Maintaining Robust Stability and Performance through Sampling and Quantization

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Outline

Robust Control: μ-Synthesis [I]

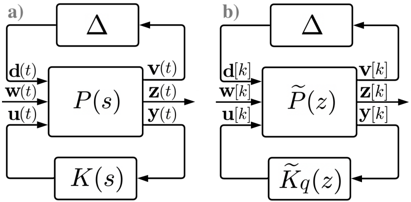
- Continuous-time augmented plant P and P-K-$\boldsymbol{\Delta}$ connection;

$$P:\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \hline \mathbf{v}(t)\\ \mathbf{z}(t)\\ \mathbf{y}(t)\end{pmatrix}=\left(\begin{array}{c|ccc}A & B_d & B_w & B_u\\ \hline C_v & D_{vd} & D_{vw} & D_{vu}\\ C_z & D_{zd} & D_{zw} & D_{zu}\\ C_y & D_{yd} & D_{yw} & D_{yu}\end{array}\right)\begin{pmatrix}\mathbf{x}(t)\\ \hline \mathbf{d}(t)\\ \mathbf{w}(t)\\ \mathbf{u}(t)\end{pmatrix}; \tag{1}$$

$$\boldsymbol{\Delta}=\{\mathrm{diag}(\Delta_1,\ldots,\Delta_f,\delta_1 I_{n_1},\ldots,\delta_s I_{n_s})\}; \tag{2}$$

$$K=(A_K,B_K,C_K,D_K)\in\mathscr{G}. \tag{3}$$

- **Sampled with fixed period** $\tau>0\Rightarrow$ discretized block equivalents: $\widetilde{P}$-$\widetilde{K}$-$\widetilde{\boldsymbol{\Delta}}$;
- **Quantization with a resolution** $q>0$ applied to $\widetilde{K}$ further symbolized by $\widetilde{K}_q$.



Notations

- Nominal process model: $G_n = (A, B_u, C_y, D_{yu})$;
- Arbitrary uncertainty block $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_\infty \leq 1$;
- Uncertain plant $G_{\mathbf{\Delta}} \equiv G$; nominal plant model becomes $G_n = G_{\mathbf{\Delta}}|_{\Delta=O}$;
- Mapping $\mathscr{T}$ between G_n and G:

$$\mathscr{T} : \mathscr{G} \times \mathscr{G} \to \mathscr{G},\ G = \mathscr{T}(G_n, U),\ \Delta \in \mathbf{\Delta},\ \|\Delta\|_\infty \leq 1. \tag{4}$$

- Mapping $\mathscr{A}$ between G and P:

$$\mathscr{A} : \mathscr{G} \times \mathscr{G} \to \mathscr{G},\ P = \mathscr{A}(G, W). \tag{5}$$

Example

SISO nominal plant $G_n = G_{n_1} G_{n_2}$ with:

- multiplicative and additive uncertainties, $U = [U_1 \quad U_2]$, i.e. $G = G_{n_1}(1+U_1\Delta_1)(G_{n_2}+U_2\Delta_2)$, $\Delta = \mathrm{diag}([\Delta_1 \quad \Delta_2])$, $\|\Delta\|_\infty \leq 1$;
- augmented plant model using the closed-loop sensitivity weight $W = W_S$ as $P = [-W_S G \quad W_S]$.

Robust Control: μ-Synthesis [II]

Structured singular value (SSV):

$$\mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P,K)) = \sup_{\omega\in\mathbb{R}_+} \frac{1}{\min\limits_{\Delta\in\mathbf{\Delta}} \{\overline{\sigma}(\Delta),\ \det(I - M_\omega\Delta)=0\}}, \tag{6}$$

with $M_\omega = \mathrm{LLFT}(P,K)(j\omega)$.
μ-synthesis problem:

$$K \text{ stabilizing s.t. } \mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P,K)) < 1. \tag{7}$$

Assumed closed-loop properties imposed through $K(s)$:

- **robust stability (RS)**;
- **robust performance (RP)**.

Discretization

The numeric regulator $\widetilde{K}_q$ is obtained through:

- **sampling with a fixed period** $\tau \in \mathbb{R}_+$, followed by a discretization of K;

$$\mathscr{D} : \mathscr{G} \times \mathbb{R}_+ \to \mathscr{G}_D,\ \widetilde{K} = \mathscr{D}\left\{K, \tau\right\}. \tag{8}$$

- **uniform quantization** of its coefficients with a fixed step $q \in \mathbb{R}_+$:

$$\mathscr{Q} : \mathscr{G}_D \times \mathbb{R}_+ \to \mathscr{G}_D,\ \widetilde{K}_q = \mathscr{Q}\left\{\widetilde{K}, q\right\}. \tag{9}$$

Denote $\xi = (\tau, q) \in \mathbb{R}_+^2$ encompassed into a single two-dimensional variable.

Example

For $K = (A_K, B_K, C_K, D_K) \in \mathscr{G}$:

- discretization using the Tustin method, leading to $\widetilde{K} = (A_d, B_d, C_d, D_d) \in \mathscr{G}_D$;
- rounding (midtread) quantizer:

$$\widetilde{K}_q = \left(q\left\lfloor\frac{A_d}{q}\right\rceil, q\left\lfloor\frac{B_d}{q}\right\rceil, q\left\lfloor\frac{C_d}{q}\right\rceil, q\left\lfloor\frac{D_d}{q}\right\rceil\right) \in \mathscr{G}_D. \tag{10}$$

Key Questions

1. How to guarantee that $\widetilde{K}_q \in \mathscr{G}_D$ still maintains RS and RP?

2. How to select such $\widetilde{K}_q \in \mathscr{G}_D$ configurations?

Contributions

Lemma

Given the continuous models G_n, U, W, P and mappings $\mathscr{T}$, $\mathscr{A}$ from (4) and (5), the discrete augmented plant counterpart $\widetilde{P}$ can be computed based on the discretization of its individual components $\widetilde{G}_n$, $\widetilde{U}$, $\widetilde{W}$ through $\mathscr{D}\{\cdot,\tau\}$, $\tau>0$. Moreover, the block $\mathbf{\Delta}$ is invariant from the continuous domain to the discrete domain. As such:

$$\widetilde{G} = \mathscr{T}\left(\mathscr{D}\{G_n,\tau\}, \mathscr{D}\{U,\tau\}\right),\ \Delta \in \mathbf{\Delta}; \tag{11}$$

$$\widetilde{P} = \mathscr{D}\{P,\tau\} = \mathscr{A}\left(\widetilde{G}, \mathscr{D}\{W,\tau\}\right). \tag{12}$$

Main advantage: Reuse the models and weights from the continuous to discrete case.
Reformulate the μ-synthesis problem in terms of the equivalent discrete-time models:

$$\mu_{\mathbf{\Delta}}(\mathrm{LLFT}(\widetilde{P},\widetilde{K}_q)) = \sup_{\omega\in\Omega_N} \frac{1}{\min\limits_{\Delta\in\mathbf{\Delta}}\{\overline{\sigma}(\Delta), \det(I-\widetilde{M}_\omega\Delta)=0\}} < \mathbf{1}\ (\textit{mandatory constraint}),$$

with $\widetilde{M}_\omega = \mathrm{LLFT}(\widetilde{P},\widetilde{K}_q)(e^{j\omega\tau})$ and domain $\Omega_N = [0,\omega_N)$, where $\omega_N = \pi/\tau$ is the Nyquist frequency for the period τ.

Optimization functionals

In addition to the mandatory SSV constraint, we are left with a separate degree of freedom to select the pairs $\xi = (\tau, q) \in \mathbb{R}^2_+$. Two dichotomic approaches are suggested:

- **Implementability functional**:

$$\min_{\xi \in \mathbb{R}^2_+} f_1(\xi) = -\xi_1 \xi_2. \tag{13}$$

- **Fidelity functional**:

$$\min_{\xi \in \mathbb{R}^2_+} f_2(\xi) = \mathscr{I}(\xi), \tag{14}$$

$$\mathscr{I}(\xi) = \int_\Omega \left| \overline{\sigma}(K) - \overline{\sigma}(\widetilde{K}_q) \right| \left(1 + \left\| \nabla^2 K \right\| \right) d\omega. \tag{15}$$

Hands-On Summary

Preconditions:

1. establish G_n, $\boldsymbol{\Delta}$, $U \to \mathscr{T}$, $W \to \mathscr{A} \Rightarrow K$;
2. define discratization methods $\mathscr{D}_p$, $\mathscr{D}_c$;
3. define quantization method $\mathscr{Q}$ for $\widetilde{K}_q$;
4. discretize plant $\widetilde{P}$ (Lemma 1);
5. define cost functional $f : \mathbb{R}^2_+ \to \mathbb{R}_+$;
6. define nonlinear constrained optimization problem:

$$\min_{\xi \in \mathbb{R}^2_+} f(\xi) \text{ s.t. } \mu_{\boldsymbol{\Delta}}(\mathtt{LLFT}(\widetilde{P}, \widetilde{K}_q)) < 1,$$

$\Leftrightarrow$ the interior-point function, $\rho > 0$:

$$F(\xi) = f(\xi) - \rho \ln\left(1 - \mu_{\Delta}(\mathtt{LLFT}(\widetilde{P}, \widetilde{K}_q))\right).$$

Algorithm 1: Optimal selection of sampling rate and quantization step for a continuous-time regulator K

Input: K, G_n, U, W, $\mathcal{T}$, $\mathcal{A}$, $\boldsymbol{\Delta}$, $F \in \{F_1, F_2\}$, Ω, ρ
Discretization operators $\mathcal{D}_p\{\cdot, \tau\}$, $\mathcal{D}_c\{\cdot, \tau\}$.
{For the plant and controller models}
Solver algorithm $\xi_{k+1} = \Sigma(F, \xi_k)$.

Output: Optimum $\xi^\star = (\tau^\star, q^\star) \in \mathbb{R}^2_+$.

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Initialize ξ ← ξ_0 = (τ_0, q_0) ∈ R^2_+.
while stopping criterion is not satisfied do
    G̃_n ← D_c {G_n, ξ_1}.
    Ũ ← D_c {U, ξ_1}; W̃ ← D_c {W, ξ_1}.
    G̃ ← T {G̃_n, Ũ}; P̃ ← A {G̃, W̃} {Lemma 1}.
    K̃_q ← Q {D {K, ξ_1}, ξ_2} {Eq. (9), (10)}.
    Compute μ_Δ (LLFT (P̃, K̃_q)) approximation.
    Compute F(ξ) {Eq. (16) or (21)}.
    Update ξ ← Σ(F, ξ).
    Verify stopping criterion.
end
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Numerical Example

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Conclusions and Future Works

Conclusions:

- mathematical tools to quantify the influence of (τ, q) on the transient response of the closed-loop system;
- **smaller (τ, q) pairs do not automatically lead to improved performance**;
- $(\tau^\star, q^\star)$ sensitivity analysis using the condition number.

Future Works:

- general-purpose functional $f(\xi)$ directly tied to hardware specifications (τ – operating frequency; q – data handling capability, i.e. memory and bandwidth);
- extensions on the modelling framework directly for discrete-time systems with uncertainties, i.e. $e^{\frac{\tau}{T}}$, $T > 0$.

Thank you for your attention!