Maintaining Robust Stability and Performance through Sampling and Quantization

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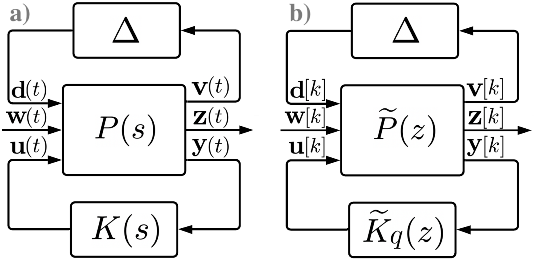
Robust Control Context



The numeric regulator $\widetilde{K}_q(z)$ is obtained through:

- **sampling with a fixed period** $\tau \in \mathbb{R}_+$, with discretization;
- **uniform coefficient quantization** with fixed step $q \in \mathbb{R}_+$.

Denote $\xi = (\tau, q) \in \mathbb{R}^2_+$ single two-dimensional variable.

Robust Control: μ-Synthesis

Structured singular value (SSV):

$$\mu_{\mathbf{\Delta}}(\mathtt{LLFT}(P,K)) = \sup_{\omega\in\mathbb{R}_+} \frac{1}{\min\limits_{\Delta\in\mathbf{\Delta}} \{\overline{\sigma}(\Delta),\ \det(I - M_\omega \Delta) = 0\}}, \tag{1}$$

with $M_\omega = \mathtt{LLFT}(P,K)(j\omega)$.
μ-synthesis problem:

$$K \text{ stabilizing s.t. } \mu_{\mathbf{\Delta}}(\mathtt{LLFT}(P,K)) < 1. \tag{2}$$

Assumed closed-loop properties imposed through $K(s)$:

- **robust stability (RS)**;
- **robust performance (RP)**.

Key Questions

1. How to guarantee that $\widetilde{K}_q \in \mathscr{G}_D$ still maintains RS and RP?

2. How to select such $\widetilde{K}_q \in \mathscr{G}_D$ configurations?

Contributions

Lemma

Given the continuous models G_n, U, W, P and mappings $\mathscr{T}$, $\mathscr{A}$, the discrete augmented plant counterpart $\widetilde{P}$ can be computed based on the discretization of its individual components $\widetilde{G}_n$, $\widetilde{U}$, $\widetilde{W}$ through $\mathscr{D}\{\cdot,\tau\}$, $\tau>0$. Moreover, the block $\boldsymbol{\Delta}$ is invariant from the continuous domain to the discrete domain. As such:

$$\widetilde{G} = \mathscr{T}\left(\mathscr{D}\{G_n,\tau\}, \mathscr{D}\{U,\tau\}\right),\ \Delta \in \boldsymbol{\Delta}; \tag{3}$$

$$\widetilde{P} = \mathscr{D}\{P,\tau\} = \mathscr{A}\left(\widetilde{G}, \mathscr{D}\{W,\tau\}\right). \tag{4}$$

Main advantage: Reuse the models and weights from the continuous to discrete case.
Reformulate the μ-synthesis problem in terms of the equivalent discrete-time models:

$$\mu_{\boldsymbol{\Delta}}(\mathrm{LLFT}(\widetilde{P},\widetilde{K}_q)) = \sup_{\omega\in\Omega_N} \frac{1}{\min\limits_{\Delta\in\boldsymbol{\Delta}}\{\overline{\sigma}(\Delta), \det(I-\widetilde{M}_\omega\Delta)=0\}} < \mathbf{1}\ (\textit{mandatory constraint}),$$

with $\widetilde{M}_\omega = \mathrm{LLFT}(\widetilde{P},\widetilde{K}_q)(e^{j\omega\tau})$ and domain $\Omega_N = [0,\omega_N)$, where $\omega_N = \pi/\tau$ is the Nyquist frequency for the period τ.

Optimization Functionals

In addition to the mandatory SSV constraint, we are left with a separate degree of freedom to select the pairs $\xi = (\tau, q) \in \mathbb{R}_+^2$. Two dichotomic approaches are suggested:

- **Implementability functional**:

$$\min_{\xi \in \mathbb{R}_+^2} f_1(\xi) = -\xi_1 \xi_2. \tag{5}$$

- **Fidelity functional**:

$$\min_{\xi \in \mathbb{R}_+^2} f_2(\xi) = \mathscr{I}(\xi), \tag{6}$$

$$\mathscr{I}(\xi) = \int_\Omega \left| \overline{\sigma}(K) - \overline{\sigma}(\widetilde{K}_q) \right| \left(1 + \left\| \nabla^2 K \right\| \right) d\omega. \tag{7}$$

Hands-On Summary

Preconditions:

1. establish G_n, $\mathbf{\Delta}$, $U \to \mathscr{T}$, $W \to \mathscr{A} \Rightarrow K$;
2. define discretization methods $\mathscr{D}_p$, $\mathscr{D}_c$;
3. define quantization method $\mathscr{Q}$ for $\widetilde{K}_q$;
4. discretize plant $\widetilde{P}$ (Lemma 1);
5. define cost functional $f : \mathbb{R}^2_+ \to \mathbb{R}_+$;
6. define nonlinear constrained optimization problem:

$$\min_{\xi \in \mathbb{R}^2_+} f(\xi) \text{ s.t. } \mu_{\mathbf{\Delta}}(\mathtt{LLFT}(\widetilde{P}, \widetilde{K}_q)) < 1,$$

$\Leftrightarrow$ the interior-point function, $\rho > 0$:

$$F(\xi) = f(\xi) - \rho \ln\left(1 - \mu_{\Delta}(\mathtt{LLFT}(\widetilde{P}, \widetilde{K}_q))\right).$$

Algorithm 1: Optimal selection of sampling rate and quantization step for a continuous-time regulator K

Input: K, G_n, U, W, $\mathcal{T}$, $\mathcal{A}$, $\mathbf{\Delta}$, $F \in \{F_1, F_2\}$, Ω, ρ
Discretization operators $\mathcal{D}_p\{\cdot, \tau\}$, $\mathcal{D}_c\{\cdot, \tau\}$.
{For the plant and controller models}
Solver algorithm $\xi_{k+1} = \Sigma(F, \xi_k)$.
Output: Optimum $\xi^\star = (\tau^\star, q^\star) \in \mathbb{R}^2_+$.

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Initialize ξ ← ξ_0 = (τ_0, q_0) ∈ R^2_+.
while stopping criterion is not satisfied do
    G̃_n ← D_c{G_n, ξ_1}.
    Ũ ← D_c{U, ξ_1}; W̃ ← D_c{W, ξ_1}.
    G̃ ← T{G̃_n, Ũ}; P̃ ← A{G̃, W̃} {Lemma 1}.
    K̃_q ← Q{D{K, ξ_1}, ξ_2} {Eq. (9), (10)}.
    Compute μ_Δ(LLFT(P̃, K̃_q)) approximation.
    Compute F(ξ) {Eq. (16) or (21)}.
    Update ξ ← Σ(F, ξ).
    Verify stopping criterion.
end
```

In our experiments: Σ $=$ `fmincon` $+$ `GlobalSearch` initialization.