

Ejercicio 1

$$\bullet P(X=k) = \binom{12}{k} (0.2)^k (0.8)^{12-k}, \quad k = 0, 1, 2, \dots, 12$$

A variables aleatorias discretas, en concreto la expresión es una distribución binomial con $n=12$ y $p=0.2$ ($\text{Bin}(12, 0.2)$).

$$\mu_X = E(X) = \sum_{k=1}^{12} k \cdot \binom{12}{k} (0.2)^k (0.8)^{12-k} = 12 \cdot 0.2 = \frac{12}{5} = 2.4$$

$$\text{Var}(X) = \sum_{k=1}^{12} (k - \mu_X)^2 \cdot P(X=k) = 12 \cdot 0.2 \cdot 0.8 = \frac{48}{25} = 1.92$$

$$\bullet P(X=k) = \exp(-0.3) \cdot \frac{(0.3)^k}{k!}, \quad k = 0, 1, 2, \dots$$

A variables aleatorias discretas, en concreto es una distribución de Poisson con parámetro $\lambda = 0.3$.

$$\mu_X = E(X) = \sum_{k=1}^{\infty} k \cdot \exp(-0.3) \cdot \frac{(0.3)^k}{k!} = 0.3$$

$$\text{Var}(X) = \sum_{k=1}^{\infty} (k - \mu_X)^2 \cdot P(X=k) = 0.3$$

$$\bullet f_X(x) = \frac{1}{\sqrt{2\pi} (0.3)} \cdot \exp\left(-\frac{(x-3)^2}{2(0.09)}\right), \quad x \in \mathbb{R}$$

A variables aleatorias continuas, distribución normal de parámetros $\mu = 3$ y $\sigma^2 = 0.3$ ($N(3, 0.09)$).

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x) dx = 3 = \mu$$

$$\text{Var}(X) = \int_{-\infty}^{+\infty} (x - \mu_X)^2 \cdot f_X(x) dx = \sigma^2 = 0.09$$

$$f_X(x) = \begin{cases} 1/3 & , \text{ si } x \in (-1, 2) \\ 0 & , \text{ en otro caso} \end{cases}$$

A variables aleatorias continuas, distribución uniforme.

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x) dx = \int_{-1}^2 x \cdot \frac{1}{3} dx = \frac{1}{3} \cdot \frac{x^2}{2} \Big|_{x=-1}^{x=2} =$$

$$= \frac{1}{6} (4 - 1) = \frac{1}{2}$$

$$\text{Var}(X) = \int_{-\infty}^{+\infty} (x - \mu_X)^2 \cdot f_X(x) dx = \int_{-1}^2 (x - \frac{1}{2})^2 \cdot \frac{1}{3} dx$$

$$= \frac{1}{3} \frac{(x - \frac{1}{2})^3}{3} \Big|_{x=-1}^{x=2} = \frac{1}{9} \cdot \left[(2 - \frac{1}{2})^3 - (-1 - \frac{1}{2})^3 \right] = \frac{3}{4}$$

Ejercicio 2.

$$Y_t = X_t - t X_1, \quad 0 \leq t \leq 1$$

Se define como el 'proceso' puente browniano.

$$\begin{aligned} \mu_Y(t) &= E(Y_t) = E(X_t - t X_1) = E(X_t) - E(t X_1) \\ &= E(X_t) - t \cdot E(X_1) = 0 - 0 = 0 \quad (X_t \sim N(0, t), X_1 \sim N(0, 1)) \end{aligned}$$

$$\begin{aligned} C_Y(t, s) &= \text{Cov}(Y_t, Y_s) = E(Y_t Y_s) - E(Y_t) \cdot E(Y_s) = \\ &= E((X_t - t X_1)(X_s - s X_1)) = E(X_t X_s - s X_t X_1 - t X_s X_1 \\ &\quad + ts X_1^2) = E(X_t X_s) - s E(X_t X_1) - t E(X_s X_1) + ts E(X_1^2) \\ &= \min(s, t) - st - st + st = \min(t, s) - st. \end{aligned}$$

- $Y_t = 6t + 0.3X_t, t \geq 0$

Se define como browniano con deriva con $\sigma = 0.3$ y $\mu = 6$.

$$\mu_Y(t) = E(Y_t) = E(6t + 0.3X_t) = 6t + 0.3 \cdot E(X_t) = 6t, \quad t \geq 0$$

$$\begin{aligned} C_Y(t) &= \text{cov}(Y_t, Y_s) = E(Y_t Y_s) - E(Y_t) E(Y_s) = \\ &= E((6t + 0.3X_t)(6s + 0.3X_s)) - 36st = \\ &= 36ts + 6t \cdot 0.3 \cdot E(X_s) + 0.3 \cdot 6s \cdot E(X_t) + 0.09 E(X_t X_s) \\ &\quad - 36st = 0.09 \cdot \min(t, s), \quad s, t \geq 0 \end{aligned}$$

Ejercicio 3

- $Y_t = X_t - tX_1, 0 \leq t \leq 1$

Y se dice adaptado al browniano X si está adaptado a la filtración natural del browniano $(\mathcal{F}_t) = (\sigma(X_s, s \leq t))$.

Cuando $t < 1$, X_1 requiere información futura, por lo tanto Y_t no está adaptado al browniano.

- $Y_t = 6t + 0.3X_t, t \geq 0$

Y_t se puede escribir como función de t y X_t de la siguiente forma:

$$Y_t = f(t, X_t) = 6t + 0.3X_t, t \geq 0$$

Luego, está adaptado al browniano.

- $Y_t = \exp(X_t)$

De la misma forma, tomando $f(t, x) = \exp(x)$,

$Y_t = f(t, X_t) = \exp(X_t)$ y está adaptado al browniano X

- $Y_t = \cos(X_{t+1} - X_t)$

En este caso, el término X_{t+1} requiere información futura y así $\nabla(Y_t)$ no está contenido en \mathcal{F}_t para todo $t \geq 0$. Así, Y_t no está adaptado al browniano. ($t+1 > t$)

- $Y_t = \min_{0 \leq s \leq 3t} X_s$

No está adaptado al browniano. En el caso que $3t > t$, Y_t requiere información futura.

- $Y_t = \max_{0 \leq s \leq t/3} X_s$

Independientemente de cual sea el máximo, el intervalo $[0, t/3] \subset [0, t]$, por lo tanto $\nabla(Y_t) \subset \mathcal{F}_t$ para todo $t \geq 0$ y está adaptado al browniano.

Ejercicio 4

$$\bullet Y = (X_t^4 - 4tX_t, t \geq 0)$$

Si tomamos la función $f(t, x) = x^4 - 4tx$ y definimos

$$Y_t = f(t, X_t), \text{ veamos que } \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial f}{\partial t} = -4x, \quad \frac{\partial f}{\partial x} = 4x^3 - 4t, \quad \frac{\partial^2 f}{\partial x^2} = 12x^2$$

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = -4x + \frac{1}{2} \cdot 12x^2 = -4x + 6x^2 \neq 0 \text{ para algún } x \in \mathbb{R}$$

Por lo tanto, no es martingala.

$$\bullet Y = (\exp(X_t), t \geq 0)$$

De forma análoga, tomamos $f(t, X_t) = \exp(X_t) = Y_t$.

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \exp(x) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = \frac{1}{2} \exp(x) \neq 0 \text{ para algún } \forall x \in \mathbb{R}$$

No es martingala.

$$\bullet Y = (\exp(\sigma X_t - 0.5\sigma^2 \cdot t), t \geq 0), \quad \sigma > 0.$$

$$f(t, x) = \exp(\sigma x - \frac{1}{2}\sigma^2 \cdot t) = Y_t, \quad \sigma > 0$$

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 \exp(\sigma x - \frac{1}{2}\sigma^2 \cdot t)$$

$$\frac{\partial^2 f}{\partial x^2} = \sigma^2 \exp(\sigma x - \frac{1}{2}\sigma^2 \cdot t)$$

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = -\frac{1}{2}\sigma^2 \exp(\sigma x - \frac{1}{2}\sigma^2 t) + \frac{1}{2}\sigma^2 \exp(\sigma x - \frac{1}{2}\sigma^2 t) = 0$$

\Rightarrow es martingala.

• un puente browniano, $Y_t = X_t - tX_1$, $0 \leq t \leq 1$

En el ejercicio 3 hemos visto que un puente browniano no está adaptado a la filtración del browniano F , por lo que no es martingale.

Ejercicio 6

• $Y_t = X_t^3$

Por el lema de Ito para $f(x) = x^3$, con $Y_t = f(X_t) = X_t^3$,

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) dX_u + \frac{1}{2} \int_s^t f''(X_u) du$$

$f'(x) = 3x^2$, $f''(x) = 6x$. De esta forma,

$$X_t^3 - X_s^3 = \int_s^t 3X_u^2 dX_u + \int_s^t 3X_u du$$

En forma diferencial,

$$dY_t = 3X_u du + 3X_u^2 dX_u$$

Por otra parte, $X_t = Y_t^{1/3}$, luego $\alpha(t, Y_t) = 3Y_t^{1/3}$

y $\beta(t, Y_t) = 3Y_t^{2/3}$.

$$\bullet Y_t = \exp(-0.3t) X_t$$

Por el lema de Ito para $f(t, X_t) = \exp(-0.3t) X_t = Y_t$,
con $f(t, x) = \exp(-0.3t) \cdot x$ se obtiene:

$$f_t(t, x) = -0.3 x \cdot \exp(-0.3t)$$

$$f_x(t, x) = \exp(-0.3t)$$

$$f_{xx}(t, x) = 0$$

$$f(t, X_t) - f(s, X_s) = \int_s^t -0.3 X_u \exp(-0.3u) du + \int_s^t \exp(-0.3u) dX_u$$

$$s=0, \quad f(t, X_t) = \int_0^t -0.3 X_u \exp(-0.3u) du + \int_0^t \exp(-0.3u) dX_u$$

$$dY_t = -0.3 X_t \exp(-0.3t) dt + \exp(-0.3t) dX_t$$

$$dY_t = -0.3 Y_t dt + \exp(-0.3t) dX_t$$

De esta forma, $\alpha(t, Y_t) = -0.3 Y_t$ y $\beta(t, Y_t) = \exp(-0.3t)$

$$\bullet Y_t = (X_t^1) (X_t^2)^3, \quad X_t^i \text{ brownianos para } i=1, 2$$

Aplicamos el lema de Ito para función de 2 procesos
de Ito, con $f(t, X_t^1, X_t^2) = (X_t^1) (X_t^2)^3$, es decir,

$$f(t, x^1, x^2) = x^1 \cdot (x^2)^3$$

$$f(t, x^1, x^2) = (x^1) \cdot (x^2)^3$$

$$f_t = 0, \quad f_{x^1} = (x^2)^3, \quad f_{x^1 x^1} = 0, \quad f_{x^1 x^2} = 3(x^2)^2$$

$$f_{x^2} = (x^1) \cdot 3(x^2)^2, \quad f_{x^2 x^2} = 6(x^1) \cdot (x^2)$$

$$(X_t^1)(X_t^2)^3 - (X_s^1)(X_s^2)^3 = \int_s^t \left(\frac{1}{2} \cdot 6 (\beta_u^2)^2 \cdot (X_u^1)(X_u^2) + 3 \beta_u^1 \beta_u^2 (X_u^2)^2 \right) du + \int_s^t (X_u^2)^3 dX_u^1 + \int_s^t 3(X_u^1)(X_u^2)^2 dX_u^2$$

$$d(Y_t) = 3(\beta_u^2)^2 (X_t^1)(X_t^2) + 3\beta_u^1 \beta_u^2 (X_t^2)^2 dt + (X_t^2)^3 d(X_t^1) + 3(X_t^1)(X_t^2)^2 d(X_t^2)$$

$$= 3(\beta_u^2)^2 (X_t^1)(X_t^2) + 3\beta_u^1 \beta_u^2 (X_t^2)^2 dt + (X_t^2)^3 \cdot (\alpha_t^1 dt + \beta_t^1 dX_t) + 3(X_t^1)(X_t^2)^2 (\alpha_t^2 dt + \beta_t^2 dX_t)$$

$$= (3(\beta_u^2)^2 (X_t^1)(X_t^2) + 3\beta_u^1 \beta_u^2 (X_t^2)^2 + (X_t^2)^3 \alpha_t^1 + 3(X_t^1)(X_t^2)^2 \alpha_t^2) dt + (X_t^2)^3 \beta_t^1 + 3(X_t^1)(X_t^2)^2 \beta_t^2 dX_t$$

Hemos usado que $dX_t^i = \alpha_t^i dt + \beta_t^i dX_t$, $i = 1, 2$

Así,

$$\alpha(t, Y_t) = 3(\beta_u^2)^2 (X_t^1)(X_t^2) + 3\beta_u^1 \beta_u^2 (X_t^2)^2 + (X_t^2)^3 \alpha_t^1 + 3(X_t^1)(X_t^2)^2 \alpha_t^2$$

$$\beta(t, Y_t) = (X_t^2)^3 \beta_t^1 + 3(X_t^1)(X_t^2)^2 \beta_t^2$$

- $Y_t = \exp(at + bX_t)$, a, b constantes

Aplicamos el lema de Ito con $f(t, x) = \exp(at + bx)$,
de forma que $Y_t = f(t, X_t)$.

$$f_t = a \cdot \exp(at + bx)$$

$$f_x = b \cdot \exp(at + bx), \quad f_{xx} = b^2 \cdot \exp(at + bx)$$

$$f(t, X_t) - f(s, X_s) = \int_s^t a \cdot \exp(at + bX_u) + \frac{1}{2} b^2 \exp(at + bX_u) du \\ + \int_s^t b \cdot \exp(at + bX_u) dX_u \Rightarrow$$

$$dY_t = a \cdot \exp(at + bX_t) + \frac{1}{2} b^2 \exp(at + bX_t) dt + \\ + b \cdot \exp(at + bX_t) dX_t = \left(\left(a + \frac{1}{2} b^2 \right) \cdot \exp(at + bX_t) \right) dt \\ + b \cdot \exp(at + bX_t) dX_t$$

De esta forma, se verifica

$$dY_t = \alpha(t, Y_t) dt + \beta(t, Y_t) dX_t$$

con $\alpha(t, Y_t) = \left(a + \frac{1}{2} b^2 \right) Y_t$ y $\beta(t, Y_t) = b Y_t$.

Ejercicio 7

Sea $f(x) = x^6$ y $Y_t = f(X_t) = X_t^6$, por el lema de Ito tenemos:

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) dX_u + \frac{1}{2} \int_s^t f''(X_u) du$$

$f'(x) = 6x^5$, $f''(x) = 30x^4$. Tomando $s=0$,

$$X_t^6 = \int_0^t 6 X_u^5 dX_u + \frac{1}{2} \int_0^t 30 X_u^4 du \Rightarrow$$

$$X_t^6 = \int_0^t 6 X_u^5 dX_u + \int_0^t 15 X_u^4 du$$

Tomando la esperanza en ambas expresiones,

$$\begin{aligned} E(X_t^6) &= 6 \cdot E \left(\underbrace{\int_0^t X_u^5 dX_u}_{\text{"0 (int. de Ito)"}} \right) + 15 \cdot E \left(\int_0^t X_u^4 du \right) \\ &= 15 \cdot \int_0^t E(X_u^4) du \stackrel{(*)}{=} 15 \cdot 3 \cdot \int_0^t u^2 du = 15 \cdot 3 \cdot \frac{u^3}{3} \Big|_{u=0}^{u=t} = 15 \cdot t^3 \end{aligned}$$

(*) Hemos utilizado que $E(X_u^4) = 3u^2$, ya que $X_u \sim N(0, u)$

Ejercicio 8. X browniano

$$\int_0^T \exp(X_t) \circ dX_t = \int_0^T Y_t \circ dX_t$$

Si definimos $Y_t = f(X_t) = \exp(X_t)$, tenemos que $f'(X_t) = f(X_t)$, luego por la regla de la cadena se tiene lo siguiente:

$$\begin{aligned} \int_0^T \exp(X_t) \circ dX_t &= \int_0^T f'(X_t) \circ dX_t = f(X_T) - f(X_0) \\ &= \exp(X_T) - \exp(X_0) = \exp(X_T) - 1 \quad (\exp(X_0) = \exp(0) = 1) \end{aligned}$$

Ejercicio 9. μ y σ dependen de t .

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dX_t, \text{ So conocido}$$

Se trata de una EDE lineal con $\alpha_1(t) = \mu(t)$, $\alpha_2(t) = \beta_2(t) = 0$ y $\beta_1(t) = \sigma(t)$ de forma que

$$dS_t = (\alpha_1(t)S_t + \alpha_2(t))dt + (\beta_1(t)Y_t + \beta_2(t))dX_t$$

Es decir, es una EDE lineal homogénea con ruido multiplicativo.

Si $S_0 = 0$, entonces $S_t = 0$ para todo t . Supongamos que

$S_0 \neq 0$ y $S_t > 0$. Consideramos el proceso

$$F_t = \ln(S_t) = f(S_t) \Rightarrow S_t = \exp(F_t)$$

Aplicamos el lema de Ito con $f_t(t, x) = 0$, $f_x(t, x) = \frac{1}{x}$
 y $f_{xx}(t, x) = -\frac{1}{x^2}$, $f(t, x) = \ln(x)$ y tenemos en cuenta
 lo siguiente:

$$\begin{aligned} (dS_t)^2 &= (\mu(t)S_t dt + \sigma(t)S_t dX_t)^2 = (\mu(t)S_t dt)^2 + \\ &+ (\sigma(t)S_t dX_t)^2 + 2(\mu(t)S_t dt)(\sigma(t)S_t dX_t) = \\ &(\mu(t)S_t)^2 dt^2 + (\sigma(t)S_t)^2 (dX_t)^2 + 2\mu(t)\sigma(t)S_t^2 dt dX_t \end{aligned}$$

los términos de orden superior, dt^2 y $dt dX_t$ son despreciables
 y $(dX_t)^2 = dt$, por lo que se obtiene que

$$(dS_t)^2 = (\sigma(t)S_t)^2 dt$$

De esta forma, para la función $F_t = f(t, S_t)$ el lema
 de Ito se puede reescribir como

$$dF_t = f_t dt + f_x dS_t + \frac{1}{2} f_{xx} (dS_t)^2$$

Entonces,

$$\begin{aligned} dF_t &= \frac{1}{S_t} dS_t + \frac{1}{2} \cdot -\frac{1}{S_t^2} (dS_t)^2 = \\ &= \frac{1}{S_t} (\mu(t)S_t dt + \sigma(t)S_t dX_t) - \frac{1}{2} \cdot \frac{1}{S_t^2} (\sigma(t)S_t)^2 dt \\ &= \mu(t) dt + \sigma(t) dX_t - \frac{1}{2} (\sigma(t))^2 dt \\ &= \left(\mu(t) - \frac{1}{2} \sigma(t)^2 \right) dt + \sigma(t) dX_t \end{aligned}$$

Integrando a ambos lados de la igualdad y considerando S_0 , se obtiene:

$$\int_0^t d\ln(S_u) = \int_0^t (\mu(u) - \frac{1}{2}\sigma(u)^2) du + \int_0^t \sigma(u) dX_u \Rightarrow$$

$$\ln(S_t) - \ln(S_0) = \int_0^t (\mu(u) - \frac{1}{2}\sigma(u)^2) du + \int_0^t \sigma(u) dX_u \Rightarrow$$

$$\frac{S_t}{S_0} = \exp\left(\int_0^t (\mu(u) - \frac{1}{2}\sigma(u)^2) du + \int_0^t \sigma(u) dX_u\right) \Rightarrow$$

$$S_t = S_0 \cdot \exp\left(\int_0^t (\mu(u) - \frac{1}{2}\sigma(u)^2) du + \int_0^t \sigma(u) dX_u\right)$$

Solución de la EDE con $\mu(t)$, $\sigma(t)$ y S_0 conocido.