08/25/15

David Gay
Lecture 1

Decompositions of 4-Manifolds (assume smooth)

Want: Combinatorial description with some sort of existence and uniqueness properties

Content

1. Handle decomposition (any dimension)
2. Heegaard splitting (3D)
3. Trisections (4D)

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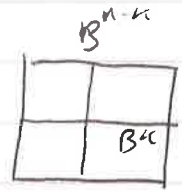
n-dimensional k-handle
$H^{n,k} = B^k \times B^{n-k}$



"Build in smoothings" but really just being imprecise about corners

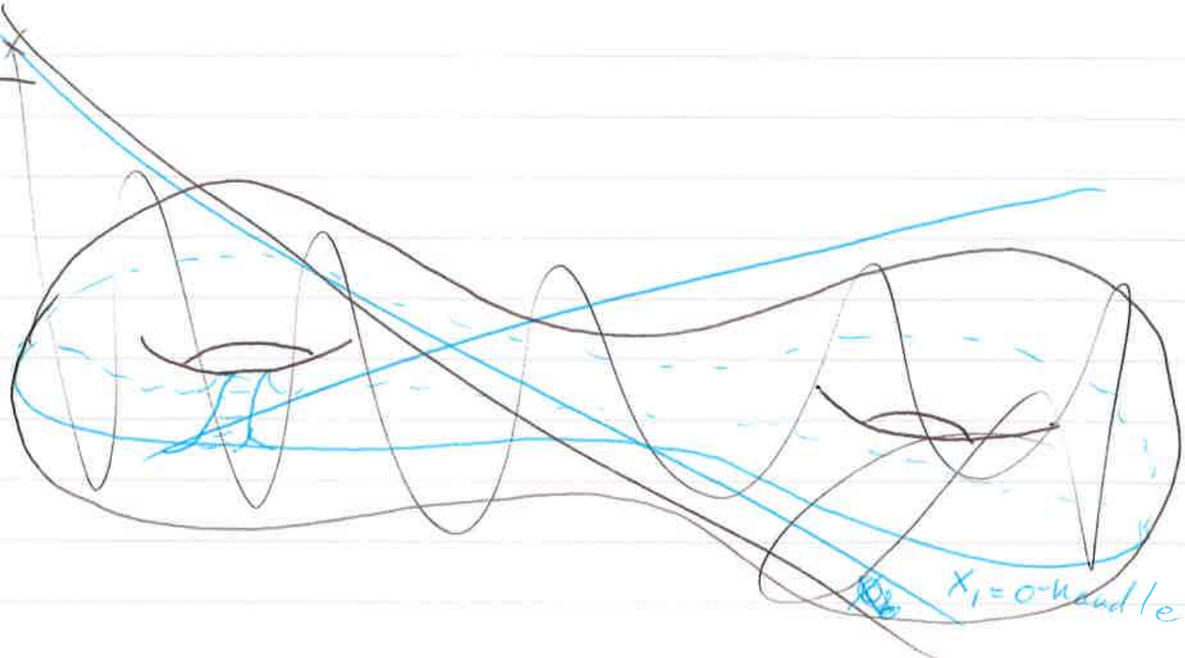
$\partial_- H^k = S^{k-1} \times B^{n-k}$ = attaching region
$\partial_+ H^k = B^k \times S^{n-k-1}$ = free region

Supress "n" from notation when dimension is clear

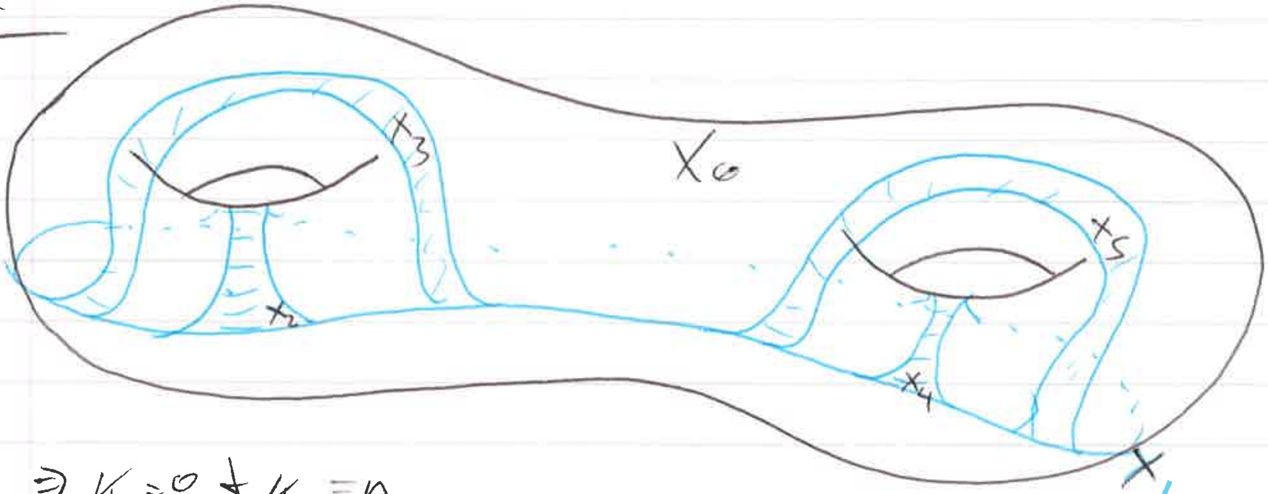
Convention: $S^{-1} = \emptyset$
So $H^0 = B^0 \times B^n = B^n$, $\partial_- H^0 = \emptyset$
$H^n = B^n \times B^0 = B^n$, $\partial_+ H^n = \emptyset$

For X^n closed: a handle decomposition
$X = X_1 \cup X_2 \cup \ldots \cup X_p$ s.t. $\forall i, \exists \varphi_i : X_i \xrightarrow{\cong} H^{n,k_i}$
and let $X_{\leq j} = X_1 \cup \ldots \cup X_j$, then
$X_i \cap (X_{\leq i-1}) = \partial_- X_i = \varphi_i^{-1}(\partial_- H^{k_i})$

Ex



Ex



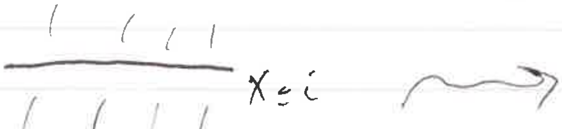
$\Rightarrow k_1 = 0 + k_p = n$

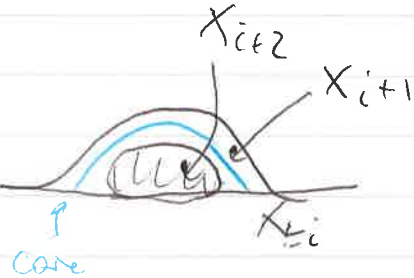
Things to do to handle decompositions:

① Isotopy (smooth 1-parameter family)
② Reorder (when conditions are maintained)

can't switch

③ Birth/death of cancelling pair

Birth 



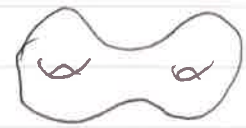
k-handle and k+1 handle

Exercise[1]: Understand how to describe this carefully in all dimensions and all indices.

Thm $\forall$ closed X^n has a handle decomposition unique up to isotopy, reordering, and birth/death.

Bonus operation!

(4) Reverse ordering and switch k_i with $n-k_i$
"Turn upside down"

Exercise[2]: digest by example e.g.  1 0-h, 4 1-h, 1 2-h

turn upside down w/ same number of each. Demonstrate uniqueness. (i.e. show decompositions are same)

Def A handle decomposition is simple if $0 = k_1 \leq k_2 \leq k_3 \leq \dots \leq k_p = n$ and if $k_i = k_j$ then $X_i \cap X_j = \phi$.

Exercise[3]: Prove by isotopy and reordering (assume only one 0-handle and n-handle) every handle decomposition can be made simple. Hint: Think about transversality in middle D.

From existence, also get: $\forall\, X^n \cong \coprod H^{n,k_i} / \sim$ ← gluing maps

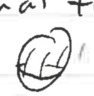
$\partial_- H^{n,k_i} \hookrightarrow \partial$(what you've already built)

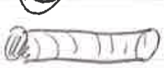
care about isotopy class of embedding = framed S^{k-1} in M^{n-1}

$\partial_- H^{n,k} = \overbrace{S^{k-1} \times B^{n-k}}$

can think of a framing of bdry of cell (an n,k handle is an n-dimensional thickening of a k-cell)

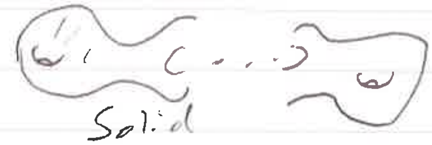
D=3

0-h 

1-h 

2-h 

Def Heegaard splitting:

$M^3 = M_1 \cup M_2$, $\quad M_i = \natural^g S^1 \times B^2 =$ 

Solid

$M_1 \cap M_2 = \Sigma_g \cong \#^g S^1 \times S^1 =$ genus g surface

Thm Every oriented closed 3-mfld has a Heegaard splitting.

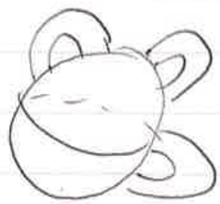
Pf idea

1, 2-handles , 2, 3-handles

turn handle decomp upside down

get another handlebody

this is simple decomp.



Def Stabilization of H.S.

$M = M_1 \cup M_2$, choose arc $a \subset M_1$, boundary parallel

$M_1' = M_1 \setminus \nu(a)$

$M_2' = M_2 \cup \nu(a)$

Thm (Reidemeister–Singer)

Any two H.S. of same 3-mflds are related by isotopy and stabilization.

Pf Uniqueness of handle decompostion

Ex S^3; $g=0$

Lens space: $g=1$ (by definition)

Dimension 4

0-h : B^4 $\quad \partial = S^3 = \mathbb{R}^3 \cup \infty$

1-h : $B^1 \times B^3$ $\quad \partial_- = S^0 \times B^3$ 

2-h : $B^2 \times B^2$ $\quad \partial_- = S^1 \times B^2$  ← Framing, tells how much B^2 "twists"

3-h : $B^3 \times B^1$ $\quad \partial_- = S^2 \times B^1$

~~Kirby calculus for~~ 4-manifolds

Trisections

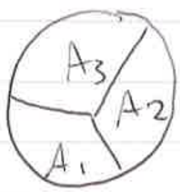
A trisection of X^4 is $X = X_1 \cup X_2 \cup X_3$ s.t.
$X_i \cong \natural^k S^1 \times B^3$, $X_i \cap X_j \cong \natural^{g \geq k} S^1 \times B^2$
$X_1 \cap X_2 \cap X_3 \cong \#^g S^1 \times S^1 = F$

regular nbhd of $\vee^k S^1$ in B^4

Ex $S^4 \subset \mathbb{R}^5$ (just project)

$$\begin{array}{ccc} S^4 & \subset & \mathbb{R}^5 \\ \downarrow \pi & & \downarrow \\ B^2 & \subset & \mathbb{R}^2 \end{array}$$

π^{-1}



Each piece is a 4 ball, they intersect along 3-balls

$X_i = \pi^{-1}(A_i)$ is a trisection of S^4

$k = 0$, $g = 0$

David Gay: Lecture 2

Thm 1 (Gay, Kirby) Every closed X^4 has a trisection. Therefore $\forall\ X \cong \amalg^3 \natural^k S^1 \times B^3 / \sim$ gluing info

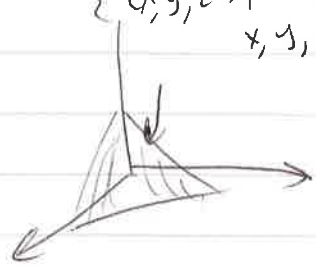
existence

Ex $\mathbb{C}P^2 = \{[z_0 : z_1 : z_2]\}$ ← homogeneous coordinates

$$\downarrow \varphi$$

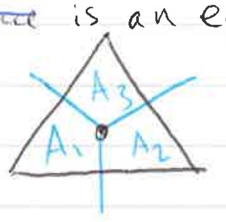
$$\left(\frac{|z_0|^2}{\sum |z_i|^2}, \frac{|z_1|^2}{\sum |z_i|^2}, \frac{|z_2|^2}{\sum |z_i|^2} \right)$$

$\{(x,y,z) \mid x+y+z=1,\ x,y,z \geq 0\}$



$\operatorname{Im} \varphi$ = standard 2-simplex in $\mathbb{R}^3$

~~One face~~ is an equilateral triangle



Claim: $X_i = \varphi^{-1}(A_i)$ is a trisection

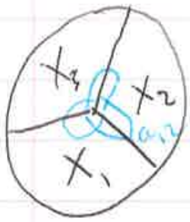
Ex Work out details.

$\boxed{k=0,\ g=1}$

$\varphi^{-1}(p) = 2$-torus

Def <u>Stabilization</u>:

Given $X = X_1 \cup X_2 \cup X_3$



Choose $a_{ij} \subset X_i \cap X_j$ ∂-parallel with disjoint endpoints.

$$X_1' = X_1 \cup \nu(a_{23}) \setminus (\nu(a_{12}) \cup \nu(a_{31}))$$

repeat cyclically (i.e. add 1 to index mod 3)

Claim $X = X_1' \cup X_2' \cup X_3'$ is a trisection with $g' = g+3$, $k' = k+1$

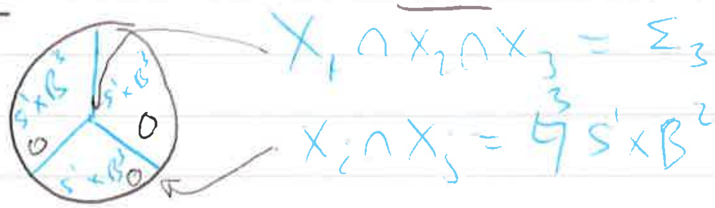
Analogy "digging a ditch doesn't change the topology of the floor"

Fact $\chi(X) = 2 + g - 3k$

Note If k is different between pieces X_i we call it an unbalanced trisection

Thm 2 (G, Kirby) Any two trisections of given X^4 are related by stabilization and isotopy. (= ambient isotopy)

Exercise: Visualize the once stabilized trisection of S^4.



Goal: Sketch proof of existence and uniqueness. (Use handles)

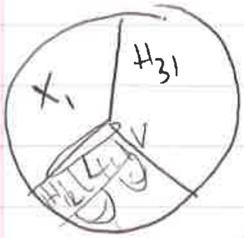
Existence: Given X^4 closed, oriented with a simple handle decomposition, add cancelling pairs if necessary so that # 1 handles = # 3 handles

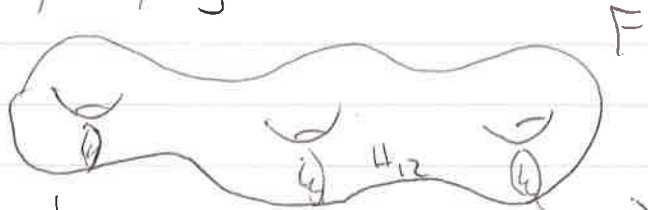
Lem If the handle decomposition has these properties:

- k 1-handles
- $g-k$ 2-handles

- Let $X_1 = $ 0-h $\cup$ 1-h's $\cong \natural^k S^1 \times B^3$ and 2-h attached along framed link $L \subset \partial X_1$ s.t. $\exists$ Heegaard splitting s.t. $\partial X_1 = H_{12} \cup H_{31}$ s.t. $L \subset \text{int } H_{12}$

Lem continued. L' has framing coming from F (~~pushoff~~ pushoff) and $\exists$ compressing disks $D_1, \dots, D_g$ for $F = \partial H_{12}$





(choice of cocores of a H.D.)

with components of L' geometrically dual to $\partial D_1, \dots, \partial D_{g-k}$

then let $X_2 = \nu(H_{12}) \cup$ 2-handles
$X_3 = X \setminus (X_1 \cup X_2)$
is a trisection

Why is X_2 what it needs to be?

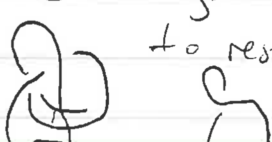
$$[0,1] \times \natural^g S^1 \times B^2 = \natural^g S^1 \times B^3$$

Claim attaching $g-k$ 2-handles kills $g-k$ of these factors
Geometric duality $\Rightarrow$ 2-handles kill $g-k$ of 1-handles in $\natural^g S^1 \times B^3$

Q: How does that change bdry?
X_3 ends up being 3,4 handles
Convince yourself that intersections are handlebodies and not more complicated 3-mflds.

To get a handle decomposition to look like this:
Start with: $X_1 = $ 0-h $\cup$ 1-h's, $\partial X_1 = \#^k S^1 \times S^2$ has a standard H.S. into two $\natural^k S^1 \times B^2$

2-handles attached along L. Flow L onto F with crossings. Stabilize Heegaard splitting to resolve crossings. What about framing?



add kinks, stabilize again to

For geometric duality, stabilize one more time.

Sketch of uniqueness

Given two trisections:
1. Turn both into handle decompositions (satisfy lemma)
2. Relate two handle decompositions by:
   - 1–2 births
   - 2–3 births
   - handle slides 1's over 1's, 2's over 2's, or 3's over 3's (isotopies)

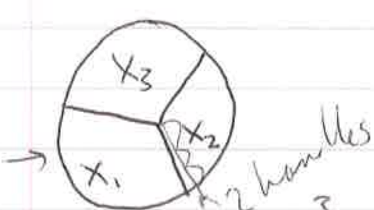
   isotopies between indices

3. Do each move preserving conditions of lemma. Interpret in terms of stabilizations.

08/26/15 Day 2
Lecture 3

Recall Trisection of (closed, oriented) X^4

$X_i \cap X_j = \natural^g S^1 \times B^2$



$F = X_1 \cap X_2 \cap X_3 \cong \Sigma_g = \#^g S^1 \times S^1$

$X_i \cong \natural^k S^1 \times B^3$

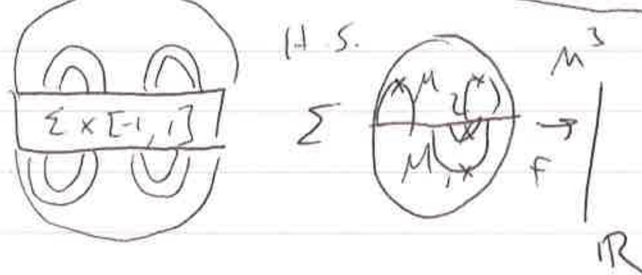
Q: Why 3 pieces?
2 causes problems with 2-handles. 3 ⇒ topology of each piece is described by one integer

Uniqueness is up to stabilization

<u>Diagrams:</u> Heegaard splittings vs. Heegaard diagram $(\Sigma, \underline{\alpha}, \underline{\beta})$ g s.c.c.

g simple closed curves

H.d.



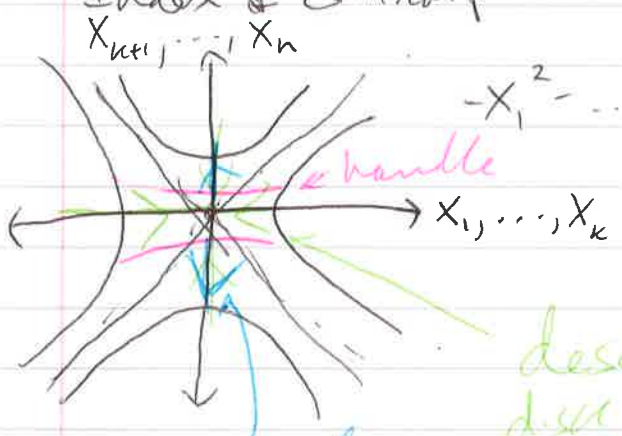
Where f is a morse function and there is a gradient like vector field.

<u>Def</u> (Morse function)
Near critical points f looks like $(x_1, \dots, x_n) \mapsto \sum \pm x_i^2$

<u>Def</u> Index = # $-$'s

<u>Def</u> (Gradient-like vector field)
V s.t. $df(V) > 0$ when $df \neq 0$ and at critical points
$V = \sum \pm 2x_i \partial x_i$
Index of critical points $\longrightarrow$ index of handles

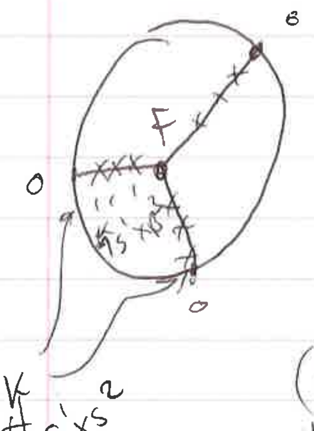
$$-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$



descending manifold
disk of dimension k

ascending manifold
S^{n-k-1} belt sphere of handle

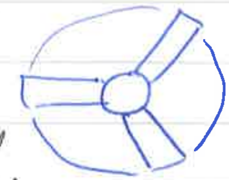
Trisection Diagrams



Put a Morse function (+V) on each 3-d handlebody

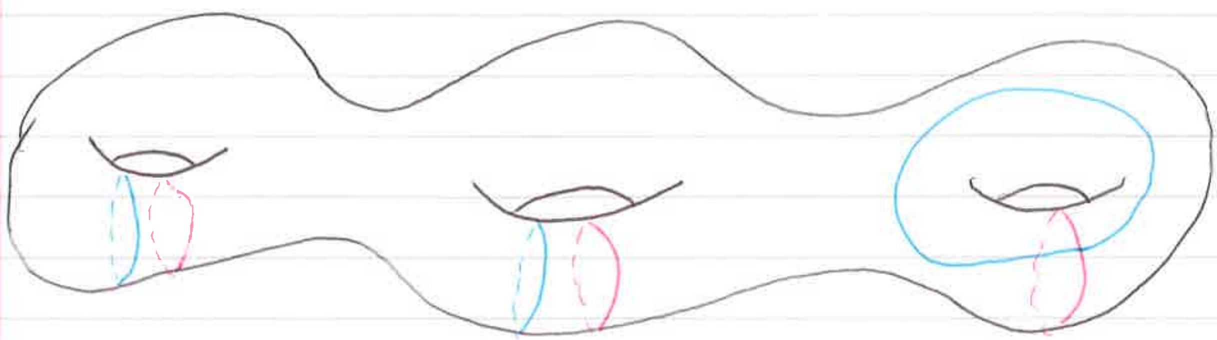
$(F, \underline{\alpha}, \underline{\beta}, \underline{\gamma})$ trisection diagram

So condition:
$(F, \underline{\beta}, \underline{\gamma})$, $(F, \underline{\alpha}, \underline{\beta})$, and $(F, \underline{\gamma}, \underline{\alpha})$ are all Heegaard diagrams for $\#^k S^1 \times S^2$



Start w/ $F \times D^2$, attach handlebodies $\times I$, then Laudenbach-Poenaru uniquely defines a 4-manifold.

Ex

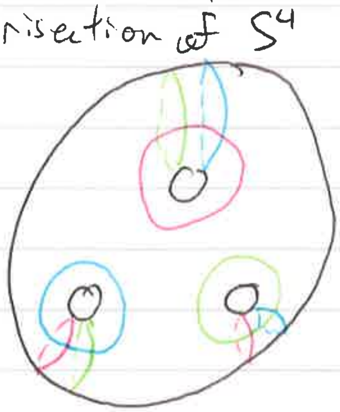


$g = 3$, $k = 2$

\# of 4-mflds = \# of diagrams
Connected Sum

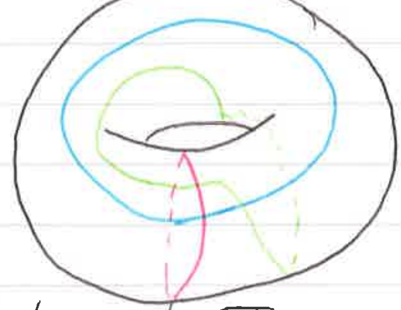
Can think of this as a special case of Nick Castro's result.

Ex stabilized standard genus 0 trisection of S^4



$g = 3$
$k = 1$

Ex $\mathbb{CP}^2$



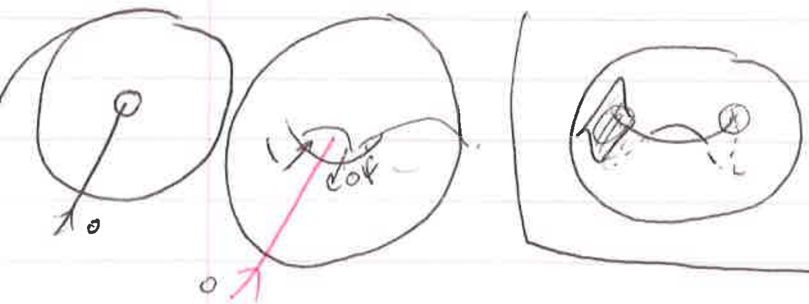
$g = 1$
$k = 0$

Reverse orientation get $\overline{\mathbb{CP}^2}$

Morse 2-functions

wiggle so that change parameterization but have same maps

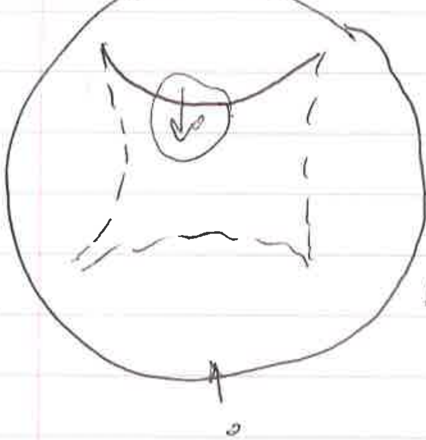
Stable, smooth map to 2-dim



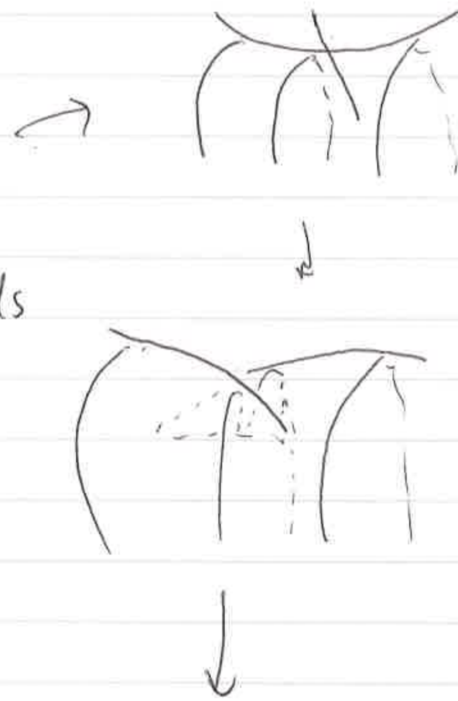
really birth or death promoted to higher dimension.

morse function coming in radially

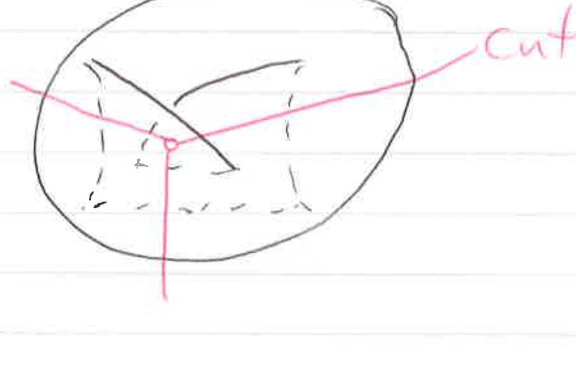
lean on make folds/cusps



folds

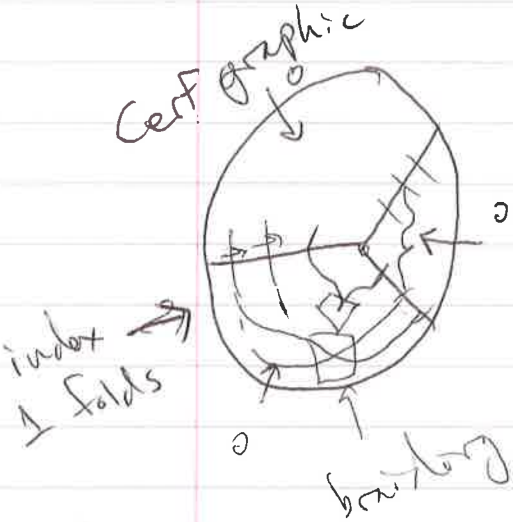


trisection of T^2



Every trisection of X^4 comes from a Morse 2-function to D^2

Cusps



Movie John Stroud UGA

Perturbations of monkey saddle

$Re(z^3) = x^3 - 3xy^2$ (single crit point at (0,0))

Could perturb by $+\varepsilon x + \varepsilon y + \delta(x^2+y^2)$

Relates to "universal unfolding"

Lecture 4

S^4 $g=1$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $S^1 \times S^3$ $g=2$, $g=3$ Meier Zupan

Questions

① Find Examples (low genus)

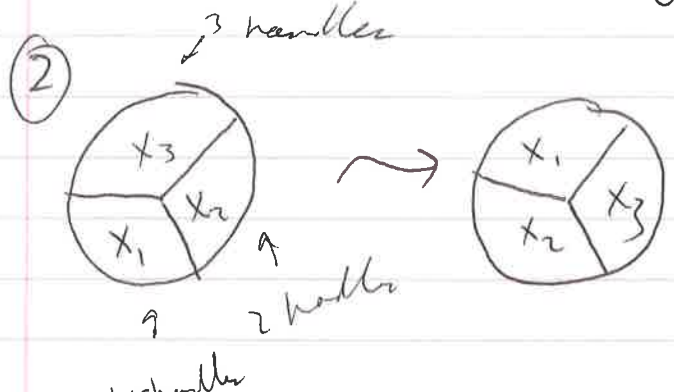
$g=3$ (nontrivial) (not a connected sum)

$\pi_1 = 0$

For 2-cpt answer is $0 \, ⚭ \, 0$

¿∃ 3-component link with surgery $= S^3$ not slideable to a split link?

②



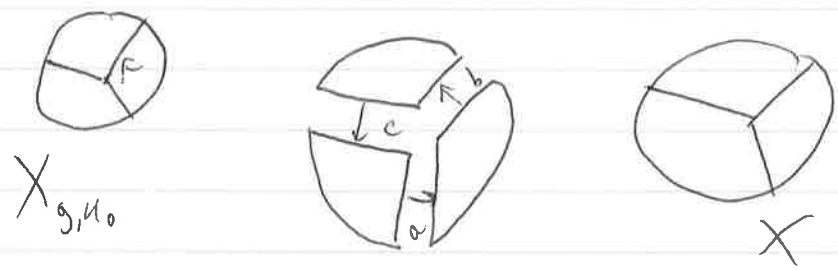
turn trisection by $2\pi/3$

understand this better. Turning a Morse function upside down is Poincaré duality, is this Poincaré "triality"?

③ Examples of inequivalent trisections of same X^4 with same g?

Waldhausen for trisections? (S^4?)

(4) Gluing maps: ($\text{Ex}\ \#^k S^1 \times S^3 \#^{g-k} \mathbb{CP}^2$)
$\forall\ g \geq k \geq 0 \quad \exists\ \overline{X}^{g,k}$ with (g,k) trisection
$\forall\ X$ w/(g,k) trisection, X can be obtained from $X^{g,k}$ by cutting open and regluing.



Fact Every self-diffeo of handlebody is determined by boundary

$\leftarrow$ Maps extending over each handlebody

$\text{Map}(F) \supset A, B, C$
Need $a \in A$, $b \in B$, $c \in C$, $abc = 1$.
Factorization
$(a,b,c) \sim (a',b',c')$ if $a=a'$ or $b=b'$ or $c=c'$

(5) Where does Heegaard-Floer theory come in?
The HF 4-mfd invariant from triples that doesn't come up here.

(6) Curve complex ($\Rightarrow$ stabilization/destabilization)

(7) Notion of symplectic trisection?
There is a notion of a contact Heegaard splitting

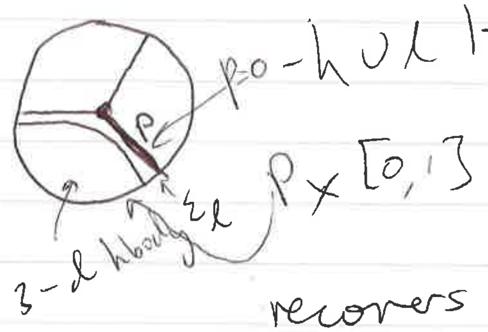
(8) Trisections of groups?
inclusions give surjections on fundamental groups and give cube of groups

Relative case:

Q: What is a trisection of a 3-manifold?

$M^3 = M_1 \cup M_2 \cup M_3$, $M_i \cong$ 0-h $\cup\, \ell$ 1-h's (3-d)
$M_i \cap M_j \cong$ 0-h $\cup\, \ell$ 1-h's (2-d)

$M_1 \cap M_2 \cap M_3$ = some circles

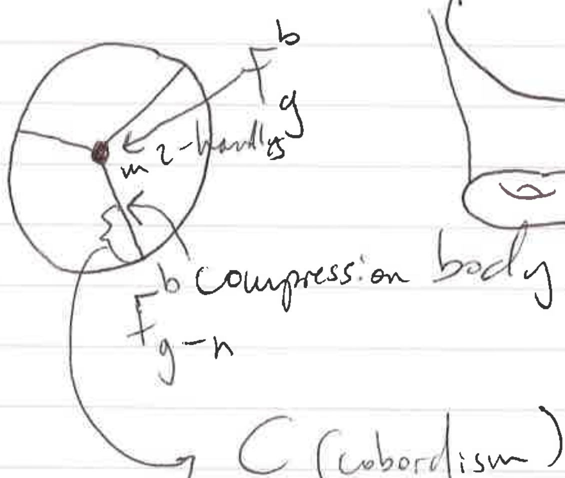


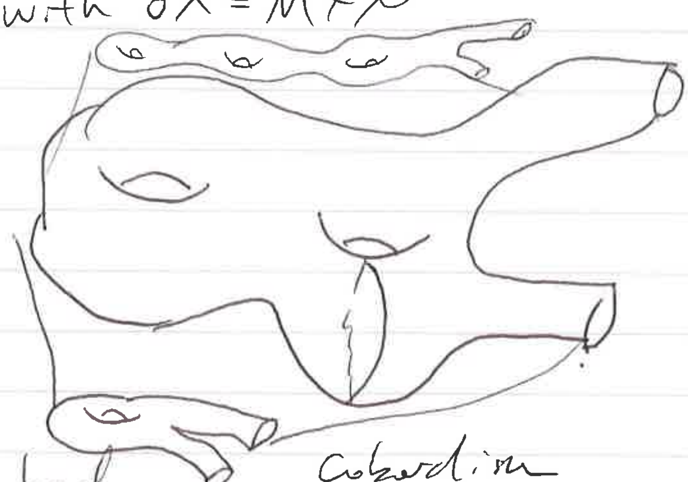
open book decomposition
P for page

3-d trisection stabilization
recovers Hopf plumbing for open books

Def of trisection of X with $\partial X = M \neq \emptyset$

~~Relative case~~
~~What is a~~





(G, Kirby)

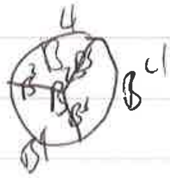
Thm $\forall X^4$ and any OBD on ∂X, $\exists$ trisection of X restricting to given OBD, any two trisections w/ same OBD are stably equivalent

Thm (Castro)

If X, X' trisections w/same OBD on $\partial X = \partial X'$, have then can glue to get trisection on closed $X \cup X'$.

Example

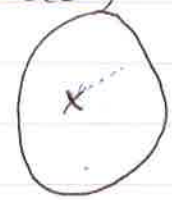
$$B^4 \xrightarrow{\text{proj}} B^2$$



$$B^4 \to B^2 \qquad (w,z) \overset{\pi}{\mapsto} w^2+z^2+\cancel{Re(z)}$$

(Lekili)

standard Lefschetz singularity



$\pi^{-1}(\text{reg pt})$ = annulus

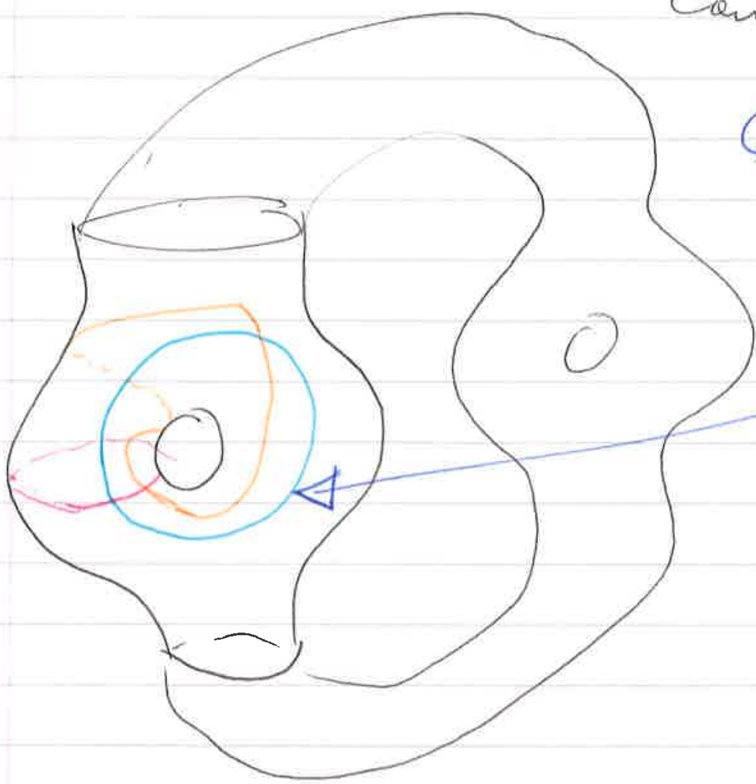
not stable as real smooth function

is as complex function

is std Lefschetz fibration

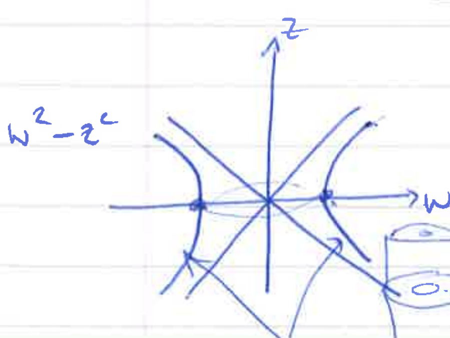
Can perturb $\pi \mapsto w^2+z^2+Re(z)$

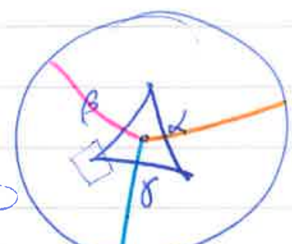
singular value $* \rightsquigarrow \triangleleft$



$\pi^{-1}(\text{reg pt})$ = annulus

$\pi^{-1}(\text{reg pt})$ = F





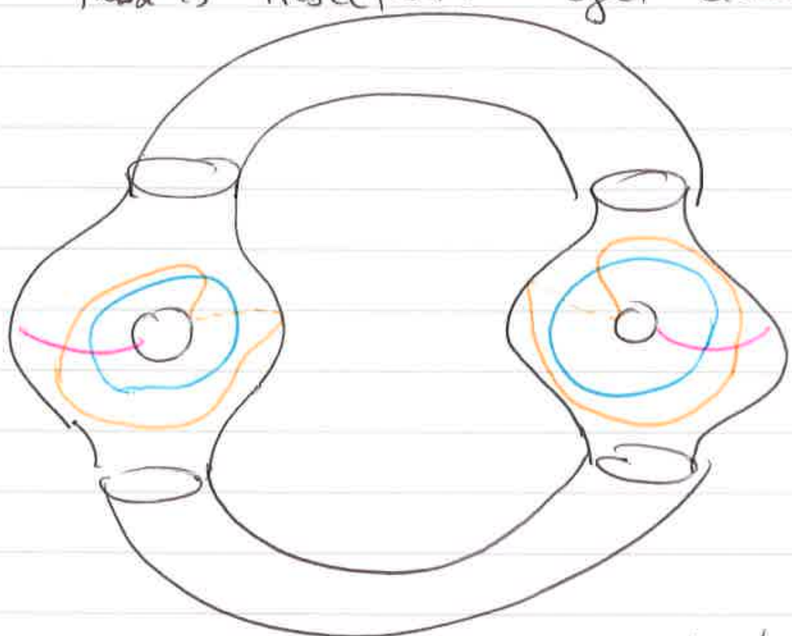
~~up to diffeo of F~~,

know $\alpha \cap \beta = 1$ pt

$\beta \cap \gamma = 1$ pt

$\gamma \cap \alpha = 1$ pt

What if we glue B^4 to $-B^4$
using Nick Castro's result &
there is trisection: get diagram



where are the third α, β, γ curves?

(need to include arcs
as well as closed curves
in relative trisection diagrams)