2021/07/08

Exercise 3 Consider the following mapping:

$$\begin{array}{cccc} (0,0,0)\to 1, & (1,0,0)\to 0, & (0,1,0)\to 0, & (0,0,1)\to 0, \\ (0,1,1)\to 1, & (1,1,0)\to 0, & (1,0,1)\to 0, & (1,1,1)\to 1. \end{array} \tag{1}$$

- Is it possible to learn the previous map using only a single perceptron ?
- Propose a multi-perceptron neural network that is able to learn the previous mapping. Compute the weights and the biases of all the perceptrons in the network.

- Linearly separable ? $W^T X + b \geq 0 \longrightarrow$ class 1

Class 0

(1) $(1\ 0\ 0)$ $w_1 + b < 0$

(2) $(0\ 1\ 0)$ $w_2 + b < 0$

(3) $(0\ 0\ 1)$ $w_3 + b < 0$

(4) $(1\ 1\ 0)$ $w_1 + w_2 + b < 0$

(5) $(1\ 0\ 1)$ $w_1 + w_3 + b < 0$

Class 1

(6) $(0\ 0\ 0)$ $b \geq 0$

(7) $(0\ 1\ 1)$ $w_2 + w_3 + b \geq 0$

(8) $(1\ 1\ 1)$ $w_1 + w_2 + w_3 \geq 0$

From (6) $b \geq 0$

From (1) $w_1 < -b \leq 0$

From (2) $w_2 < -b \leq 0$

From (3) $w_3 < -b \leq 0$

$\longrightarrow w_1, w_2, w_3 < 0$

From (7) $w_2 + w_3 + b \geq 0$

$w_2 + w_3 < -2b \longrightarrow w_2 + w_3 + b < -b \longrightarrow w_2 + w_3 + b < -b < 0$

but $w_2 + w_3 + b$ should be ≥ 0

Contradiction!

- MLP proposed (use 2 hidden neurons)

~ model output as $f(x) = (x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3)$

$h_1 = x_2 \wedge x_3$

$h_2 = \bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3$

output = step $(w_1 x_1 + w_2 x_2 + w_3 x_3 + b)$

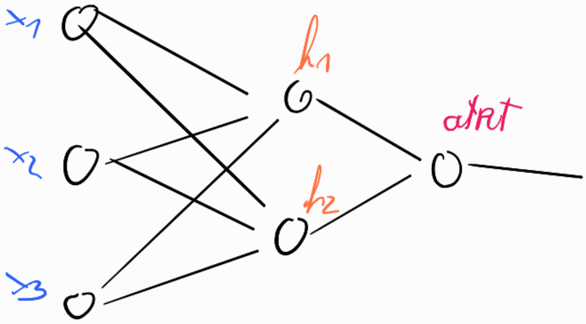
$$\text{step}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$h_1 = x_2 \wedge x_3$ can be implemented as $w = [0 \ 1 \ 1]$ $b = -1.5$

$h_2 = \bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3$ can be implemented as $w = [-1 \ -1 \ -1]$ $b = 0.5$

output should be 1 if either $h_1 = 1$ or $h_2 = 1$

↳ output neuron $w = [1 \ 1]$ $b = -0.5$



Layer 1: $W_1 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$, $b_1 = \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix}$

out : $W_{out} = [1 \ 1]$ $b_{out} = -0.5$