01_validation_my_solution

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1 Machine Learning LAB 1: MODEL SELECTION

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The notebook contains a simple learning task over which we will implement **MODEL SELECTION AND VALIDATION**.

Complete all the **required code sections** and **answer all the questions**.

1.0.1 IMPORTANT for the exam:

The functions you might be required to implement in the exam will have the same signature and parameters as the ones in the labs

1.1 Polynomial Classification on Signal to Noise Ratios

In this notebook we are going to explore the use of polynomial classification with polynomial regression. We are going to use the Numpy **polyfit** function, which performs polynomial regression.

Our use case is a communication problem: we have a set of measurements of the Signal to Noise Ratio (SNR), i.e., the quality of the communication link, in various positions. The SNR depends on two components: firstly, the noise level (which is a random variable that does not depend on position) and the signal attenuation (usually modeled as a polynomial function of the distance).

Our transmitter is in (0,0), and coordinates are in meters. In urban scenarios, the attenuation usually follows a third-degree polynomial, but it might be a fourth- or fifth-degree polynomial in more complex cases. How do we choose between different degrees? We will try with a maximum of **6**

1.2 Import all the necessary Python libraries

[1]:
```python
import numpy as np
import scipy as sp
import pandas as pd
import itertools
from matplotlib import pyplot as plt
import numpy.random as npr
```

1.3 Load the data

In this case, x and y are the two coordinates, and the SNR is the thing we are trying to predict

DO NOT CHANGE THE PRE-WRITTEN CODE UNLESS OTHERWISE SPECIFIED

[2]:
```
df = pd.read_csv('data/snr_measurements.csv',sep=';')
x = df['x'].to_numpy()
y = df['y'].to_numpy()
SNR = df['SNR'].to_numpy()
```

1.4 Helper functions

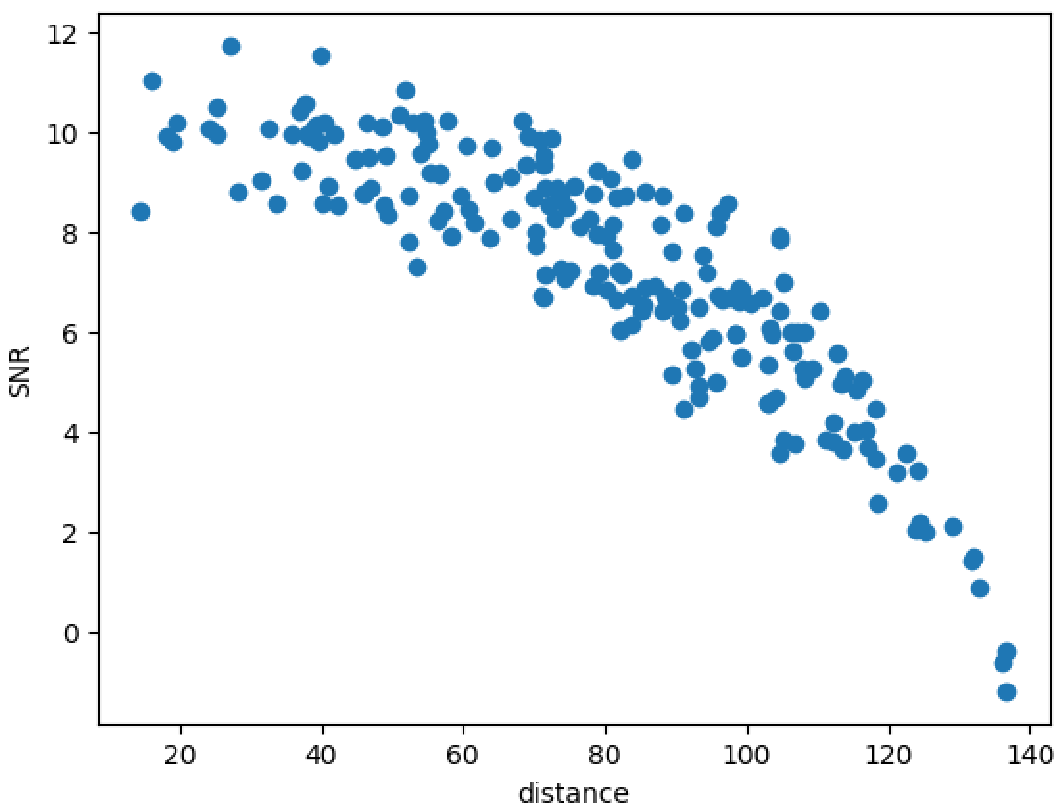
These functions will help us evaluate the results

[3]:
```
def compute_distance(x: np.ndarray, y: np.ndarray) -> np.ndarray:
    # TODO: Compute the Euclidean distance from the origin
    distance = np.sqrt( x**2 + y**2 )
    return distance

def fit(distance: np.ndarray, SNR: np.ndarray, degree: int) -> np.ndarray:
    return np.polyfit(distance, SNR, deg=degree)
# p: polinomial coefficients, highest degree first

def predict(distance: np.ndarray, poly_coeffs: np.ndarray) -> np.ndarray:
    # TODO:  Predict the SNR from a given model
    poly_coeffs = poly_coeffs[::-1] # lowest degree first
    SNR_predicted = np.zeros(len(distance))
    for k, coefficient in enumerate(poly_coeffs):
        SNR_predicted += distance**k * coefficient
    return SNR_predicted

def evaluate(distance: np.ndarray, SNR: np.ndarray, poly_coeffs: np.ndarray) ->␣
↪np.ndarray:
    # TODO: Compute the error of the polynomial fit on the chosen data
    SNR_predicted = predict(distance, poly_coeffs)
    mse_array = ( SNR_predicted - SNR )**2
    mse = np.mean(mse_array)
    return mse

def separate_test(distance: np.ndarray, SNR: np.ndarray, test_points: int):
    # TODO: Return a training set and a test set (the test_points parameter␣
↪controls the number of test points).
    # The points should be selected randomly
    indexes = np.arange(0, len(distance))
    test_indexes = np.random.choice(indexes, size= test_points, replace= False)
    train_indexes = np.array( list( set(indexes) - set(test_indexes) ), dtype=␣
↪int)
```

```
    x_train = distance[train_indexes]
    y_train = SNR[train_indexes]
    x_test = distance[test_indexes]
    y_test = SNR[test_indexes]

    return x_train, y_train, x_test, y_test
```

[4]:
```
# TODO: separate the training and test sets and create a scatter plot of the␣
 ↪SNR as a function of the distance
distance = compute_distance(x, y)
x_train, y_train, x_test, y_test = separate_test(distance, SNR, int␣
 ↪(len(distance)/2 ) )

import matplotlib.pyplot as plt
plt.scatter(distance, SNR)
plt.xlabel("distance")
plt.ylabel("SNR")
```

[4]: Text(0, 0.5, 'SNR')



1.5 A. K-fold cross-validation

In this case, x and y are the two coordinates, and the SNR is the thing we are trying to predict

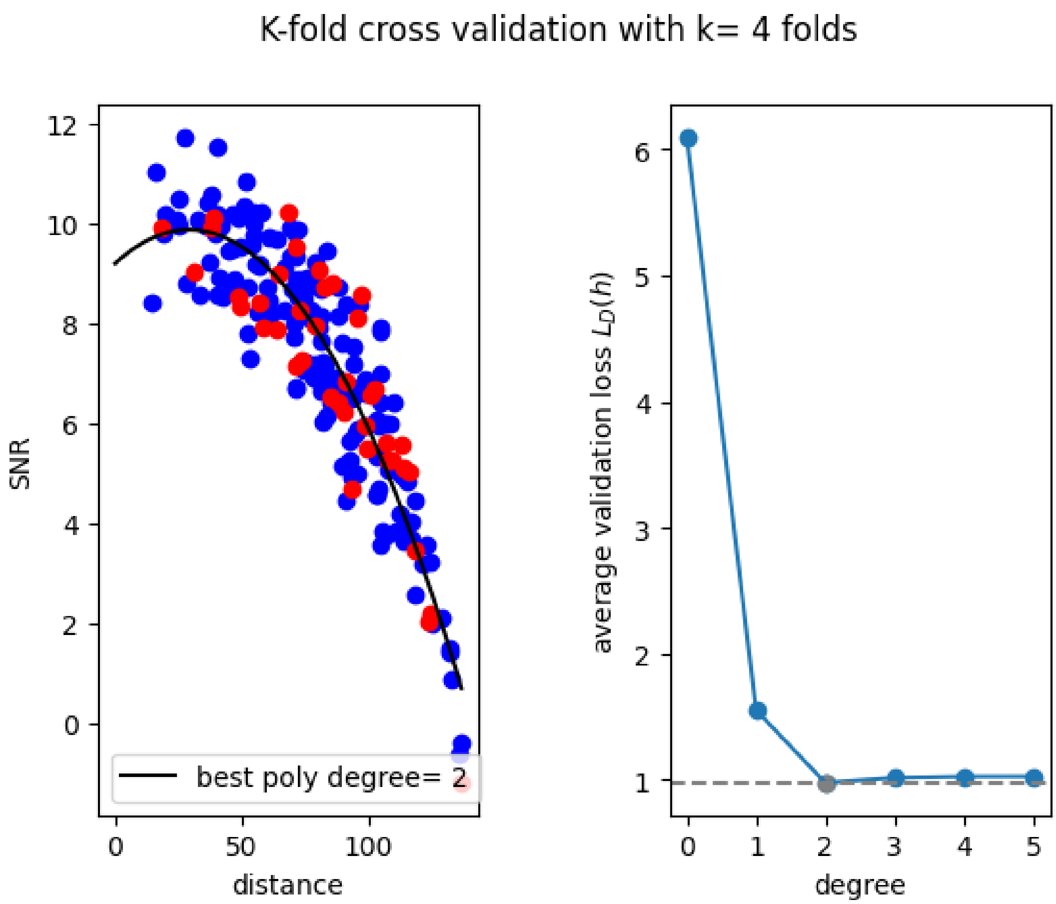
[5]:
```
# Function to perform the K-fold cross validation
def k_fold_cross_validation(x_train: np.ndarray, y_train: np.ndarray, k: int,␣
↪max_degree: int) -> tuple[tuple, tuple]:
    # TODO: Perform K-fold cross-validation on the training set.
    # The two returned values are the best model and the list of results(=␣
↪class scores) for all degrees up to max_degree.
    # The points should be selected randomly.
    # The inputs and labels are already in terms of distance and SNR
    """
    Returns:
    - best: (np.ndarray) coefficients of the best polynomial
    - results: (np.ndarray) score of each degree subclass, defined as the␣
↪average validation loss
    """
    idxs = np.arange(0, len(x_train))
    npr.shuffle(idxs)
    folds_idxs = np.array_split( idxs , k)


    results = [] #score of each degree subclass

    for degree in range(max_degree + 1):
        fold_results = []
        for i in range(k):
            validation_idxs = folds_idxs[i]

            training_idxs = np.array( list( set(idxs) - set(validation_idxs)) ,␣
↪dtype = int)
            x_validation_fold = x_train[validation_idxs]
            y_validation_fold = y_train[validation_idxs]
            x_train_fold = x_train[training_idxs]
            y_train_fold = y_train[training_idxs]

            poly = fit(distance= x_train_fold, SNR= y_train_fold, degree=␣
↪degree)
            mse = evaluate(distance= x_validation_fold, SNR= y_validation_fold,␣
↪poly_coeffs= poly)
            fold_results.append(mse)
        # Compute average validation loss
        results.append(np.mean(fold_results))

    # Find best degree
    best_degree = np.argmin(results)
```

```python
    # Fit again with that degree, using whole training dataset now
    best = fit(distance= x_train, SNR= y_train, degree= best_degree)

    return best, results
```

[6]:

```python
# TODO: run the training with K-fold cross-validation with 40 test points and 4␣
 ↪folds Plot the validation score as a function of the degree
k = 4



x_train, y_train, x_test, y_test = separate_test(distance, SNR, test_points= 40␣
 ↪)
best, results = k_fold_cross_validation(x_train, y_train, k, max_degree= 5)
#print("best degree: \n", len(best) -1, '\n', best, '\n\n')
#print(results)

fig, axs = plt.subplots(nrows= 1, ncols= 2)
plt.suptitle(f"K-fold cross validation with k= {k} folds")
plt.subplots_adjust(wspace= 0.5)
# Best fit plot
axs[0].scatter(x_train, y_train, color = 'blue')
axs[0].scatter(x_test, y_test, color = 'red')
x_fine_spaced = np.linspace(0, np.max(distance), 1000)
y_fine_spaced = predict(x_fine_spaced, best)
axs[0].plot(x_fine_spaced, y_fine_spaced, color = 'black', label= f'best poly␣
 ↪degree= {len(best) - 1}')
axs[0].legend(loc= "best")
axs[0].set_xlabel("distance")
axs[0].set_ylabel("SNR")
# Validation loss w.r.t. degree
degrees = np.arange(len(results), dtype= int)
axs[1].scatter(degrees, results)
axs[1].plot(degrees, results)
axs[1].axhline(np.min(results), linestyle = 'dashed', color = 'grey')
axs[1].scatter(degrees[np.argmin(results)], np.min(results), color = 'grey')
axs[1].set_xticks(degrees)
axs[1].set_xticklabels([f"{degree}" for degree in degrees])
axs[1].set_xlabel("degree")
axs[1].set_ylabel(r"average validation loss $L_D(h)$")


test_error = evaluate(x_test, y_test, best)
print("Test error:", test_error)
```

```
Test error: 1.0733822062179983
```




```
[7]: # TODO: get the test performance of the best model and plot the model output
     ↪and test points.
     # Try running the program multiple times, changing the values of K and the
     ↪number of test points: is the output always the same?
```

Answer: no, the output changes: the best degree fluctuates between 3 and 5

1.6 B. Tikhonov regularization (RLM learning rule)

Change the loss function to include a Tikhonov regularization term, as an alternative to cross-validation (try $\lambda = 0.01$)

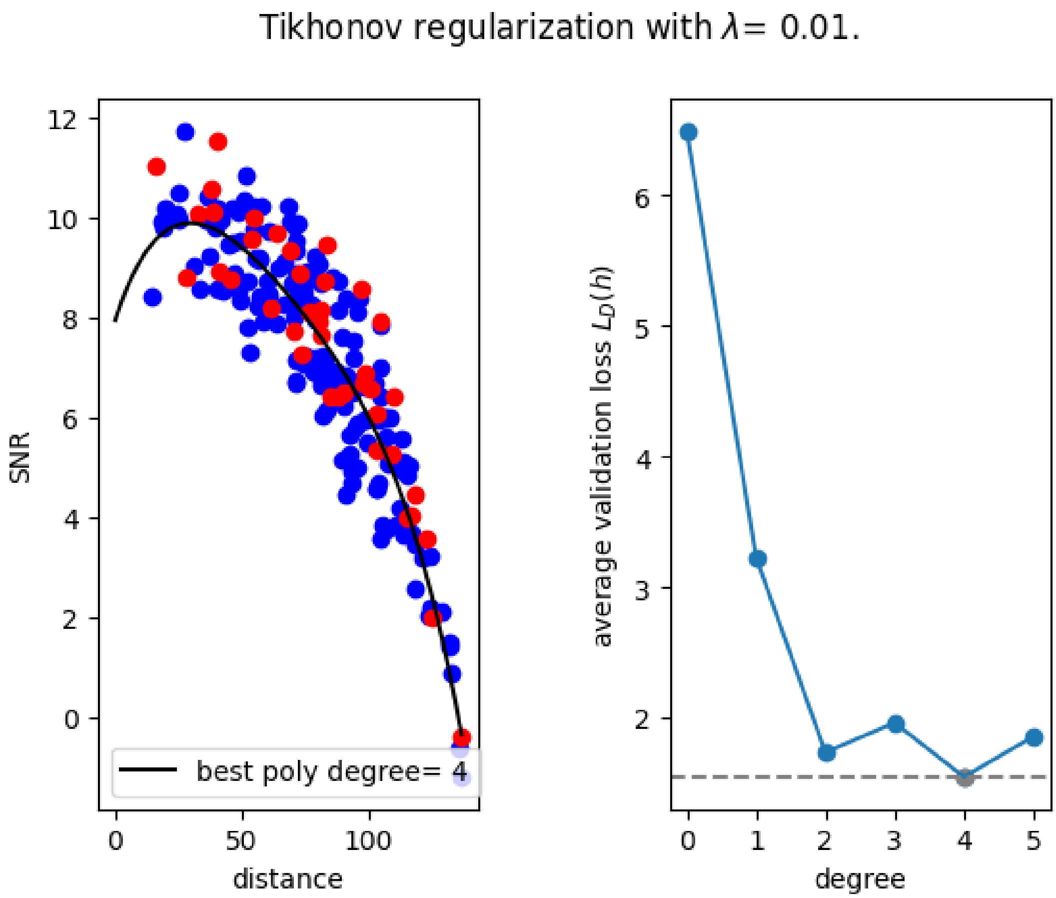
```
[8]: def evaluate_tikhonov(x_train: np.ndarray, y_train: np.ndarray, lambda_par:
     ↪float, max_degree: int) -> tuple[tuple, tuple]:
         # TODO: apply Tikhonov regularization AFTER the fitting process
         results = []
         for degree in range(max_degree + 1):
```

```
        poly = fit(distance= x_train, SNR= y_train, degree= degree)
        mse = evaluate(distance= x_train, SNR= y_train, poly_coeffs= poly)
        regularized_loss = mse + lambda_par * np.sum( poly**2)
        results.append(regularized_loss)
    # Find best model
    best_degree = np.argmin(results)
    best = fit(distance= x_train, SNR= y_train, degree= best_degree)
    return best, results
```

[9]:

```
# TODO: run the training with Tikhonov regularization and plot the loss as a␣
↪function of the degree
lambda_par = 0.01

x_train, y_train, x_test, y_test = separate_test(distance, SNR, test_points= 40␣
↪)
best, results = evaluate_tikhonov(x_train, y_train, lambda_par= lambda_par,␣
↪max_degree= 5)
#print("best degree: \n", len(best) -1, '\n', best, '\n\n')
#print(results)

fig, axs = plt.subplots(nrows= 1, ncols= 2)
plt.suptitle(fr"Tikhonov regularization with $\lambda$= {lambda_par}.")
plt.subplots_adjust(wspace= 0.5)
# Best fit plot
axs[0].scatter(x_train, y_train, color = 'blue')
axs[0].scatter(x_test, y_test, color = 'red')
x_fine_spaced = np.linspace(0, np.max(distance), 1000)
y_fine_spaced = predict(x_fine_spaced, best)
axs[0].plot(x_fine_spaced, y_fine_spaced, color = 'black', label= f'best poly␣
↪degree= {len(best) - 1}')
axs[0].legend(loc= "best")
axs[0].set_xlabel("distance")
axs[0].set_ylabel("SNR")
# Validation loss w.r.t. degree
degrees = np.arange(len(results), dtype= int)
axs[1].scatter(degrees, results)
axs[1].plot(degrees, results)
axs[1].axhline(np.min(results), linestyle = 'dashed', color = 'grey')
axs[1].scatter(degrees[np.argmin(results)], np.min(results), color = 'grey')
axs[1].set_xticks(degrees)
axs[1].set_xticklabels([f"{degree}" for degree in degrees])
axs[1].set_xlabel("degree")
axs[1].set_ylabel(r"average validation loss $L_D(h)$")



test_error = evaluate(x_test, y_test, best)
print("Test error:", test_error)
```

Test error: 0.9335117640369359



1.7 C. Minimum description length regularization (SRM learning rule)

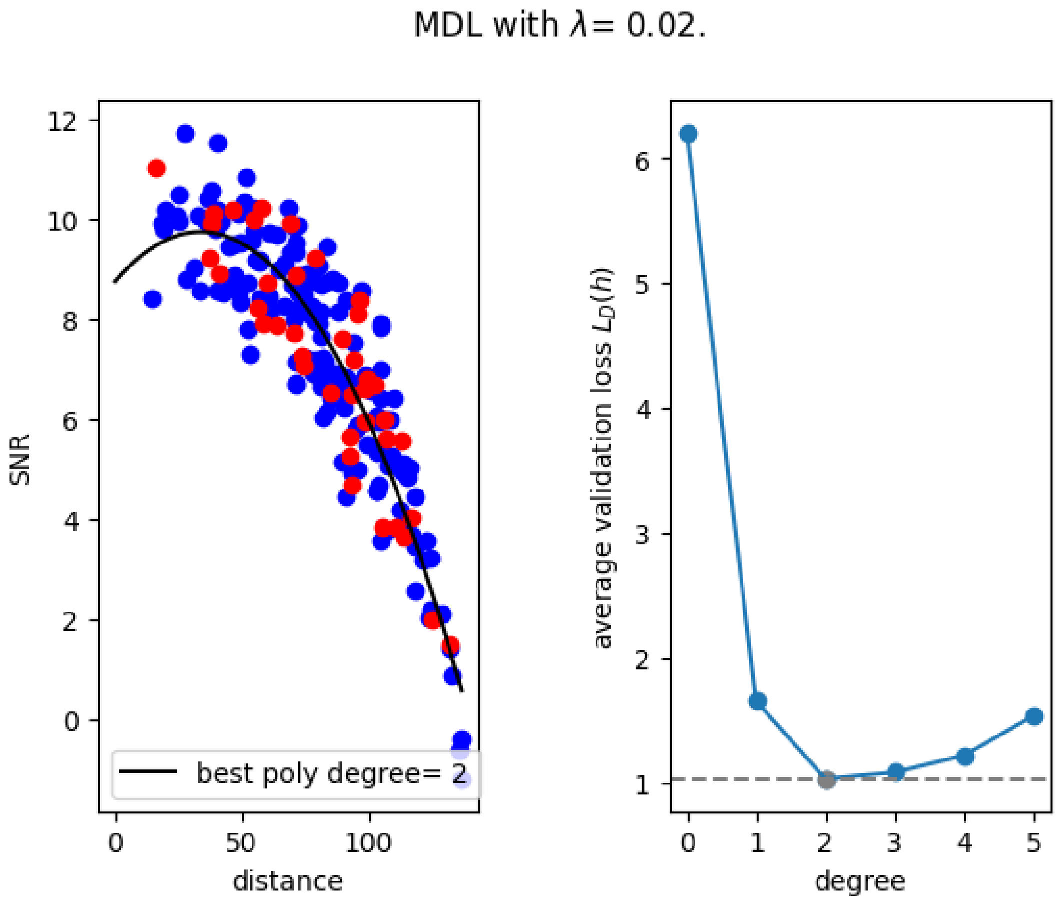
Change the loss function to include a representation length regularization term, as an alternative to cross-validation. The minimum description length of a polynomial of degree N is O(2^N) - try $\lambda = 0.02$

[10]:
```
def evaluate_representation(x_train: np.ndarray, y_train: np.ndarray,␣
↪lambda_par: float, max_degree: int) -> tuple[tuple, tuple]:
    # TODO: apply MDL regularization AFTER the fitting process
    results = []
    for degree in range(max_degree + 1):
        poly = fit(distance= x_train, SNR= y_train, degree= degree)
        mse = evaluate(distance= x_train, SNR= y_train, poly_coeffs= poly)
        regularized_loss = mse + lambda_par * 2**degree
        results.append(regularized_loss)
    # Find best model
    best_degree = np.argmin(results)
```

```
    best = fit(distance= x_train, SNR= y_train, degree= best_degree)
    return best, results
```

[11]:
```
# TODO: run the training with MDL regularization and plot the loss as a␣
↪function of the degree

lambda_par = 0.02

x_train, y_train, x_test, y_test = separate_test(distance, SNR, test_points= 40␣
↪)
best, results = evaluate_representation(x_train, y_train, lambda_par=␣
↪lambda_par, max_degree= 5)
#print("best degree: \n", len(best) -1, '\n', best, '\n\n')
#print(results)

fig, axs = plt.subplots(nrows= 1, ncols= 2)
plt.suptitle(fr"MDL with $\lambda$= {lambda_par}.")
plt.subplots_adjust(wspace= 0.5)
# Best fit plot
axs[0].scatter(x_train, y_train, color = 'blue')
axs[0].scatter(x_test, y_test, color = 'red')
x_fine_spaced = np.linspace(0, np.max(distance), 1000)
y_fine_spaced = predict(x_fine_spaced, best)
axs[0].plot(x_fine_spaced, y_fine_spaced, color = 'black', label= f'best poly␣
↪degree= {len(best) - 1}')
axs[0].legend(loc= "best")
axs[0].set_xlabel("distance")
axs[0].set_ylabel("SNR")
# Validation loss w.r.t. degree
degrees = np.arange(len(results), dtype= int)
axs[1].scatter(degrees, results)
axs[1].plot(degrees, results)
axs[1].axhline(np.min(results), linestyle = 'dashed', color = 'grey')
axs[1].scatter(degrees[np.argmin(results)], np.min(results), color = 'grey')
axs[1].set_xticks(degrees)
axs[1].set_xticklabels([f"{degree}" for degree in degrees])
axs[1].set_xlabel("degree")
axs[1].set_ylabel(r"average validation loss $L_D(h)$")


test_error = evaluate(x_test, y_test, best)
print("Test error:", test_error)
```

```
Test error: 0.9972427932927858
```



1.7.1 TEST

Check the performance of the three solutions on the test set: which one does best?

[40]:
```
# TODO: plot the results
x_train, y_train, x_test, y_test = separate_test(distance, SNR, test_points= 40␣
↪)
best_kfold, _ = k_fold_cross_validation(x_train, y_train, k= 4, max_degree= 5)
best_tikhonov, _ = evaluate_tikhonov(x_train, y_train, lambda_par= 0.01,␣
↪max_degree= 5)
best_mdl, _ = evaluate_representation(x_train, y_train, lambda_par= 0.02,␣
↪max_degree= 5)
best_models = [best_kfold, best_tikhonov, best_mdl]
best_models_labels = ['k fold', 'tikhonov', 'MDL']
test_losses = [evaluate(x_test, y_test, poly_coeffs= best) for best in␣
↪best_models]
print("Test empirical losses (mse):", test_losses)
lines = ['solid', 'dashed', 'dotted']
```

```python
fig, ax = plt.subplots()
ax.scatter(x_test, y_test, color = 'black')
x_fine_spaced = np.linspace(0, np.max(distance), 1000)
for i,best in enumerate(best_models):
    y_fine_spaced = predict(x_fine_spaced, best)
    ax.plot(x_fine_spaced, y_fine_spaced, label= best_models_labels[i],␣
 ↪linestyle = lines[i])
    ax.legend(loc= "best")
    ax.set_xlabel("distance")
    ax.set_ylabel("SNR")
```

```
Test empirical losses (mse): [np.float64(0.9771992100275378),
np.float64(0.9771992100275378), np.float64(0.9355392932807245)]
```