Trabalho extra de Cálculo 2

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Exercício 1: Sabe-se que:

$$\int \frac{1}{\cos^2 x}\,dx \;=\; \int 1+\tan^2 x\;dx$$

Usando:

$$u=\tan x \implies du = sec^2 x\;dx$$

Sabe-se que:

$$\sec^2 x = \tan^2 x + 1$$

Portanto:

$$du=\tan^2 x+1\;dx \implies du=u^2+1\;dx \implies dx=\frac{du}{u^2+1}$$

Substituindo na integral:

$$\begin{aligned}\int 1+\tan^2 x\;dx \;&=\; \int \frac{1+u^2}{u^2+1}\,du\\ &=\int du = u+C\\ Resposta: \;&=\; \tan x + C\end{aligned}$$

Exercício 2:

$$\int \frac{\sin x \cdot \cos x}{\sqrt{\cos 2x}}\, dx$$

Usando:

$$u = \cos 2x \implies du = -2 \sin 2x \; dx$$

Sabe-se que:

$$\sin 2x = 2 \sin x \cdot \cos x$$

Portanto:

$$du = -4 \sin x \cdot \cos x \; dx \implies -\frac{1}{4}\, du = \sin x \cdot \cos x \; dx$$

Substituindo na integral:

$$\begin{aligned}
\int \frac{\sin x \cdot \cos x}{\sqrt{\cos 2x}}\, dx &= -\frac{1}{4}\int \frac{du}{\sqrt{u}} \\
&= -\frac{1}{4}\int u^{-\frac{1}{2}}\, du \\
&= -\frac{1}{4}\left(2u^{\frac{1}{2}}\right) + C \\
&= -\frac{\sqrt{u}}{2} + C \\
\mathit{Resposta:} &= -\frac{\sqrt{\cos 2x}}{2} + C
\end{aligned}$$

Exercício 3:

$$\int \left(1-x^2\right)^2 \ dx$$

Pelo item A do exercício 3.4.7.1:

$$\begin{aligned}
\int \left(1-x^2\right)^2 \ dx &= x\left(1-x^2\right)^2 \ + 4\int x^2\left(1-x^2\right) dx \\
&= x\left(1-x^2\right)^2 \ + 4\int \left(x^2-x^4\right) dx \\
&= x\left(1-x^2\right)^2 \ + 4\left(\frac{x^3}{3}-\frac{x^5}{5}\right)+C \\
Resposta: &= x\left(1-x^2\right)^2 \ + \frac{4x^3}{3}-\frac{4x^5}{5}+C
\end{aligned}$$

Exercício 4: Sabe-se que:

$$\int \left(1-x^2\right) \ dx = \int \cos^5 u \ du$$

Pelo item A do exercício 3.4.7.3:

$$\int \cos^5 u \ du = \frac{\sin u \cdot \cos^4 u}{5}+\frac{4}{5}\int \cos^3 u \ du$$

Aplicando novamente:

$$\begin{aligned}
\int \cos^5 u \ du &= \frac{\sin u \cdot \cos^4 u}{5}+\frac{4}{5}\left(\frac{\sin u \cdot \cos^2 u}{3}+\frac{2}{3}\int \cos u \ du\right) \\
&= \frac{\sin u \cdot \cos^4 u}{5}+\frac{4 \ sin \ u \cdot \cos^2 u}{15}+\frac{2}{3}\sin u + C \\
Resposta: &= \frac{x \cdot \left(1-x^2\right)^2}{5}+\frac{4x\left(1-x^2\right)}{15}+\frac{2}{3}x+C
\end{aligned}$$

Exercício 5: Sabe-se que:

$$\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx = \int \left(1 - \sin^2 x\right)^2 \cdot \cos x \, dx$$

Usando:

$$u = \sin x \quad e \quad du = \cos x \, dx$$

Portanto:

$$\begin{aligned} \int \left(1 - \sin^2 x\right)^2 \cdot \cos x \, dx \ &= \ \int \left(1 - u^2\right)^2 \ du \\ = \ \int \left(1 - 2u^2 + u^4\right) \ du &= u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\ Resposta := \ \sin x &- \frac{2}{3} \sin^3 x + \frac{sen^5 x}{5} + C \end{aligned}$$

Exercício 6:

$$\int \cos^3 x \, dx$$

Pelo item A do exercício 3.4.7.3:

$$\begin{aligned} \int \cos^3 x \, dx \ &= \ \frac{\sin x \cdot \cos^2 x}{3} + \frac{2}{3} \int \cos x \, dx \\ Resposta : \ &= \ \frac{\sin x \cdot \cos^2 x}{3} + \frac{2}{3} \sin x + C \end{aligned}$$

Exercício 7: Sabe-se que:

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int \left(1 - \sin^2 x\right) \cdot \cos x \, dx$$

Pela substituição simples:

$$u = \sin x \quad e \quad du = \cos x \, dx$$

Portanto:

$$\begin{aligned} \int \left(1 - \sin^2 x\right) \cos x \, dx &= \int \left(1 - u^2\right) du \\ &= u - \frac{u^3}{3} + C \\ Resposta: &= \sin x - \frac{sen^3 x}{3} + C \end{aligned}$$

Exercício 8:

$$\int \frac{1}{\left(1 + x^2\right)^2} \, dx$$

Pelo item A do exercício 34.7.2:

$$\begin{aligned} \int \frac{1}{\left(1 + x^2\right)^2} \, dx &= \frac{1}{2}\left(\frac{x}{1 + x^2}\right) + \frac{1}{2} \int \frac{dx}{1 + x^2} \\ Resposta: &= \frac{x}{2 + 2x^2} + \frac{1}{2} \arctan x + C \end{aligned}$$

Exercício 9: Sabe-se que:

$$\begin{aligned}
\int \frac{1}{\left(1+x^2\right)^2}\,dx &= \int \left(\frac{1}{1+x^2} - \frac{x^2}{\left(1+x^2\right)^2}\right)\,dx \\
&= \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{\left(1+x^2\right)^2}\,dx \\
&= \arctan x - \int \frac{x^2}{\left(1+x^2\right)^2}\,dx
\end{aligned}$$

Calculando por partes, usando:

$$\begin{aligned}
&u = x \quad e \quad du = dx \\
&dv = \frac{x}{\left(1+x^2\right)^2} \qquad v = -\frac{1}{2\left(1+x^2\right)}
\end{aligned}$$

Portanto:

$$\begin{aligned}
\int \frac{1}{\left(1+x^2\right)^2}\,dx &= \arctan x - \left(\frac{-x}{2\left(1+x^2\right)} + \frac{1}{2}\int \frac{dx}{1+x^2}\right) \\
&= \arctan x + \frac{x}{2\left(1+x^2\right)} - \frac{1}{2}\int \frac{dx}{1+x^2} \\
&= \arctan x + \frac{x}{2\left(1+x^2\right)} - \frac{1}{2}\arctan x + C \\
\mathit{Resposta}: &= \frac{1}{2}\arctan x + \frac{x}{2\left(1+x^2\right)} + C
\end{aligned}$$

Exercício 10:

$$\int x^3\sqrt{1+x^2}\;dx$$

Usando:

$$u = x^2+1 \implies du = 2x\;dx \implies \frac{du}{2} = x\;dx$$
$$u = x^2+1 \implies x^2 = u-1$$

Substituindo:

$$\int x^3\sqrt{1+x^2}\;dx = \int x^2\sqrt{1+x^2}\cdot x\;dx = \int (u-1)\cdot\sqrt{u}\;\;\frac{du}{2}$$
$$= \frac{1}{2}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{1}{2}\left(\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right) + C = \frac{u^{\frac{5}{2}}}{5} - \frac{u^{\frac{3}{2}}}{3} + C$$
$$Resposta := \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + C$$

Exercício 11:

$$\frac{1}{2}\int x^2\cdot 2x\;\sqrt{1+x^2}\;dx$$

Aplicando a integral por partes:

$$u = x^2 \quad e \quad du = 2x\;dx$$

$$dv = 2x\sqrt{1+x^2}\;dx \quad e \quad v = \frac{2}{3}\sqrt{(x^2+1)^3}$$

Tem-se:

$$\int x^2\cdot 2x\cdot\sqrt{1+x^2}\;dx = \frac{2}{3}x^2\sqrt{(x^2+1)^3} - \frac{2}{3}\int 2x\sqrt{(x^2+1)^3}\;dx$$

Aplicando novamente a integral por partes:

$$u = 2 \quad e \quad du = 0$$

$$dv = x\sqrt{(x^2+1)^3}\;dx \quad e \quad v = \frac{\sqrt{(x^2+1)^5}}{5}$$

Tem-se:

$$\begin{aligned}\int x^2 \cdot 2x\sqrt{1+x^2}\;dx &= \frac{2}{3}x^2\sqrt{(x^2+1)^3} - \frac{2}{3}\left(\frac{2}{5}\sqrt{(x^2+1)^5} - 0\right)\\ &= \frac{2}{3}x^2\sqrt{(x^2+1)^3} - \frac{4}{15}\sqrt{(x^2+1)^5}\end{aligned}$$

Portanto:

$$Resposta: \frac{1}{2}\int x^2 \cdot 2x \cdot \sqrt{1+x^2}\;dx = \frac{x^2\sqrt{(x^2+1)^3}}{3} - \frac{2}{15}\sqrt{(x^2+1)^5} + C$$

Exercício 12: Sabe-se que:

$$\int \frac{1}{\cos^3 x}\;dx = \int \frac{1}{\cos x} \cdot \frac{1}{cos^2 x}\;dx$$

Aplicando a integral por partes:

$$u = \frac{1}{\cos x} \quad e \quad du = \sec x \cdot \tan x\;dx$$

$$dv = \frac{1}{\cos^2 x}\;dx \quad e \quad v = \tan x$$

Então:

$$\begin{aligned}\int \frac{1}{\cos^3 x}\,dx &= \frac{\tan x}{\cos x} - \int \sec x \cdot \tan^2 x\,dx \\ &= \frac{\tan x}{\cos x} - \int \sec x \cdot \left(\sec^2 x - 1\right)\,dx \\ &= \frac{\tan x}{\cos x} - \int \sec^3 x\,dx + \int \frac{1}{\cos x}\,dx \\ &= \frac{\tan x}{\cos x} + \log\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}} - \int \sec^3 x\,dx\end{aligned}$$

Tem-se:

$$2\int \frac{1}{\cos^3 x}\,dx = \frac{\tan x}{\cos x} + \log\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}} + C$$

Portanto:

$$Resposta: \int \frac{1}{\cos^3 x}\,dx = \frac{1}{2}\left(\frac{\tan x}{\cos x} + \log\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\right) + C$$

Exercício 13: Sabe-se que:

$$\int \frac{1}{\cos^4 x}\,dx = \int \frac{1}{\cos^2 x} \cdot \int \frac{1}{\cos^2 x}\,dx$$

Aplicando a integral por partes:

$$u = \frac{1}{\cos^2 x} \quad e \quad du = 2\sec^2 x \cdot \tan x$$

$$dv = \frac{1}{\cos^2 x}\,dx \quad e \quad v = \tan x$$

Então:

$$\int \frac{1}{\cos^4 x}\ dx = \frac{\tan x}{\cos^2 x} - 2\int \sec^2 x \tan^2 x\ dx$$

Usando:

$$w = \tan x \quad e \quad dw = \sec^2 x\ dx$$

Portanto:

$$\begin{aligned}
\int \frac{1}{\cos^4 x}\ dx &= \frac{\tan x}{\cos^2 x} - 2\int w^2\ dw \\
&= \frac{\tan x}{\cos^2 x} - \frac{2}{3}w^3 + C \\
Resposta: &= \frac{\tan x}{\cos^2 x} - \frac{2}{3}\tan^3 x + C
\end{aligned}$$

Exercício 14: Sabe-se que:

$$\begin{aligned}
\int \frac{1}{\cos^4 x}\ dx = \int \sec^4 x\ dx = & \int \sec^2 x \cdot \sec^2 x\ dx \\
= & \int \left(\tan^2 x + 1\right)\sec^2 x\ dx \\
= & \int \tan^2 x \cdot \sec^2 x\ dx + \int \sec^2 x\ dx
\end{aligned}$$

Usando:

$$u = \tan x \quad e \quad du = \sec^2 x\ dx$$

Portanto:

$$\begin{aligned}
\int \frac{1}{\cos^4 x}\ dx &= \int u^2\ du + \tan x \\
&= \frac{u^3}{3} + \tan x + C \\
Resposta: &= \frac{\tan^3 x}{3} + \tan x + C
\end{aligned}$$

Exercício 15:

$$\begin{aligned}\int \frac{1}{\sin^4 x}\, dx = \int \csc^4 x\, dx &= \int \csc^2 x \cdot \csc^2 x\, dx \\ &= \int \left(\cot^2 x + 1\right) \cdot \csc^2 x\, dx \\ &= \cot^2 x \cdot \csc^2 x\, dx + \int \csc^2 x\, dx\end{aligned}$$

Usando:

$$u = \cot x \quad e \quad du = \csc^2 x\, dx$$

Portanto:

$$\int \frac{1}{\sin^4 x}\, dx = -\int u^2\, du - \cot x = -\frac{u^3}{3} - cotx + C$$

$$Resposta : -\frac{\cot^3 x}{3} - \cot x + C$$

Exercício 16: Sabe-se que:

$$\int \frac{1}{\sin^4 x}\, dx = \int \frac{1}{\cos^4\left(\frac{\pi}{2} - x\right)}\, dx$$

Usando:

$$u = \frac{\pi}{2} - x \quad e \quad du = -dx$$

Portanto (consultando o exercício 14):

$$\int \frac{1}{\sin^4 x}\, dx = -\int \frac{1}{\cos^4 u}\, du = -\left(\frac{\tan^3 u}{3} + \tan u\right) + C$$

$$Resposta := -\frac{\tan^3\left(\frac{\pi}{2} - x\right)}{3} + \tan\left(\frac{\pi}{2} - x\right) + C$$

Exercício 17: Sabe-se que:

$$\int \frac{\ln(\cos x)}{\cos^2 x}\,dx = \tan x \cdot \ln(\cos x) + \int \tan^2 x\,dx$$

Usando:

$$u = \tan x \implies x = \arctan u \implies dx = \frac{1}{u^2+1}\,du$$

Então:

$$\int \frac{\ln(\cos x)}{\cos^2 x}\,dx = \tan x \cdot \ln(\cos x) + \int \frac{u^2}{u^2+1}\,du$$

Usando a divisão de polinômios:

$$\begin{aligned}
\int \frac{\ln(\cos x)}{\cos^2 x}\,dx &= \tan x \cdot \ln(\cos x) + \int \left(1 - \frac{1}{u^2+1}\right)\,du \\
&= \tan x \cdot \ln(\cos x) + \int du - \int \frac{1}{u^2+1}\,du \\
&= \tan x \cdot \ln(\cos x) + u - \arctan u + C \\
\textit{Resposta}: &= \tan x \cdot \ln(\cos x) + \tan x - x + C
\end{aligned}$$

Exercício 18:

$$\begin{aligned}
\int \frac{\ln(\cos x)}{\cos^2 x}\,dx = \tan x \cdot \ln(\cos x) + \int \tan^2 x\,dx & \\
&= \tan x \cdot \ln(\cos x) + \int \frac{\sin^2 x}{\cos^2 x}\,dx \\
&= \tan x \cdot \ln(\cos x) + \int \frac{1-\cos^2 x}{\cos^2 x}\,dx \\
&= \tan x \cdot \ln(\cos x) + \int \frac{1}{\cos^2 x}\,dx - \int dx \\
Resposta: &= \tan x \cdot \ln(\cos x) + \tan x - x + C
\end{aligned}$$

Exercício 19: Usando a substituição

$$u = 1 - x^2 \implies du = -2x\,dx \implies -\frac{du}{2} = x\,dx$$

resulta:

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int (1-u)u^{1/2}\,du \\
&= -\frac{1}{2}\int \left(u^{1/2} - u^{3/2}\right)\,du \\
&= -\frac{1}{2}\left(\frac{2}{3}\,u^{3/2} - \frac{2}{5}u^{5/2}\right) \\
&= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3} \\
&= \frac{(1-x^2)^{5/2}}{5} - \frac{(1-x^2)^{3/2}}{3}.
\end{aligned}$$

Exercício 20: Usando a substituição

$$x = \sin u \implies dx = \cos u \, du$$

tem-se:

$$\int x^3 \sqrt{1-x^2}\, dx \;=\; \int \sin^3 u \cdot \cos u \cdot \cos u \, du \;=\; \int \sin^3 u \cdot \cos^2 u \, du. \qquad (1)$$

Considere f como

$$f(u) = \cos u \implies f'(u) = -\sin u$$

e a função g dada por:

$$g'(u) = \sin^3 u \cdot \cos u \implies g(u) = \frac{\sin^4 u}{4}.$$

Logo, integrando por partes tem-se

$$\begin{aligned} \int \sin^3 u \cdot \cos u \cdot \cos u \, du \;&=\; \frac{\sin^4 u}{4} \cos u \;+\; \frac{1}{4} \int \sin^5 u \, du \\ &=\; \frac{\sin^4 u}{4} \cos u \;+\; \frac{1}{4} \int (1-\cos^2 u) \sin^3 u \, du, \end{aligned}$$

de onde segue que:

$$5 \int \sin^3 u \cdot \cos^2 u \, du \;=\; \sin^4 u \cdot \cos u \;+\; \int \sin^3 u \, du. \qquad (2)$$

Agora, considere f como

$$f(u) = \sin^2 u \implies f'(u) = 2 \sin u \cdot \cos u$$

e a função g dada por:

$$g'(u) = \sin u \implies g(u) = -\cos u.$$

Logo, integrando por partes tem-se

$$\begin{aligned} \int \sin^3 u \, du \;=\; \int \sin^2 u \cdot \sin u \, du \;&=\; -\sin^2 u \cdot \cos u \;+\; 2 \int \sin u \cdot \cos^2 u \, du \\ &=\; -\sin^2 u \cdot \cos u \;+\; 2 \int (1-\sin^2 u) \sin u \, du, \end{aligned}$$

de onde segue que:

$$
\begin{aligned}
3\int \sin^3 u\, du \;&=\; -\sin^2 u \cdot \cos u \;+\; 2\int \sin u\, du \\
&=\; -\sin^2 u \cdot \cos u \;-\; 2\cos u. \quad (3)
\end{aligned}
$$

Finalmente, combinando as identidades (1), (2) e (3) resulta:

$$
15\int x^3\sqrt{1-x^2}\, dx \;=\; 3\sin^4 u \cdot \cos u \;-\; \left(\sin^2 u \cdot \cos u \;+\; 2\cos u\right).
$$

O último termo acima pode ser expressado em função do cosseno observando

$$
\begin{aligned}
\sin^4 u \cdot \cos u \;&=\; (1-\cos^2 u)^2 \cos u \\
&=\; (1 \;-\; 2\cos^2 u \;+\; \cos^4 u)\cos u \;=\; \cos u \;-\; 2\cos^3 u \;+\; \cos^5 u,
\end{aligned}
$$

como também:

$$
\begin{aligned}
&\sin^2 u \cdot \cos u \;+\; 2\cos u \\
&\qquad =\; (1-\cos^2 u)\cos u \;+\; 2\cos u \;=\; 3\cos u \;-\; \cos^3 u.
\end{aligned}
$$

Logo, tem-se

$$
15\int x^3\sqrt{1-x^2}\, dx \;=\; -5\cos^3 u \;+\; 3\cos^5 u
$$

de onde resulta:

$$
\int x^3\sqrt{1-x^2}\, dx \;=\; \frac{\cos^5 u}{5} \;-\; \frac{\cos^3 u}{3} \;=\; \frac{(1-x^2)^{5/2}}{5} \;-\; \frac{(1-x^2)^{3/2}}{3}.
$$