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Exercise 3

Consider the following matrix

$$A = \begin{bmatrix} 3 & 6 \\ 0 & 1 \\ 4 & 8 \end{bmatrix}. \tag{2}$$

1. Compute, by hand, the QR factorization of the matrix A using the Gram-Schmidt procedure.
2. Use the computed factorization to build the projection matrix on the range(A) (or column space of A).
3. How can you use the projection matrix to determine whether the vector $\mathbf{b} = [1, 1, 0]^T$ belongs to the column space of A ?
4. Use the QR factorization to find a solution (or best-fit solution) to $A\mathbf{x} = \mathbf{b}$.
5. Does the solution exist to $A^T\mathbf{x} = \mathbf{c}$ where $\mathbf{c} = [2, 2]^T$? If no solution exists, find the best-fit. If one or more solution exist, find the one for which $\|\mathbf{x}\|$ is as small as possible.

• Gram-schmidt Procedure

$$A = \begin{bmatrix} 3 & 6 \\ 0 & 1 \\ 4 & 8 \end{bmatrix} \qquad a_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix}$$

$$\|a_1\| = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{25}$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix}$$

$$r_{12} = q_1^T a_2 = \begin{bmatrix} 0.6 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix} = 0.6 \cdot 6 + 0 \cdot 1 + 0.8 \cdot 8 = 3.6 + 0 + 6.4 = 10$$

$$\text{Proj}_{q_1}(a_2) = r_{12} \cdot q_1 = 10 \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$$

$$v_2 = a_2 - \text{Proj}_{q_1}(a_2) = \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\|v_2\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$q_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \\ 0.8 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \quad \text{where} \quad r_{11} = \|a_1\| = 5 \quad r_{12} = 10 \quad r_{22} = \|v_2\| = 1$$

$$R = \begin{bmatrix} 5 & 10 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \\ 0.8 & 0 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 0 & 1 \end{bmatrix}$$

• Projection onto $C(A)$

$$P = Q\,Q^T = \begin{bmatrix} 0.36 & 0 & 0.48 \\ 0 & 1 & 0 \\ 0.48 & 0 & 0.64 \end{bmatrix}$$

• Is $b = [1 \; 1 \; 0]^T \in C(A)$

Projected vector $\hat{b} = Pb = \begin{bmatrix} 0.36 \\ 1 \\ 0.48 \end{bmatrix}$ $\qquad b \neq \hat{b} \quad \rightarrow \quad b \notin C(A)$

• $Ax = b \implies QRx = b \rightarrow Rx = Q^T b$

$$Q^T b = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$$

$$\hookrightarrow \begin{cases} x_2 = 1 \\ 5x_1 + 10 \cdot 1 = 0.6 \end{cases} \qquad 5x_1 = 0.6 - 10 = -9.4 \implies x_1 = -1.88$$

$$x_1 = \begin{bmatrix} -1.88 \\ 1.0 \end{bmatrix}$$

• $A^T x = c$ when $c = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 4 \\ 6 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Rank (A^T) = Rank $(A^T|c)$? yes, solution exist (infinite solutions)

↳ You find the minimum norm : Moore-Penrose Pseudoinverse

$$x = A^{T*} c$$

$$A^T x = c \Rightarrow x = A(A^T A)^{-1} c$$

$$\dots \quad x = \begin{bmatrix} 0.24 \\ -2 \\ 0.32 \end{bmatrix}$$