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Exercise 3
Consider the quadratic function

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}, \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$ is SPD and $\mathbf{b} \in \mathbb{R}^n$.

1. Compute the gradient and the Hessian of J.
2. Verify that J is strictly convex and find the unique global minimum of J.
3. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^n$ a direction s.t. $\nabla J(\mathbf{x})^T\mathbf{q} < 0$. Compute analytically the step length α that solve the following exact line-search problem

$$\min_{\alpha>0} J(\mathbf{x} + \alpha\mathbf{q}). \qquad (2)$$

$\nabla J(x) = Ax - b$
$\nabla^2 J(x) = A$

Hessian positive definite $\rightarrow$ Function strictly convex. : A is SPD $\rightarrow$ function is convex

Minimum $\nabla J(x) = Ax - b = 0 \rightarrow Ax = b \qquad x^* = A^{-1}b$

Exact line search acode $\min_{\alpha>0} J(x + \alpha q)$

define $\phi(\alpha) = J(x + \alpha q) = \frac{1}{2}(x + \alpha q)^T A (x + \alpha q) - b^T (x + \alpha q)$

$\hookrightarrow \phi(\alpha) = \frac{1}{2} x^T A x + \alpha x^T A q + \frac{1}{2}\alpha^2 q^T A q - b^T x - \alpha b^T q$
$= J(x) + \alpha (x^T A q - b^T q) + \frac{1}{2}\alpha^2 q^T A q$

$\hookrightarrow$ Minimize wrt α $\quad \frac{d\phi}{d\alpha} = 0 = x^T A q - b^T q + \alpha q^T A q = 0$

$\hookrightarrow \alpha^* = \dfrac{b^T q - x^T A q}{q^T A q}$

(using $\nabla J(x) = Ax - b$
$\nabla J(x)^T q = (Ax - b)^T q$
$= x^T A q - b^T q$

$\rightarrow \alpha^* = -\dfrac{\nabla J(x)^T q}{q^T A q}$