Discrete Structures 2824 -- A brief walk through the semester

Note that this is an incomplete guide through the material we have covered in CSCI 2824. It is **not** meant to be an exhaustive listing of material to study for the final exam. Also note that the "Example problems" are not meant to indicate level of difficulty or particularly important applications or problem types. They are just meant to jog your memory.

- **Binary Arithmetic**
    - Converting integers and fractions from decimal to binary
    - Originally, we started using these tools because operating bit-wise makes sense from a computational perspective. But it turns out that binary is also a useful introduction to **propositional logic**, where propositions take one of **two** values, True or False
    - How many bits does it take to encode a piece of information? (E.g., the homework problem about discovering who is leaking pizza secrets)
    - **Example problem:** How long of a bit string do we need in order to give each of 30 people an ID number?
- **Propositional Logic and Applications**
    - This was our first step into formally "thinking like a computer" (at least for this class). Computers don't see any shade of gray - that bit is either a 0 or a 1 - it's either True or False. And so, using propositional logic, we were able to construct propositional statements and assess their truth values. We used these to solve puzzles and represent **declarative statements**, which we later used to construct arguments and proofs.
    - Logical connectives and compound propositions
    - Truth values and truth tables
    - Conditionals, and conditionals in English language (necessary and/or sufficient conditions, "only if")
    - Logic Puzzles -- Knights and knaves! Who wants coffee?
    - **Example problem:** Knights and knaves. Satisfiability (first exam).
- **Logical Equivalences** -- from propositional logic, we found out that - just like in the spoken word - two statements might appear/sound different, but they mean the same thing
    - Logical equivalences were our first step into the world of manipulating propositional statements, which ultimately led to us writing out full-blown proofs.
    - **Propositional Satisfiability** -- a great application of our propositional logic skills, we were able to deduce that the Scooby Doo gang really ought to just stop being friends with Fred because he is so darn picky. But also, satisfiability connects our propositional statements to *reality* -- is it actually *possible* for this [possibly very complicated compound] proposition to be true?
    - **Example problem:** Logical equivalences proofs, like using truth tables to show that $p \rightarrow q \equiv \neg p \vee q$ (... or much more complicated things, using chains of logical equivalences)

- **Predicate Logic**
    - Turns out, propositional logic lacked the flexibility to make many statements that we would find useful in computational sciences. For example, it would be cumbersome to say “all computer science instructors ride a unicorn to work every morning and enjoy asparagus” via propositional logic. But predicate logic - by combining propositional functions and quantifiers, can handle these things!
    - Nested Quantifiers
        - Get comfortable translating to/from quantifiers from/to English.
    - **Example problem:** You and Tony are at the dog park, and he tells you this about the dogs: “For all of the dogs in this park, it is the case that dogs with brown fur always have a long tail.” Let the domain of discourse be all dogs in the park, and translate Tony’s rule into a quantifier statement. Be sure to define all propositional functions you might use.
- **Rules of Inference**
    - Convenient “mini-proofs” that we found to be useful again and again in the next section (Proofs), especially by linking together common sets of propositions and compound propositions.
    - **Example problem:** Pretty much any of our proofs - these and our logical equivalences are the steps that we keep writing out as justification in the right-hand column of the proof “stencil”.
- **Proofs**
    - Proof Methods and Strategies (direct/conditional proof, proof by contradiction, proof by cases, contrapositive proof, proof by construction)
    - We brought everything from propositional logic and predicate logic together here.
    - **Example problem:** Prove that n is odd if and only if $n^2 - 4n + 3$ is even.
- **Set Theory and Set Operations**
    - Sets are among the most useful discrete structures. The propositional/predicate logic that we learned previously all used sets (at least implicitly). For example, if we said the domain of discourse was all CU students, we were actually saying that our propositional *function* has the domain that is the set of all CU students.
    - This was also a natural place to go after quantifiers and proofs because we sometimes would like to prove something is true (or false) about **all** or **some** element(s) in a set (universal and existential quantifiers, respectively)
    - Set unions, intersections, complements, differences
    - Empty sets, subsets, elements of sets vs subsets
    - Set identities
    - **Example problem:** Let $A=\{1,2,3,4,5\}$ and $B=\{0,3,6\}$. Find $B-A$.

---------- MIDTERM EXAM 1 ----------

- **Functions and Cardinality**
    - One-to-One and Onto Functions

        - **Example problem:** Consider the function $f(n) = 2n^3$, where $f : Z \rightarrow Z$. Which of the following choices best describes the function f(n)? (i) one-to-one, (ii) onto, (iii) neither, (iv) both.
    - Inverse Functions and the composition of functions
    - We studied **sets** and then **functions** because functions operate on elements of one set (the **domain**) and map them into another set (the **codomain**).
    - Functions also turned out to be a useful tool to measure **how large sets are** (i.e., their **cardinality**)
    - Other Important Functions (floor, ceil, etc.) -- because these appear all over the place, and turn out to be very useful, especially when defining a one-to-one function from the set of natural numbers to a set of interest (be sure you know why on Earth would we do that)
    - **Countable and Uncountable Sets** -- oh right, *this* is why we would do that
    - **Example problem:** Let B be the set of all bit-strings that do not contain any 0's. What is the cardinality of B?
- **Sequences**
    - A natural next stop from functions because determining that a set is countable means we can define a sequence of its elements. Sequences are like functions from the natural numbers to some set, that we decided should be in some kind of an order (a listing, anyone?)
    - **Example problem:** Find a closed form solution to the recurrence relation:
      $a_n = 2a_{n-1} + 3$, $a_0 = 1$
- **Algorithms**
    - Searching Algorithms -- Linear Search, Binary Search
    - Sorting Algorithms -- Bubble Sort, Insertion Sort
    - Greedy Algorithms
    - Get used to interpreting code, and thinking algorithmically.
    - This followed nicely from sequences and functions because an algorithm uses the elements along a sequence as input to a function:

      Example:
      ```
      for ii in range(0,3):
          output[ii] = func(ii)
      ```
      This little algorithm does the function "func(...)" to elements from the set {0, 1, 2}.
- **Complexity**
    - Growth of Functions -- big-O, big-Omega, big-Theta
    - Worst-Case, Best-Case, Average-Case
    - **Example problem:** What is the smallest integer p such that $f(n) = 5n^3 - \log(n^2) + n^2 \log(n^4)$ is $\boldsymbol{O}(n^p)$?
- **Matrix and Matrix Operations**
    - Matrix Addition and Multiplication (including matrix-vector multiplication)
    - Complexity of Matrix Addition and Multiplication
    - We care a great deal about these operations because they're so common in scientific computing, and potentially computationally expensive.

- **Example problem:**

  Compute $A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 1 & 5 \end{bmatrix}, \text{and } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- **Induction**
  - We wanted to be able to prove things that should be true for an infinite number of cases, but we don't want to have to actually sit here and prove each and every case
  - So induction allows us to show that a statement is true in general by showing:
    - (1) the statement is true for a Base Case (or set of base cases), and
    - (2) if the statement is true for Case *k*, then it must be true for Case *k*+1
    - The **induction hypothesis** is "assume the statement is true for Case *k*" (weak induction), or "assume the statement is true for every case up through Case *k*" (strong induction)
  - Two forms: strong and weak -- know which is which
  - **Example problem:** Use induction to prove that 1 + 2 + 3 + … + n = n(n+1)/2. Did you use strong or weak induction?
- **Combinatorics/counting**
  - Sum rule/product rule
  - Pigeonhole principle
    - **Example problem:** Suppose you own a gift shop that sells candles, cards, and clocks. The color options for each type of gift are gold, green, or rose. What is the minimum number of shoppers who each purchase one item needed to guarantee that at least three shoppers buy the same item in the same color?
  - Permutations and combinations
  - Binomial theorem
  - **Example problem:** How many distinct rearrangements of the 8 letters ABCDEFGH contain the string FADE?
- **Discrete probability**
  - Basics, probability of events from counting
  - Properties of probability distributions, computing distributions

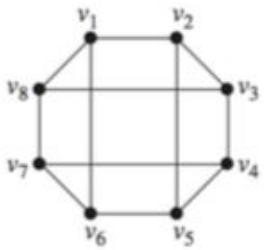
---------- **MIDTERM EXAM 2** ----------

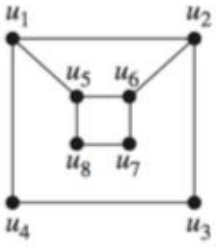
  - Law of Total Probability
  - Bayes' Theorem
  - Bayesian Spam Filters

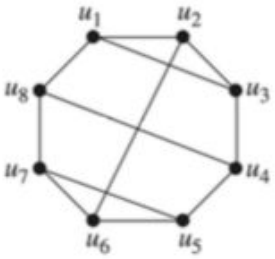
- Independence/dependence, conditional independence: we saw a number of different tests for independence, and a bunch of counterintuitive results
- **Example problem:** Consider rolling two 6-sided dice. One of them is a fair die. The other is unfair, where the numbers 1-4 are all equally likely to be rolled, but the number 5 is twice as likely as the number 1 to be rolled, and the number 6 is 3 times as likely as the number 1 to be rolled. What are these values from the probability distribution for the set of possible outcomes for the sum of the two dice? Are the two events independent? How can you tell?

- **Recurrence relations**
  - Solving recurrences:
    - Recursive-plugging-back-in (also known as iteration or unrolling)
    - Combining the homogeneous and particular nonhomogeneous solutions
      - Characteristic polynomial
      - What are our "usual" guess(es) for the form of the particular solution?
      - What is the "usual" guess for the homogeneous solution?
      - What do we do in the repeated roots case? What about if part of the homogeneous solution ends up being part of our guess for the particular solution?
      - How do we solve for the constants? *When* do we solve for the constants?
  - Counting using recurrence relations
  - Solving counting problems using recurrences vs using dynamic programming
    - What would the solution to the same problem look like done each way?
  - **Example problem:** In the kingdom of Hyrule, currency comes in denominations of 1, 2 and 5 rupees. After purchasing a pair of iron boots from a shop, you are owed 17 rupees in change. If the shopkeeper places each gem on the counter one at a time, how many ways are there for him to make your change?
- **Relations**
  - We liked functions, but there are a lot of restrictions on functions. Relations are like functions, but more flexible. We like that, because it allows us to model more things using relations
  - Equivalence relations, equivalence classes, partitioning of sets
  - **Example problem:** Determine whether each of the following relations $R \subseteq A \times A$, where A is the set of all CU students, is reflexive, symmetric, transitive, and/or an equivalence relation. Briefly justify each conclusion.
    - (a) $(a, b) \in R$ if and only if a shares at least one class with b.
    - (b) $(a,b) \in R$ if and only if *a* has a higher GPA than *b*.
    - (c) $(a,b) \in R$ if and only if a is roommates with b
  -
- **Graphs**
  - Degree of vertices (undirected), in/out-degree (directed)
  - Walks, paths, Eulerian tours

- Graph coloring and scheduling problems; chromatic number
- **Example problem:** For each of the graphs below, determine if the graph has an Eulerian Tour. If it does, give one such tour. If it does not, explain why. For graphs that do not contain Eulerian tours, can you add a small number of edges so that they do contain one?







- **Trees**
  - Binary trees, *m*-ary trees
  - leaves, structure
  - Structural induction on trees
  - **Example problem:** Prove that if T is a full binary tree (FBT), then T has an odd number of vertices.

What should I be doing to prepare?

1. **ACTUALLY *DO* PROBLEMS.** Solve them all the way out **by hand**. This is what you will need to do during the exam, so why would you prepare any other way?
2. Review the homework and exam problems from this semester. Reattempt tricky ones, and make sure you understand how to do any problems you couldn't before.
   a. Solutions are posted under Piazza, Resources tab
3. Review old exams from previous semesters. Solutions are mostly available, there may be some errata, but by all means ask questions if something doesn't make sense!
4. Review the Workgroup worksheets (Piazza, Resources tab)
   a. Solutions are **partially available**. We're filling them in as time permits - the workgroup TAs are already working (probably) more hours than they can get paid for.
   b. If a solution is not available for a problem you are tackling (and you should tackle problems even if solutions are not available!), then you are encouraged to discuss/check solutions on Piazza!
5. Read the slides.
   a. Can you do the standard examples from the lecture slides?
   b. Can you do the challenge examples from the lecture slides?
6. If any concept is unclear, go to an Instructor or a CA for help.
   a. Office Hours are posted on Piazza.
   b. Or you can **post to Piazza** (we strongly encourage this)
   c. You can contact CA's too using Piazza.

7. Use Piazza for constructive discussion.
    a. The problems you're studying to prepare for the midterm aren't graded work. We **strongly encourage** you to discuss the problems with each other via Piazza! If you're stumped, that means you've found a nice, challenging problem and others would probably benefit from studying it.
8. Attend the review session (Friday 14 December, time and location to-be-determined).
    a. They will follow a Question-and-Answer format, so **bring questions** if you have them. If your questions are not all answered, either post to Piazza or attend office hours.
9. Attend class (this should be a no-brainer, but is worth mentioning)
10. Look at the textbook for additional problems.
11. Look at the "All Moodle" problem set (to be linked from the Course Calendar, and on Moodle).
12. Use this guide as a checklist, not as your only study tool.

Related to the exam:

13. You may use a calculator, provided it cannot access the internet. **You may NOT use a smartphone as a calculator.**
14. **Format:** part multiple choice, part short answer (with some justification), part free response (like the written homework problems; extended justification)

You will **not** need to write any code or use a computer for the exams. You **may** need to interpret code, fill in some blank lines of code by hand, or deduce what will be the output of Python code.