Moment Fitting with Phase–Type Distributions: Closed-Form Constructions and SCV Validity Ranges

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Abstract

This note compiles closed-form, ready-to-use constructions for fitting the first two moments (mean and squared coefficient of variation, SCV) with classical phase–type (PH) families: Erlang, hypoexponential, hyperexponential, Coxian, and general PH. For each family we give: (i) formulas for the mean, variance and SCV, (ii) closed-form parameters that match a target mean m and SCV c^2, and (iii) the range of c^2 values each construction can realize.

1 Preliminaries: PH notation and moments

A nonnegative random variable X is *phase–type* (PH) if it is the absorption time of a finite-state Markov chain with subgenerator $\mathbf{T}$ and initial distribution $\boldsymbol{\alpha}$ over the transient states. Let $\mathbf{1}$ be the vector of ones and $\mathbf{t} = -\mathbf{T}\mathbf{1}$ the exit rates. The Laplace–Stieltjes transform is

$$\phi_X(s) = \boldsymbol{\alpha}\,(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}, \tag{1}$$

and the moments exist for all orders with the closed form

$$\mathbb{E}[X^n] = n!\,\boldsymbol{\alpha}\,(-\mathbf{T})^{-n}\mathbf{1}, \quad n = 1, 2, \ldots \tag{2}$$

In particular, $m = \mathbb{E}[X] = \boldsymbol{\alpha}(-\mathbf{T})^{-1}\mathbf{1}$ and $\mathrm{Var}(X) = 2\,\boldsymbol{\alpha}(-\mathbf{T})^{-2}\mathbf{1} - m^2$; the squared coefficient of variation (SCV) is $c^2 = \mathrm{Var}(X)/m^2$.

Goal. Given a target (m, c^2), construct PH parameters in closed form (no numerical solves) that *exactly* match the first two moments, and state the range of c^2 covered by each family.

2 Erlang $E_k(\mu)$

The Erlang distribution is the sum of $k \in \mathbb{N}$ i.i.d. exponentials with rate $\mu > 0$.

$$\mathbb{E}[X] = \frac{k}{\mu}, \qquad \mathrm{Var}(X) = \frac{k}{\mu^2}, \qquad c^2 = \frac{1}{k}. \tag{3}$$

Closed-form fit. Exact two-moment matching is possible iff $1/c^2$ is an integer. Then set

$$k = \frac{1}{c^2} \in \mathbb{N}, \qquad \mu = \frac{k}{m}. \tag{4}$$

SCV validity. $c^2 = 1/k \in (0, 1]$. If $1/c^2 \notin \mathbb{N}$, pure Erlang cannot match (m, c^2) exactly; use a hypoexponential (below) for $c^2 < 1$ or a hyperexponential for $c^2 > 1$.

3 Hypoexponential (series of k exponentials)

Let $X = \sum_{i=1}^{k} Y_i$ with independent $Y_i \sim \mathrm{Exp}(\mu_i)$ (rates $\mu_i > 0$, not necessarily equal). Then

$$\mathbb{E}[X] = \sum_{i=1}^{k} \frac{1}{\mu_i}, \qquad \mathrm{Var}(X) = \sum_{i=1}^{k} \frac{1}{\mu_i^2}, \qquad c^2 = \frac{\sum_{i=1}^{k} \mu_i^{-2}}{\left(\sum_{i=1}^{k} \mu_i^{-1}\right)^2} \in \Big[\frac{1}{k}, 1\Big). \tag{5}$$

Two-phase case ($k = 2$): exact (m, c^2) match for $\frac{1}{2} \le c^2 < 1$

Let $m = \mathbb{E}[X]$ and $c^2 \in [1/2, 1)$. Define

$$s := \sqrt{2c^2 - 1} \in [0, 1), \qquad a = \frac{m}{2}(1 - s), \quad b = \frac{m}{2}(1 + s), \tag{6}$$

and set the rates $\mu_1 = 1/a$, $\mu_2 = 1/b$. Then $\mathbb{E}[X] = m$ and c^2 is matched exactly. (When $s = 0$ this reduces to Erlang-2 with $c^2 = \frac{1}{2}$.)

General $k \ge 2$: a closed-form (m, c^2) fit (all $c^2 \in [1/k, 1)$)

Pick any k with $k \ge \lceil 1/c^2 \rceil$; equivalently ensure $c^2 \ge 1/k$. Define

$$\delta := \sqrt{\frac{k\,c^2 - 1}{k - 1}} \in [0, 1), \qquad a = \frac{m}{k}(1 - \delta), \qquad b = \frac{m}{k}\big(1 + (k - 1)\delta\big). \tag{7}$$

Take $k - 1$ "fast" phases with mean a (rate $\mu_f = 1/a$) and one "slow" phase with mean b (rate $\mu_s = 1/b$). This construction *exactly* matches $\mathbb{E}[X] = m$ and c^2 for any $c^2 \in [1/k, 1)$. For $k = 2$ this reduces to the formula above with $s = \delta = \sqrt{2c^2 - 1}$.

SCV validity. For fixed k, the achievable range is $c^2 \in [1/k, 1)$ with the minimum $1/k$ at equal rates (Erlang-k) and $c^2 \uparrow 1$ as rates become more uneven.

4 Hyperexponential H_2 (mixture of two exponentials)

Let X be an exponential with rate μ_1 w.p. p and rate μ_2 w.p. $1 - p$. Then

$$\mathbb{E}[X] = \frac{p}{\mu_1} + \frac{1 - p}{\mu_2}, \tag{8}$$

$$\mathrm{Var}(X) = 2\Big(\frac{p}{\mu_1^2} + \frac{1 - p}{\mu_2^2}\Big) - \Big(\frac{p}{\mu_1} + \frac{1 - p}{\mu_2}\Big)^2, \tag{9}$$

$$c^2 > 1 \ \text{ (with no upper bound).} \tag{10}$$

Balanced Rates (BR): exact (m, c^2) for all $c^2 > 1$

Given target (m, c^2) with $c^2 > 1$, define

$$\gamma = \sqrt{\frac{c^2-1}{c^2+1}} \in (0,1), \qquad p = \frac{1+\gamma}{2}, \qquad \mu_1 = \frac{1+\gamma}{m}, \quad \mu_2 = \frac{1-\gamma}{m}. \tag{11}$$

Then $\mathbb{E}[X] = m$ and $c^2 = \dfrac{1+\gamma^2}{1-\gamma^2}$ equals the target. This closed form works for *every* $c^2 > 1$.

Balanced Means (BM): exact (m, c^2) for $1 < c^2 < 3$

Fix $p = \frac{1}{2}$ and let

$$\alpha = \sqrt{\frac{c^2-1}{2}} \in (0,1), \qquad \mu_1 = \frac{1+\alpha}{m(1-\alpha^2)}, \quad \mu_2 = \frac{1-\alpha}{m(1-\alpha^2)}. \tag{12}$$

Then $\mathbb{E}[X] = m$ and $c^2 = 1 + 2\alpha^2$; the restriction $\alpha < 1$ translates to $c^2 < 3$.

SCV validity. Hyperexponential mixtures realize any $c^2 > 1$; the BR form covers the full range; the BM form covers $1 < c^2 < 3$.

5 Coxian-2 (sequential with optional completion)

Start in phase 1 with rate μ_1. With probability $p \in (0,1]$ proceed to phase 2 (rate μ_2); otherwise absorb. One can write $X = T_1 + JT_2$ with $T_i \sim \mathrm{Exp}(\mu_i)$ independent and $J \sim \mathrm{Bernoulli}(p)$ independent.

$$\mathbb{E}[X] = \frac{1}{\mu_1} + \frac{p}{\mu_2}, \tag{13}$$

$$\mathrm{Var}(X) = \frac{1}{\mu_1^2} + \frac{p(2-p)}{\mu_2^2}, \tag{14}$$

$$c^2 = \frac{\mu_2^{-2} p(2-p) + \mu_1^{-2}}{\left(\mu_1^{-1} + p\,\mu_2^{-1}\right)^2} \in \Big[\frac{1}{2}, \infty\Big). \tag{15}$$

Closed-form fitting to (m, c^2)

Underdispersion $(\frac{1}{2} \le c^2 < 1)$**.** Set $p = 1$ and use the hypoexponential 2-phase fit:

$$s = \sqrt{2c^2-1}, \quad a = \tfrac{m}{2}(1-s), \quad b = \tfrac{m}{2}(1+s), \quad \mu_1 = \tfrac{1}{a}, \ \mu_2 = \tfrac{1}{b}. \tag{16}$$

Overdispersion $(c^2 \ge 1)$**.** Choose any $p \in (0,1]$ (smaller p gives larger c^2) and compute

$$b = \frac{m}{2}\left(1 + \sqrt{1 - \frac{2(1-c^2)}{p}}\right), \qquad a = m - p\,b, \qquad \mu_1 = \frac{1}{a}, \ \mu_2 = \frac{1}{b}. \tag{17}$$

This matches $\mathbb{E}[X] = m$ and the target c^2 exactly for all $c^2 \ge 1$. For $c^2 < 1$, feasibility requires $p \ge 2(1-c^2)$; in particular Coxian-2 *cannot* achieve $c^2 < \frac{1}{2}$ (more phases are needed).

6 General PH (acyclic)

For a k-phase acyclic PH one typically parameterizes with per-phase rates and continuation probabilities (e.g., a Coxian-k). The moment formula $\mathbb{E}[X^n] = n!\,\boldsymbol{\alpha}(-\mathbf{T})^{-n}\mathbf{1}$ is closed form, but an explicit closed-form mapping from (m, c^2) to all parameters is generally *not* available beyond the special families above. Practical recipes are:

- For $c^2 < 1$, pick the smallest k with $1/k \le c^2$ and use the hypoexponential construction with $k-1$ equal fast phases plus one slow phase.
- For $c^2 > 1$, use the H_2 Balanced Rates fit. For additional moment matching (e.g., also skewness), increase branches (hyper-3) or use Coxian-k and solve a small nonlinear system.

At-a-glance summary

Family	SCV range realized	Closed-form 2-moment fit
Erlang $E_k(\mu)$	$c^2 = 1/k$	$k = 1/c^2 \in \mathbb{N}$, $\mu = k/m$
Hypoexp (k in series)	$[1/k,\,1)$	$\delta = \sqrt{\frac{kc^2-1}{k-1}}$, $a = \frac{m}{k}(1-\delta)$ & $b = \frac{m}{k}(1+(k-1)\delta)$
Hyperexp H_2 (mixture)	$(1,\,\infty)$	BR: $\gamma = \sqrt{\frac{c^2-1}{c^2+1}}$, $p = \frac{1+\gamma}{2}$, $\mu_{1,2} = \frac{1\pm\gamma}{m}$
Coxian-2	$[1/2,\,\infty)$	$c^2 < 1$: set p=1 and use hypo-2; $c^2 \ge 1$: pick $p \in (0,1]$, then: $b = \frac{m}{2}\left(1+\sqrt{1-\frac{2(1-c^2)}{p}}\right)$, $a = m - pb$
General PH	$(0,\,\infty)$	Use hypo-k (for $c^2 < 1$) or hyper-2 (for $c^2 > 1$); higher k for extra moments

Notes

- All formulas assume $m > 0$ and $c^2 > 0$; logarithms are not required and all square roots above are taken over nonnegative radicands, which imposes the stated SCV ranges.
- For hyperexponential H_2, two common closed forms exist: Balanced Rates (valid $\forall c^2 > 1$) and Balanced Means (valid $1 < c^2 < 3$).
- For very small c^2 one must increase the number of serial phases k (since the minimum SCV with k phases in series is $1/k$).

Quick recipe (given m and c^2)

1. If $c^2 = 1/k$ for an integer k: use Erlang E_k with $\mu = k/m$.
2. If $c^2 < 1$: choose $k = \lceil 1/c^2 \rceil$ and use the hypoexponential k-phase construction.
3. If $c^2 > 1$: use hyperexponential H_2 with the Balanced Rates parameters.
4. If you want one family on both sides of 1: use Coxian-2; for $c^2 < 1$ set $p = 1$ (hypo-2), and for $c^2 \ge 1$ pick any $p \in (0,1]$ and compute (μ_1, μ_2) as above.