Quantization Aligned with LQR Lyapunov Geometry for Networked Control

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***Abstract*—TODO**

***Index Terms*—TODO**

I. Introduction

MULTI-agent systems (MAS) ...

II. Notations and Necessary Tools

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III. Problem Statement

We consider the discrete-time linear time-invariant (LTI) system

$$x_{k+1} = Ax_k + Bu_k, \qquad u_k = -K\hat{x}_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, (A, B) is stabilizable, and K is a stabilizing state-feedback gain. The controller receives a quantized state $\hat{x}_k = Q(x_k)$ transmitted over a rate-limited channel.

Objective: Given a fixed bit budget, design a quantization strategy $Q(\cdot)$ that minimizes the degradation of closed-loop performance relative to the ideal (unquantized) controller.

Key question: How should quantization resolution be distributed in the state space *given the closed-loop dynamics induced by the control law*?

IV. Closed-Loop Geometry and Performance Metric

Let $Q \succeq 0$ and $R \succ 0$ be given weighting matrices, and let $S \succ 0$ denote the solution of the discrete-time algebraic Riccati equation associated with (A, B, Q, R). The quadratic function

$$V(x) = x^\top Sx \tag{2}$$

has two fundamental interpretations:

- it is the optimal infinite-horizon LQR value function;
- it is a Lyapunov function for the closed-loop system $x_{k+1} = (A - BK)x_k$.

As a consequence, state estimation errors are not equally important in Euclidean coordinates. The control-relevant distortion induced by quantization is naturally measured by the S-metric

$$d_S(x, \hat{x}) = (x - \hat{x})^\top S(x - \hat{x}). \tag{3}$$

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Implication: Standard uniform or logarithmic quantizers, which allocate resolution based on Euclidean amplitude and axis-aligned coordinates, are generally mismatched to the closed-loop geometry encoded by S.

V. Proposed Idea: LQR-Shaped Quantization

Let $S = L^\top L$ be a Cholesky factorization of the Riccati matrix. Define the shaped coordinates

$$z = Lx. \tag{4}$$

In these coordinates, the value function becomes isotropic:

$$V(x) = x^\top Sx = \|z\|_2^2. \tag{5}$$

Design principle:

- transform the state using $z = Lx$;
- apply a quantizer that minimizes Euclidean distortion in z;
- reconstruct $\hat{x} = L^{-1}\hat{z}$ at the controller.

This construction yields a *control-aware quantizer* whose resolution is aligned with the LQR value geometry.

VI. Algorithm: LQR-Shaped Product K-Means

Offline training:

1) simulate closed-loop trajectories under ideal state feedback;
2) collect samples of shaped states $z_k = Lx_k$;
3) train independent one-dimensional K-means codebooks for each coordinate of z under a fixed bit budget.

Online operation:

$$\hat{z}_{k,i} = \arg\min_{c \in \mathcal{C}_i} |z_{k,i} - c|, \qquad \hat{x}_k = L^{-1}\hat{z}_k, \tag{6}$$

where $\mathcal{C}_i$ denotes the learned codebook for the ith shaped coordinate.

Baselines for comparison:

- uniform scalar quantization;
- logarithmic (μ-law) companding;
- unshaped (Euclidean) K-means quantization.

VII. Performance Analysis

Theorem 1 (Excess LQR cost induced by quantization): Let $A_c = A - BK$ be Schur and let $S \succ 0$ be the Riccati solution. Consider the quantized-feedback system $u_k = -K\hat{x}_k$ with quantization error $e_k = x_k - \hat{x}_k$. Then the excess LQR cost relative to the ideal controller satisfies

$$\Delta J = \sum_{k=0}^{\infty} e_k^\top \Xi e_k, \qquad \Xi \succeq 0, \tag{7}$$

and there exist constants $\underline{\alpha}, \bar{\alpha} > 0$ such that

$$\underline{\alpha} \sum_{k=0}^{\infty} \|e_k\|_S^2 \;\leq\; \Delta J \;\leq\; \bar{\alpha} \sum_{k=0}^{\infty} \|e_k\|_S^2. \tag{8}$$

Interpretation: Closed-loop performance degradation is governed by the accumulation of quantization errors measured in the S-metric. Reducing $\|e_k\|_S^2$ directly reduces excess LQR cost.

Corollary 1 (Uniform quantization): Uniform scalar quantization yields bounded Euclidean error but does not control distortion in the S-metric, resulting in conservative bounds on ΔJ proportional to $\lambda_{\max}(S)$.

Corollary 2 (μ-law companding): μ-law quantization increases resolution near the origin in each coordinate, but remains axis-aligned and does not account for state coupling encoded by S.

Corollary 3 (Cholesky shaping): In shaped coordinates $z = Lx$, Euclidean distortion equals S-metric distortion, i.e., $\|e_k\|_S^2 = \|e_k^z\|_2^2$.

Corollary 4 (Shaped K-means): Product K-means quantization in shaped coordinates minimizes an empirical approximation of $\mathbb{E}\|e_k\|_S^2$, and therefore reduces $\mathbb{E}\Delta J$ when training and operating distributions are matched.

VIII. Case Study

A two-dimensional unstable LTI plant with LQR feedback is considered under a fixed bit budget. Monte Carlo simulations demonstrate that LQR-shaped K-means quantization consistently achieves lower excess LQR cost than uniform, μ-law, and unshaped K-means quantizers. Geometric visualizations confirm alignment between quantization cells and Lyapunov level sets.

IX. Conclusion

This work shows that quantization should be designed in the geometry induced by the closed-loop control objective. Riccati-based shaping provides a principled metric, while data-driven quantization adapts resolution to the relevant state distribution without sacrificing stability guarantees. The approach naturally bridges optimal control, learning, and networked systems, making it well suited for L-CSS/CDC dissemination.