Quantum Information and Computation

Superposition, Entanglement, and Applications

Antonije Mirkovic

Contents

1 Superposition as a Resource: Quantum Key Distribution (BB84)

1.1 Motivation from Classical Cryptography

Classical cryptography offers two major paradigms:

- **One-time pad:** Alice and Bob share a long random key K in advance. Encryption and decryption are both done by bitwise XOR:

$$C = M \oplus K, \qquad M = C \oplus K.$$

  This scheme is *perfectly secure*, but impractical because K must be as long as the message and securely exchanged beforehand.

- **Public-key cryptography (e.g. RSA):**

  (i) Choose two large primes p, q and compute $n = pq$.

  (ii) Compute Euler's totient $\varphi(n) = (p-1)(q-1)$.

  (iii) Choose a public exponent e coprime to $\varphi(n)$.

  (iv) Compute the private exponent d such that $ed \equiv 1 \pmod{\varphi(n)}$.

  The public key is (n, e), the private key is d. Security relies on the hardness of factoring n.

Both methods rely on classical assumptions. Quantum mechanics offers a fundamentally different approach through *quantum key distribution (QKD)*.

1.2 The BB84 Protocol

The BB84 protocol (Bennett & Brassard, 1984) uses quantum superposition and measurement disturbance to establish a shared random key.

Step 1: Encoding random bits. Alice chooses two random strings:

- A random bit string $a = (a_1, \ldots, a_n)$.
- A random basis choice string $b = (b_1, \ldots, b_n)$ with $b_j \in \{Z, X\}$.

She encodes each bit according to:

Bit	Z basis	X basis
0	$\|0\rangle$	$\|+\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
1	$\|1\rangle$	$\|-\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$

Step 2: Transmission. Alice sends the n qubits to Bob.

Step 3: Random measurement. Bob chooses his own random basis string $b' = (b'_1, \ldots, b'_n)$ and measures each qubit in basis b'_j. His results form a bit string a'.

Step 4: Basis reconciliation. Alice and Bob publicly compare their basis choices b and b'. They discard positions where $b_j \neq b'_j$. The remaining bits are the *sifted key*.

Step 5: Eavesdropping test. They sacrifice a random subset of the sifted key, compare outcomes, and compute the error rate. If it is below a threshold, the remaining bits form a secure key.

1.3 Security Intuition

Eavesdropper strategy. Suppose Eve intercepts qubits and measures them in a random basis. If she measures in the wrong basis, she introduces errors with probability 1/2.

Error rate. On average, half of the positions where Alice and Bob's bases match are measured by Eve in the wrong basis. This leads to a 25% error rate in the sifted key.

Detection probability. If Alice and Bob test t bits, the probability Eve escapes detection is

$$P_{\text{undetected}} = \left(\tfrac{3}{4}\right)^t.$$

For $t = 50$, this is approximately 6×10^{-9}, essentially zero.

1.4 Summary

BB84 shows how quantum superposition and projective measurement can be turned into a cryptographic resource. Unlike RSA, its security does not rely on computational assumptions but on fundamental quantum principles: measurement disturbance and the impossibility of cloning unknown quantum states.

2 Entanglement: Definitions, Entropy, and Examples

2.1 Definition of Entanglement

Definition 2.1 (Product and entangled states)**.** A bipartite pure state $|\psi\rangle_{AB}$ is a *product state* if it can be written as

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B .$$

It is *entangled* if it cannot be written in this form.

For mixed states, separability is defined in terms of convex mixtures:

Definition 2.2 (Separable mixed states)**.** A bipartite mixed state ρ_{AB} is *separable* if it can be expressed as

$$\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)},$$

with $p_k \geq 0$, $\sum_k p_k = 1$. Otherwise ρ_{AB} is entangled.

2.2 Schmidt Decomposition and Entanglement Entropy

Theorem 2.3 (Schmidt decomposition)**.** *Every bipartite pure state $|\psi\rangle_{AB}$ admits a decomposition*

$$|\psi\rangle_{AB} = \sum_{j=1}^{r} \sqrt{\lambda_j} \; |u_j\rangle_A \otimes |v_j\rangle_B ,$$

where $\{|u_j\rangle\}$ and $\{|v_j\rangle\}$ are orthonormal sets, $\lambda_j \geq 0$, and $\sum_j \lambda_j = 1$.

The Schmidt coefficients $\{\lambda_j\}$ determine the entanglement. Define the reduced density operator $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. Its eigenvalues are precisely $\{\lambda_j\}$.

Definition 2.4 (Entropy of entanglement)**.** For a pure state $|\psi\rangle_{AB}$, the entropy of entanglement is

$$E(|\psi\rangle) = S(\rho_A) = -\operatorname{Tr}\left(\rho_A \ln \rho_A\right) = -\sum_j \lambda_j \ln \lambda_j .$$

2.3 Bell States

The four maximally entangled two-qubit states, known as *Bell states*, are

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Each of these states has reduced density matrices

$$\rho_A = \rho_B = \frac{\mathbb{1}}{2}.$$

Hence their entanglement entropy is

$$S(\rho_A) = -\operatorname{Tr}\left(\tfrac{\mathbb{1}}{2}\ln\tfrac{\mathbb{1}}{2}\right) = \ln 2,$$

the maximal possible for two qubits.

2.4 Example: Partially Entangled State

Consider

$$|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle\,, \qquad 0 < \theta < \tfrac{\pi}{4}.$$

The reduced state is

$$\rho_A = \cos^2\theta\,|0\rangle\langle 0| + \sin^2\theta\,|1\rangle\langle 1|\,.$$

Thus the entanglement entropy is

$$S(\rho_A) = -\cos^2\theta\ln\Big(\cos^2\theta\Big) - \sin^2\theta\ln\Big(\sin^2\theta\Big).$$

For $\theta = \pi/4$, we recover a maximally entangled Bell state with $S = \ln 2$. For θ close to 0, S approaches 0.

Example 2.5 (Numeric case)**.** Take $\theta = 0.1$. Then $\cos^2\theta \approx 0.99$, $\sin^2\theta \approx 0.01$, and

$$S(\rho_A) \approx -0.99\ln(0.99) - 0.01\ln(0.01) \approx 0.068,$$

showing a very small but nonzero entanglement.

2.5 Summary

Entanglement is a uniquely quantum correlation that cannot be captured by classical probability theory. The entropy of entanglement provides a quantitative measure, ranging from 0 (product state) to $\ln 2$ (maximally entangled Bell state).

3 The EPR Paradox and Bell's Inequalities

3.1 Einstein–Podolsky–Rosen (EPR) Argument

In 1935, Einstein, Podolsky, and Rosen (EPR) considered a pair of particles in an entangled state. For example, take the Bell state

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

If Alice and Bob are spacelike separated, then:

- If Alice measures her qubit in the Z basis and obtains $|0\rangle$, she knows instantly that Bob's qubit is in $|0\rangle$.
- If Alice instead measures in the X basis and gets $|+\rangle$, Bob's qubit must be $|+\rangle$.

EPR argued that either (i) quantum mechanics is incomplete, because Bob's particle must have "hidden variables" specifying both outcomes in advance, or (ii) there is nonlocal influence between the particles.

3.2 Local Hidden Variable (LHV) Models

An LHV model assumes:

(a) **Realism:** Measurement outcomes are determined by pre-existing properties of particles, described by a hidden variable λ.

(b) **Locality:** Alice's outcome depends only on her choice of measurement and λ, not on Bob's choice, and vice versa.

Bell's theorem shows that these two assumptions lead to testable inequalities that quantum mechanics can violate.

3.3 The CHSH Inequality

Consider two parties (Alice and Bob) with two measurement settings each: Alice chooses A or A', Bob chooses B or B'. Each observable has outcomes ± 1.

Define the *CHSH operator*:

$$S = A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B'.$$

Theorem 3.1 (CHSH inequality)**.** *For any local hidden variable model,*

$$\langle S \rangle \leq 2.$$

Sketch of proof. For any hidden variable λ, the outcomes $A(\lambda), A'(\lambda), B(\lambda), B'(\lambda) \in \{\pm 1\}$. Consider

$$S(\lambda) = A(\lambda)B(\lambda) + A(\lambda)B'(\lambda) + A'(\lambda)B(\lambda) - A'(\lambda)B'(\lambda).$$

Factor:

$$S(\lambda) = A(\lambda)\big(B(\lambda) + B'(\lambda)\big) + A'(\lambda)\big(B(\lambda) - B'(\lambda)\big).$$

Since $B(\lambda) \pm B'(\lambda) \in \{-2, 0, 2\}$, it follows that $|S(\lambda)| \leq 2$. Taking expectation values over λ preserves this bound. □

3.4 Quantum Mechanical Violation

Quantum mechanics predicts violations of the CHSH inequality. For the state $|\Phi^+\rangle$, choose measurement operators:

$$A = \sigma_z, \quad A' = \sigma_x, \quad B = \tfrac{1}{\sqrt{2}}(\sigma_z + \sigma_x), \quad B' = \tfrac{1}{\sqrt{2}}(\sigma_z - \sigma_x).$$

Then

$$\langle S \rangle_{\Phi^+} = 2\sqrt{2},$$

known as the *Tsirelson bound*. This exceeds the classical bound of 2, demonstrating that no local hidden variable theory can reproduce all quantum predictions.

3.5 Experimental Tests

Bell inequalities have been tested extensively, most recently in loophole-free experiments (2015). The 2022 Nobel Prize in Physics was awarded to Aspect, Clauser, and Zeilinger for their foundational experiments demonstrating entanglement and violation of Bell inequalities.

3.6 Summary

- EPR raised doubts about the completeness of quantum mechanics.
- Bell showed that quantum predictions conflict with the joint assumptions of locality and realism.
- CHSH inequality provides a concrete test, with quantum mechanics predicting stronger correlations.
- Experiments confirm quantum predictions, forcing us to abandon either locality, realism, or both.

4 Entanglement as a Resource

Entanglement is not only a peculiar feature of quantum theory but also a powerful *resource* that enables communication and computation tasks impossible in classical settings. In this section we discuss three canonical examples: superdense coding, quantum teleportation, and gate teleportation.

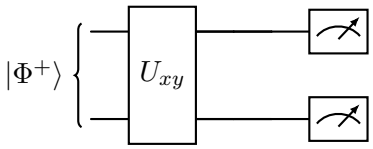
4.1 Superdense Coding

Idea. Classically, one qubit (a two-level system) can carry at most one bit of information. With prior entanglement, however, Alice can transmit *two* classical bits to Bob by sending only one qubit.

Protocol.

1. Alice and Bob share a maximally entangled pair, say $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
2. To send the classical bits $(x, y) \in \{0, 1\}^2$, Alice applies one of four unitary operations to her qubit:
$$(x, y) = \begin{cases} (0,0): & I \\ (0,1): & \sigma_x \\ (1,0): & \sigma_z \\ (1,1): & i\sigma_y \end{cases}$$
This transforms $|\Phi^+\rangle$ into one of the four Bell states.
3. Alice sends her qubit to Bob. Now Bob has both qubits.
4. Bob performs a Bell basis measurement, distinguishing perfectly between the four Bell states, and recovers (x, y).

Circuit Representation.



where $U_{xy} \in \{I, X, Z, iY\}$ encodes the two classical bits.

Outcome. Two classical bits are transmitted by sending only one qubit, thanks to shared entanglement.

4.2 Quantum Teleportation

Idea. Teleportation uses entanglement and two classical bits of communication to transfer an *unknown* qubit state from Alice to Bob, without physically sending the qubit.

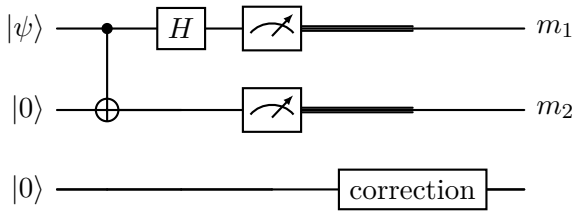
Protocol.

1. Alice wants to send $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob.
2. Alice and Bob share a Bell state $|\Phi^+\rangle_{AB}$.
3. Alice applies a CNOT and Hadamard to her two qubits (the unknown state and her half of $|\Phi^+\rangle$).
4. Alice measures her two qubits in the computational basis, obtaining two classical bits (m_1, m_2).
5. She sends (m_1, m_2) to Bob over a classical channel.
6. Depending on (m_1, m_2), Bob applies a correction:

$$(m_1, m_2) = \begin{cases} (0,0): & I \\ (0,1): & \sigma_x \\ (1,0): & \sigma_z \\ (1,1): & \sigma_z\sigma_x \end{cases}$$

After this correction, Bob holds $|\psi\rangle$.

Circuit.



Remarks.

- No faster-than-light communication: Bob must wait for Alice's classical bits.
- No-cloning: Alice's measurement destroys her copy of $|\psi\rangle$.

4.3 Gate Teleportation and Teleported CNOT

Motivation. Some two-qubit gates are difficult to implement directly in hardware. Gate teleportation provides a way to implement them using only entanglement, Bell measurements, and single-qubit corrections.

Example: Teleported CNOT.

1. Suppose we want to implement a CNOT between qubits in two separate locations.
2. Prepare entangled Bell pairs as resources.
3. Use quantum teleportation to transfer the state of the control and target qubits into an ancilla space, where the CNOT is applied.
4. Teleport the results back, applying corrections depending on measurement outcomes.

Key point. The combination of teleportation and entanglement allows us to *simulate* a nonlocal CNOT gate. If the measurement results are unfavorable, the protocol can be retried until it succeeds.

4.4 Summary

- **Superdense coding:** Send 2 classical bits by transmitting 1 qubit (requires shared entanglement).
- **Quantum teleportation:** Send 1 unknown qubit by transmitting 2 classical bits (requires shared entanglement).
- **Gate teleportation:** Use entanglement and teleportation to effectively implement otherwise difficult gates, such as CNOT, across separate systems.

Entanglement thus acts as a versatile resource, enabling powerful communication and computational primitives.

5 Three-Qubit Entanglement: GHZ and W States

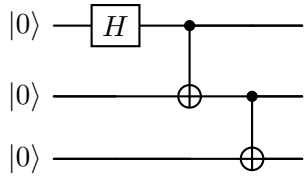
So far, we have studied entanglement in two-qubit systems. Multi-qubit entanglement introduces new classes of states that are not reducible to combinations of Bell pairs. Two canonical examples are the GHZ state and the W state.

5.1 The GHZ State

Definition. The Greenberger–Horne–Zeilinger (GHZ) state for three qubits is

$$|\Psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(\,|000\rangle + |111\rangle\,).$$

Circuit preparation. One way to prepare $|\Psi_{\text{GHZ}}\rangle$ is:



Measurement properties. If we measure the first qubit in the computational (Z) basis:

- With probability 1/2 we obtain outcome 0, leaving $|000\rangle$.
- With probability 1/2 we obtain outcome 1, leaving $|111\rangle$.

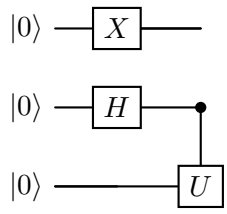
In either case, the remaining state is a *product state*, so measuring one qubit collapses the entanglement.

5.2 The W State

Definition. The W state is

$$|\Psi_W\rangle = \frac{1}{\sqrt{3}}(\,|001\rangle + |010\rangle + |100\rangle\,).$$

Circuit preparation. One possible preparation circuit uses an open-controlled NOT (inverted control) and a single-qubit unitary U:



where

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

Measurement properties. If we measure one qubit in the computational basis:

- If the outcome is 0, the remaining two qubits are still entangled (e.g. $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$).
- If the outcome is 1, the remaining two qubits are in a product state $|00\rangle$.

Unlike the GHZ state, the W state retains bipartite entanglement even after the loss of one qubit, making it more robust.

5.3 Comparison: GHZ vs. W

- **GHZ:** Maximally entangled, but fragile — measuring or tracing out one qubit destroys entanglement.
- **W:** Less entangled globally, but robust — entanglement survives if one qubit is lost.
- They represent two inequivalent classes of three-qubit entanglement; no local operations and classical communication (LOCC) can convert one into the other.

5.4 Summary

- Multi-qubit systems exhibit qualitatively new entanglement structures.
- GHZ states: strong three-way entanglement, useful for nonlocality tests.
- W states: robust distributed entanglement, useful for fault-tolerant protocols.

6 Three-Party Quantum Teleportation

We now extend the standard two-party teleportation protocol to a scenario with three parties: Alice, Bob, and Charlie. The shared resource is no longer a Bell pair, but a GHZ state.

6.1 Setup

Alice, Bob, and Charlie share the GHZ state

$$|\Psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(\,|000\rangle + |111\rangle\,),$$

where each holds one qubit. Charlie additionally possesses a fourth qubit in the unknown state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,$$

which he wants to teleport to Alice, with Bob's help.

Thus, the total four-qubit system (ordered as Alice–Bob–Charlie–input) is initially

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(\,|000\rangle + |111\rangle\,)_{ABC} \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)_D.$$

6.2 Circuit Operations

1. Charlie applies a CNOT gate between his two qubits (control = input D, target = his GHZ share C).
2. Charlie applies a Hadamard gate H to his input qubit D.
3. Bob applies a Hadamard gate H to his qubit B.

This sequence entangles Charlie's input with the GHZ resource and distributes the information across the system.

6.3 Measurement and Communication

- Charlie measures his two qubits (C and D) in the computational basis.
- Bob measures his qubit B in the computational basis.
- Together, they have three classical bits of outcomes (m_B, m_C, m_D).
- They communicate these results to Alice using classical channels.

6.4 Alice's Correction Operations

Based on (m_B, m_C, m_D), Alice applies a correction operator to her qubit A to reconstruct $|\psi\rangle$ (up to a global phase). The required corrections are combinations of Pauli X and Z operators.

m_B	m_C	m_D	Alice's operation
0	0	0	I
0	0	1	X
0	1	0	Z
0	1	1	XZ
1	0	0	Z
1	0	1	XZ
1	1	0	I
1	1	1	X

6.5 Discussion

- The three-party teleportation scheme generalizes the two-party case but requires a three-qubit entangled resource (GHZ).
- Charlie cannot send $|\psi\rangle$ to Alice without Bob's participation: all three parties' measurement outcomes are needed.
- This protocol demonstrates how multipartite entanglement enables distributed quantum communication tasks.

6.6 Summary

- A GHZ state can act as a three-party teleportation resource.
- Teleportation requires local gates (CNOT, Hadamard), projective measurements, and classical communication.
- Alice's corrections depend on the measurement outcomes of both Bob and Charlie.

7 Approximate Quantum Cloning

7.1 The No-Cloning Theorem

Quantum mechanics forbids the existence of a universal device that perfectly clones an arbitrary quantum state.

Theorem 7.1 (No-cloning)**.** *There is no unitary U and fixed state $|a\rangle$ such that*

$$U\big(\,|\psi\rangle \otimes |a\rangle\,\big) = |\psi\rangle \otimes |\psi\rangle$$

for all qubit states $|\psi\rangle$.

Sketch. Suppose such U exists. For two distinct states $|\phi\rangle, |\psi\rangle$ we must have

$$\langle\phi\rangle\,\psi = \langle\phi\rangle\,\psi^2,$$

which implies $\langle\phi\rangle\,\psi = 0$ or 1. Thus, cloning cannot hold for all non-orthogonal states. □

7.2 Motivation for Approximate Cloning

Although perfect cloning is impossible, one can design circuits that produce *approximate* copies of a qubit. The quality of these copies is quantified by their *fidelity* with the original state.

7.3 An Approximate Cloning Circuit (Exercise)

Consider a circuit acting on three qubits, defined on basis states:

$$\begin{aligned}
|000\rangle &\mapsto \sqrt{\tfrac{2}{3}}\,|000\rangle + \sqrt{\tfrac{1}{6}}\,|011\rangle + \sqrt{\tfrac{1}{6}}\,|101\rangle\,,\\
|100\rangle &\mapsto \sqrt{\tfrac{2}{3}}\,|111\rangle + \sqrt{\tfrac{1}{6}}\,|010\rangle + \sqrt{\tfrac{1}{6}}\,|100\rangle\,.
\end{aligned}$$

The action on the other six basis states is irrelevant.

Interpretation. The first qubit is the *input*, and the next two are blank ancillas. The circuit attempts to spread the information in the input across the three qubits.

7.4 Output State for Arbitrary Input

Let $|\chi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. The input to the circuit is $|\chi\rangle \otimes |00\rangle$. By linearity, the output is

$$\begin{aligned}
|\Psi_{\text{out}}\rangle = \alpha\Big(&\sqrt{\tfrac{2}{3}}\,|000\rangle + \sqrt{\tfrac{1}{6}}\,|011\rangle + \sqrt{\tfrac{1}{6}}\,|101\rangle\Big)\\
&+ \beta\Big(\sqrt{\tfrac{2}{3}}\,|111\rangle + \sqrt{\tfrac{1}{6}}\,|010\rangle + \sqrt{\tfrac{1}{6}}\,|100\rangle\Big).
\end{aligned}$$

7.5 Density Matrix Description

The full output state is $\rho = |\Psi_{\text{out}}\rangle\langle\Psi_{\text{out}}|$. To analyze individual qubits, we take partial traces.

Reduced states. The reduced density matrix of qubit 1 is

$$\rho_1 = \mathrm{Tr}_{23}(\rho),$$

and similarly for qubit 2.

A detailed calculation (exercise) shows:

$$\rho_1 = \rho_2 = \frac{5}{6} |\chi\rangle\langle\chi| + \frac{1}{6} |\chi^\perp\rangle\langle\chi^\perp|,$$

where $|\chi^\perp\rangle$ is orthogonal to $|\chi\rangle$.

7.6 Fidelity of Approximate Cloning

The fidelity of each approximate clone with the original state is

$$F = \langle\chi| \rho_1 |\chi\rangle = \frac{5}{6}.$$

Interpretation. Each of the first two qubits individually has fidelity 5/6 with the input state $|\chi\rangle$. This is the best achievable for universal $1 \to 2$ cloning of qubits.

7.7 Summary

- Perfect cloning is impossible due to the linearity of quantum mechanics.
- Approximate cloners exist, producing imperfect but high-fidelity copies.
- The universal $1 \to 2$ quantum cloner achieves fidelity $F = 5/6$.
- Such approximate cloning machines are relevant in quantum cryptography, where they model eavesdroppers' optimal strategies.

8 Wrap-Up and Recap

In this lecture, we explored how the strange aspects of quantum mechanics— superposition, measurement disturbance, and entanglement—become powerful *resources* for communication and computation.

8.1 Summary of Topics

- **Quantum Key Distribution (BB84):**
  - Uses superposition and random bases (Z, X) to distribute a secret key.
  - Eavesdroppers cannot avoid disturbing the system, leading to detectable errors.
- **Entanglement Basics:**
  - Defined as states that cannot be written as products.
  - Quantified using entanglement entropy.
  - Bell states are maximally entangled two-qubit states with entropy $\ln 2$.
- **EPR and Bell Inequalities:**
  - The EPR paradox questioned the completeness of quantum mechanics.

  - Bell's theorem showed that local hidden variable models are incompatible with quantum predictions.
  - The CHSH inequality demonstrates this conflict: classical bound $|S| \leq 2$, quantum maximum $2\sqrt{2}$.
  - Experiments confirm quantum mechanics (2022 Nobel Prize).

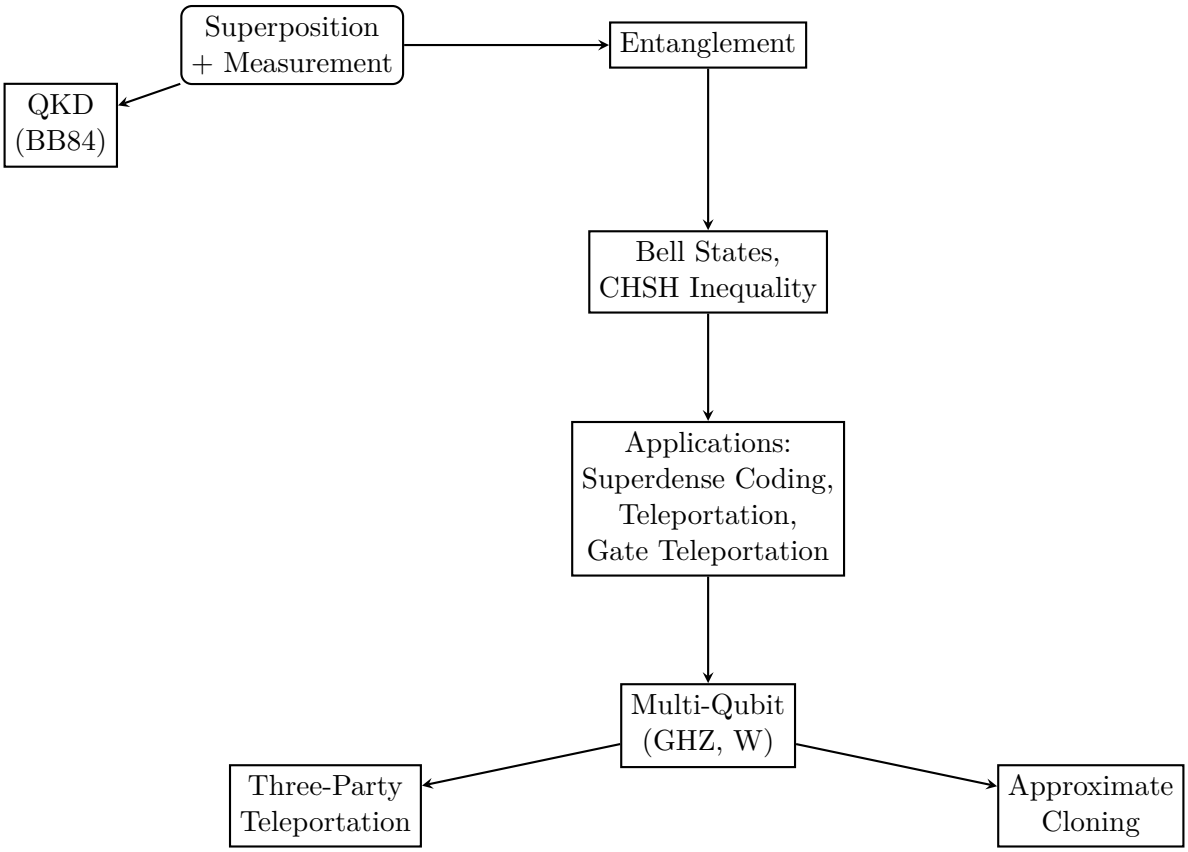
- **Entanglement as a Resource:**
  - *Superdense coding:* Transmit two classical bits using one qubit and shared entanglement.
  - *Quantum teleportation:* Transmit an unknown qubit using two classical bits and shared entanglement.
  - *Gate teleportation:* Implement difficult gates (e.g. CNOT) using entanglement and teleportation primitives.

- **Multi-Qubit Entanglement:**
  - GHZ state: maximally entangled three-qubit state, fragile to measurement or loss of qubits.
  - W state: robust entangled state, retains bipartite entanglement even if one qubit is lost.
  - GHZ and W represent inequivalent entanglement classes.

- **Three-Party Teleportation:**
  - GHZ states enable teleportation across three parties.
  - Charlie's input qubit is teleported to Alice with the cooperation of Bob.
  - Alice's correction depends on measurement results from both Bob and Charlie.

- **Approximate Cloning:**
  - Perfect cloning of unknown quantum states is forbidden.
  - Approximate cloning machines can produce imperfect copies.
  - Optimal $1 \to 2$ universal cloner achieves fidelity $F = 5/6$.

8.2 Conceptual Map



8.3 Concluding Remarks

This material demonstrates how the "weirdness" of quantum mechanics is not an obstacle but a resource. Superposition enables secure communication, entanglement powers enhanced communication and computation, and even fundamental no-go theorems (like no-cloning) inspire new concepts such as approximate cloning and fidelity analysis.