8. Polynomial Interpolation - summary

LAGRANGE

- classic
- barycentric
- Aitken / Neville

HERMITE DOUBLE NODES

Newton's form can be applied to both types (+ divided differences)

HERMITE **BIRKHOFF**

fundamental (basis) polynomials

Schematic example

Lagrange

x	x_0	x_1
f	$f(x_0)$	$f(x_1)$
f'	X	X

Hermite (double nodes)

x	x_0	x_1
f	$f(x_0)$	$f(x_1)$
f'	$f'(x_0)$	$f'(x_1)$

Hermite classic

x	x_0	x_1	x_2
f	$f(x_0)$	$f(x_1)$	$f(x_2)$
f'	X	$f'(x_1)$	$f'(x_2)$
f''	X	$f''(x_1)$	X

Birkhoff (incomplete Hermite)

x	x_0	x_1	x_2
f	$f(x_0)$	X	$f(x_2)$
f'	X	$f'(x_1)$	X
f''	X	X	$f''(x_2)$

Exercises :

1. $f:(-1,\infty)\to \mathbb{R}$, $f(x)=\dfrac{1}{1+x}$

$x_0=0$, $x_1=1$, $x_2=2$

x	0	1	2
$f(x)$	1	1/2	1/3

A. Lagrange

A.1. Classic : n = degree of the polynomial
= number of nodes − 1

(e.g. $\underbrace{x_0, x_1, \dots, x_m}_{(m+1)\text{ nodes}} \Rightarrow n=m$)

Here, $\boxed{n=2} \Rightarrow L_2f(x)=\sum_{i=0}^{2} l_i(x)f(x_i)$

$$\Rightarrow L_2f(x)=l_0(x)f(x_0)+l_1(x)f(x_1)+l_2(x)f(x_2)$$
$$=1\cdot l_0(x)+\frac{1}{2}\cdot l_1(x)+\frac{1}{3}\cdot l_2(x)$$

- $l_0(x)=\dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\dfrac{(x-1)(x-2)}{(0-1)(0-2)}=\dfrac{(x-1)(x-2)}{2}$

- $l_1(x)=\dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\dfrac{(x-0)(x-2)}{(1-0)(1-2)}=-x(x-2)$

- $\ell_2(x) = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \dfrac{(x-0)(x-1)}{(2-0)(2-1)} = \dfrac{x(x-1)}{2}$

Then

$$L_2f(x) = {}^{3/}\frac{(x-1)(x-2)}{2} - {}^{3/}\frac{x(x-2)}{2} + \frac{x(x-1)}{6}$$

$$= \frac{3x^2 - 9x + 6 - 3x^2 + 6x + x^2 - x}{6} = \frac{1}{6}x^2 - \frac{2}{3}x + 1$$

$$\Rightarrow f(x) \approx L_2f(x) = \frac{1}{6}x^2 - \frac{2}{3}x + 1 .$$

A.2. Newton form : $n = m$

$$\underset{\substack{\| \\ L_2f(x)}}{N_2f(x)} = \underset{\substack{\| \\ f[x_0]}}{f(x_0)} + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2]\underbrace{(x-x_0)(x-x_1)}_{x^2 = \text{degree } 2}$$

$$= f(0) + f[x_0,x_1]\,x + f[x_0,x_1,x_2]\,x(x-1)$$

divided differences

x	$f(x)$	$D^1 f$	$D^2 f$
0	1	$f[x_0,x_1] = \frac{1/2 - 1}{1-0} = -1/2$	$f[x_0,x_1,x_2] = \frac{-1/6 + 1/2}{2-0} = 1/6$
1	1/2	$f[x_1,x_2] = \frac{1/3 - 1/2}{2-1} = -1/6$	
2	1/3		

Hence, $N_2f(x) = 1 - \frac{1}{2}x + \frac{1}{6}x(x-1) = \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

B. Hermite (double nodes)

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$\Rightarrow f'(x) = \left[(1+x)^{-1}\right]' = -1\cdot(1+x)^{-2}\cdot\underbrace{(1+x)'}_{1} = \frac{-1}{(1+x)^2}$$

x	0	1	2
$f(x)$	1	1/2	1/3
$f'(x)$	-1	$-1/4$	$-1/9$

Pattern: time - x
distance - f
speed - f'

- In this case, n = degree of the polynomial

$$\boxed{n = 2\cdot m + 1}$$

Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ $\Rightarrow m = 2$ $\Rightarrow n = 2\cdot 2 + 1 = 5$

$$\boxed{n = 5}$$

- We apply the Newton's form for Hermite polynomial with double nodes

$$N_5 f(z) = f(z_0) + \sum_{i=1}^{5} (z-z_0)\cdot\ldots\cdot(z-z_{i-1})D^i f(z_0)$$

$$= f(z_0) + f[z_0, z_1](z-z_0) + f[z_0, z_1, z_2](z-z_0)(z-z_1) +$$

$$+ f[z_0, \ldots, z_3](z-z_0)(z-z_1)(z-z_2) +$$

$$+ f[z_0, \ldots, z_4](z-z_0)\cdot\ldots\cdot(z-z_3)$$

$$+ f[z_0, \ldots, z_5](z-z_0)\cdot\ldots\cdot(z-z_4)$$

	z	$f(z)$	$D^1 f$	$D^2 f$	$D^3 f$	$D^4 f$	$D^5 f$
x_0	$0 = z_0$	1	-1	$1/2$	$-1/4$	$1/12$	$-1/36$
	$0 = z_1$	1	$-1/2$	$1/4$	$-1/12$	$1/36$	
x_1	$1 = z_2$	$1/2$	$-1/4$	$1/12$	$-1/36$		
	$1 = z_3$	$1/2$	$-1/6$	$1/18$			
x_2	$2 = z_4$	$1/3$	$-1/9$				
	$2 = z_5$	$1/3$					

derivatives

- In this table, $f(z_0) = 1$

$$f[z_0, z_1] = -1 = D^1 f(z_0)$$

$$f[z_0, z_1, z_2] = 1/2 = D^2 f(z_0)$$

$$\vdots$$

$$\Rightarrow H_5 f(z) = N_5 f(z) = 1 - 1(z-0) + \frac{1}{2} z(z-0) -$$

$$-\frac{1}{4} z^2(z-1) + \frac{1}{12} z^2(z-1)^2 - \frac{1}{36} z^2(z-1)^2(z-2)$$

$$= \frac{-z^2(z-1)^2(z-2)}{36} + \frac{z^2(z-1)^2}{12} - \frac{z^2(z-1)}{4} + \frac{z^2}{2} - z + 1$$

$$= -\frac{1}{36} z^5 + \frac{7}{36} z^4 - \frac{5}{9} z^3 + \frac{8}{9} z^2 - z + 1$$

2

t (time)	0	3	5	8	13
$f(t)$ = distance	0	225	383	623	993
$f'(t)$ = speed	0	77	80	74	72

5 nodes

$\Downarrow$

$m = 4$

$\Downarrow$

$\boxed{n = 9}$

3 Example for Birkhoff (in particular, for Hermite classic)

$x_0 = 0$, $f(0) = 2$, $f'(0) = 1$
$x_1 = 1$, ______ , $f'(1) = 3$

($f(1)$ is unknown $\Rightarrow$ Birkhoff problem)

3.1. Direct method

m = degree of the polynomial
$= |I_0| + \dots + |I_m| - 1$

(I_k is a set associated to x_k
(I_k is a set of the derivatives for x_k,
↳ contains the orders of the derivatives where $f(x_k) = f^{(0)}(x_k)$ is the derivative of order 0)

$f(x_0)$

• In our case, $I_0 = \{0, 1\}$ $f'(x_0)$

$I_1 = \{1\}$ $f'(x_1)$

• $|I_0| = \text{card}(I_0)$ = number of elements

$\Rightarrow m = |I_0| + |I_1| - 1 = 2 + 1 - 1 = 2$

$\boxed{m = 2}$

so we have a Birkhoff polynomial of degree 2

$$B_2 f(x) = ax^2 + bx + c \quad \text{(the standard form)}$$

$$(B_2 f)'(x) = 2ax + b$$

We have that

$$B_2 f(x_0) = f(x_0)$$

$$\cancel{B_2 f(x_1) = f(x_1)} \quad \text{is unknown}$$

$$(B_2 f)'(x_0) = f'(x_0)$$

$$(B_2 f)'(x_1) = f'(x_1)$$

$$\Rightarrow \quad B_2 f(0) = f(0) \Leftrightarrow c = f(0) = 2$$

$$(B_2 f)'(0) = f'(0) \Leftrightarrow b = f'(0) = 1$$

$$(B_2 f)'(1) = f'(1) \Leftrightarrow 2a + b = f'(1) = 3$$

$$\Leftrightarrow 2a + f'(0) = f'(1)$$

$$\Leftrightarrow a = \frac{1}{2}\left[f'(1) - f'(0)\right] = 1$$

Hence,

$$B_2 f(x) = \frac{1}{2}\left[f'(1) - f'(0)\right] x^2 + f'(0) \cdot x + f(0)$$

$$= f(0) + \frac{1}{2} x(2-x) f'(0) + \frac{1}{2} x^2 \cdot f'(1)$$

$$= 1 \cdot f(0) + \frac{1}{2} x(2-x) f'(0) + \frac{1}{2} x^2 f'(1)$$

data

fundamental (basis) polynomials

In our example, $B_2 f(x) = 1 \cdot x^2 + 1 \cdot x + 2 = x^2 + x + 2$

3.2. Basis (fundamental) polynomials

- this method gives us a way to determine the basis polynomials (one by one)

$$B_m f(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k)$$

Here,

$$B_2 f(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k)$$

$$= \underbrace{b_{00}(x) f(x_0) + b_{01}(x) f'(x_0)}_{\substack{k=0 \\ j \in I_0 = \{0,1\}}} + \underbrace{b_{11}(x) f'(x_1)}_{\substack{k=1 \\ j \in I_1 = \{1\}}}$$

$$= \underline{b_{00}(x)} f(0) + \underline{b_{01}(x)} f'(0) + \underline{b_{11}(x)} f'(1)$$

Since $B_2 f$ is of degree 2, we start with basis polynomials of the same degree.

! All the data we know for f is transfered to the basis polynomials.

(i) $b_{00}(x) = ax^2 + bx + c$

$b'_{00}(x) = 2ax + b$

We know $\begin{cases} b_{00}(x_0) \\ b'_{00}(x_0) \\ b'_{00}(x_1) \end{cases}$ and based on the rule

$b^{(j)}_{kj}(x_k) = 1,$
otherwise 0

we obtain $\begin{cases} b_{00}(x_0) = \boxed{1} \\ b'_{00}(x_0) = 0 \\ b'_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} c = 1 \\ b = 0 \\ 2a + b = 0 \end{cases}$

$\Rightarrow \boxed{b_{00}(x) = 1}$

(ii) $b_{01}(x) = ax^2 + bx + c$

$\begin{cases} b_{01}(x_0) = 0 \\ b'_{01}(x_0) = \boxed{1} \\ b'_{01}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} c = 0 \\ b = 1 \\ a = -1/2 \end{cases} \Rightarrow \boxed{b_{01}(x) = \frac{1}{2}x(2-x)}$

(iii) $b_{11}(x) = ax^2 + bx + c$

$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_0) = 0 \\ b'_{11}(x_1) = \boxed{1} \end{cases} \Leftrightarrow \begin{cases} c = 0 \\ b = 0 \\ a = 1/2 \end{cases} \Rightarrow \boxed{b_{11}(x) = \frac{1}{2}x^2}$

Hence, $B_2 f(x) = 1 \cdot f(0) + \frac{1}{2}x(2-x)f'(0) + \frac{1}{2}x^2 \cdot f'(1)$ $\square$