

# THE HESTON STOCHASTIC VOLATILITY MODEL: TAYLOR-BASED SCHEMES AND MONTE CARLO SIMULATION

Assignment for *Advanced derivatives* class

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### Abstract

The Heston model is a stochastic volatility model that tries to overcome the disadvantages of the Black & Scholes framework. Its stochastic differential equation does not yield an exact solution that can be directly simulated. Thus, discretized schemes are introduced to allow the simulation of the stochastic paths under this model. The mathematical finance literature has proposed various discretization schemes that deal both with bias issues and with the positive probability of having a negative variance. This study describes some Taylor-based schemes and provides briefly some numerical results.

## 1 The Heston model

In the Black and Scholes model the stock path follows a geometric Brownian motion and its  $\mathbb{Q}$ -dynamics is governed by the following stochastic differential equation (SDE):

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

An important pitfall of the celebrated model is the assumption of constant volatility. The model is useful for educational purposes, but once it is pushed onto the market it fails to capture features such as volatility skews or volatility smiles. The Heston model extends the Black & Scholes model and allows such features to exist by defining the volatility as a stochastic process of its own.

Let the triple  $(\Omega, \mathcal{F}, \mathbb{Q})$  be a probability space, with  $\mathbb{Q}$  the risk neutral measure. The Heston model defines the  $\mathbb{Q}$ -dynamics of the stock  $S$  with stochastic variance  $\nu$  through the two-dimensional SDE:

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^S \quad (1.1)$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \eta\sqrt{\nu_t}dW_t^\nu \quad (1.2)$$

The variance process follows a mean-reverting square root process. The speed of reversion  $\kappa$ , the long-term variance  $\theta$  and the volatility of volatility  $\eta$  are strictly positive constants. In order for the variance process to keep being positive the Feller condition must hold:  $2\kappa\theta > \eta^2$ . Moreover,  $dW_t^S$  and  $dW_t^\nu$  are correlated Brownian motions with instantaneous correlation  $\rho$ , so that

$$dW_t^S dW_t^\nu = \rho dt$$

## 2 Generating correlated random variables

The Heston stochastic volatility model requires two correlated Brownian motions for Monte Carlo modelling purposes. Let us consider the general case.

Let  $Y_i \sim \mathcal{N}(\mu, \sigma^2)$  be a Gaussian random variable (RV). If we need to generate  $N$  such correlated random variables

$$\mathbf{Y} \sim \mathcal{N}(\mu, \Sigma)$$

where  $\mathbf{Y} = (Y_1, \dots, Y_N)$  is the vector of correlated random variables,  $\mu = (\mu_1, \dots, \mu_N)$  the vector of means and  $\Sigma$  the covariance matrix, the target vector  $\mathbf{Y}$  is given by

$$\mathbf{Y} = \mu + C\mathbf{Z},$$

with  $\mathbf{Z}$  a vector of uncorrelated Gaussian random variables and  $C$  the square root of  $\Sigma$ , i.e. a matrix such that  $CC^T = \Sigma$ . A popular choice for  $C$  is the Cholesky decomposition of  $\Sigma$ .

In particular, let us pick the case of standard normal random variables, which suits to our study. Since we are dealing with standard Gaussians, the covariance matrix resumes to the correlation matrix. Let  $L$  be the lower triangular matrix coming from the Cholesky decomposition of the correlation matrix  $\Sigma$ . Now, if  $\mathbf{Z}$  is a vector of uncorrelated standard normal RVs, then  $L\mathbf{Z}$  will be a vector of standard normal RVs with correlation matrix  $\Sigma$ . To show this holds, we notice that  $L\mathbf{Z}$  will be normally distributed due to the fact that the normal distribution is closed under transformations. Then we compute the first two moments:

$$\begin{aligned} \mu_{LZ} &= \mathbb{E}[L\mathbf{Z}] = L\mathbb{E}[\mathbf{Z}] = 0 \\ \Sigma_{LZ} &= \mathbb{E}[(L\mathbf{Z} - \mathbb{E}[L\mathbf{Z}])(L\mathbf{Z} - \mathbb{E}[L\mathbf{Z}])^T] = \mathbb{E}[L\mathbf{Z}(L\mathbf{Z})^T] \\ &= \mathbb{E}[L\mathbf{Z}\mathbf{Z}^T L^T] = L\mathbb{E}[\mathbf{Z}\mathbf{Z}^T]L^T = LIL^T = LL^T = \Sigma \end{aligned}$$

In conclusion, it is possible to generate a vector of correlated standard Gaussian random variables  $\epsilon$  via a vector of uncorrelated such variables  $\mathbf{x}$ , by multiplying it by the Cholesky decomposition  $L$  of the correlation matrix  $\Sigma$ :

$$\epsilon = L\mathbf{x}$$

### 3 The dynamics of the variance

Let us take a closer look at the stochastic process followed by the variance:

$$d\nu_t = \kappa(\theta - \nu_t)dt + \eta\sqrt{\nu_t}dW_t^\nu \quad (3.1)$$

We observe that for large values of  $\eta$  the variance process becomes less stable and we wonder if this can push it towards negative values. Feller investigated the phenomenon in 1951. More precisely we have  $P(\nu_t = 0) = 0$  when  $2\kappa\theta > \eta^2$ . We denote this the Feller condition, which points to the fact that the variance process will stick away from zero when the mean-reverting term is strong enough.

However, consider a Forward Euler discretization of 3.1:

$$\nu_{i+1} = \nu_i + \kappa(\theta - \nu_i)\Delta t + \eta\sqrt{\nu_i}\sqrt{\Delta t}Z_i^\nu \quad (3.2)$$

with  $Z_\nu$  a standard normal random variable. Given the finite discretization of a continuous stochastic process, the next step variance can turn negative due to discretization errors. In fact:

$$\begin{aligned} P(\nu(t + \Delta t) < 0 \mid \nu(t)) &= P\left(Z^\nu < \frac{\kappa(\nu(t) - \theta)\Delta t - \nu(t)}{\eta\sqrt{\nu(t)\Delta t}} \mid \nu(t)\right) \\ &= \Phi\left(\frac{\kappa(\nu(t) - \theta)\Delta t - \nu(t)}{\eta\sqrt{\nu(t)\Delta t}}\right) \end{aligned} \quad (3.3)$$

with  $\Phi(x)$  the cumulative distribution function (CDF) of a standard normal RV. The argument of  $\Phi$  in 3.3 is strictly positive, hence there is a positive probability of negative discretized path using Forward Euler, even if the Feller condition holds. This is physically incorrect and it breaks down the scheme, since in subsequent steps the square root of the variance ought to be used. The chance of breaking down the discretization scheme increases with  $\Delta t$  and with  $\eta$ , i.e. when the discretization is less accurate and when the vol of vol is higher.

### 4 Discretization schemes

The mathematical finance literature has proposed various numerical schemes that handle the problem of negative variance over the last years, in response to the question whether the Feller condition is strong enough in practice.

## 4.1 Euler-Maruyama

In order to overcome the possibility of negative values of variance showing up we need to modify the discretization formula 3.2. Let us introduce three functions  $f_1, f_2, f_3$  that will handle negative values. The Forward Euler discretization becomes:

$$\nu_{i+1} = f_1(\nu_i) + \kappa(\theta - f_2(\nu_i))\Delta t + \eta\sqrt{f_3(\nu_i)}\sqrt{\Delta t}Z_i^\nu \quad (4.1)$$

To prevent  $\nu(t + \Delta t)$  to become negative one technique is to project negative values to zero. Instead of projection, reflection (taking the absolute value) could be used for the same purpose. This leads to separate discretization schemes:

Scheme	$f_1$	$f_2$	$f_3$
Reflection	$ x $	$ x $	$ x $
Partial truncation	$x$	$x$	$x^+$
Full truncation	$x$	$x^+$	$x^+$

$$\text{with } x^+ = \max(x, 0)$$

In Lord et. al [8] it has been shown that the Full Truncation scheme contains the least discretization bias. In this case the discretized Heston equations become the following:

$$\nu_{i+1} = \nu_i + \kappa(\theta - \nu_i^+)\Delta t + \eta\sqrt{\nu_i^+}\sqrt{\Delta t}Z_i^\nu \quad (4.2)$$

$$S_{i+1} = S_i \exp\left(\left(r - \frac{\nu_i^+}{2}\right)\Delta t + \sqrt{\nu_i^+}\sqrt{\Delta t}Z_i^S\right) \quad (4.3)$$

The scheme is straightforward, but contains bias that increases with the size of the time step. The Full truncation's accuracy decreases in  $\Delta t$ , as it happens with all the Forward Euler schemes. The bias increases also with the number of applied projections, which is significant when the Feller condition is close to being violated.

## 4.2 Milstein

The Milstein method includes an additional term from the Itô-Taylor expansion. Consider the following SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

The Milstein approximation is recursively defined as

$$X_{n+1} = X_n + a(X_n)\Delta t + b(X_n)\Delta W_n + \frac{1}{2}b(X_n)b'(X_n)((\Delta W_n)^2 - \Delta t),$$

where  $b'$  denotes the derivative of  $b(x)$  with respect to  $x$ . When the diffusion term does not depend on  $X_t$ ,  $b'(X_t) = 0$  and the method is equivalent to the Euler scheme. The Milstein scheme has both weak and strong order of convergence of  $\Delta t$ , while the Euler-Maruyama discretization shows the same weak order of convergence, but inferior strong convergence of  $\sqrt{\Delta t}$ .

As for the Euler scheme, the Milstein method maintains a positive probability of generating negative values of  $X_t$  and therefore has to incorporate, for example, a full truncation adjustment. Applying the Milstein scheme to the variance process we obtain the following discretization for the Heston model:

$$\begin{aligned} \nu_{i+1} &= \nu_i + \kappa(\theta - \nu_i^+)\Delta t + \eta\sqrt{\nu_i^+}\sqrt{\Delta t}Z_i^\nu + \frac{\eta^2}{4}\Delta t ((Z_i^\nu)^2 - 1) \\ S_{i+1} &= S_i \exp\left(\left(r - \frac{\nu_i^+}{2}\right)\Delta t + \sqrt{\nu_i^+}\sqrt{\Delta t}Z_i^S\right) \end{aligned} \quad (4.4)$$

### 4.3 Kahl-Jäckel

Kahl and Jäckel [7] suggested an implicit Milstein scheme together with an alternative discretization for the stock price:

$$\nu_{i+1} = \frac{\nu_i + \kappa\theta\Delta t + \eta\sqrt{\nu_i}\sqrt{\Delta t}Z_i^\nu + \frac{\eta^2}{4}\Delta t((Z_i^\nu)^2 - 1)}{1 + \kappa\Delta t} \quad (4.5)$$

$$\begin{aligned} S_{i+1} &= S_i \exp\left(\left(r - \frac{\nu_i + \nu_{i+1}}{4}\right)\Delta t + \rho\sqrt{\nu_i}\sqrt{\Delta t}Z_i^\nu + \right. \\ &\quad \left. + \frac{1}{2}(\sqrt{\nu_i} + \sqrt{\nu_{i+1}})(Z_i^S - \rho Z_i^\nu)\sqrt{\Delta t} - \frac{\rho\eta\Delta t}{4}((Z_i^\nu)^2 - 1)\right) \end{aligned} \quad (4.6)$$

In this scheme, the variance stays positive provided that  $4\kappa\theta > \eta^2$ . As usually is the case, the condition is rarely met in practice, so one has to apply adjustments once again. Kahl and Jäckel do not tackle this issue, but Andersen [1] applies a full truncation, following Lord et. al [8]. He suggests that, whenever the variance process drops below zero, 4.2 shall be used in place of 4.5 and  $\nu_i^+$  and  $\nu_{i+1}^+$  shall be taken in 4.6.

## 4.4 Transformed Volatility

Zhu [11] prevents the breakdown of the computation of the square-root process in a different way: one that avoids the approximation of the process with the term  $\sqrt{\nu(t)}$ . The Transformed Volatility (TV) scheme discretizes the dynamics of the volatility instead of one for the variance. Let  $\gamma_t = \sqrt{\nu_t}$  denote the stochastic volatility in the Heston model. Applying Itô's lemma on 3.1 we obtain the following dynamics for the stochastic volatility:

$$\begin{aligned} d\gamma_t &= \frac{1}{2}\kappa \left( \left( \theta - \frac{\eta^2}{4\kappa} \right) \gamma_t^{-1} - \gamma_t \right) dt + \frac{\eta}{2} dW_t^\gamma \\ &= \kappa_\gamma (\theta_\gamma - \gamma_t) dt + \eta_\gamma dW_t^\gamma \end{aligned} \quad (4.7)$$

Compared to the process of  $\nu_t$  given in 3.1, now the volatility process  $\gamma_t$  is mean-reverting and does not have any square-root term. The speed of reversion and the vol of vol are half the values of the corresponding variance process. It is worth taking a look at the long-term volatility  $\theta_\gamma$ , that is now stochastic. This means that the volatility will not converge to a given value.

The stochastic processes 1.1 and 4.7 form a stochastic volatility model that coincides to the Heston model, but expresses the dynamics in terms of volatility, not variance. The stochastic volatility process is not restricted to any parameter constraint, while the variance process has to meet the Feller condition. Notice that the volatility can turn negative, but without affecting the distribution of the stock dynamics, by symmetry of the Brownian motion.

The Transformed Volatility scheme looks then as follows:

$$\gamma_{i+1} = \gamma_i + \frac{1}{2}\kappa(\theta_\gamma - \gamma_i)\Delta t + \frac{1}{2}\eta\sqrt{\Delta t}Z_i^\gamma \quad (4.8)$$

$$S_{i+1} = S_i \exp \left( \left( r - \frac{\gamma_i^2}{2} \right) \Delta t + \gamma_i \sqrt{\Delta t} Z_i^S \right) \quad (4.9)$$

The most important pitfall of the TV scheme is that the mean level of stochastic volatility

$$\theta_\gamma = \left( \theta - \frac{\eta^2}{4\kappa} \right) \gamma_t^{-1} \quad (4.10)$$

has a term  $\gamma_t^{-1}$  that makes it stochastic. The discretization scheme cannot capture the erratic behaviour of  $\gamma_t^{-1}$  in the time interval  $[t, t + \Delta t]$ . This

causes  $\theta_\gamma$  to jump between large positive and large negative values in a short time. The value of  $\theta_\gamma$  at time  $t$  is not representative for the whole interval. Zhu proposes the following approximations for  $\theta_\gamma$ .

### Central discretization

The idea is to evaluate  $\theta_\gamma$  using an average of  $\gamma_t$  in the time interval  $[t, t + \Delta t]$ . We define

$$\gamma_t^* = \frac{1}{\Delta} \int_t^{t+\Delta t} \gamma(s) ds \approx \frac{1}{2}(\gamma_t + \gamma_{t+\Delta t})$$

But  $\gamma_{t+\Delta t}$  is unknown at time  $t$ , so it is replaced by an expected value  $u(t + \Delta t)$

$$u_{t+\Delta t} = \gamma_t + \frac{1}{2}\kappa \left( \left( \theta - \frac{\eta^2}{4\kappa} \right) \gamma_t^{-1} - \gamma_t \right) \Delta t$$

Therefore the value for  $\gamma_t^*$  is the approximated average

$$\gamma_t^* \approx \frac{1}{2}(\gamma_t + u_{t+\Delta t}) \quad (4.11)$$

The long-term mean level is then calculated with  $\gamma_t^*$ :

$$\theta_\gamma^* = \left( \theta - \frac{\eta^2}{4\kappa} \right) / \gamma_t^* \quad (4.12)$$

The modified Euler scheme becomes

$$\gamma_{i+1} = \gamma_i + \frac{1}{2}\kappa(\theta_\gamma^* - \gamma_i)\Delta t + \frac{1}{2}\eta\sqrt{\Delta t}Z_i^\gamma \quad (4.13)$$

### Moment matching

The author proposes a better way to obtain  $\theta_\gamma$  through moment matching. Observing that  $\mathbb{E}[\gamma_t^2] = \mathbb{E}[\nu_t]$ , we have

$$\mathbb{E}[\gamma_{t+\Delta t}^2] = \mathbb{E}[\nu_{t+\Delta t}] = \mathbf{Var}[\gamma_{t+\Delta t}] + \mathbb{E}[\gamma_{t+\Delta t}]^2$$

where,

$$\begin{aligned} \mathbb{E}[\nu_{t+\Delta t}] &= m_1(t) = \theta + (\nu_t - \theta)e^{-\kappa\Delta t} \\ \mathbf{Var}[\gamma_{t+\Delta t}] &= m_2(t) = \frac{\eta^2}{4\kappa}(1 - e^{-\kappa\Delta t}) \\ \mathbb{E}[\gamma_{t+\Delta t}] &= \theta_\gamma + (\gamma_t - \theta_\gamma)e^{-\frac{1}{2}\kappa\Delta t} \end{aligned}$$



Rearranging yields an equation in the unknown variable  $\theta_\nu$  which shows the following expression:

$$\theta_\gamma = \frac{\beta - \gamma_t e^{-\frac{1}{2}\kappa\Delta t}}{1 - e^{-\frac{1}{2}\kappa\Delta t}}$$

with

$$\beta = \sqrt{[m_1(t) - m_2(t)]^+}$$

Zhu recommends the moment matching method in practical applications, although it is computationally more demanding.

## 5 Numerical results

This section provides briefly efficiency tests of the discretized schemes. The tests consider the pricing of an European call option, whose true price and its Heston parameters are taken from Broadie and Kaya [3]. The tests are conducted using a simple Monte Carlo simulation, taking as estimate for the option price the sample mean, an unbiased and consistent estimator. However, since we are discretizing a continuous-time process, we are introducing bias due to discretization errors. In order to lower the bias, the discretization mesh shall be finer, but this will lead to more computational effort. Due to computational budget constraints, a finer mesh will bring to smaller numbers of simulations, thus increasing the variance of the estimator. The measure that catches the contribution of both bias and variance is the Root Mean Square Error (RMSE). Let  $\hat{C}$  denote the Monte Carlo estimate for the European call option and  $C$  its true price. The RMSE is defined as:

$$\text{RMSE} = \sqrt{\text{bias}(\hat{C})^2 + \text{variance}(\hat{C})} \quad (5.1)$$

with

$$\text{bias}(\hat{C}) = \mathbb{E}[\hat{C} - C] \quad (5.2)$$

$$\text{variance}(\hat{C}) = \mathbb{E}[(\hat{C} - \mathbb{E}[\hat{C}])^2] \quad (5.3)$$

The bias for a specific number of time steps is estimated using 20 million simulations to estimate  $\mathbb{E}[\hat{C}]$ , and then taking the difference with the true price. The variance of each simulation is simply the sample variance of the simulation output.

The experiments are performed sequentially on CINECA’s supercomputer *Galileo* with an Intel(R) Xeon(R) CPU E5-2630 v3 @ 2.40GHz. The code is written in C++ and more info about it can be found in Section 6.

Duffie and Glynn [4] study the optimal allocation between number of simulations and number of time steps. Their results imply that it is optimal to pick a number of time steps proportional to the square root of the number of simulation trials, although the constant of proportionality is not defined by them. All the simulations in this analysis are carried out with a constant of proportionality equal to one. Results can be observed in the tables below and in Figure 1.

Number of simulations	Number of time steps	Bias	Standard Error	RMSE	Time (sec.)
40000	200	0.0385	0.0372	0.0724	1.1668
250000	500	0.0337	0.0148	0.0299	18.1053
1000000	1000	0.0221	0.0074	0.0199	142.534
9000000	3000	0.0126	0.0025	0.0110	3945.28

Table 1: Euler - Full truncation

Number of simulations	Number of time steps	Bias	Standard Error	RMSE	Time (sec.)
40000	200	0.0347	0.0368	0.0414	1.2078
250000	500	0.0412	0.0148	0.0602	18.6175
1000000	1000	0.0316	0.0074	0.0271	124.183
9000000	3000	0.0158	0.0024	0.0181	3250.11

Table 2: Milstein

Number of simulations	Number of time steps	Bias	Standard Error	RMSE	Time (sec.)
40000	200	0.0386	0.0373	0.0377	1.43441
250000	500	0.0416	0.0148	0.0484	24.0214
1000000	1000	0.0320	0.0074	0.0257	170.565
9000000	3000	0.0196	0.0025	0.0143	4421.28

Table 3: Kahl-Jäckel

Number of simulations	Number of time steps	Bias	Standard Error	RMSE	Time (sec.)
40000	200	0.0366	0.0374	0.0541	1.50487
250000	500	0.0355	0.0149	0.0559	20.9216
1000000	1000	0.0227	0.0074	0.0237	201.239
9000000	3000	0.0127	0.0025	0.0152	4713.92

Table 4: Transformed Volatility with moment matching

Observing the bias column, we notice that it decreases in a well-behaved manner for the Euler and the Transformed Volatility scheme, but not so for the other two schemes. With 9 million simulations and 3000 time steps, the two end up with similar bias. However, a comparison on the computational time points to the fact that the Euler scheme is the way to go. Switching to the RMSE, and keeping an eye on Figure 1, it can be seen that the Euler-Maruyama scheme has the highest slope throughout the simulations, which points to a better convergence than the other schemes. Although the other discretizations show somewhat bigger complexity, the Euler scheme remains a simple but nevertheless powerful discretized scheme.

A few more words have to be spent with respect to the optimal allocation between number of simulations and number of time steps. As the parameters are pushed up, the bias dominates the standard error. This points to the fact that the constant of proportionality equal to one is not appropriate. Ideally, one should choose a number of time steps for a specific number of trials in such a way that the bias and the standard error will stay close. This way the estimate converges with neither the errors dominating the other.

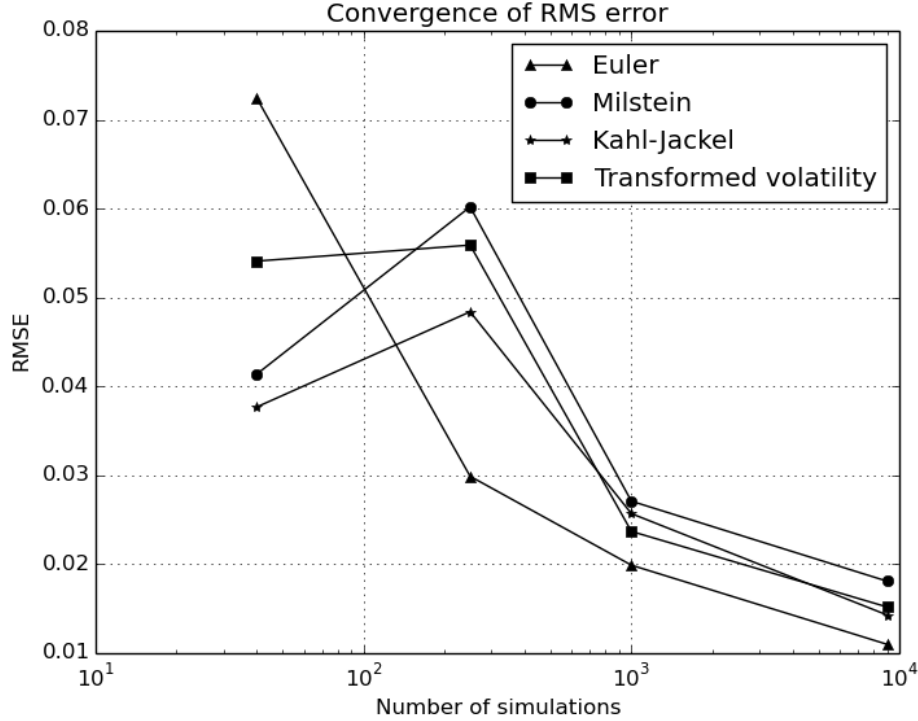


Figure 1: Convergence of RMS error

## 6 The code

The code is written in C++ and is freely available under the GNU General Public License at <https://github.com/mirceas08/heston-mc>. You need to be on a *UNIX*-type system to be able to run it. The following dependencies must be met:

- C++ compiler with C++11 support
- Armadillo: high quality C++ linear algebra library used throughout the whole code. It integrates with LAPACK and BLAS. Use Armadillo without installation and link against BLAS and LAPACK instead
- OpenBLAS: multi-threaded replacement of traditional BLAS. Recommended for significantly higher performance. Otherwise link with traditional BLAS and LAPACK.

The code is written in an object-oriented manner to improve maintainability. *PayOff* and *HestonDiscretization* are abstract classes that allow to plug-in different option types and different discretization schemes. We make use of the operator() to turn the payoff classes into a functor. This means that we can call the object just as we would do with a function. Calling a *PayOff* object has the effect of calculating the payoff and returning it. Each discretized scheme is a class derived from the *HestonDiscretization* abstract class. Each of them implements their own *calculateVariancePath()* and *calculateStockPath()* functions. Polymorphism is introduced in the code by taking advantage of the fact that a pointer to a derived class is type-compatible with a pointer to the base class.

To run the code you must first compile it. In the *Makefile* edit the entries regarding the location of headers and libraries. Then compile using

```
make
```

Check the file *parameters.dat* in the *data* directory and modify entries to your pleasure. Run the code with the following command

```
./heston-mc data/parameters.dat
```

## 6.1 Editing the parameters.dat file

In this section documentation of all the possible arguments that can be set in the *parameters.dat* file is provided.

parameter	description	arguments
numSims	number of simulations	positive integer
timeSteps	number of time steps	positive integer
S0	initial value of the stock price	positive real
strike	option strike	positive real
r	risk free interest rate	positive real
v0	initial value of variance	positive real
maturity	option maturity (in years)	positive real
kappa	speed of mean reversion	real
theta	long-term variance	positive real
eta	volatility of volatility	positive real
rho	correlation of Brownian motions	$\rho \in [-1, 1]$

### 6.1.1 Optional parameter

parameter	string	scheme
discretizationScheme	euler-ft (default) euler-pt euler-reflection milstein kahl-jackel tv_cd tv_mm	Euler with Full Truncation Euler with Partial Truncation Euler with Reflection Milstein Kahl-Jäckel TV with Central Discretization TV with Moment Matching

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