{2}\right)$