

02_NumberRepresentation

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1 Integers

1. Integer numbers are represented by N bit words.
2. Python3 allows you to store integers with **(practically) unlimited precision**: the only limitation comes from the (contiguous) space available in memory.
3. In Python2 (deprecated), N depended on the PC architecture, which is N=64 in modern computers.

Let's check if our computer is 64-bits: this information, as well as many others, can be retrieved using Python library `sys`:

```
[2]: import sys
print(sys.maxsize)
#An integer giving the maximum value a variable of type Py_ssize_t can take.
# It's usually 2**31 - 1 on a 32-bit platform and 2**63 - 1 on a 64-bit
↪platform.

print("Is your system a 64 bit one?", 2**63 - 1 == sys.maxsize)
```

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Is your system a 64 bit one? True

Now lets check that, indeed, Python3 can represent integers even larger than that:

```
[ ]: very_large_integer = sys.maxsize + 1000
print(very_large_integer) #no problem at all
```

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1.1 Binary and Hexadecimal representations

The common assumption is that numbers (in Python as in all the other languages) are expressed as decimal numbers. Built-in functions allows explicitly to convert from one base to another.

In the binary representation, typically 1 bit (j) is dedicated to specifying the sign of the number, and the conversion between binary and decimal representation is:

$$d = (-1)^j \sum_{i=0}^{N-1} \alpha_i 2^i$$

where α_i are either 0 or 1. $b = \alpha_{N-1}\alpha_{N-2}..\alpha_0$ is the binary representation of the number.

Example: an 8-bit integer in binary representation with one bit for the sign:

j	6	5	4	3	2	1	0
0	0	0	1	0	1	1	1

corresponds to:

$$d = (-1)^j \sum_{i=0}^{N-1} \alpha_i 2^i = (-1)^0 [(1) \cdot 2^0 + (1) \cdot 2^1 + (1) \cdot 2^2 + (0) \cdot 2^3 + (1) \cdot 2^4 + (0) \cdot 2^5 + (0) \cdot 2^6] = 0 + 1 + 2 + 4 + 16 = 23$$

```
[2]: # an integer in decimal representation
a = 23

# its binary representation
a_bin = bin(a)
print(type(a_bin))
print('Binary representation of', a, ': ', a_bin)

# its hexadecimal representation
a_hex = hex(a)
print('Hexadecimal representation of', a, ': ', a_hex)

# converting back to integer
print('Decimal representation of', a_bin, ': ', int(a_bin, 2))
print('Decimal representation of', a_hex, ': ', int(a_hex, 16))
```

```
<class 'str'>
Binary representation of 23 : 0b10111
Hexadecimal representation of 23 : 0x17
Decimal representation of 0b10111 : 23
Decimal representation of 0x17 : 23
```

1.2 Bitwise operators

1.2.1 Logical operators

```
[4]: a = 60          # 60 = 0011 1100
b = 13          # 13 = 0000 1101

print ("a:", bin(a)) # standard formatting, 0b.. ; not print the leading zeros
print("a:", f"{a:08b}") # print only bit values, including as many bits as
    ↪ requested
print("b:", f"{b:08b}")
```

```

c = a & b
print("Bitwise AND ", f"{c:08b}", c)

c = a | b
print("Bitwise OR ", f"{c:08b}", c)

c = a ^ b
print("Bitwise XOR ", f"{c:08b}", c)

```

```

a: 0b111100
a: 00111100
b: 00001101
Bitwise AND  00001100 12
Bitwise OR   00111101 61
Bitwise XOR   00110001 49

```

1.2.2 Unary operators

bitwise NOT given an integer a :

$\sim a = \sim \text{bin}(a) = -(\text{bin}(a)+1)$

i.e. it returns the complement to (-) 1 of that number

```

[12]: print("-61 ", f"{-61:09b}") #one extra "bit" for the "-" symbol
      print("~60 ", f"{~a:09b}")

```

```

-61  -00111101
~60  -00111101

```

1.2.3 Shifts

```

[5]: a = 60           # 60 = 0011 1100

c = a << 2
print("Left shift (towards most significant) of two positions ", f"{c:08b}",c)

c = a >> 2
print("Right shift (towards least significant) of two positions ", f"{c:08b}",c)

```

```

Left shift (towards most significant) of two positions  11110000 240
Right shift (towards least significant) of two positions  00001111 15

```

why we need these shift operators? because typically digital data is a sequence like XXXXYYYYZZZZ XXXX is a header containing some information of the measure, then YYYYYY is the actual information we need, ZZZZ is a trail. So when reading binary files, we need to locate where the actual information is.

e.g. measure pressure, temperature and volume. the binary file for measurement will be of the kind "XXXXYYYYZZZ" and you need to know how many bits are reserved for pressure, how many for

volume etc. So if, for example, you are interested in volume and corresponding bits are YYYY, you shift those bits to the beginning of the sequence and then read.

Also, a floating point number is encoded as sign+exponent+mantissa. If you need to read the mantissa only, you shift and then apply a mask.

For more details check the python [documentation](#).

2 Floating point numbers

1. Non-integer number **cannot be represented with infinite precision** on a computer. Single precision (also known as *float*) and double precision numbers use 32 and 64 bits respectively.
2. Note that all floating point numbers in python are double precision (64 bits).
3. A standard has been developed by IEEE such that the *relative precision* (see later) is the same across the whole validity range.
4. The 32 or 64 bits are divided among 3 quantities uniquely characterizing the number:

$$x_{float} = (-1)^s \times 1.f \times 2^{e-bias}$$

where s is the sign, f the fractional part of the mantissa and e the exponent. In order to get numbers in modulo smaller than 1, a constant *bias* term is added to the exponent, such *bias* is typically equal to half of the max value of e . The mantissa is defined as:

$$mantissa = 1.f = 1 + m_{n-1}2^{-1} + m_{n-2}2^{-2} + .. + m_02^{-n}$$

where n is the number of bits dedicated to f (see below) and m_i are the binary coefficients.

5. Numbers exceeding the maximum allowed value are *overflows* and the calculations involving them provide incorrect answers. Numbers smaller in absolute value than the minimum allowed value are *underflows* and simply set to zero, also in this case incorrect results are yielded.

2.0.1 Single

For single precision floating point numbers, $0 \leq e \leq 255$ and $bias = 127$. Bits are arranged as follows:

	s	e	f
Bit position	31	30-23	22-0

An example is given below:

```
[1]: from IPython.display import Image
Image(url='http://www.dspguide.com/graphics/F_4_2.gif')
```

```
[1]: <IPython.core.display.Image object>
```

Special values are also possibles. N.B.: those are not numbers that can be used in the mathematical sense!

	conditions	value
$+\infty$	$s=0, e=255, f=0$	+INF
$-\infty$	$s=1, e=255, f=0$	-INF
not a number	$e=255, f>0$	NaN

The largest value is obtained for $f \sim 2$ and $e = 254$, i.e. $2 \times 2^{127} \sim 3.4 \times 10^{38}$.

The value closest to zero is obtained instead for $f = 2^{-23}$ and $e = 0$, i.e. $2^{-149} \sim 1.4 \times 10^{-45}$.

2.0.2 Double

For double precision floating point numbers, $0 \leq e \leq 2047$ and $bias = 1023$. Bits are arranged as follows:

	s	e	f
Bit position	63	62-52	51-0

Special values are also possibles. N.B.: those are not numbers that can be used in the mathematical sense!

	conditions	value
$+\infty$	$s=0, e=2047, f=0$	+INF
$-\infty$	$s=1, e=2047, f=0$	-INF
not a number	$e=2047, f>0$	NaN

The validity range for double numbers is $2.2^{-308} - 1.8^{308}$

Serious scientific calculations almost always requires at least double precision floating point numbers.

2.0.3 Floating point numbers on your system

Information about the floating point representation on your system can be obtained from `sys.float_info`. Definitions of the stored values are given on the [python doc page](#)

```
[16]: import sys
      print(sys.float_info)
```

```
sys.float_info(max=1.7976931348623157e+308, max_exp=1024, max_10_exp=308,
min=2.2250738585072014e-308, min_exp=-1021, min_10_exp=-307, dig=15,
mant_dig=53, epsilon=2.220446049250313e-16, radix=2, rounds=1)
```

3 Accuracy and the perils of calculations with floats

Floats can only have a limited number of meaningful decimal places, on the basis of how many bits are allocated for the fractional part of the mantissa: 6-7 decimal places for singles, 15-16 for

doubles. In particular this means that calculations involving numbers with more than those decimal places involved do not yield the correct result, simply because the binary representation of those numbers does not allow to store them with sufficient accuracy.

```
[15]: # Adding an increasingly small number to 7
      for e in [14, 15, 16]: print (7+1.0*10**-e)
```

```
7.000000000000001
7.000000000000001
7.0
```

It should never been forgotten that computers store numbers in binary format. In the same way it is not possible to express the fraction $1/3$ with a finite decimal places, analogously fraction well represented in the decimal base cannot be represented in binary, e.g. $1/10$ is the infinitely repeating number:

0.00011001100110011001100110011001100110011001100110011...

corresponding to $3602879701896397/2^{55}$ which is close to but not exactly equal to the true value of $1/10$ (even though it is even printed to be like that!). Similarly 0.1 is not $1/10$, and making calculations assuming that exactly typically yield to wrong results:

4 *what is printed is not what is stored*

```
[1]: print (1 / 10, 0.1)

# is 1/10 the same of 0.1?
print (1 / 10 == 0.1) #Take home message: never use "==" with floats

# but then watch out! Does it work for 0.3, too?
print (0.1 + 0.2 == 0.3)
```

```
0.1 0.1
True
False
```

4.0.1 A lesson of paramount importance is that you must never compare floating point numbers with the “==” operator as *what is printed is not what is stored!*

The function `float.hex()` yield the exact value stored for a floating point number:

```
[18]: import math
      x = math.pi
      print(x)
      print(x.hex())
```

```
3.141592653589793
0x1.921fb54442d18p+1
```

where `0x` is prefix, indicating a hexadecimal value, `1.921fb54442d18` is the hexadecimal representation of the mantissa and `p+1` is the exponent that scales the mantissa to the correct power of

2.

There are ways to print floats (e.g. filling data into an output file) controlling the number of decimals:

```
[4]: from math import pi

print(format(pi, '.23f')) # give 13 significant digits

print('%0.15f' % (0.1 * 0.1 * 100))

# now repeat trying with >15 digits!

print('%0.30f' % (0.1 * 0.1 * 100)) # different result!
```

```
3.14159265358979311599796
1.0000000000000000
1.0000000000000000222044604925031
```

There is no bug here, this is simply due to the fact that the mantissa is represented by a limited amount of bits, therefore calculations can only make sense if an appropriate number of decimal digits are concerned:

```
[35]: # 23 bits are used for f in single precision floating point
print("Single precision:", 2**-23)

# 53 bits are used for f in double precision floating point
print("Double precision:", 2**-53)
```

```
Single precision: 1.1920928955078125e-07
Double precision: 1.1102230246251565e-16
```

A typical case is subtraction of numbers very close by in value (e.g. when dealing with spectral frequencies). The same happens with functions evaluated near critical points (see later)

```
[37]: 6.022e23 - 6.022e23 + 1
```

```
[37]: 1.0
```

Associative law does not necessarily hold:

```
[6]: print (6.022e23 - 6.022e23 + 1)
print (1 + 6.022e23 - 6.022e23)
```

```
1.0
0.0
```

Distributive law does not hold

```
[39]: import math
a = math.exp(1)
b = math.pi
```

```
c = math.sin(1)
a*(b + c) == a*b + a*c
```

[39]: False

Also identities after casting large numbers may not yield the expected result

```
[7]: x = 287475839859383374
print(float(x))
print(int(float(x)))
print(x == int(float(x)))
```

```
2.8747583985938336e+17
287475839859383360
False
```

4.1 From numbers to functions: conditioning and stability

Function conditioning A mathematical function $f(x)$ is well-conditioned if $f(x + \epsilon) \simeq f(x)$ for all small perturbations ϵ .

In other words, the function $f(x)$ is **well-conditioned** if the solution varies gradually as the input varies. For a well-conditioned function, small perturbations in the input result in small effects in the output. However, a **poorly-conditioned problem** only needs some small perturbation to have large effects. For example, inverting a nearly singular matrix (a matrix whose determinant is close to zero) is a poorly conditioned problem.

example: calculation of $y = \tan(x)$ near $x = \frac{\pi}{2}$ is a poorly conditioned problem.

Algorithm stability Suppose we have a computer algorithm $g(x)$ that numerically evaluates the mathematical function $f(x)$. The algorithm $g(x)$ is **numerically stable** if $g(x + \epsilon) \simeq f(x)$ and it is called **unstable** if large changes in the output are produced. Analyzing an algorithm for stability is more complicated than determining the condition of an expression, even if the algorithm simply evaluates the expression. This is because an algorithm consists of many basic calculations and each one must be analyzed and, due to roundoff error, we must consider the possibility of small errors being introduced in every computed value.

Numerically unstable algorithms tend to amplify approximation errors due to computer arithmetic over time. If we used an infinite precision numerical system, stable and unstable algorithms would have the same accuracy. However, as we see below (e.g. variance calculation), *when using floating point numbers, algebraically equivalent algorithms can give different results.*

In general, we need both a well-conditioned problem and an algorithm with sufficient numerical stability to obtain reliably accurate answers. In most of the cases, the solution to stability issues is solved by properly rewriting the function, as in the examples below.

1. Example of a poorly conditioned function: the tangent of an angle

$\tan(x)$ is usually implemented as a series of Chebyshev polynomials $T_n(x)$, which grants faster convergence than the Taylor expansion. They are defined in $[-1, 1]$ in the following recurrent form:

$$T_0(x) = 1, \quad T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad \text{for } n > 0$$

For the tangent we have:

$$\tan(x) \approx c_0 T_0\left(\frac{4x}{\pi}\right) + c_1 T_1\left(\frac{4x}{\pi}\right) + \dots + c_5 T_5\left(\frac{4x}{\pi}\right)$$

with $c_i = 0$, i even; $c_1 = 0.99986$, $c_3 = 0.3333314$, $c_5 = 0.1333923$

```
[8]: import math
# Define two numbers x and x + epsilon very close to pi/2 (singularity:
#    ↪ divergence)
x1 = 1.57078
x2 = 1.57079
# Calculate the tangent of the x1 and x2 angles
t1 = math.tan(x1)
t2 = math.tan(x2)

print ('tan(x1) =', t1)
print ('tan(x2) =', t2)
print ('percent change in x =', 100.0*(x2-x1)/x1, '%')
print ('percent change in tan(x) =', (100.0*(t2-t1)/t1), '%')
```

```
tan(x1) = 61249.00853150303
tan(x2) = 158057.9134162482
percent change in x = 0.0006366263894271296 %
percent change in tan(x) = 158.05791343536953 %
```

2. Example of a numerically unstable algorithm: the limit $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2}$

we know that $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$. However, if we do naively the calculations like...

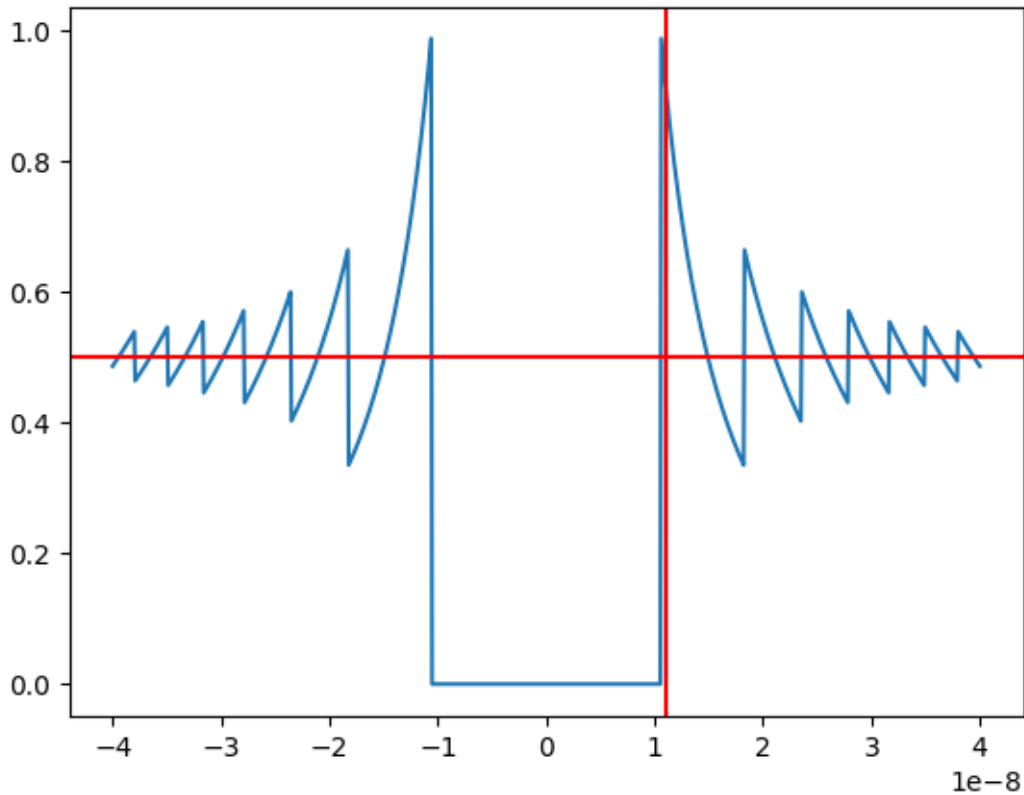
```
[8]: # Catastrophic cancellation occurs when subtracting
# two numbers that are very close to one another
# Here is another example

import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return (1 - np.cos(x))/(x*x)

x = np.linspace(-4e-8, 4e-8, 1000)
plt.plot(x, f(x))
plt.axvline(1.1e-8, color='red')
plt.axhline(0.5, color='red')
```

[8]: <matplotlib.lines.Line2D at 0x11c5dcdd0>



The problem is not only at $|x| < 1.1 \times 10^{-8}$: also those fluctuations do not make sense.

The instability of this algorithm arises from the fact that 1 and $\cos(x)$ are very close for $\lim_{x \rightarrow 0}$, therefore $1 - \cos(x)$ becomes inaccurate and eventually equal to 0.

```
[26]: x
print('%.30f' % np.cos(1.1e-8))
print('%.30f' % (1 - np.cos(1.1e-8))) # failure point: the exact answer is 6.
                                     ↪ 05e-17
print('%2f' % ((1 - np.cos(1.1e-8))/(1.1e-8*1.1e-8)))

0.9999999999999999888977697537484
0.00000000000000000111022302462516
0.917540
```

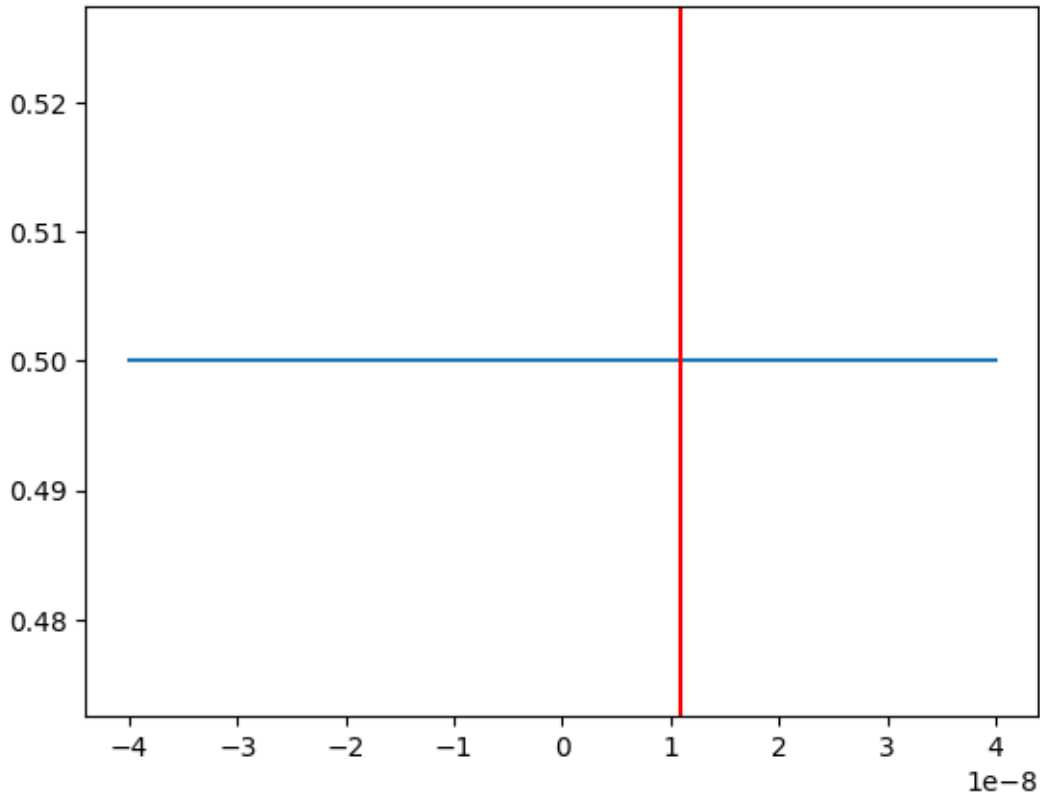
Solution: rewrite the function using sin instead of cos: $1 - \cos(x) = 2 \sin^2(\frac{x}{2})$

```
[9]: def f1(x):
      return 2*np.sin(x/2)**2/(x*x)

      #x = np.linspace(-4e-1, 4e-1, 1000)
```

```
x = np.linspace(-4e-8, 4e-8, 1000)
plt.plot(x, f1(x))
plt.axvline(1.1e-8, color='red')
```

[9]: <matplotlib.lines.Line2D at 0x11c61cb90>



3. Another common example of a numerically unstable algorithm. The stable and unstable version of the (unbiased sample) variance:

$$s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$$

Stable version:

$$s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$$

Unstable version:

$$s^2 = \mathbb{E}(x^2) - \mathbb{E}(x)^2 = \frac{1}{n(n-1)} \cdot n * \sum (x^2) - (\sum x)^2$$

This algorithm becomes unstable when the data is very large, because the subtraction of two close large numbers is likely to return 0 (*remember: this is due to the fact that the absolute spacing in the float representation is not homogeneous on the real axis: it gets larger for larger values*)

```
[11]: # direct method
# squaring occurring after subtraction
def direct_var(x):
    n = len(x)
    xbar = np.mean(x)
    return 1.0/(n-1)*np.sum((x - xbar)**2)

# sum of squares method (vectorized version)

# pay attention to the subtraction of two large numbers
def sum_of_squares_var(x):
    n = len(x)
    return (1.0/(n*(n-1)))*(n*np.sum(x**2) - (np.sum(x))**2))

# Welford's method
# an optimized method
def welford_var(x):
    s = 0
    m = x[0]
    for i in range(1, len(x)):
        m += (x[i]-m)/i
        s += (x[i]-m)**2
    return s/(len(x) - 1)

# check the performances with an array
# of randomly distributed data around 1e12
x_ = np.random.uniform(0, 1, int(1e3))
x = 1e12 + x_
# correct answer from a purpose-built function in numpy
print("Numpy:", np.var(x_))
print("Direct:", direct_var(x))
print("Sum of squares:", sum_of_squares_var(x))
print("Welford's:", welford_var(x))
```

Numpy: 0.08553336358204569

Direct: 0.08561777941755823

Sum of squares: 0.0

Welford's: 0.08496583219584045

4. The example of the Likelihood: $\mathcal{L} = \prod_{i=0}^N \text{prob}(x, \mu)$

```
[13]: # loss of precision can be a problem when calculating Likelihoods
probs = np.random.random(1000) # Generating 1000 random numbers between 0 and 1
    ↪ 1, as if they were probabilities
#print(probs)
print("L =", np.prod(probs))
```

```
# when multiplying lots of small numbers, work in log space  
print("log L =", np.sum(np.log(probs)))
```

L = 0.0

log L = -983.293993349005

Summary:

- **Well/Ill-Conditioned** refers to the problem; **Stable/Unstable** refers to an algorithm or numerical process.
- If the problem is well-conditioned then there is a stable way to solve it.
- If the problem is ill-conditioned then there is no reliable way to solve it in a stable way.
- Mixing roundoff-error with an unstable process is a recipe for disaster.
- With exact arithmetic (no roundoff-error), stability would not be a concern (but we do not have exact arithmetic, so it is).