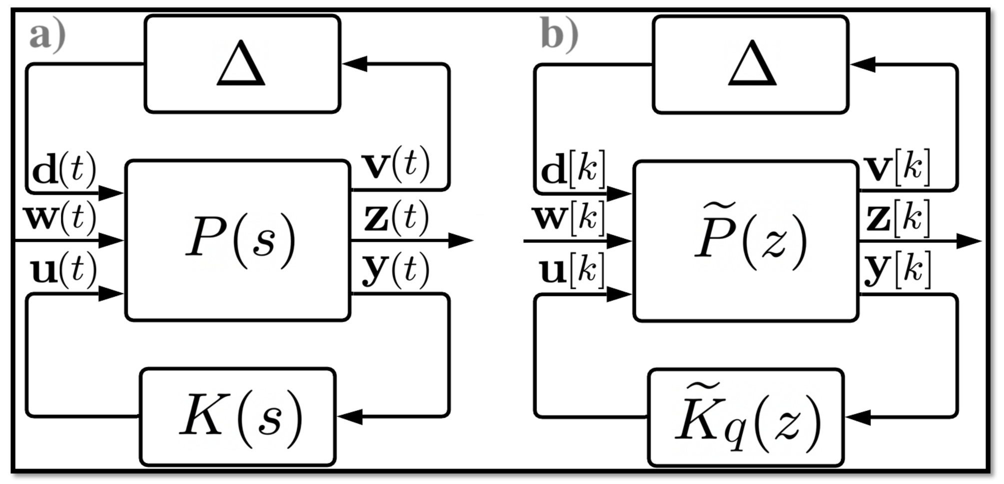
Maintaining Robust Stability and Performance through Sampling and Quantization

Mircea Șușcă, Vlad Mihaly, Petru Dobra
Technical University of Cluj-Napoca, Romania

American Control Conference
May 31 - June 2, 2023 | San Diego, CA, USA

INTRO

- Robust Control Framework P-K-Δ;
- Impose RS and RP metrics through K(s).
- **Fixed sampling rate $\tau > 0$.**
- **Uniform quantization $q > 0$.**



Generalized plant structures: continuous and discrete cases

METHOD

- Discretize P, Δ, K $\Rightarrow \widetilde{P}, \widetilde{\Delta}, \widetilde{K}$, $\tau > 0$.
- Quantize regulator $\widetilde{K} \Rightarrow \widetilde{K}_q$, $q > 0$.
- Reuse continuous-time models G, P, U, W, through discretization (Lemma 1).
- **Maintain SSV less than 1:** $\mu_\Delta(\widetilde{P}, \widetilde{K}_q) < 1$.
- **Extra DOF**: select functional F_1 for *implementability* or F_2 for *fidelity*.
- Perform optimization with solver Σ.

RESULTS

- Graphical analysis for RS/RP behaviour.
- (τ^*, q^*) Sensitivity analysis using the relative condition number κ_F.
- Monte Carlo analysis to obtain a least guaranteed sensitivity bound.

DISCUSSION

- Sensitivity analysis favorable to the **implementability solution** as opposed to the high-fidelity approach;
- High fidelity ⇒ Numerical difficulties.

THE TECHNICAL UNIVERSITY of CLUJ-NAPOCA

SAN DIEGO ACC 2023

KEY TAKEAWAYS

Smaller sampling rates τ and quantizer resolutions q do not automatically lead to better closed-loop performance.

In a Robust Control context, the RS and RP may be lost by „improving" the sampling and quantization processes.

How to guarantee that a continuous regulator K(s) maintains RS/RP after discretization?

NUMERICAL EXAMPLE

$$G_n(s) = k_0 \frac{1-\alpha_0 s}{(s+1)^3}, \quad G(s) = k\frac{1-\alpha s}{(s+1)^3}$$

Parameters and tolerances:
$\alpha_0 = 0.1, tol_\alpha = \pm 10[\%]$
$k_0 = 2, tol_k = \pm 10[\%]$

$$P = \mathcal{A}(G, W) = \begin{bmatrix} W_S & 0 & 0 & 1 \\ -W_S G & W_R & W_T G & -G \end{bmatrix}^\top$$

$$W_S(s) = \frac{\frac{1}{M}s + \omega_B}{s + \omega_B A}, \quad W_T(s) = \frac{s + \omega_{BT}}{A_T s + \omega_{BT} M_T}, \quad W_R(s) = 1/A_R$$

$$\omega_B = 1.5,\ \mathrm{M} = 2,\ \mathrm{A} = 10^{-2}, M_T = 2, A_T = 2, A_R = 10^5$$

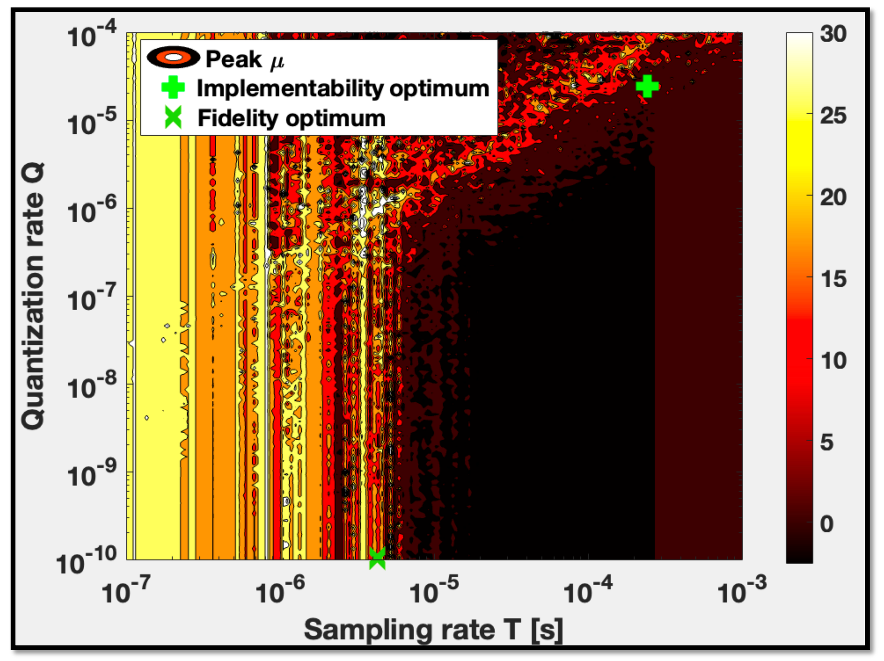
$\Rightarrow$ K(s) (closed-loop shaping)

$$(\tau^*, q^*) \in \mathbf{D} = [10^{-7}, 10^{-3}] \times [10^{-10}, 10^{-4}] \subset \mathbb{R}_+^2$$

$$\min_{\xi \in \mathbb{R}_+^2} f(\xi) \text{ s.t. } \mu_\Delta(\mathrm{LLFT}(\widetilde{P}, \widetilde{K}_q)) < 1$$

$\Updownarrow$ interior-point function

$$\boxed{F(\xi) = f(\xi) - \rho \ln\left(1 - \mu_\Delta(\mathrm{LLFT}(\widetilde{P}, \widetilde{K}_q))\right)}$$



Feasibility constraint: peak SSV [dB] as a function of (τ, q)

Implementability functional f_1

$$\min_{\xi \in \mathbb{R}_+^2} f_1(\xi) = -\xi_1 \xi_2.$$

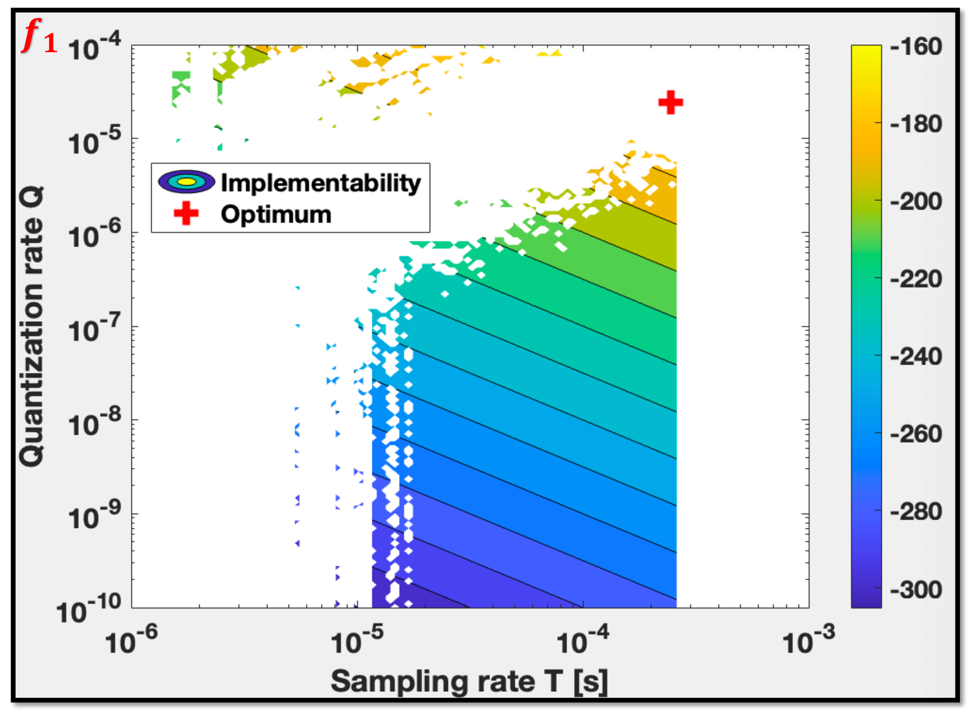
Fidelity functional f_2

$$\min_{\xi \in \mathbb{R}_+^2} f_2(\xi) = \mathscr{I}(\xi),$$

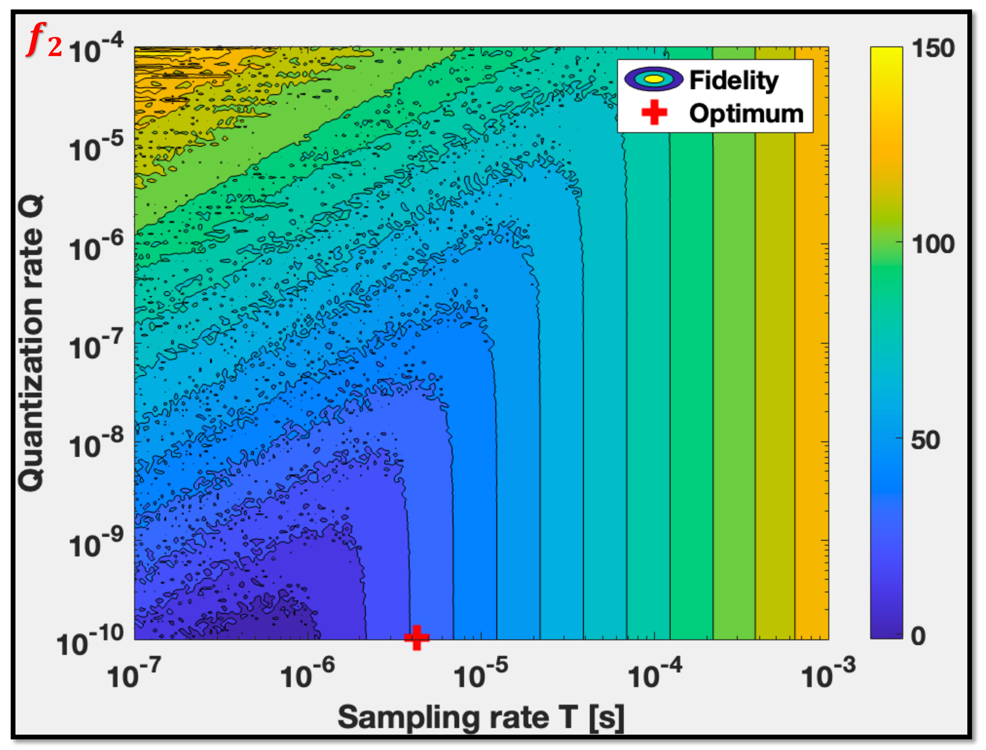
$$\mathscr{I}(\xi) = \int_\Omega \left| \bar{\sigma}(K) - \bar{\sigma}(\widetilde{K}_q) \right| \left(1 + \|\nabla^2 K\|\right) d\omega.$$

Numerical implementation details (MATLAB):

- `GlobalSearch` routine;
- `fmincon` nonlinear solver with nonlinear constraints;
- `interior-point` algorithm.



Feasible region contour plot illustration of functional f_1, leading to the optimal solution (τ_1^*, q_1^*)



Functional f_2 contour plot, leading to the optimal solution (τ_2^*, q_2^*)

Optimized solutions:

$\xi_1^* = (244.20, 24.42) \times 10^{-6}$

$\xi_2^* = (4.27 \times 10^{-6}, 1.0244 \times 10^{-10})$

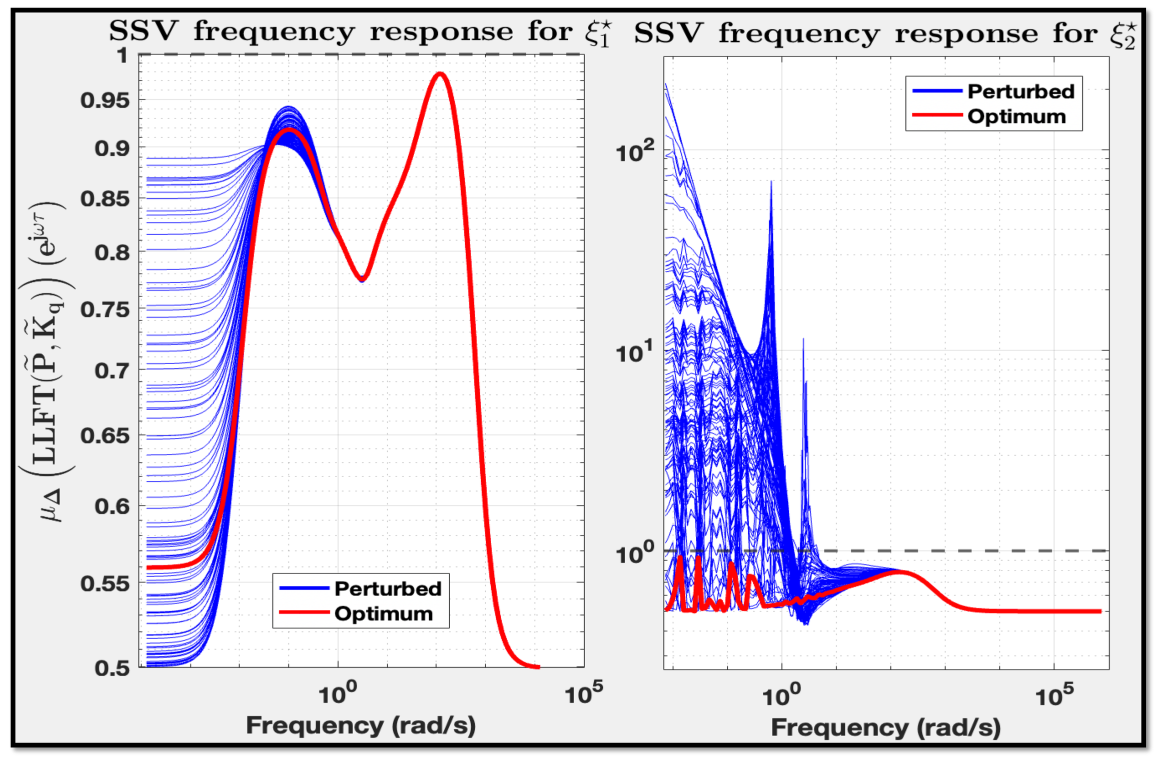
Sensitivity analysis (2D perturbation):

$h_1 = 3.2 \times 10^{-6} \Rightarrow N_{F_2}(\xi_1^*) \approx 7.85 \times 10^{-10}$

$h_2 = 0.25 \times 10^{-15} \Rightarrow N_{F_2}(\xi_2^*) \approx 1.06 \times 10^{-21}$

$\xi_1^* \Rightarrow \kappa_{F_2}(x) \in [964, 7812]$

$\xi_2^* \Rightarrow \kappa_{F_2}(x) \in [8, 84] \times 10^{17}$

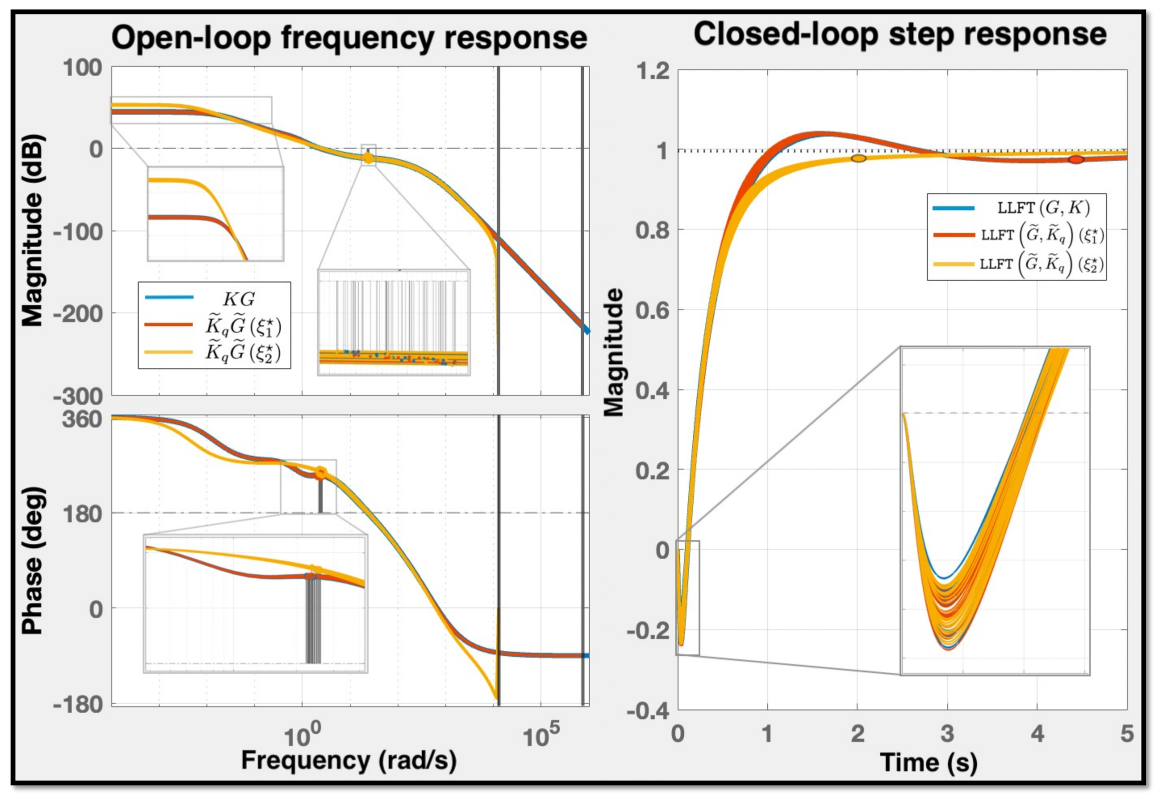


SSV plot for the two solutions: (τ_1^*, q_1^*) – feas. and (τ_2^*, q_2^*) – impl.

Open-loop and closed-loop performance analysis

⇒

Better performance with the easier to implement solution



Open-loop frequency responses using solutions (τ_1^*, q_1^*) – feasibility and (τ_2^*, q_2^*) – implementability

TECHNICAL DETAILS

Algorithm 1: Optimal selection of sampling rate and quantization step for a continuous-time regulator K

Input: K, G_n, U, W, $\mathcal{T}$, $\mathcal{A}$, $\mathbf{\Delta}$, $F \in \{F_1, F_2\}$, Ω, ρ
Discretization operators $\mathcal{D}_p\{\cdot, \tau\}$, $\mathcal{D}_c\{\cdot, \tau\}$.
{For the plant and controller models}
Solver algorithm $\xi_{k+1} = \Sigma(F, \xi_k)$.
Output: Optimum $\xi^\star = (\tau^\star, q^\star) \in \mathbb{R}_+^2$.

```
Initialize ξ ← ξ_0 = (τ_0, q_0) ∈ R^2_+.
while stopping criterion is not satisfied do
    G̃_n ← D_c{G_n, ξ_1}.
    Ũ ← D_c{U, ξ_1}; W̃ ← D_c{W, ξ_1}.
    G̃ ← T{G̃_n, Ũ}; P̃ ← A{G̃, W̃} {Lemma 1}.
    K̃_q ← Q{D{K, ξ_1}, ξ_2} {Eq. (9), (10)}.
    Compute μ_Δ(LLFT(P̃, K̃_q)) approximation.
    Compute F(ξ) {Eq. (16) or (21)}.
    Update ξ ← Σ(F, ξ).
    Verify stopping criterion.
end
```

$$\kappa_F(\xi) = \lim_{\varepsilon \to 0} \sup_{\|\delta\xi\| \le \varepsilon} \frac{\|F(\xi + \delta\xi) - F(\xi)\| / \|\delta\xi\|}{\|F(\xi)\| / \|\xi\|}$$

Relative condition number

$$\mathcal{E}(\xi, \delta\xi) = \left\{ x \in \mathbb{R}_+^2 \;\middle|\; \frac{|\xi_1 - x_1|^2}{\delta\xi_1^2} + \frac{|\xi_2 - x_2|^2}{\delta\xi_2^2} < 1 \right\}$$

2D ellipse

$$N_F(\xi) = \inf_{\varepsilon > 0} \{\kappa_F(x) \in \mathbb{R} \mid \forall x \in \mathcal{E}(\xi, \delta\xi),\ \|\delta\xi\| < \varepsilon\}$$

Least guaranteed sensitivity bound

$$K(s): \left(\begin{array}{ccccccc|c} -1.372 & 613.4 & 8.111 & -0.01228 & 7.242 & 0.2304 & -1.163 & 198.4 \\ -613.4 & -1234 & -198.3 & 0.2579 & -233.3 & -7.376 & 37.27 & 5564 \\ -8.111 & -198.3 & -72.1 & 0.145 & -150.4 & -4.655 & 23.65 & 512.1 \\ -0.01299 & -0.4204 & -0.2688 & -0.006131 & -12.27 & 0.5647 & -1.327 & 0.9463 \\ 7.242 & 233.3 & 150.4 & 23.69 & -4357 & -483.1 & 1452 & -524.8 \\ -0.2303 & -7.373 & -4.651 & -0.8474 & 483 & -10.25 & 94.98 & 16.64 \\ 1.161 & 37.27 & 23.64 & 4.143 & -1452 & 95 & -2903 & -84.06 \\ \hline -198.4 & 5564 & 512.1 & -0.5376 & 524.8 & 16.65 & -84.06 & 0 \end{array}\right)$$

Continuous-time regulator for case study

TABLE I
HIGH NUMERICAL SENSITIVITY OF THE CHARACTERISTIC POLYNOMIAL OF $\widetilde{G}_n$ FOR THE SOLUTION $\xi_2^\star = (4.2739 \times 10^{-6}, 1.02448 \times 10^{-10})$

Case	Parameter	Computed value
1	Ideal $\hat{z}_{1,2,3}$	**$0.999995726091887 < 1$**
2	Analytic a_1	-2.999987178275663
	Analytic a_2	2.999974356606124
	Analytic a_3	-0.999987178330461
2	Analytic $\hat{z}_1$	**$1.000005221571111 > 1$**
	Analytic $\hat{z}_2$	$0.999990978352276 + 0.000008223234706j$
	Analytic $\hat{z}_3$	$0.999990978352276 - 0.000008223234706j$
3	`c2d` a_1	-2.999987178275662
	`c2d` a_2	2.999974356606122
	`c2d` a_3	-0.999987178330461
3	`c2d` $\hat{z}_1$	**$0.999995723377957 < 1$**
	`c2d` $\hat{z}_2$	$0.999995727448852 + 0.000000002350638j$
	`c2d` $\hat{z}_3$	$0.999995727448852 - 0.000000002350638j$

GitHub: ACC paper and poster, source code