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Exercise 3

Consider the following function $g : \{-1,+1\}^N \to \{-1,1\}$:

$$g(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{N} x_i \in [S_{min}, S_{max}], \\ -1 & \text{otherwise,} \end{cases} \tag{1}$$

where $S_{min}, S_{max} \in \mathbb{Z}$ and $-N \leq S_{min} \leq S_{max} \leq N$.

1. Show that in general $g(\mathbf{x})$ cannot be reproduced using a single perceptron.
2. Show that the function $g(\mathbf{x})$ can be reproduced using a network with one hidden layer and two neurons using

$$\sigma(z) = \text{sign}(z) = \begin{cases} +1 & \text{if } z \geq 0, \\ -1 & \text{otherwise,} \end{cases} \tag{2}$$

   with all weights and biases integers.
3. Show that the function $g(\mathbf{x})$ can be reproduced using a network with one hidden layer and two neurons using $\sigma(z) = ReLU(z)$.

- Single perceptron $f(x) = \text{sign}(w^T x + b)$: linear decision boundary

  → Function $g(x)$

  - $g(x) = 1$ only when the sum of inputs is inside $[S_{min}, S_{max}]$
  - we need basically 2 decision boundaries : one at S_{min}, one at S_{max}

  ↳ single perceptron can only model one threshold

- $g(x)$ as one hidden layer and two sign neurons

  - define $s = \sum_{i=1}^{N} x_i$, $s \in [-N, N]$ and all values are odd or even integers depending on N
  - Neuron 1 : $h_1 = \text{sign}(s - S_{min} + 0.5)$ → $= 1$ when $s \geq S_{min}$
  - Neuron 2 : $h_2 = \text{sign}(S_{max} - s - 0.5)$ → $= 1$ when $s \leq S_{max}$
  - output neuron $z = \text{sign}(h_1 + h_2 - 1.5)$ → 1 both hidden neurons output 1, -1 otherwise

  since $s = \sum x_i$ all weights can be set to 1 or -1 (integer weights and biases)

$y = \text{sign}(z)$

• $g(x)$ using ReLU $\quad (\text{ReLU}(z) = \max(0, z))$

$s = \sum x_i$

$h_1 = \text{ReLU}(s - s_{min})$

$h_2 = \text{ReLU}(s_{max} - s)$

Define $g(x) = \begin{cases} 1 & \text{if } h_1 > 0 \text{ and } h_2 > 0 \\ \emptyset & \text{otherwise.} \end{cases}$

↳ since outputs of ReLU are values ≥ 0

$out = \text{ReLU}(1 - \text{ReLU}(1 - h_1) - \text{ReLU}(1 - h_2))$

↳ $out = 1$ only when $h_1, h_2 \geq 1$ i.e. $s \in [s_{min} + 1, s_{max} - 1]$

if $h_1 > 0 \quad h_2 > 0 \quad out = (1 - 0 - 0) = 1$

if $h_1 \leq 0$ or $h_2 \leq 0 \quad out = (1 - 1 - \ldots) = \emptyset$