Direction

1. The exam is for 65 minutes.
2. The total score is 90. There are 8 problems.

1 Choose all correct statements.

pt 10

(a) If $A^T A$ and AA^T are both invertible then A is square.

(b) If P is the standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto a two dimensional subspace W, then the rank of $I - P$ is two.

(c) If A is a 3×3 matrix with two pivot positions, then the equation $Ax = 0$ has a nontrivial solution.

(d) If $\text{null}(A) = \{0\}$, then A is invertible.

(e) If A is a 3×4 matrix with $\text{nullity}(A) = 1$, the equation $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ has infinitely many solutions.

2 Choose all correct statements.

pt 10

(a) If A is diagonalizable, then the characteristic polynomial of A has n distinct roots.

(b) If A is diagonalizable, then so is A^k for every positive integer k.

(c) If the sum of the algebraic multiplicities of eigenvalues of A is n, then A is diagonalizable.

(d) If A is diagonalizable, then A is orthogonally diagonalizable.

(e) Let A be a symmetric and invertible matrix. If $\mathbf{x}^T A\mathbf{x}$ is a negative definite quadratic form, then so is $\mathbf{x}^T A^{-1}\mathbf{x}$.

3 Choose all correct statements.

pt 10

(a) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is a a diagonal matrix.

(b) If an invertible matrix A is diagonalizable, then A^{-1} is also diagonalizable.

(c) If A is a square matrix, then AA^T is orthogonally diagonalizable.

(d) Let A be a symmetric matrix. If there is a square matrix C such that $A = C^TC$, then A is positive definite.

(e) Let $U\Sigma V^T$ be the singular value decomposition of an $m \times n$ matrix A. Then we can find orthonormal bases of the fundamental spaces of A.

4 For positive integers m, n, let $M_{m\times n}(\mathbb{R})$ be the set of matrices of size $m \times n$ with entries in $\mathbb{R}$. Choose all correct statements.

pt 10

(a) Given a matrix $A \in M_{m\times n}(\mathbb{R})$. If both A^TA and AA^T are invertible, then $m = n$.

(b) Given a matrix $A \in M_{m\times n}(\mathbb{R})$. If $\mathrm{rank}(A) = \mathrm{rank}(A^T)$, then $m = n$.

(c) Given a matrix $A \in M_{m\times n}(\mathbb{R})$. The matrix A and AA^T have the same column space.

(d) There exists a matrix $A \in M_{5\times 3}(\mathbb{R})$ such that $\mathrm{tr}(A^TA) = -2021$.

(e) Given a matrix $A \in M_{3\times 5}(\mathbb{R})$, the column vectors of A are linearly dependent.

5 Let $A \in M_{8\times 9}(\mathbb{R})$ with $\mathrm{rank}(A) = 6$. Choose all true statements.

pt 10

(a) $\dim(\mathrm{col}(A)) = 6$.

(b) $\mathrm{nullity}(A) = 3$.

(c) $\dim(\mathrm{col}(A)^{\perp}) = 3$.

(d) $\mathrm{nullity}(A^T) = 3$.

(e) $\dim(\mathrm{null}(A^T)^{\perp}) = 6$.

6 pt 10

Let $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{C} = \{w_1, w_2, w_3\}$ be two bases of $\mathbb{R}^3$, where

$$\begin{aligned} v_1 &= (1,0,1), \quad v_2 = (0,1,1), \quad v_3 = (1,1,0), \\ w_1 &= (1,0,0), \quad w_2 = (1,1,0), \quad w_3 = (1,1,1). \end{aligned}$$

Let $\mathbf{x}$ be a vector in $\mathbb{R}^3$ whose coordinate with respect to $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = (1,2,3)$. If $[\mathbf{x}]_{\mathcal{C}} = (a,b,c)$, which of the following is $a+b+c$?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

7 pt 10

Find $a+b$ where $y = ax+b$ is the least square fit line to the following four points: (2, 1), (3, 1), (5, 3), (6, 4).

(a) $\frac{1}{8}$

(b) $\frac{2}{9}$

(c) $\frac{3}{20}$

(d) $\frac{7}{40}$

(e) $\frac{1}{4}$

8

pt 10

(a) (10 points, 2 points for each)

Let A be a 7×8 matrix given by

$$A = \begin{bmatrix} 1 & 8 & 0 & 0 & 0 & 0 & 0 & 50 \\ 2 & 9 & 0 & 0 & 0 & 0 & 44 & 51 \\ 3 & 10 & 0 & 0 & 0 & 0 & 45 & 52 \\ 4 & 11 & 0 & 0 & 0 & 0 & 46 & 53 \\ 5 & 12 & 0 & 0 & 0 & 0 & 47 & 54 \\ 6 & 13 & 0 & 0 & 0 & 0 & 48 & 55 \\ 7 & 0 & 0 & 0 & 0 & 0 & 49 & 56 \end{bmatrix}.$$

Write **True** for codes that generate the same matrix A as above, or **False** for codes that do not.

(a)
```
A = reshape(1:56, 7, 8);
A(14 <= A & 43 >= A) = 0;
```

(b)
```
A = reshape(1:56, 7, 8);
A = A .* ((14 > A) + (43 < A));
```

(c)
```
A(1:13) = 1:13;   A(44:56) = 44:56;
A = reshape(A, 7, 8);
```

(d)
```
A = reshape(1:56, 7, 8);
A = A(14 <= A) .* A(43 >= A);
```

(e)
```
A = zeros(7, 8);
A(:) = [1:13, zeros(1, 30), 44:56];
```

8
pt 10

(b) (10 potins)

Let A be an $m \times n$ matrix with full column rank. The following MATLAB function performs a Gram-Schmidt process and produces an $m \times n$ orthogonal matrix Q whose columns form an orthonormal basis for $\mathrm{col}(A)$. Fill in the blanks (1) - (5).

(In this problem, assume that the input A is a matrix that gives the same result for the both of classical Gram-Schmidt and the modified Gram-Schmidt.)

```
% ------- The following is the script file ‘GramSchmidt.m’. -------
___(1)___ Q = GramSchmidt(A)
[m, n] = size(A);

% Initialize the matrix Q as an m*n zero matrix.
Q = zeros(m, n);
for i = 1 : n
% v starts with a column of A.
v = ___(2)___;
for j = 1 : i-1
% Subtract orthogonal projections of v onto the subspaces
% spanned by the previously generated orthonormal vectors.
q = ___(3)___;
v = v - ___(4)___;
end
% Normalize v by its Euclidean norm.
Q(:, i) = v / ___(5)___;
end
```

(a) (1) `def`, (2) `A(i, :)`, (3) `Q(:, j)`, (4) `(q’ * A(:, i)) * q`, (5) `norm(v)`

(b) (1) `function`, (2) `A(:, i)`, (3) `Q(:, i)`, (4) `(q’ * v) * q`, (5) `norm(v)`

(c) (1) `def`, (2) `A(:, i)`, (3) `Q(:, i)`, (4) `(q’ * v) * q`, (5) `sqrt(sum(v^2))`

(d) (1) `function`, (2) `A(:, i)`, (3) `Q(:, j)`, (4) `(q’ * A(:, i)) * q`, (5) `norm(v)`

(e) (1) `function`, (2) `A(:, i)`, (3) `Q(:, j)`, (4) `(q’ * v) * q`, (5) `sqrt(sum(v^2))`