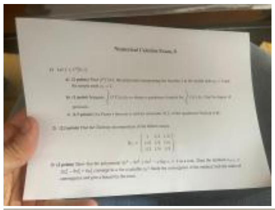
Exam 8

Wednesday, May 28, 2025 4:48 PM



$$L f(x) = \int_a^b K_n(x,t) f^{(n+1)}(t)\,dt,$$

$$K_n(x,t) = \frac{1}{n!} L\left((x-t)_+^n\right)$$

$$L f = \frac{1}{(n+1)!} f^{(n+1)}(\xi) L e_{n+1},$$

$$\int_a^b w(x) f(x)\,dx = \sum_{j=0}^{m} A_j F_j(f) + R(f),$$

Hermite Interpolation **Lecture 4**

double nodes $H_n f(x) = \sum_{i=0}^{m} \left[h_{i0}(x) f(x_i) + h_{i1}(x) f'(x_i)\right]$ (1.3)

$$h_{i0}(x) = \left[1 - 2l_i'(x_i)(x-x_i)\right]\left[l_i(x)\right]^2, \quad (1.4)$$
$$h_{i1}(x) = (x-x_i)\left[l_i(x)\right]^2,\ i = 0,\dots,m.$$

Newton's divided difference form

$$\begin{aligned} N_n(x) &= f(x_0) + f[x_0,x_0](x-x_0) + f[x_0,x_0,x_1](x-x_0)^2 \\ &+ f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1) + \dots \quad (1.6) \\ &+ f[x_0,x_0,\dots,x_m,x_m](x-x_0)^2\dots(x-x_{m-1})^2(x-x_m) \end{aligned}$$

$$R_n(x) = f(x) - H_n f(x) = [u_m(x)]^2 \frac{f^{(n+1)}(\xi_x)}{(n+1)!},\ \xi_x \in (a,b). \quad (1.9)$$

$$h_{ij}(x) = \frac{(x-x_i)^j}{j!}\left[\sum_{k=0}^{r_i-j} \frac{(x-x_i)^k}{k!}\left[\frac{1}{u_i(x)}\right]^{(k)}_{x=x_i}\right] u_i(x). \quad (1.13)$$

$$N_n f(x) = f(z_0) + f[z_0,z_1](x-z_0) + \dots \quad (1.14)$$
$$+ f[z_0,\dots,z_n](x-z_0)\dots(x-z_{n-1}),$$

$R_n(x)$ (1.15)

$$= \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi_x),\ \xi_x \in (a,b).$$

1) Let $f \in C^2[0,1]$.

a) **(2 points)** Find $(Pf)(x)$, the polynomial interpolating the function f at the double node $x_0 = 0$ and the simple node $x_1 = 1$.

b) **(1 point)** Integrate $\int_0^1 (Pf)(x)dx$ to obtain a quadrature formula for $\int_0^1 f(x)\,dx$. Find its degree of precision.

c) **(1.5 points)** Use Peano's theorem to find the remainder $R(f)$ of the quadrature formula in b).

$x_0 = 0$ double node => știm $f(0), f'(0)$ $\quad I_0 = \{0, 1\}$

$x_1 = 1$ single node => știm $f(1)$ $\quad I_1 = \{0\}$

$n = 1 + 1 + 1 - 1 = 2$ = nr. de informații pe care le avem $\{f(0), f'(0), f(1)\} - 1$

z	$f(z)$	$\Delta^1 f$	$\Delta^2 f$
$z_0 = x_0$	$f(0)$	$\frac{f(0)-f(0)}{x_0 - x_0} = f'(0)$	$\frac{f(1)-f(0) - f'(0)}{x_1 - x_0} = \frac{f(1)-f(0)-f'(0)}{1}$
$z_1 = x_0$	$f(0)$	$\frac{f(1)-f(0)}{x_1 - x_0} = \frac{f(1)-f(0)}{1}$	
$z_2 = x_1$	$f(1)$		

$(z_2 - z_0)$

Newton's divided difference form

$$\begin{aligned} N_n(x) &= f(x_0) + f[x_0,x_0](x-x_0) + f[x_0,x_0,x_1](x-x_0)^2 \\ &+ f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1) + \dots \quad (1.6) \\ &+ f[x_0,x_0,\dots,x_m,x_m](x-x_0)^2\dots(x-x_{m-1})^2(x-x_m) \end{aligned}$$

$$N_2(x) = f(0) + f'(0)(x - x_0) + (f(1) - f(0) - f'(0))(x - x_0)^2$$

$$= f(0) + f'(0)x + (f(1) - f(0) - f'(0))x^2$$

b) $\int_0^1 f(0) + \int_0^1 f'(0)x + (f(1) - f(0) - f'(0))x^2 =$

$$f(0)\cdot x\Big|_0^1 + f'(0)\frac{x^2}{2}\Big|_0^1 + (f(1) - f(0) - f'(0))\frac{x^3}{3}\Big|_0^1$$

$$= f(0)(1-0) + f'(0)\left(\frac{1}{2} - 0\right) + (f(1) - f(0) - f'(0))\left(\frac{1}{3} - \frac{0}{3}\right)$$

$$= f(0) + \frac{1}{2}f'(0) + \frac{1}{3}(f(1) - f(0) - f'(0))$$

$$= {}^{3)}f(0) - \frac{f(0)}{3} + {}^{3)}\frac{f'(0)}{2} - {}^{2)}\frac{f'(0)}{3} + \frac{f(1)}{3} = \frac{2f(0)}{3} + \frac{f'(0)}{6} + \frac{f(1)}{3}$$

$$= {}^{2)}\frac{2f(0) + f(1)}{3} + \frac{f'(0)}{6} = \frac{4f(0) + 2f(1) + f'(0)}{6}$$

$R(f) = \int_0^1 f(x)dx - \int_0^1 (Pf)(x)dx$ $\qquad \int_0^1 (Pf)(x)dx + R(f)$

$$\boxed{R(f) = \int_0^1 f(x)\,dx - \int_0^1 (Pf)(x)\,dx}$$

$$\underline{\int_0^1 (Pf)(x)\,dx + R(f)}$$
quadrature formula

$$\Rightarrow R(f) = \int_0^1 f(x)\,dx - \frac{4f(0) + 2f(1) + f'(0)}{6}$$

$$\boxed{f(x) = e_i(x) = x^i}$$

$$R(e_0(x)) = \int_0^1 x^0\,dx - \frac{4\overbrace{e_0(0)}^{1} + 2e_0(1) + \overbrace{e_0'(0)}^{0}}{6}$$

$$= x\Big|_0^1 - \frac{4+2+0}{6} = 1 - 0 - 1 \equiv 0$$

$$R(e_1(x)) = \int_0^1 x^1\,dx - \frac{4e_1(0) + 2e_1(1) + e_1'(0)}{6}$$

$$= \frac{x^2}{2}\Big|_0^1 - \frac{0+2+1}{6} = \frac{1}{2} - 0 - \frac{3}{6} = \frac{1}{2} - \frac{1}{2} \equiv 0$$

$$R(e_2(x)) = \int_0^1 x^2\,dx - \frac{4e_2(0) + 2e_2(1) - e_2'(0)}{6}$$

$$= \frac{x^3}{3}\Big|_0^1 - \frac{0+2-0}{6} = \frac{1}{3} - \frac{2}{6} = 0$$

$$R(e_3(x)) = \int_0^1 x^3\,dx - \frac{4e_3(0) + 2e_3(1) - e_3'(0)}{6}$$

$$\frac{x^4}{4}\Big|_0^1 - \frac{0+2-0}{6} = \frac{1}{4} - \frac{2}{6} = \frac{1}{4} - \frac{1}{3} \neq 0$$

$$= \frac{3-4}{12} = -\frac{1}{12}$$

degree of precision $= 2$

c) $f(x) = H_2(f(x)) + R(f(x)) \Rightarrow R(f(x)) = f(x) - H_2(f(x))$

$$Lf(x) = \int_a^b K_n(x,t) f^{(n+1)}(t)\,dt,$$

$\Rightarrow$ using Peano's theorem for our rest function $\Rightarrow$

$$K_n(x,t) = \frac{1}{n!} L\left((x-t)_+^n\right)$$

$$Lf = \frac{1}{(n+1)!} f^{(n+1)}(\xi) L e_{n+1},$$

$$\int_a^b w(x) f(x)\,dx = \sum_{j=0}^{m} A_j F_j(f) + R(f),$$

$$\Rightarrow R(f(x)) = \int_0^1 K_2(x,t) f'''(t)\,dt$$

$$K_2(x,t) = \frac{1}{2!} R\left((x-t)_+^2\right)$$

$$\boxed{H_2(f(x)) = f(0) + f'(0)x + (f(1) - f(0) - f'(0))x^2}$$

$$\dots t)^2 + \frac{\partial}{} (x-t)^2 \Big| \cdot x +$$

$$R\left((x-t)_+^2\right) = (x-t)_+^2 - \underbrace{(0-t)_+^2}_{=0} + \frac{\partial}{\partial x}(x-t)_+^2\Big|_{x=0} \cdot x +$$

$$+\left((1-t)_+^2 - \underbrace{(0-t)_+^2}_{=0} - \frac{\partial}{\partial x}(x-t)_+^2\Big|_{x=0} \right) x^2$$

$$= (x-t)_+^2 + \left(2(x-t)_+\Big|_{x=0}\right)\cdot x + \left((1-t)^2 - 0 - 2(x-t)_+\Big|_{x=0}\right.$$

$$= (x-t)_+^2 + \left(2\underbrace{(0-t)_+}_{=0}\right)\cdot x + \left((1-t)^2 - 2\underbrace{(0-t)_+}_{=0}\right) x^2$$

$$= (x-t)_+^2 + 0 + \left((1-t)^2 - 0\right) x^2$$

$$= (x-t)_+^2 + x^2(1-t)^2 > 0$$

$$0 \le x \le t \le 1 \Rightarrow (x-t)_+ = 0$$

$$\hookrightarrow 0^2 + x^2(1-t)^2 > 0$$

$$0 \le t \le x \le 1 \Rightarrow (x-t)_+ = x-t$$

$$\hookrightarrow (x-t)^2 + x^2(1-t)^2 > 0$$

$\Rightarrow$

$\Rightarrow K$ has constant sign on $[0,1]$ $\Rightarrow$

corollary Peano $\Rightarrow Rf = \frac{1}{3!} f^{(3)}(\xi)\cdot Re_3$

$$R_2 f(x) = \int_0^1 \frac{1}{3!} f^{(3)}(\xi)\cdot Re_3 = \frac{1}{6} f^{(3)}(\xi) \int_0^1 Re_3$$

$$= \frac{1}{6} f^{(3)}(\xi)\cdot R_2 e_3(x) = \frac{1}{6} f^{(3)}(\xi)\cdot\left(-\frac{1}{12}\right)$$

$= \frac{1}{6} f^{(3)}(\xi) \cdot R_2 e_3(x) = \frac{1}{6} f^{(3)}(\xi) \cdot (-1/2)$

$= -\frac{1}{12} f^{(3)}(\xi)$

2) (2.5 points) Find the Cholesky decomposition of the Hilbert matrix

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & & \\ 1/2 & 1/3 & \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \sim \begin{bmatrix} \sqrt{1} & \\ 1/2 & \\ 1/3 & \end{bmatrix}$$

$$A' - ww^T/a_{11} = \begin{bmatrix} 1/3 & \\ 1/4 & 1/5 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & \\ 1/4 & 1/5 \end{bmatrix} - \begin{bmatrix} 1/4 & \\ 1/6 & 1/9 \end{bmatrix} = \begin{bmatrix} 1/12 & \\ 1/12 & 4/45 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & & \\ 1/2 & 1/12 & \\ 1/3 & 1/12 & 4/45 \end{bmatrix} \sim \begin{bmatrix} 1 & & \\ 1/2 & 1/\sqrt{12} & \\ 1/3 & \underbrace{1/12 \cdot \frac{\sqrt{12}}{1}}_{\frac{\sqrt{12}}{12}} & \end{bmatrix}$$

$$A' - ww^T/a_{21} = \begin{bmatrix} 4/45 \end{bmatrix} - \begin{bmatrix} 1/12 \end{bmatrix} \cdot \begin{bmatrix} 1/12 \end{bmatrix} / 1/12$$

$$= \begin{bmatrix} 4/45 \end{bmatrix} - \left(1/144 \cdot \frac{12}{1}\right) = \begin{bmatrix} 4/45 \end{bmatrix} - \begin{bmatrix} \frac{1}{12} \end{bmatrix} = \ldots$$

$$A \sim \begin{bmatrix} 1 & & \\ 1/2 & 1/\sqrt{12} & \\ 1/3 & \frac{1}{\sqrt{12}} & \frac{1}{180} \end{bmatrix} \Rightarrow R^T \sim \begin{bmatrix} 1 & 0 & \\ 1/2 & 1/\sqrt{12} & \\ 1/3 & 1/\sqrt{12} & \end{bmatrix}$$

$$R \sim \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & 0 & 1/\sqrt{180} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & w^* \\ v & A' \end{bmatrix}$$

$w^* \to \begin{bmatrix} a_{12} & a_{13} \end{bmatrix}$, $v \to \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & w^* \\ v & A' \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & w^* \\ v/a_{11} & A' - v \cdot w^*/a_{11} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1/2 & -1 & 1 \\ 1/2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix} / 2 =$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} / 2 =$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \end{bmatrix} - \begin{bmatrix} -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} / \begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 4 & 2 \\ 1/2 & -1 & 1 \\ 1/2 & 1 & -1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$$

Cholesky $\Rightarrow$ $A = \begin{bmatrix} a_{11} & w^T \\ w & A' \end{bmatrix}$ $\quad A = R^T \cdot R$

Cholesky $\Rightarrow$ $A = \begin{bmatrix} a_{11} & w \\ w & A' \end{bmatrix}$ $\quad A = R^T R$

$\Rightarrow \begin{bmatrix} \sqrt{a_{11}} & \\ w/\sqrt{a_{11}} & A' - w \cdot w^T / a_1 \end{bmatrix}$

2) (2.5 points) Find the Cholesky decomposition of the Hilbert matrix

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & & \\ 1/2 & 1/3 & \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \to \left[\begin{array}{c|cc} 1 & & \\ \hline 1/2 & 1/3 & \\ 1/3 & 1/4 & 1/5 \end{array}\right] \to$$

$$\to \left[\begin{array}{c|cc} \sqrt{1} & & \\ \hline (1/2)/\sqrt{1} & 1/3 & \\ (1/3)/\sqrt{1} & 1/4 & 1/5 \end{array}\right] \quad \begin{bmatrix} 1/3 & \\ 1/4 & 1/5 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/3 \end{bmatrix} / 1 =$$

$$= \begin{bmatrix} 1/3 & \\ 1/4 & 1/5 \end{bmatrix} - \begin{bmatrix} 1/4 & \\ 1/6 & 1/9 \end{bmatrix} = \begin{bmatrix} 1/12 & \\ 1/12 & 4/45 \end{bmatrix}$$

$$\left[\begin{array}{c|c} 1/12 & \\ \hline 1/12 & 4/45 \end{array}\right] \to \left[\begin{array}{c|c} 1/\sqrt{12} & \\ \hline \frac{1}{\sqrt{12}} & \end{array}\right]$$

$$\begin{bmatrix} 4/45 \end{bmatrix} - \begin{bmatrix} 1/12 \end{bmatrix} \cdot \begin{bmatrix} 1/12 \end{bmatrix} / \frac{1}{12} = \frac{\overset{4/}{4}}{45} - \frac{\overset{15/}{1}}{12} = \frac{1}{180} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & & \\ 1/2 & 1/\sqrt{12} & \\ 1/3 & 1/\sqrt{12} & 1/\sqrt{180} \end{bmatrix} \Rightarrow$$

$$\boxed{R^T = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2\sqrt{3} \\ 1/3 & 1/2\sqrt{3} \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2\sqrt{3} \\ 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1/\sqrt{12} & \\ 1/\sqrt{12} & \boxed{\frac{1}{180}} \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1/\sqrt{12} & \\ 1/\sqrt{12} & 1/\sqrt{180} \end{bmatrix}$$

$$\boxed{\begin{matrix} \begin{bmatrix} 0 \\ 0 \\ 1/6\sqrt{5} \end{bmatrix} \\ \begin{bmatrix} 1/3 \\ 1/2\sqrt{3} \\ 1/6\sqrt{5} \end{bmatrix} \end{matrix}}$$