Seminar 5

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Def: V - K. v. space

$X \subseteq V$

$<X> = \bigcap_{S \leq V} = \{ \sum_{i=1}^{n} a_i x_i \mid n \in \mathbb{N}, a_i \in K, x_i \in V \}$ → the subspaces generated by x $S \supseteq x$

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

i) $<1, X, X^2>$ in $\mathbb{R}[X]$

$\sum_{k=1}^{n} = a_0 + a_1 X + a_2 x^2$

$<1, X, X^2>$ in $\mathbb{R}[X] = \{ a_0 + a_1 X + a_2 x^2 \mid a_i \in \mathbb{R}, i = \overline{0,2} \} = \mathbb{R}_2[X]$

ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} c + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d \mid a,b,c,d \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \right\} = M_2(\mathbb{R})$

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

i) $A = \{(0,y,z)\} = \{(0,y,0) + (0,0,z)\} = \{y(0,1,0) + z(0,0,1)\} = <(0,1,0),(0,0,1)>$

ii) $B = \{(x, y, -(x+y))\} = \{(x,0,-x) + (0,y,-y)\} = \{x(1,0,-1) + y(0,1,-1)\} = <(1,0,-1),(0,1,-1)>$

iii) $C = \{(x,x,x)\} = \{x(1,1,1)\} = <(1,1,1)>$

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b\rangle = \langle c, d, e\rangle$.

$\begin{cases} c = a + b \\ d = a - b \\ e = 3a - b \end{cases}$

$<a,b> = \{ a_0 \cdot a + a_1 b \mid a_0, a_1 \in \mathbb{R} \}$

$<c,d,e> = \{ a_2 \cdot (a+b) + a_3(a-b) + a_4(3a-b) \mid a_2,a_3,a_4 \in \mathbb{R} \} = \{ a(a_2 + a_3 - 3a_4) + b(a_2 - a_3 - a_4) \}$

Let $a_2 = a_3 = a_4 = 1 \Rightarrow a_0 = -1$

$a_1 = -1$

$<a,b> = <c,d,e>$

For a vector space V over K, we denote by $S(V)$ the set of all subspaces of V. Sometimes, this set is denoted by $S_K(V)$ if we like to emphasize the field K.

> **Theorem 2.3.1** *Let V be a vector space over K and let $(S_i)_{i\in I}$ be a family of subspaces of V. Then $\bigcap_{i\in I} S_i \in S(V)$.*

Proof. For each $i \in I$, we have $S_i \in S(V)$, hence $0 \in S_i$. Then $0 \in \bigcap_{i\in I} S_i \neq \emptyset$. Now let $k_1, k_2 \in K$ and $x, y \in \bigcap_{i\in I} S_i$. Then $x, y \in S_i$, $\forall i \in I$. But $S_i \in S(V)$, $\forall i \in I$. It follows that $k_1x + k_2y \in S_i$, $\forall i \in I$, hence $k_1x + k_2y \in \bigcap_{i\in I} S_i$. Therefore, $\bigcap_{i\in I} S_i \in S(V)$. □

$<X> = \{ k_1 v_1 + k_2 v_2 + \dots + k_n v_n \mid k_i \in K, v_i \in X, i = \overline{1,n}, n \in \mathbb{N}^* \}$

Proof: $L = \{ k_1 v_1 + k_2 v_2 + \dots + k_n v_n \mid k_i \in K, v_i \in X, i = 1,\dots,n, n \in \mathbb{N}^* \}$

i) Let $v \in X$. Then $v = 1 \cdot v \in L \Rightarrow L \neq \emptyset$

Let $k, k' \in K$, $v, v' \in V$ $\Rightarrow v = \sum_{i=1}^{n} k_i v_i$, $k_1, \dots, k_n, k'_1, \dots, k'_m \in K$

$v' = \sum_{j=1}^{m} k'_j v'_j$, $v_1, \dots, v_n, v'_1, \dots, v'_m \in X$

$kv + k'v' = k \sum_{i=1}^{m} k_i v_i + k' \sum_{j=1}^{m} k'_j v'_j = \sum_{i=1}^{m} (k k_i) v_i + \sum_{j=1}^{m} (k' k'_j) v'_j \in L$ (finite linear combination of vectors of x) $\Rightarrow L \leq V$

ii) For $n=1$, $k_1 = 1$, $kv + k'v' = kv_1 + \sum_{i=1}^{m} (k' k_j) v_j \in V$

iii) Let $S \leq V$ s.t. $X \subseteq S$, $k_1, k_2, \dots, k_n \in K$, $v_1, v_2, \dots, v_n \in X$ $\Rightarrow k_1 v_1 + k_2 v_2 + \dots + k_n v_n \in S \Rightarrow L \subseteq S$

Thus $<X> = L$

Def: W, V K.V.S.

$f: V \to W$ is a homomorphism of K v.s. or a K-linear map if

$\forall v_1, v_2 \in V, k_1, k_2 \in \mathbb{R}$

$f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$ (Endomorphism if $V = W$)

The longer version: $\forall v_1, v_2 \in V$, $f(v_1 + v_2) = f(v_1) + f(v_2)$

$\forall k \in K, \forall v \in V$: $f(kv) = k f(v)$

The kernel: $\text{Ker} f = \{ v \in V \mid f(v) = 0_W \}$

$\text{Im} f = \{ f(v) \mid v \in V \} = \{ w \in W \mid \exists v \in V \text{ s.t. } f(v) = w \}$

$\text{Ker} f \leq_K V$

$\text{Im} f \leq_K W$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

9. Determine the kernel and the image of the endomorphisms from Exercise **6.** + find system of generators

f: $\forall (x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$

$\forall k_1, k_2 \in \mathbb{R}$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = k_1 f(x_1,y_1) + k_2 f(x_2,y_2)$

$k_1(x_1,y_1) + k_2(x_2,y_2) = (k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) \mid f$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = f(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) = (k_1 x_1 + k_2 x_2 + k_1 y_1 + k_2 y_2, k_1 x_1 + k_2 x_2 - k_1 y_1 - k_2 y_2) = (k_1(x_1+y_1) + k_2(x_2+y_2), k_1(x_1-y_1) + k_2(x_2-y_2))$

$k_1 f(x_1,y_1) = k_1(x_1+y_1, x_1-y_1) = (k_1 x_1 + k_1 y_1, k_1 x_1 - k_1 y_1)$

$k_2 f(x_2,y_2) = k_2(x_2+y_2, x_2-y_2) = (k_2 x_2 + k_2 y_2, k_2 x_2 - k_2 y_2)$

$\Rightarrow k_1 f(x_1,y_1) + k_2 f(x_2,y_2) = (k_1 x_1 + k_1 y_1 + k_2 x_2 + k_2 y_2, k_1(x_1 - y_1) + k_2(x_2 - y_2))$

$\Rightarrow f(k_1(x_1,y_1) + k_2(x_2,y_2)) = k_1 f(x_1,y_1) + k_2 f(x_2,y_2)$

$\text{Ker} f = \{ v \in V \mid f(v) = (0,0) \}$

$f(x,y) = (x+y, x-y) = (0,0) \iff \begin{cases} x + y = 0 \\ x - y = 0 \end{cases} \iff (x,y) = (0,0)$

$\text{Ker} f = \{(0,0)\}$

$\text{Im} f = \{ f(x,y) \mid (x,y) \in \mathbb{R}^2 \} = \{ (x+y, x-y) \mid (x,y) \in \mathbb{R}^2 \} = \{ x(1,1) + y(1,-1) \mid (x,y) \in \mathbb{R}^2 \} = <(1,1),(1,-1)>$

g) Same as above

$\text{Ker}\, g = \{ v \mid f(v) = (0,0) \} = \{ (x,y) \mid (2x - 4y, 4x - 2y) = (0,0) \}$

$\begin{cases} 2x - 4y = 0 \\ 4x - 2y = 0 \end{cases} \iff \begin{cases} x = 2y \\ 2x = y \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\text{Ker}\, g = \{(0,0)\}$

$\text{Im}\, g = \{ f(x,y) \mid (x,y) \in \mathbb{R}^2 \} = \{ (2x - 4y, 4x - 2y) \mid (x,y) \in \mathbb{R}^2 \} = \{ 2x(1,2) - 2y(2,1) \mid (x,y) \in \mathbb{R}^2 \} = <(1,2),(2,1)>$

h) Same as above

$\text{Ker}\, h = \{ (x,y,z) \mid f(x,y,z) = (0,0,0) \} = \{ (x,y,z) \mid (x-y, y-z, z-x) = (0,0,0) \} = \{ (x,x,x) \mid x \in \mathbb{R} \} = <(1,1,1)>$

$\text{Im}\, h = \{ f(x,y,z) \mid (x,y,z) \in \mathbb{R}^3 \} = \{ (x-y, y-z, z-x) \mid (x,y,z) \in \mathbb{R}^3 \} = \{ x(1,0,-1) + y(-1,1,0) + z(0,-1,1) \mid x,y,z \in \mathbb{R} \} = <(1,0,-1),(-1,1,0),(0,-1,1)>$

> **Corollary 2.3.8** *Let V be a vector space over K and let $x_1, \dots, x_n \in V$. Then*
> $$\langle x_1, \dots, x_n\rangle = \{k_1x_1 + \dots + k_nx_n \mid k_i \in K, x_i \in X, i = 1, \dots, n\}.$$

Def: V K v.s.

$S, T \leq V$

$S + T = \{ s + t \mid s \in S, t \in T \}$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

S and T are subspaces of $\mathbb{R}^3$ (proven in Seminar 3 exercise 2)

$S = <(-1,1,0),(-1,0,1)>$

$T = <(1,1,1)>$

To prove that $S \oplus T = \mathbb{R}^3$, we prove that $S + T = \mathbb{R}^3$ and $S \cap T = \{0\}$

$\forall v \in \mathbb{R}^3\ \exists!\ a, b, c \in \mathbb{R}$ s.t. $v = a(-1,1,0) + b(-1,0,1) + c(1,1,1) = (-a-b+c, a+c, b+c)$

$\begin{cases} -a - b + c = x \\ a + c = y \\ b + c = z \end{cases}$ $\quad b = z - c$ $\quad -a - z + y - a = x \iff -3a = x - 2y + z \iff a = \frac{2y - x - z}{3}$

$c = y - \frac{2y - x - z}{3} = \frac{x + y + z}{3}$

$b = z - \frac{x + y + z}{3} = \frac{-x - y + 2z}{3}$

(a, b, c) are unique

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

f - odd $\iff f(x) = -f(-x)$

f - even $\iff f(-x) = f(x)$

$S \neq \emptyset$, $\cos(x) \in S$

$T \neq \emptyset$, $f(x) = -2x$

S subspace of $\mathbb{R}^{\mathbb{R}} \iff \forall f, g \in S, \forall a, b \in \mathbb{R}$

$(af + bg)(x) = af(x) + bg(x) = af(-x) + bg(-x)$

$\iff (af + bg)(x) = (af + bg)(-x)$

$\iff (af + bg)(x) = (af + bg)(-x) \Rightarrow S \leq \mathbb{R}^{\mathbb{R}}$

T subspace of $\mathbb{R}^{\mathbb{R}} \iff \forall f, g \in T, \forall a, b \in \mathbb{R}$

$(af + bg)(x) = af(x) + bg(x) = -af(-x) - bg(-x)$

$\iff (af + bg)(-x) = -(af + bg)(x) \Rightarrow T \leq \mathbb{R}^{\mathbb{R}}$

Let $f: \mathbb{R} \to \mathbb{R}$, $g \in S$, $h \in T$, $f(x) = g(x) + h(x)$

$f(-x) = g(-x) + h(-x)$

$f(-x) = g(x) - h(x)$

$f(x) = g(x) + h(x)$

$\Rightarrow g(x) = \frac{1}{2}(f(x) + f(-x)) \in S$

$h(x) = \frac{1}{2}(f(x) - f(-x)) \in T$

$\Rightarrow g$ and h are unique functions that can be written in any way of $f(x)$

Intersection

Let $f(x) \in S \cap T$

Then $\begin{cases} f(x) = f(-x) \\ f(x) = -f(-x) \end{cases} \Rightarrow f(-x) = -f(-x) \iff f(x) = -f(x) \Rightarrow f(x) = \theta(x)$

So $S \cap T = \{\theta(x)\}$

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

i) $\forall (x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$

$\forall k_1, k_2 \in \mathbb{R}$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = k_1 f(x_1,y_1) + k_2 f(x_2,y_2)$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = f(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) = (a k_1 x_1 + a k_2 x_2 + b k_1 y_1 + b k_2 y_2, c k_1 x_1 + c k_2 x_2 + d k_1 y_1 + d k_2 y_2)$

$k_1 f(x_1,y_1) + k_2 f(x_2,y_2) = k_1(a x_1 + b y_1, c x_1 + d y_1) + k_2(a x_2 + b y_2, c x_2 + d y_2) = (a k_1 x_1 + a k_2 x_2 + b k_1 y_1 + b k_2 y_2, c k_1 x_1 + c k_2 x_2 + d k_1 y_1 + d k_2 y_2)$

$\Rightarrow f \in End_{\mathbb{R}}(\mathbb{R}^2)$

ii) For $k_1 = k_2 = 1$

$g((x_1,y_1) + (x_2,y_2)) = g(x_1,y_1) + g(x_2,y_2)$

$g(x_1 + x_2, y_1 + y_2) = (a + x_1, b + y_1) + (a + x_2, b + y_2)$

$(a + x_1 + x_2, b + y_1 + y_2) = (2a + x_1 + x_2, 2b + y_1 + y_2)$ (F)

$\Rightarrow g \notin End_{\mathbb{R}}(\mathbb{R}^2)$

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a).$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

$\forall k_1, k_2 \in \mathbb{R}$, $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = k_1 f(x_1,y_1) + k_2 f(x_2,y_2)$

$k_1(x_1,y_1) + k_2(x_2,y_2) = (k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) \mid f$

$f(k_1(x_1,y_1) + k_2(x_2,y_2)) = ((k_1 x_1 + k_2 x_2)\cos a - (k_1 y_1 + k_2 y_2)\sin a, (k_1 x_1 + k_2 x_2)\sin a + (k_1 y_1 + k_2 y_2)\cos a)$

$k_1 f(x_1,y_1) = k_1(x_1 \cos a - y_1 \sin a, x_1 \sin a + y_1 \cos a) = (k_1 x_1 \cos a - k_1 y_1 \sin a, k_1 x_1 \sin a + k_1 y_1 \cos a)$

$k_2 f(x_2,y_2) = k_2(x_2 \cos a - y_2 \sin a, x_2 \sin a + y_2 \cos a) = (k_2 x_2 \cos a - k_2 y_2 \sin a, k_2 x_2 \sin a + k_2 y_2 \cos a)$

$k_1 f(x_1,y_1) + k_2 f(x_2,y_2) = ((k_1 x_1 + k_2 x_2)\cos a - (k_1 y_1 + k_2 y_2)\sin a, (k_1 x_1 + k_2 x_2)\sin a + (k_1 y_1 + k_2 y_2)\cos a)$

$\Rightarrow f \in End_{\mathbb{R}}(\mathbb{R}^2)$

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

$S \neq \emptyset$, $f(0) = 0 \in S$

$\forall x_1, x_2 \in V$, $x_1 + x_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in S$ (f endomorphism)

$\forall a \in K, \forall x \in S \Rightarrow ax = f(ax) \in S$, f endomorphism

$S \leq V$