

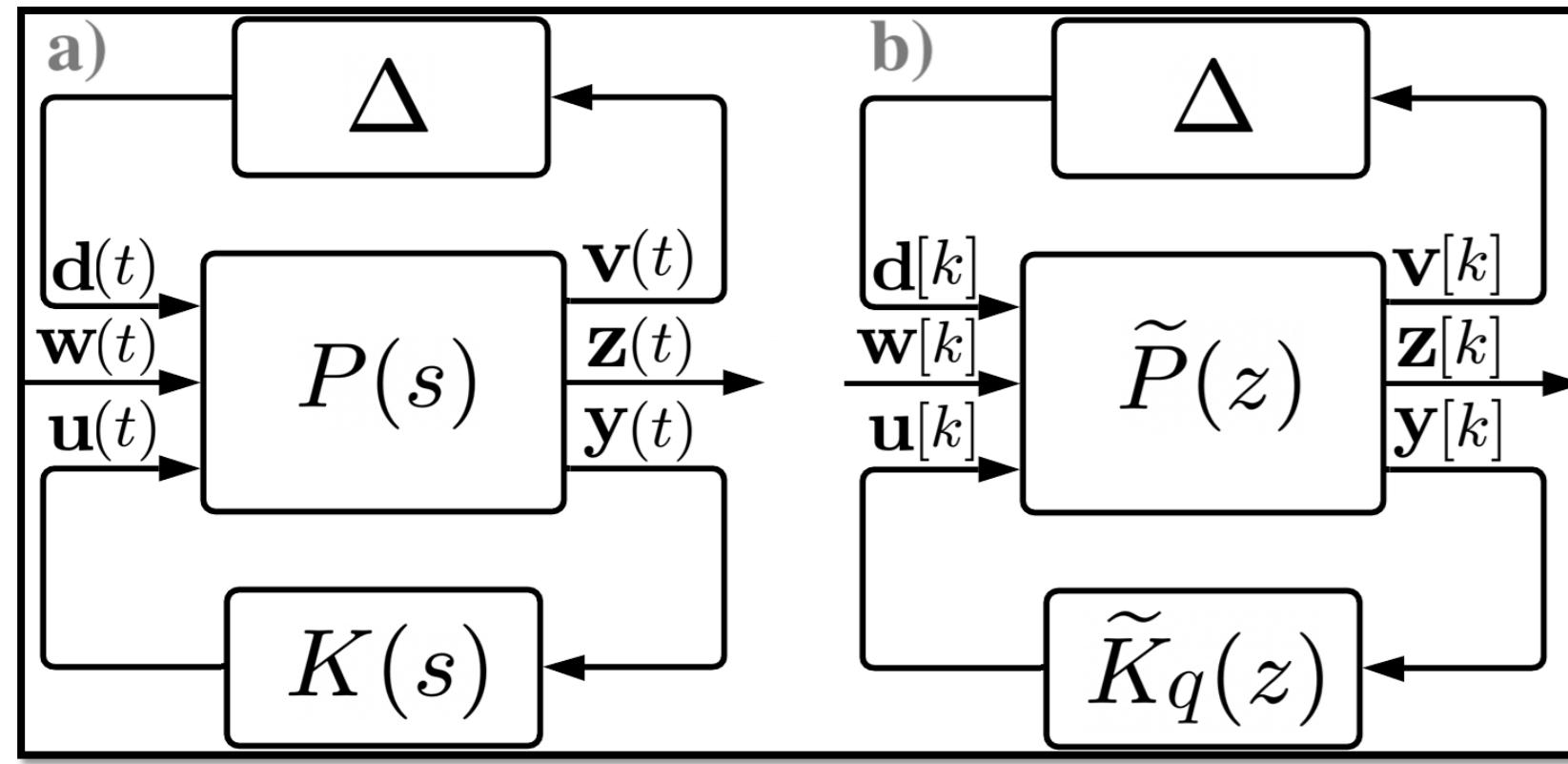
Maintaining Robust Stability and Performance through Sampling and Quantization

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INTRO

- Robust Control Framework P-K-Δ;
- Impose RS and RP metrics through $K(s)$.
- **Fixed sampling rate $\tau > 0$.**
- Uniform quantization $q > 0$.



Generalized plant structures: continuous and discrete cases

METHOD

- Discretize $P, \Delta, K \Rightarrow \tilde{P}, \tilde{\Delta}, \tilde{K}, \tau > 0$.
- Quantize regulator $\tilde{K} \Rightarrow \tilde{K}_q, q > 0$.
- Reuse continuous-time models G, P, U, W , through discretization (Lemma 1).
- **Maintain SSV less than 1:** $\mu_{\Delta}(\tilde{P}, \tilde{K}_q) < 1$.
- **Extra DOF:** select functional F_1 for implementability or F_2 for fidelity.
- Perform optimization with solver Σ .

RESULTS

- Graphical analysis for RS/RP behaviour.
- (τ^*, q^*) Sensitivity analysis using the relative condition number κ_F .
- Monte Carlo analysis to obtain a least guaranteed sensitivity bound.

DISCUSSION

- Sensitivity analysis favorable to the **implementability solution** as opposed to the high-fidelity approach;
- High fidelity \Rightarrow Numerical difficulties.



KEY TAKEAWAYS

Smaller sampling rates τ and quantizer resolutions q do not automatically lead to better closed-loop performance.

In a Robust Control context, the RS and RP may be lost by „improving“ the sampling and quantization processes.

How to guarantee that a continuous regulator $K(s)$ maintains RS/RP after discretization?

NUMERICAL EXAMPLE

$$G_n(s) = k_0 \frac{1 - \alpha_0 s}{(s + 1)^3}, \quad G(s) = k \frac{1 - \alpha s}{(s + 1)^3}$$

$$P = \mathcal{A}(G, W) = \begin{bmatrix} W_S & 0 & 0 \\ -W_S G & W_R & W_T G \\ 0 & W_R & -G \end{bmatrix}^T$$

$$W_S(s) = \frac{\frac{1}{M}s + \omega_B}{s + \omega_B A}, \quad W_T(s) = \frac{s + \omega_B T}{A_T s + \omega_B T M_T}, \quad W_R(s) = 1/A_R$$

$$\omega_B = 1.5, \quad M = 2, \quad A = 10^{-2}, \quad M_T = 2, \quad A_T = 2, \quad A_R = 10^5$$

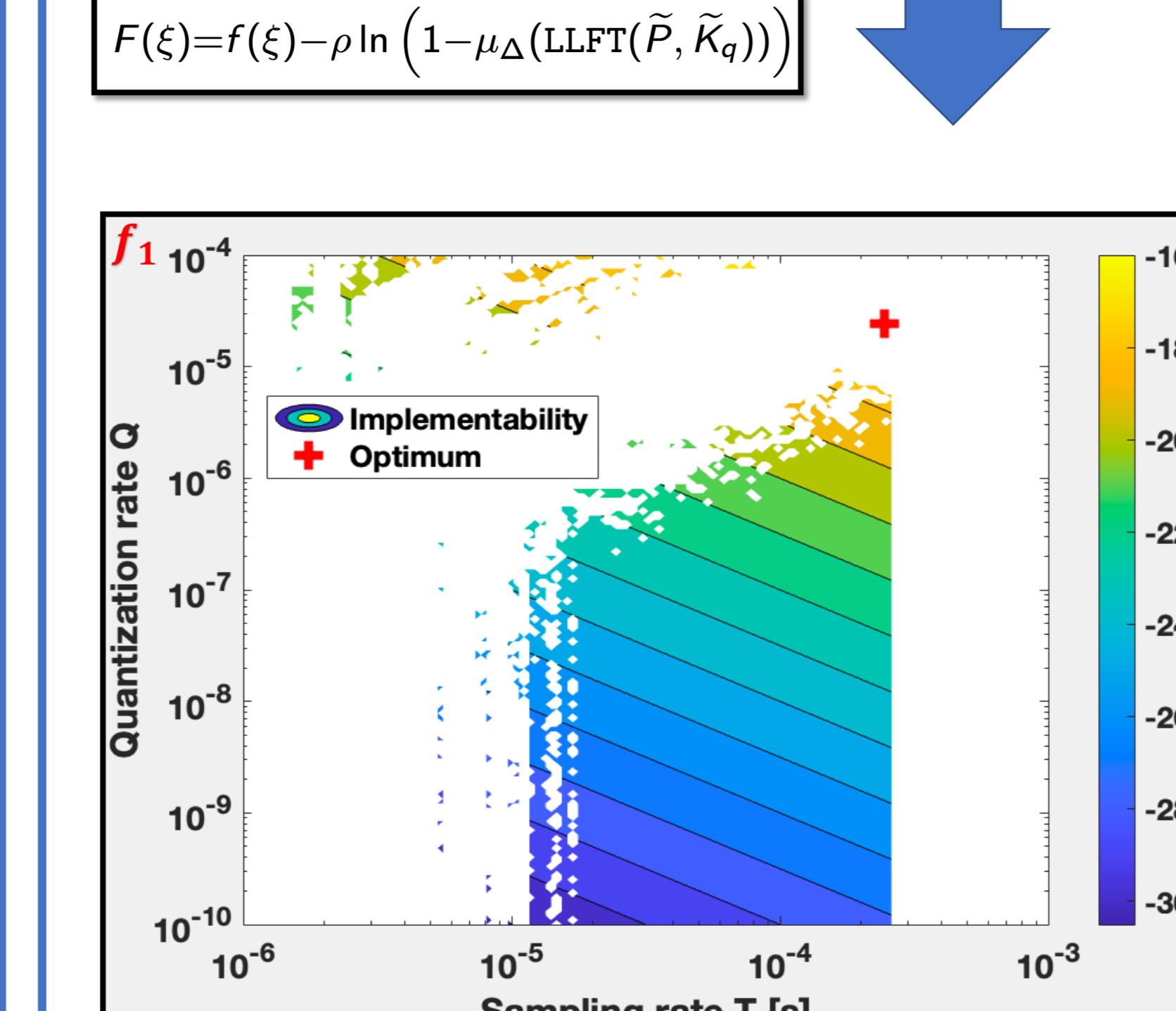
$$\Rightarrow K(s) \text{ (closed-loop shaping)}$$

$$(\tau^*, q^*) \in \mathbb{D} = [10^{-7}, 10^{-3}] \times [10^{-10}, 10^{-4}] \subset \mathbb{R}^2$$

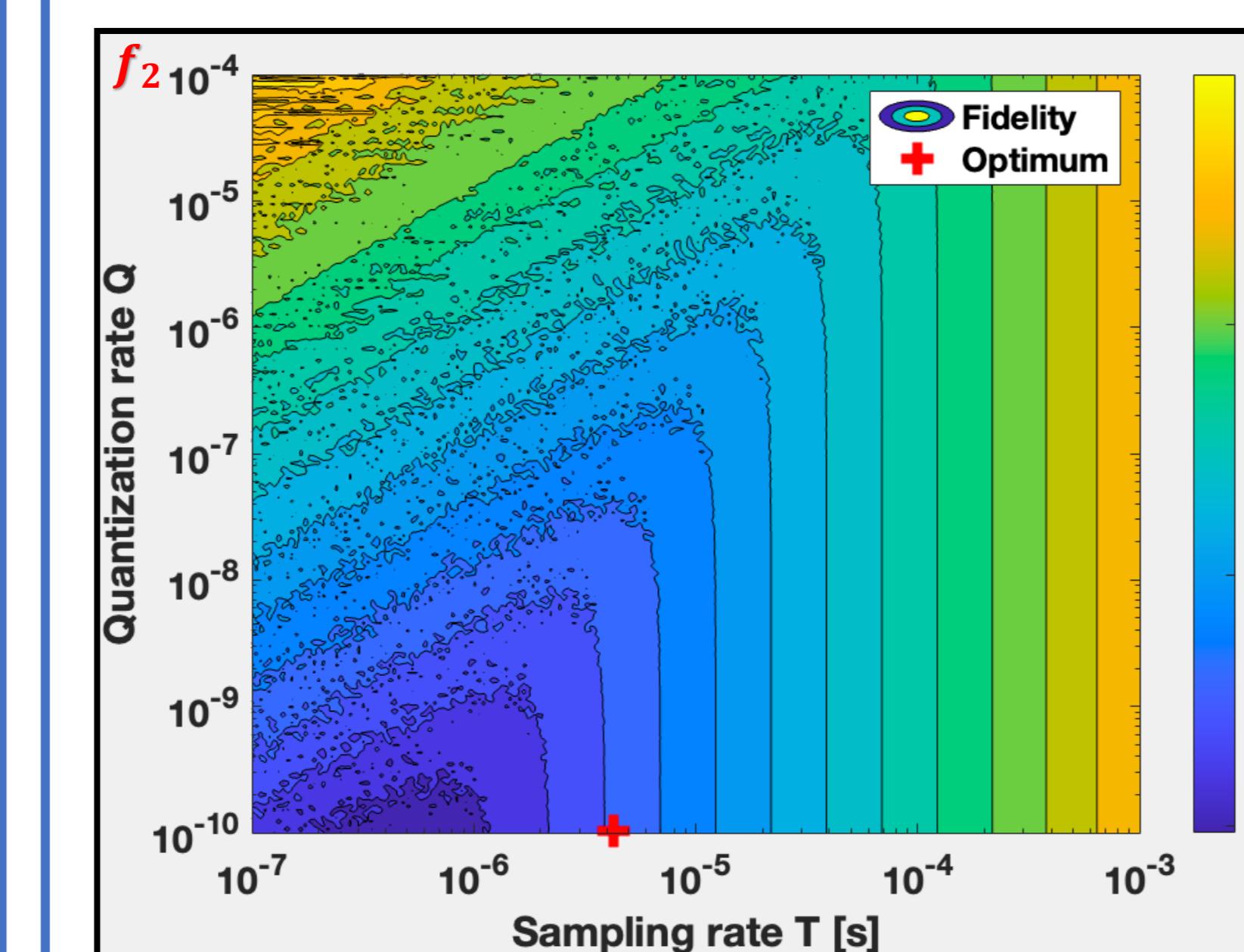
$$\min_{\xi \in \mathbb{R}^2_+} f(\xi) \text{ s.t. } \mu_{\Delta}(\text{LLFT}(\tilde{P}, \tilde{K}_q)) < 1$$

\Leftrightarrow interior-point function

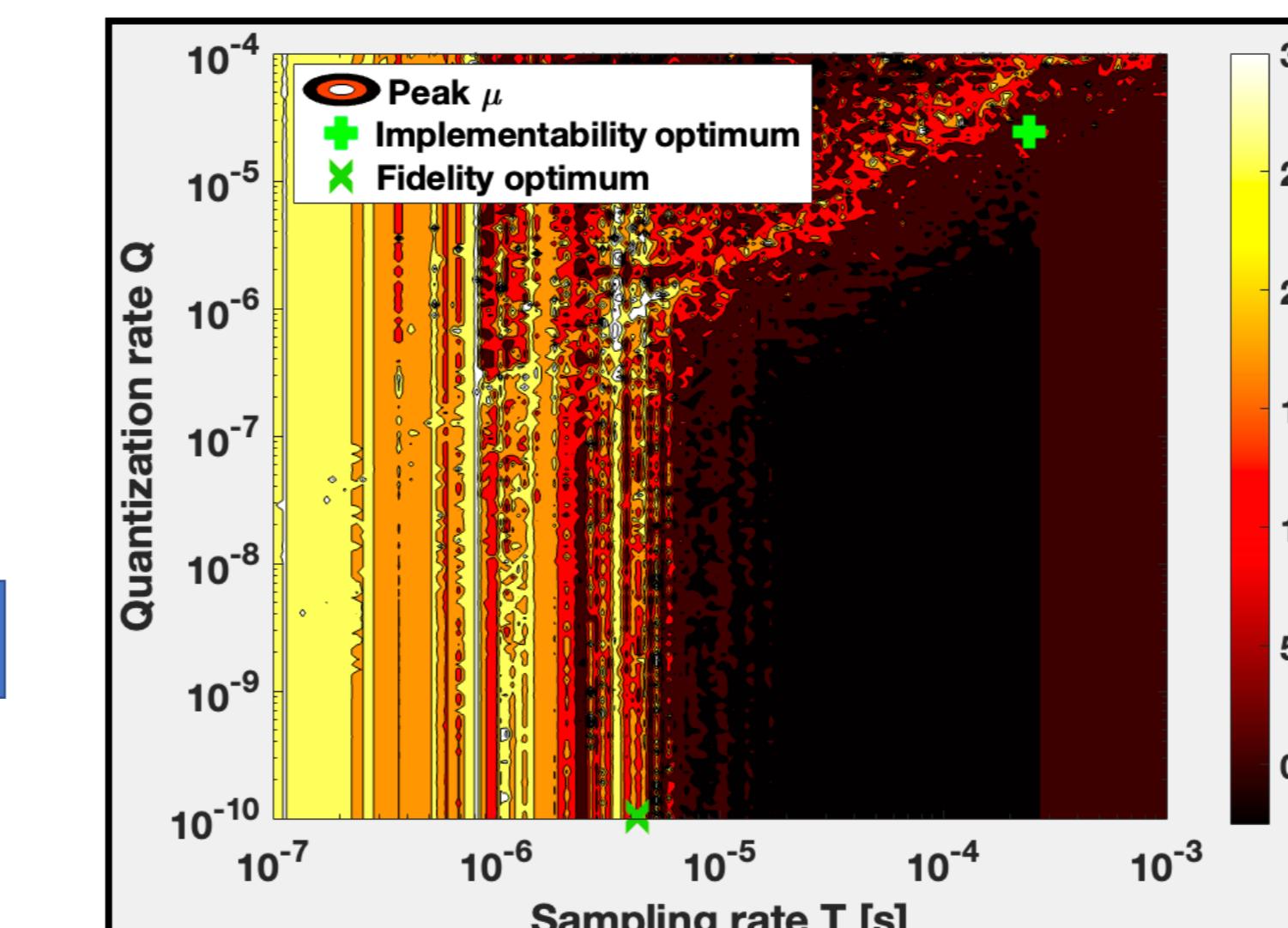
$$F(\xi) = f(\xi) - \rho \ln(1 - \mu_{\Delta}(\text{LLFT}(\tilde{P}, \tilde{K}_q)))$$



Feasible region contour plot illustration of functional f_1 , leading to the optimal solution (τ_1^*, q_1^*)



Functional f_2 contour plot, leading to the optimal solution (τ_2^*, q_2^*)



Feasibility constraint: peak SSV [dB] as a function of (τ, q)

Implementability functional f_1	Fidelity functional f_2
$\min_{\xi \in \mathbb{R}^2_+} f_1(\xi) = -\xi_1 \xi_2$	$\min_{\xi \in \mathbb{R}^2_+} f_2(\xi) = \mathcal{J}(\xi)$,
	$\mathcal{J}(\xi) = \int_{\Omega} \bar{\sigma}(K) - \bar{\sigma}(\tilde{K}_q) (1 + \ \nabla^2 K\) d\omega$.

Numerical implementation details (MATLAB):

- GlobalSearch routine;
- fmincon nonlinear solver with nonlinear constraints;
- interior-point algorithm.

Optimized solutions:

$$\xi_1^* = (244.20, 24.42) \times 10^{-6}$$

$$\xi_2^* = (4.27 \times 10^{-6}, 1.0244 \times 10^{-10})$$

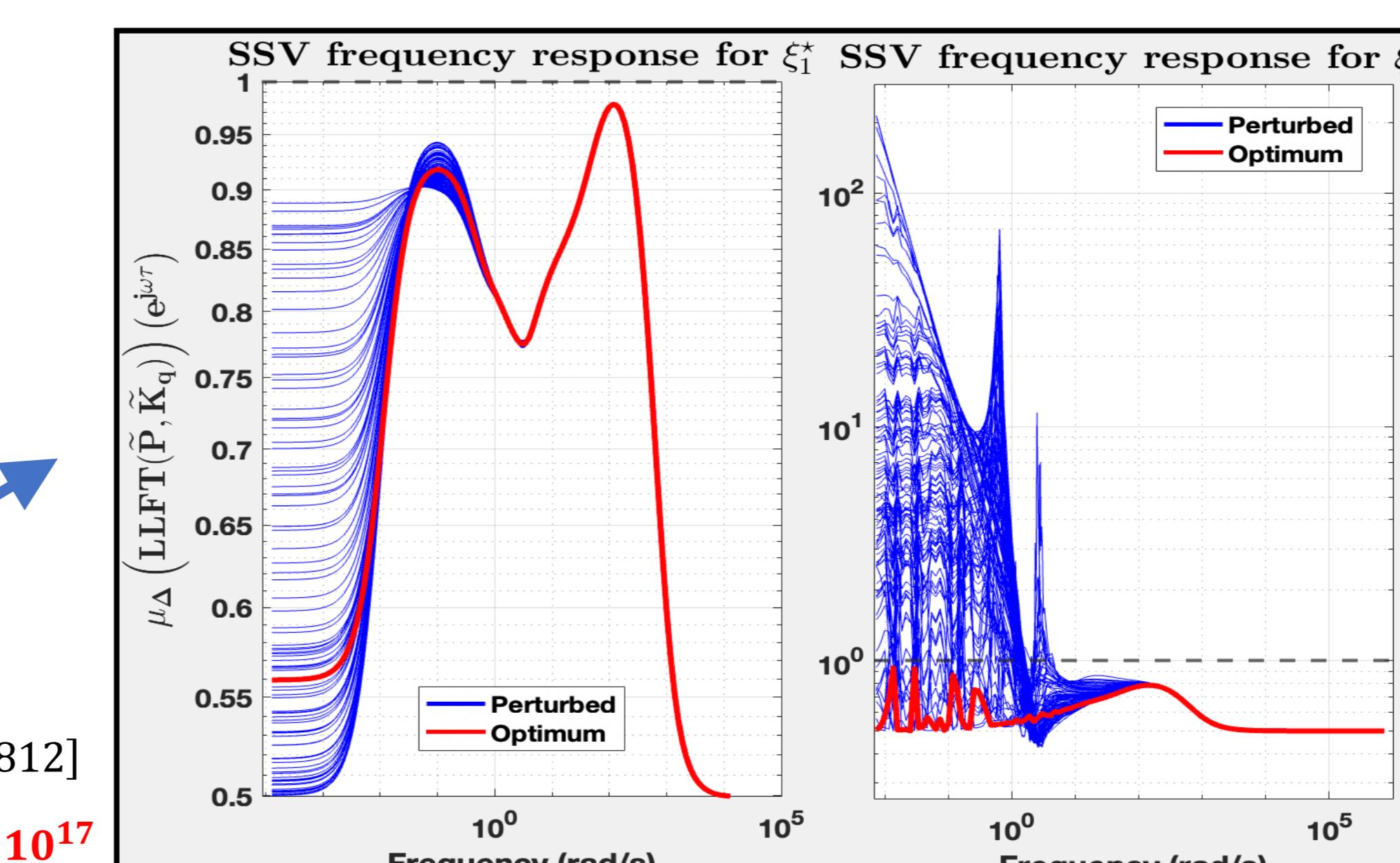
Sensitivity analysis (2D perturbation):

$$h_1 = 3.2 \times 10^{-6} \Rightarrow N_{F_2}(\xi_1^*) \approx 7.85 \times 10^{-10}$$

$$h_2 = 0.25 \times 10^{-15} \Rightarrow N_{F_2}(\xi_2^*) \approx 1.06 \times 10^{-21}$$

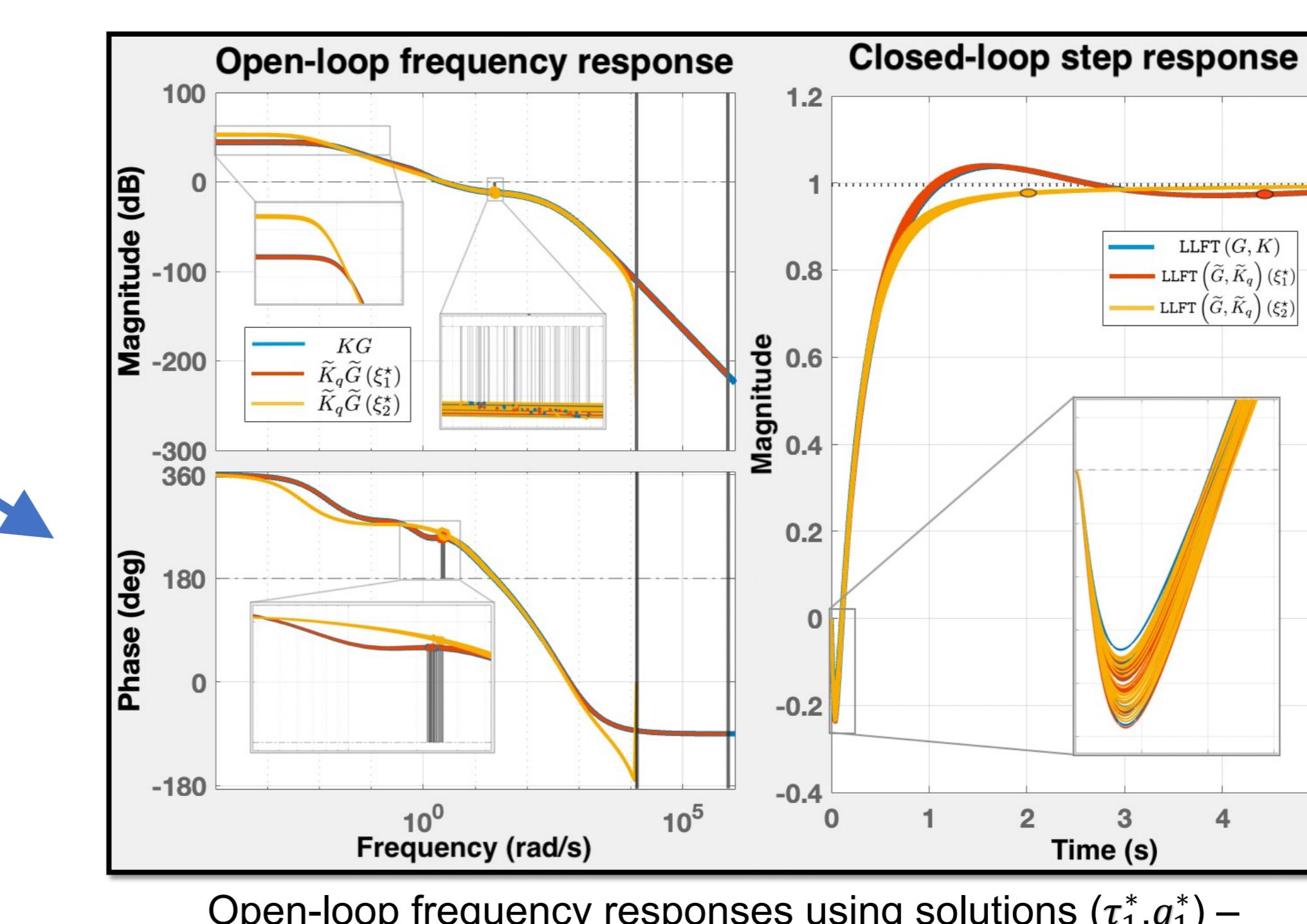
$$\xi_1^* \Rightarrow \kappa_{F_2}(x) \in [964, 7812]$$

$$\xi_2^* \Rightarrow \kappa_{F_2}(x) \in [8, 84] \times 10^{17}$$



SSV plot for the two solutions: (τ_1^*, q_1^*) – feas. and (τ_2^*, q_2^*) – impl.

Open-loop and closed-loop performance analysis
 \Rightarrow
 Better performance with the easier to implement solution



Open-loop frequency responses using solutions (τ_1^*, q_1^*) – feasibility and (τ_2^*, q_2^*) – implementability

TECHNICAL DETAILS

Algorithm 1: Optimal selection of sampling rate and quantization step for a continuous-time regulator K

Input: $K, G_n, U, W, \mathcal{T}, \mathcal{A}, \Delta, F \in \{F_1, F_2\}, \Omega, \rho$,
 Discretization operators $\mathcal{D}_p \{\cdot, \tau\}, \mathcal{D}_c \{\cdot, \tau\}$.
 {For the plant and controller models}
 Solver algorithm $\xi_{k+1} = \Sigma(F, \xi_k)$.
Output: Optimum $\xi^* = (\tau^*, q^*) \in \mathbb{R}^2_+$.

- 1 Initialize $\xi \leftarrow \xi_0 = (\tau_0, q_0) \in \mathbb{R}^2_+$.
- 2 **while** stopping criterion is not satisfied **do**
- 3 $\tilde{G}_n \leftarrow \mathcal{D}_c \{G_n, \xi_1\}$.
- 4 $\tilde{U} \leftarrow \mathcal{D}_c \{U, \xi_1\}; \tilde{W} \leftarrow \mathcal{D}_c \{W, \xi_1\}$.
- 5 $\tilde{G} \leftarrow \mathcal{T} \{\tilde{G}_n, \tilde{U}\}; \tilde{P} \leftarrow \mathcal{A} \{\tilde{G}, \tilde{W}\}$ {Lemma 1}.
- 6 $\tilde{K}_q \leftarrow \mathcal{Q} \{\mathcal{D} \{K, \xi_1\}, \xi_2\}$ {Eq. (9), (10)}.
- 7 Compute $\mu_{\Delta}(\text{LLFT}(\tilde{P}, \tilde{K}_q))$ approximation.
- 8 Compute $F(\xi)$ {Eq. (16) or (21)}.
- 9 Update $\xi \leftarrow \Sigma(F, \xi)$.
- 10 Verify stopping criterion.
- 11 **end**

$$\kappa_F(\xi) = \lim_{\varepsilon \rightarrow 0} \sup_{\|\delta\xi\| \leq \varepsilon} \frac{\|F(\xi + \delta\xi) - F(\xi)\| / \|\delta\xi\|}{\|F(\xi)\| / \|\xi\|}$$

Relative condition number

$$\mathcal{E}(\xi, \delta\xi) = \left\{ x \in \mathbb{R}^2_+ \mid \frac{|\xi_1 - x_1|^2}{\delta\xi_1^2} + \frac{|\xi_2 - x_2|^2}{\delta\xi_2^2} < 1 \right\}$$

2D ellipse

$$N_F(\xi) = \inf_{\varepsilon > 0} \{ \kappa_F(x) \in \mathbb{R} \mid \forall x \in \mathcal{E}(\xi, \delta\xi), \|\delta\xi\| < \varepsilon \}$$

Least guaranteed sensitivity bound

$\begin{pmatrix} -1.372 & 613.4 & 8.111 & -0.01228 & 7.242 & 0.2304 & -1.163 & 198.4 \\ -613.4 & -1234 & -198.3 & 0.2579 & -233.3 & -7.376 & 37.27 & 556.1 \\ -8.111 & -198.3 & 72.1 & 0.145 & -150.4 & -4.655 & 23.65 & 512.1 \\ -1.163 & 198.3 & 233.3 & 0.2688 & -0.000131 & -12.27 & 0.5647 & -1.327 & 104.463 \\ -1.1209 & 4.2024 & 1.0244 & 0.2688 & -0.000131 & -4.655 & -435.7 & -483.1 & 1452 & -324.8 \\ 7.242 & 233.3 & 18.4 & 0.2688 & 23.68 & 483.1 & -10.25 & 94.98 & 16.64 \\ -0.2303 & -3.737 & -4.651 & 0.8474 & 483 & 4.143 & -1452 & 95 & -2903 & -84.06 \\ 1.161 & 37.27 & 23.64 & 0.8474 & 483 & 4.143 & -1452 & 95 & -2903 & -84.06 \\ -198.4 & 556.4 & 512.1 & 0.5376 & 524.8 & 16.65 & -84.06 & 95 & -2903 & -84.06 \end{pmatrix}$
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Continuous-time regulator for case study

TABLE I

HIGH NUMERICAL SENSITIVITY OF THE CHARACTERISTIC POLYNOMIAL OF \tilde{G}_n FOR THE SOLUTION $\xi_2^* = (4.2739 \times 10^{-6}, 1.02448 \times 10^{-10})$

Case	Parameter	Computed value
1	Ideal $\hat{z}_{1,2,3}$	0.999995726091887 < 1
2	Analytic a_1	-2.99987178275663
2	Analytic a_2	2.99974356606124
2	Analytic a_3	-0.999987178330461
2	Analytic \hat{z}_1	1.000005221571111 > 1
2	Analytic \hat{z}_2	0.999990978352276 + 0.000008223234708j
2	Analytic \hat{z}_3	0.999990978352276 - 0.000008223234708j
3	c2d a_1	-2.999987178275662
3	c2d a_2	2.99974356606122
3	c2d a_3	-0.999987178330461
3	c2d \hat{z}_1	0.999995723377957 < 1
3	c2d \hat{z}_2	0.999995723448852 + 0.00000002350638j
3	c2d \hat{z}_3	0.999995723448852 - 0.00000002350638j



GitHub: ACC paper and poster, source code