Numerical Calculus - Q&A - 28.05.2025

$$R(f) = \int_a^b K_n(t) f^{(n+1)}(t)\,dt$$

$\hookrightarrow$ kernel

degree of precision $= n$

$d = n$

! $(x-t)_+ = \begin{cases} x-t & , x \geq t \\ 0 & , x \leq t \end{cases}$

$$K_n(t) = \frac{1}{n!} R_n\left((x-t)_+^n\right)$$

$f(x) = L_n f(x) + R_n(f)$

$R_n(f) = f - L_n f$

$R_n(e_j)$

$$\int f = Aprox(f) + R(f)$$

$$R(f) = \int f - Aprox(f)$$

$$R\left((x-t)_+^n\right) = \int (x-t)_+^n - Aprox\left((x-t)_+^n\right)$$

$-2f'(-1) + 2f(1)$

$-2\left[(x-t)_+^n\right]'_{x=-1} + 2\left[(x-t)_+^n\right]_{x=1}$

$\det(A - \lambda I) = 0$

$\hookrightarrow$ sol. λ = eigenvalues A

Examples

- QR
- Cholesky

$A v = \lambda v$

eigenvector

eigenvalue

x_k	$f(x_k)$
0	1
1	-3
2	7

$(0,1)$, $(1,-3)$, $(2,7)$

Linear polyn. $= \boxed{P(x) = ax + b}$

$$E(a,b) = \sum_{k=1}^{3} \left[f(x_k) - P(x_k) \right]^2$$

$$= \sum_{k=1}^{3} \left[f(x_k) - a x_k - b \right]^2$$

$$= (1 - a \cdot 0 - b)^2 + (-3 - a \cdot 1 - b)^2 + (7 - a \cdot 2 - b)^2$$

$$= (1-b)^2 + (-3 - a - b)^2 + (7 - 2a - b)^2$$

$$\begin{cases} \dfrac{\partial E}{\partial a}(a,b) = 0 \\ \dfrac{\partial E}{\partial b}(a,b) = 0 \end{cases}$$

Hermite : $x_0, \ldots, x_n$

$x_0 = 0$, $f^{(0)}(x_0) = 1$, $f^{(1)}(x_0) = 2$

$x_1 = 1$, $f^{(0)}(x_1) = 0$

$k_0 = 1$

$k_1 = 0$

$$H_m f(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k)$$

$r_k \to r_0 = 1$, $r_1 = 0$

$m = 1$

$$m = \sum_{k=0}^{m} (r_k + 1) - 1 \quad \text{degree of } H_m f$$

$$\boxed{m = n + r_0 + \dots + r_n - 1}$$

$$= (r_0+1) + (r_1+1) - 1$$
$$= 1 + 1 + 0 + \cancel{1} - \cancel{1} = 2$$

$$\Rightarrow H_2 f(x) = \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k)$$

$$= \underbrace{h_{00}(x) f^{(0)}(x_0) + h_{01}(x) f^{(1)}(x_0)}_{k=0:\ j=0 \ + \ j=1} + \underbrace{h_{10}(x) f^{(0)}(x_1)}_{k=1,\ j=0}$$

$$= \underbrace{h_{00}(x)}\cdot 1 + \underbrace{h_{01}(x)}\cdot 2 + \cancel{h_{10}(x)\cdot 0}$$

Hermite fundamental polynomials

$$h_{00}(x) = ax^2 + bx + c$$
$$h'_{00}(x) = 2ax + b$$

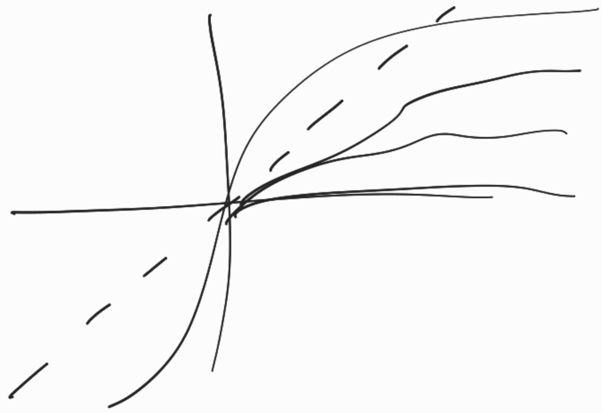
$$\begin{cases} h_{00}(x_0) = 1 \\ h'_{00}(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} ax_0^2 + bx_0 + c = 1 \\ 2ax_0 + b = 0 \\ ax_1^2 + bx_1 + c = 0 \end{cases} \Leftrightarrow \begin{cases} c = 1 \\ b = 0 \\ a + b + c = 0 \end{cases}$$

$$\Rightarrow a = -1$$

$$\Rightarrow \boxed{h_{00}(x) = -x^2 + 1}$$

$$g(x) = \frac{1-\sin x}{2}$$

$$g(x) = x \Leftrightarrow 1 - \sin x = 2x$$



$$g : [0,1] \to \mathbb{R}, \quad g(x) = \frac{1-\sin x}{2}$$

i). $g \in C^1[0,1]$

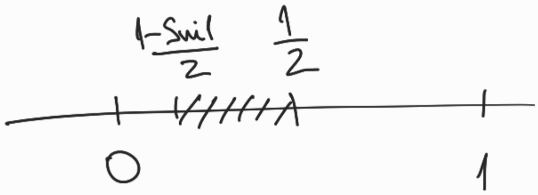
ii). $\forall x \in [0,1] : g(x) \in [0,1]$

$$g'(x) = \frac{-\cos x}{2}$$

x	0		1
$g(x)$	$1/2$	$\searrow$	$\frac{1-\sin 1}{2}$
$g'(x)$	$-$	$-$	$-$

$\forall x \in [0,1]$:

$$g(x) \in \left[\frac{1-\sin 1}{2}, \frac{1}{2}\right] \subseteq [0,1]$$



iii). $\max\limits_{x \in [0,1]} |g'(x)| = \max\limits_{x \in [0,1]} \left| \frac{-\cos x}{2} \right| \leq \frac{1}{2} \cdot 1 = \frac{1}{2} < 1$

$$|\cos x| \leq 1$$

T.B.

$\Rightarrow \exists! \ \alpha \in [0,1]$ s.t. $g(\alpha) = \alpha$

$$\alpha = \lim_{n \to \infty} x_n, \quad x_n = g(x_{n-1}), \ \forall n \geq 1$$

$$x_0 \in [0,1]$$

$x_0 = 0$, $f(x_0)$, — , —

$x_1 = 1$, $\cancel{\ }$, $f'(x_1)$, —

$x_2 = 2$, $f(x_2)$, $\cancel{\ }$, $f''(x_2)$

Birkhoff (Hermite lacunary)

x_0 : $I_0 = \{0\}$

x_1 : $I_1 = \{1\}$

x_2 : $I_2 = \{0, 2\}$

$m = |I_0| + |I_1| + |I_2| - 1$

m ← degree of $B_m f$

$\mathrm{card}(I_0)$ = nb. of elements

$M = 2$ (index of the last point)

$$B_m f(x) = \sum_{k=0}^{M} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k)$$

Here, $m = 1 + \cancel{1} + 2 - \cancel{1} = 3 \Rightarrow m = 3$

$$\Rightarrow B_3 f(x) = \sum_{k=0}^{2} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k)$$

$$= b_{00}(x) f^{(0)}(x_0) + b_{11}(x) f'(x_1) + b_{20}(x) f^{(0)}(x_2) + b_{22}(x) f''(x_2)$$

$k=0$, $j \in I_0 = \{0\}$, $j = 0$

$k=1$, $j \in I_1 = \{1\}$, $j = 1$

$k=2$, $j \in I_2 = \{0, 2\}$, $j = 0$, $j = 2$

Spline cubic

(2.9) + (2.10)

Natural (2.13)

$\rightarrow m_k$

$\downarrow$

(2.4) + (2.5)