Markov Decision Process

"In which we examine methods for deciding what to do today,
given that we may decide again tomorrow"

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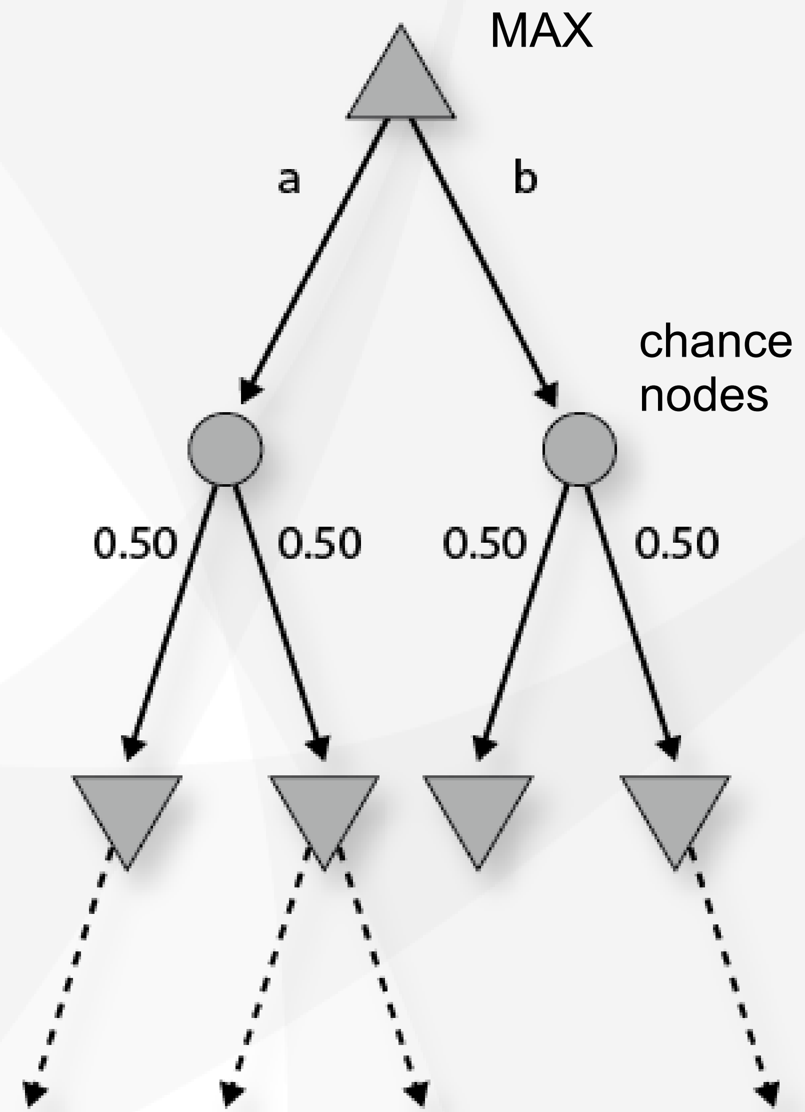
Uncertain outcomes

- Outcomes controlled by a **chance node**
  - Randomness
  - Is not guarantee that you'll follow the optimal path
- Utility of states are uncertain
  - We must calculate an **expected value**
  - We can do a weighted average of possible outcomes (expectation)



Expectiminimax search

- Calculate the expected utility in the case we play optimally
  - weighted average of the children of a chance node

def $expmm(s)$:

$$\max_a expmm(move(s,a)) \text{ if } s = MAX$$

$$\min_a expmm(move(s,a)) \text{ if } s = MIN$$

$$\sum_r P(r)\, expmm(move(s,r)) \text{ if } s = CHANCE$$

r is the label of a possible arc from a chance node

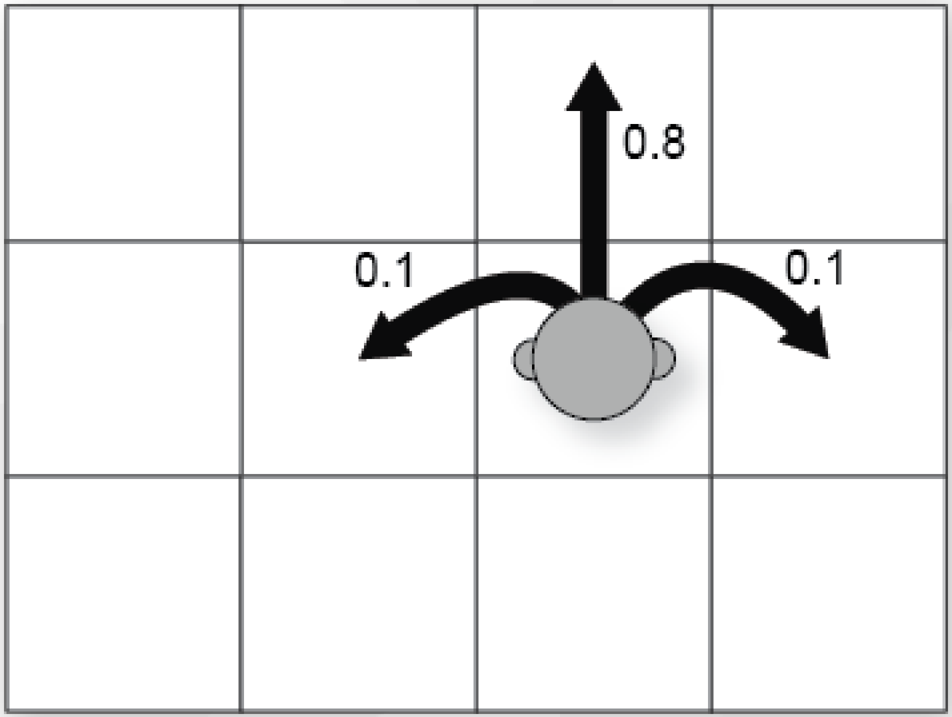
- With trees without MIN nodes, the algorithms is called expectimax search.
  - We'll work with these trees

More info at "Artificial Intelligence: a modern approach 3rd ed" (pag. 171+)

Grid World example

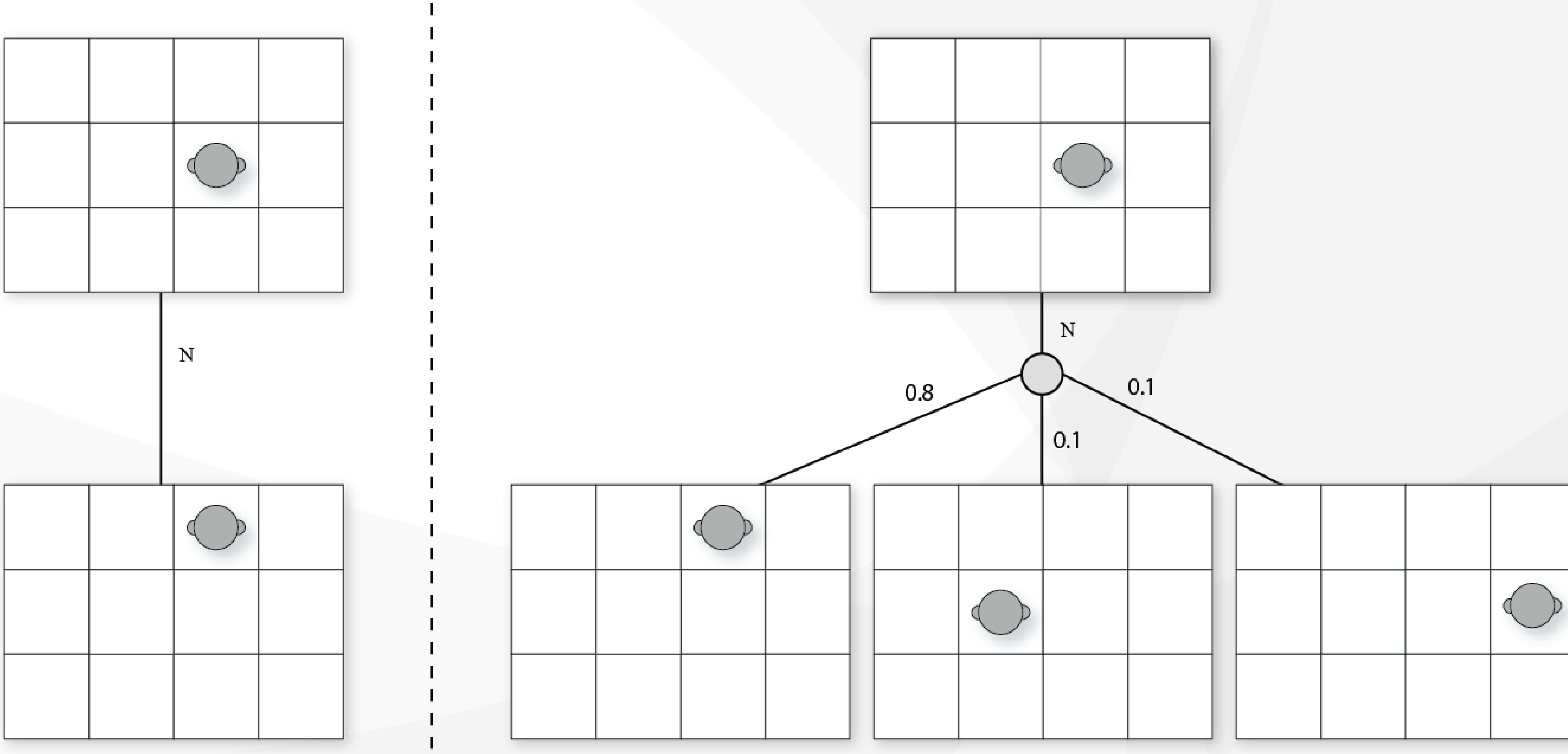
- Environment: 4x3 grid
  - Fully observable
  - Stochastic (we know the probabilities)

For example the action NORTH can lead in three different cells



This example can be found at “Artificial Intelligence: a modern approach 3rd ed” (pag. 645)

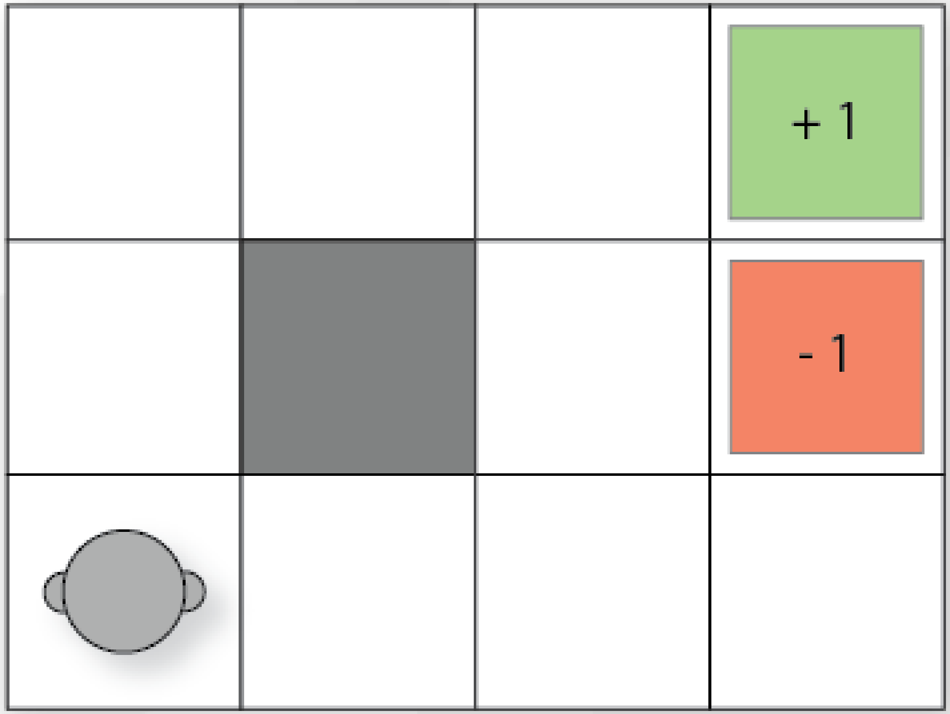
Deterministic vs. Stochastic



The action are unreliable, we have a chance point that execute an arbitrary action

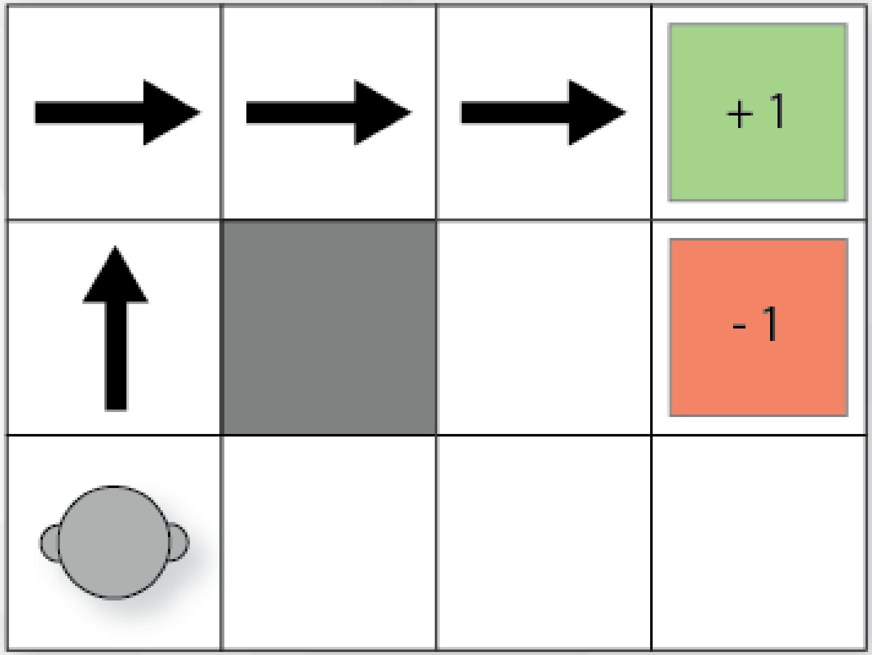
Grid World example (2)

- Various types of cell
  - **Exit**: our goal, with a positive reward, no actions
  - **Pit**: what we want to avoid, with a negative reward, no actions
  - **Wall**: unattainable cell, no reward
  - **Void**: a negative reward to represent the step cost
- Goal:
  - Maximize the sum of rewards
  - Hence, reach a terminal state (i.e. an Exit or a Pit)



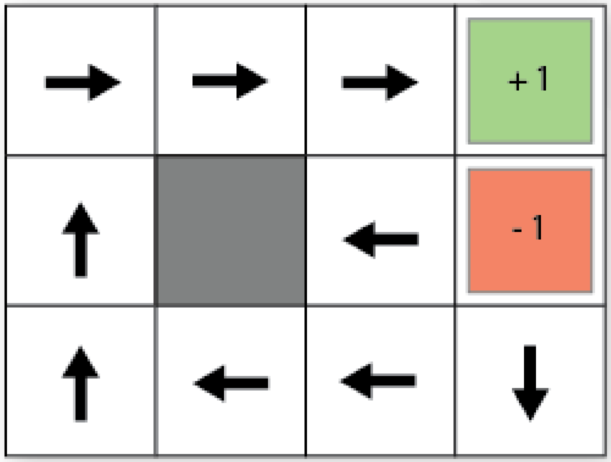
Needs for a policy

- The plan [North, North, East, East, East] is not a solution
  - Caused by the non-deterministic environment
  - A sequence of actions is not sufficient

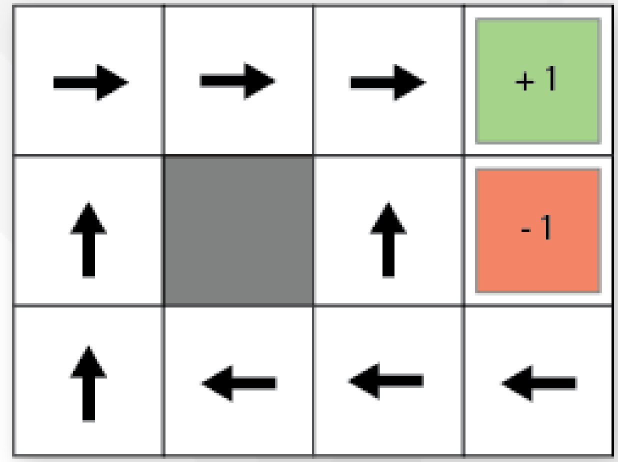


- The solution can't be a plan, but a **Policy**
  - For each state s, the policy π says the action we should do

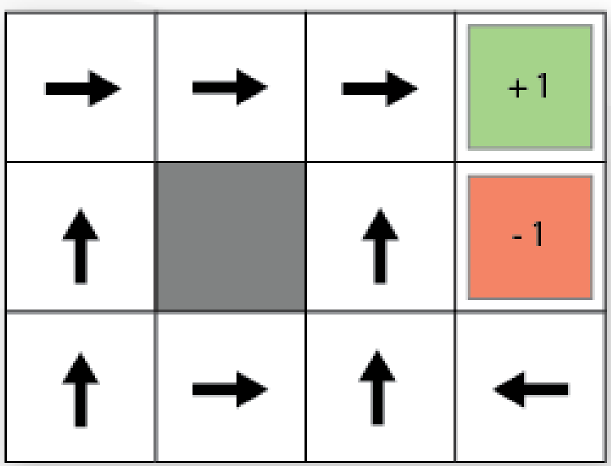
Policy examples



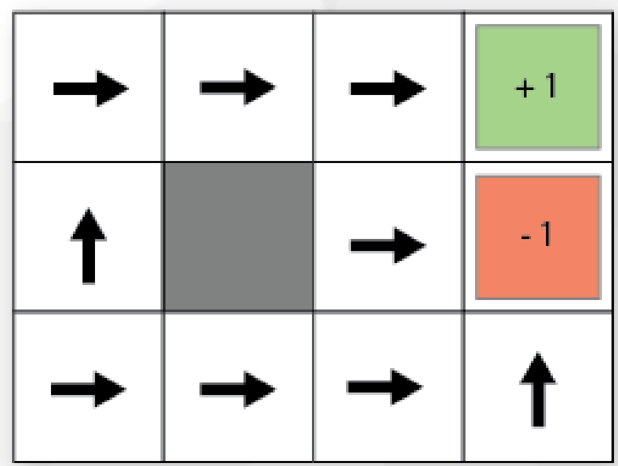
R(s) = -0.01
where s is a void cell



R(s) = -0.04
where s is a void cell



R(s) = -0.09
where s is a void cell



R(s) = -2.00
where s is a void cell

Markov Decision Process

- An MDP is defined by:
  - A set of action $a \in A$
  - A set of states $s \in S$
  - A transition function $T(s, a, s') = P(s' | s, a)$
  - A reward function $R(s, a, s')$
    - Sometimes it is simply: $R(s')$ or $R(s)$
  - An initial state s_0
  - Some terminal state $(s_{1,} s_{2,} s_{3,} \ldots, s_n)$
    - No action allowed from a terminal state

- Why we use the term "Markov"?

Markov chain

- A sequence of random variables

$$X_{1,} X_{2,} \ldots, X_n$$

- All variables satisfy the Markov property

$$P\left(X_{n+1}=x \middle| X_1=x_{1,} X_2=x_{2,} \ldots, X_n=x_n\right)=P\left(X_{n+1}=x \middle| X_n=x_n\right)$$

that is, given the present state, the future and past states are independent

Markov chain and MDP

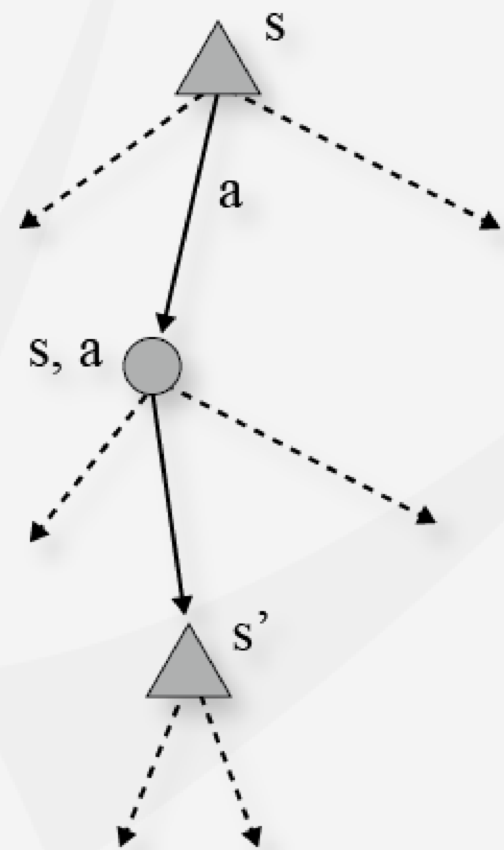
- With the MDP the property is similar

$$P\left(S_{n+1}=s_{n+1}|S_0=s_0, A_0=a_0, S_1=s_1, A_1=a_1, \ldots, S_n=S_n, A_n=a_n,\right)$$
$$=P\left(S_{n+1}=s_{n+1}|S_n=s_n, A_n=a_n\right)$$

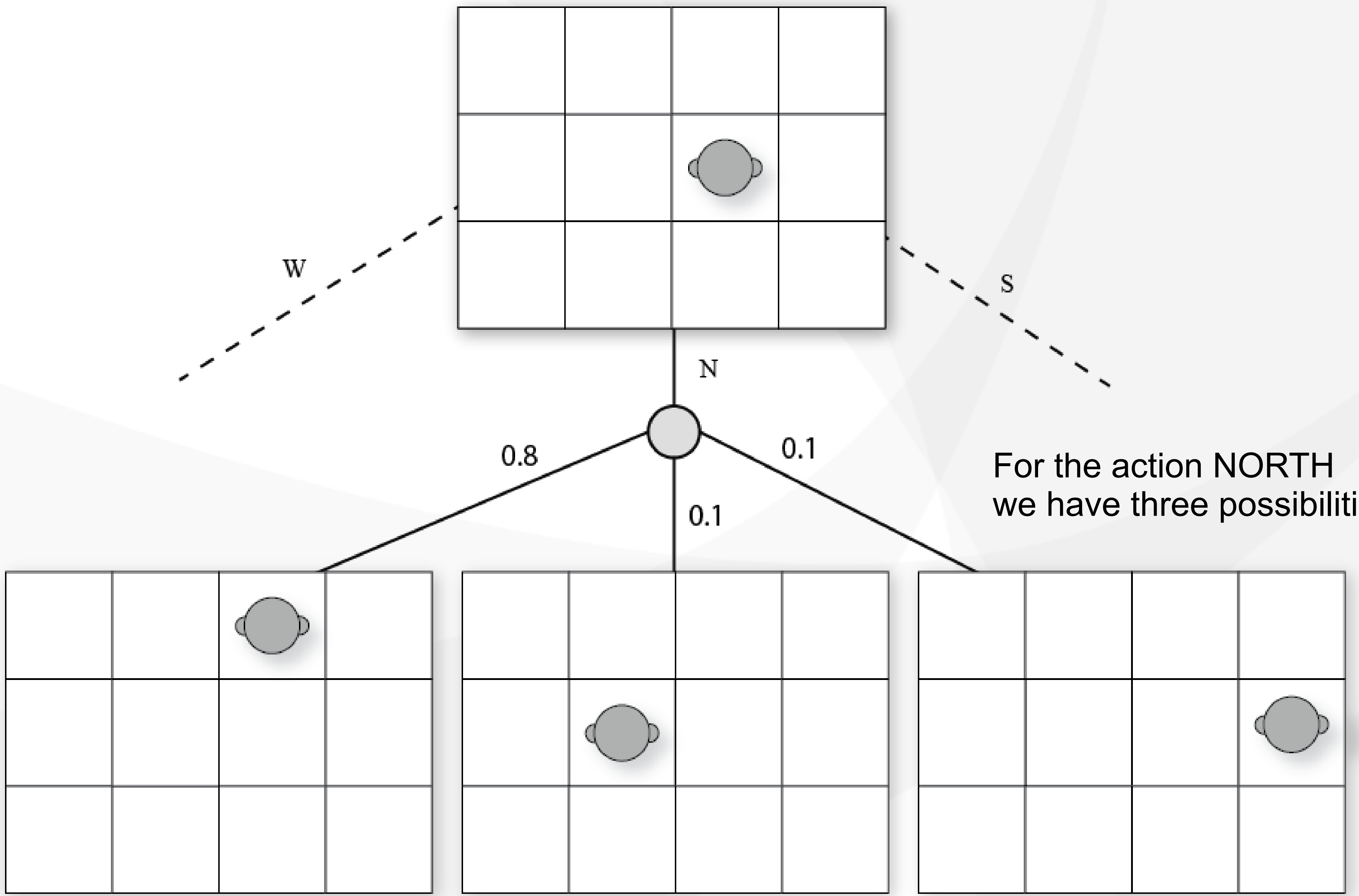
- From a state s_n, The outcome of an action a_n only depends on the current state s_n and not the history.

MDP Search tree

- From state s you can do some actions
- The action a leads us to a **q-state** s,a
  - The q-state represent a state where we have decided the action to do, but we haven't done the action yet, we don't know in which state we'll land.
- With some probability we'll land to the state s' where we can do another action
  - Each arc from a q-state s,a has a probability: $P(s'|s,a)$



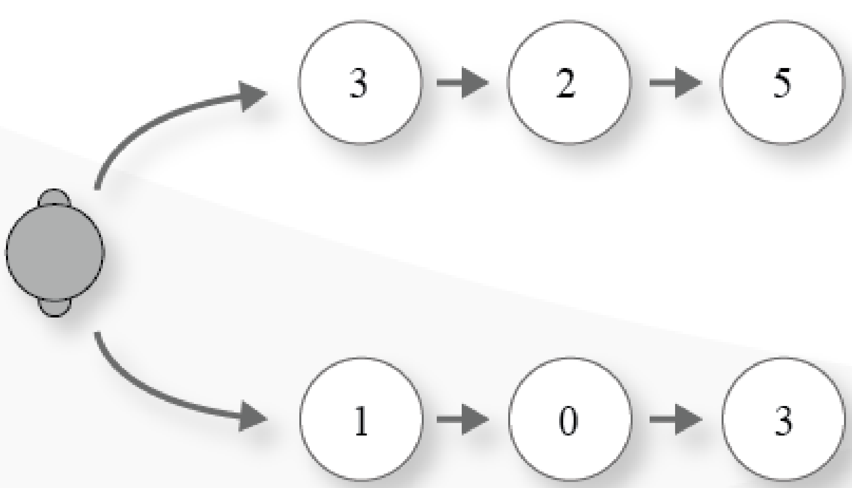
MDP Search tree (2)

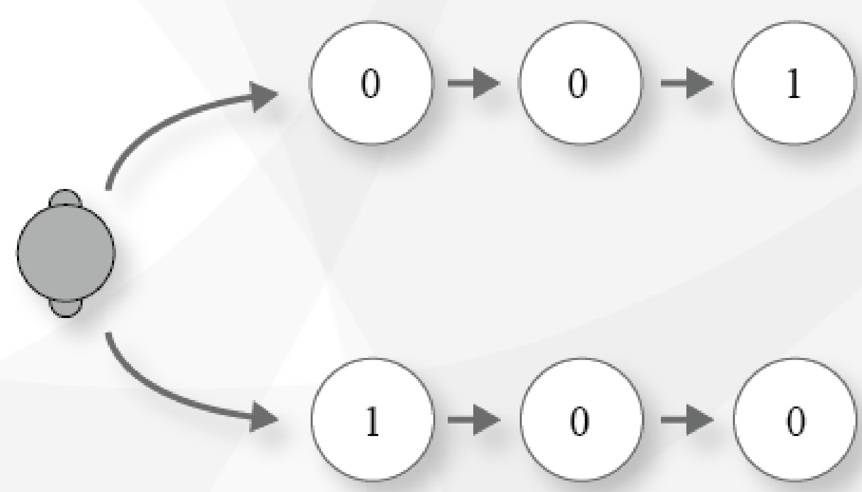


For the action NORTH
we have three possibilities

Utilities of state sequences

- The agent should prefer the "best" sequences, two cases.





- We need "More" because we want to maximize the sum of rewards
- We need "Now" because the non-determinism influences our outcomes

Utilities of state sequences (2)

- Two ways to define the utility of a state sequence:
  - Additive rewards:

    $$U([s_0, s_1, s_2, \dots, s_n]) = \sum_{i=0}^{n} R(s_i)$$

    don't satisfy our “Now” concept

  - Discounted rewards:

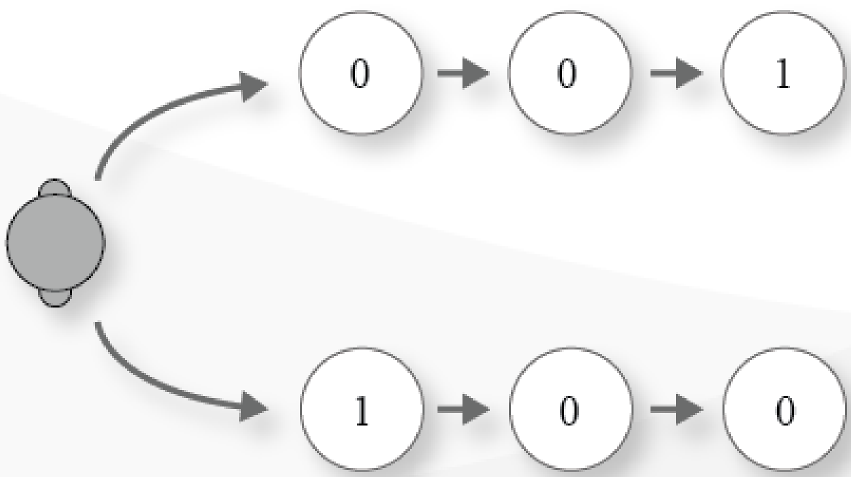
    $$U([s_0, s_1, s_2, \dots, s_n]) = \sum_{i=0}^{n} \gamma^i R(s_i)$$

    satisfies the two concepts “Now” and “More”

    γ is the **discount factor**, a value between 0 and 1

Discounted rewards

- With the discounted rewards:
  - with discount factor of 0.5



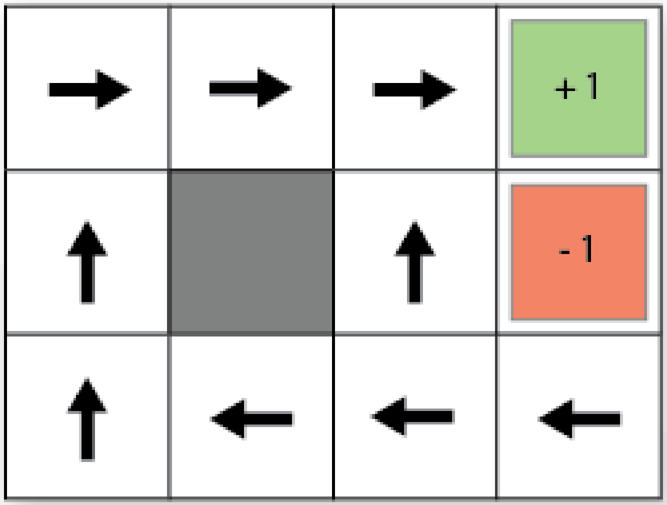
$$U([0,0,1]) = 0*1 + 0*0.50 + 1*0.25 = 0.25$$

$$U([1,0,0]) = 1*1 + 0*0.50 + 0*0.25 = 1$$

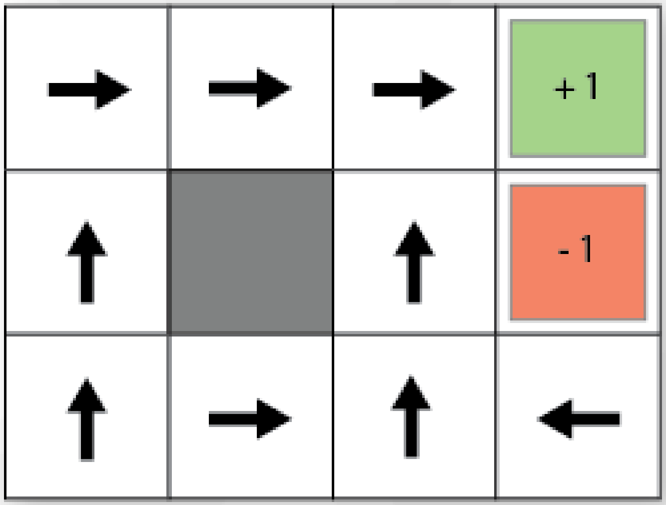
$$U([0,0,1]) < U([1,0,0])$$

Discounted rewards (2)

- In our example:
  - with $R(s) = -0.04$ for the void cells



discount factor = 1



discount factor = 0.8

- with 0.8 the agent prefers to take the risk to fall into the pit. in the other path, the big reward is too far

Discounted rewards (3)

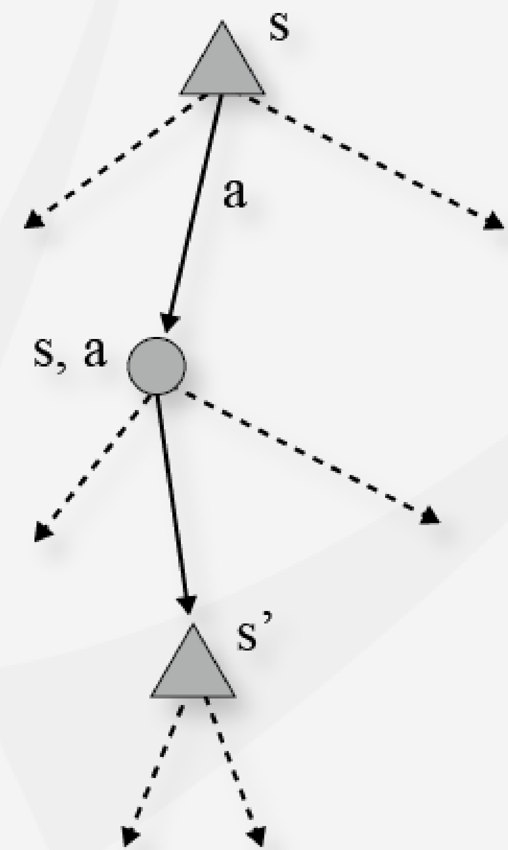
- What if the game doesn't finish?
  - We can use an approach similar to the “depth-limited search”
    - Gives a **non-stationary policy** (i.e. π depends on the time left)
  - We can use the discounted rewards
    - With $\gamma < 1$ we have:

$$U([s_0, s_1, \ldots]) = \sum_{i=0}^{\infty} \gamma^i R(s_i)$$

      converges (it's just a geometric series)
    - With γ we set our “horizon”: the smaller is γ and the smaller is the horizon.
    - It's a **stationary policy**

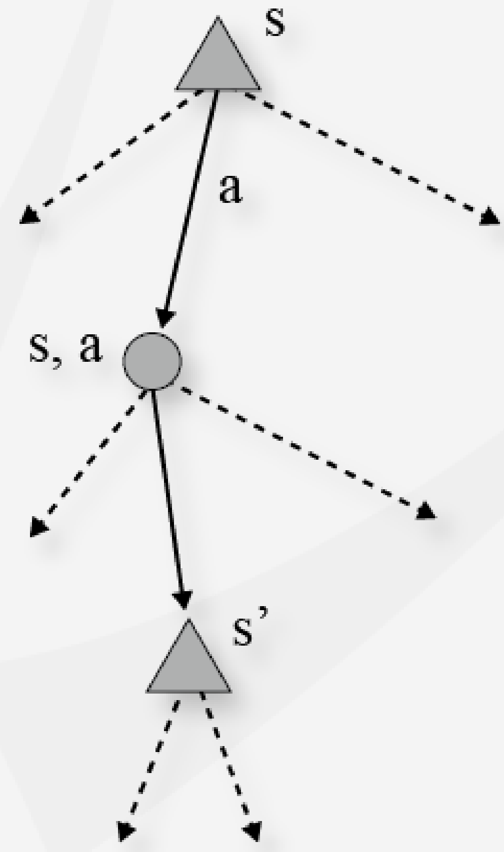
Utilities

- The utility $U^{\pi}(s)$ of a state s:
  - Is the expected utility starting from s and acting as the policy π *wants.*
- The utility $Q^{\pi}(s,a)$ of a q-state s,a:
  - Is the expected utility starting from s and doing the action a
  - It's called **q-value**
- With π^* we denote an optimal policy
  - $U^*(s)$ and $Q^*(s,a)$ denote the latter utility definitions, but using an optimal policy



Calculate Q^*

- From s,a we can go into some states
  - i.e. in the grid world we can go forward with a probability of 0.8, left or right with a probability of 0.1
  - to compute the $Q^*(s,a)$ we need a sum average
  - we define it recursively, using the utilities of the states below (U^*)



$$Q^*(s,a) = R(s) + \gamma * \sum_{s'} P(s'|s,a) * U^*(s')$$

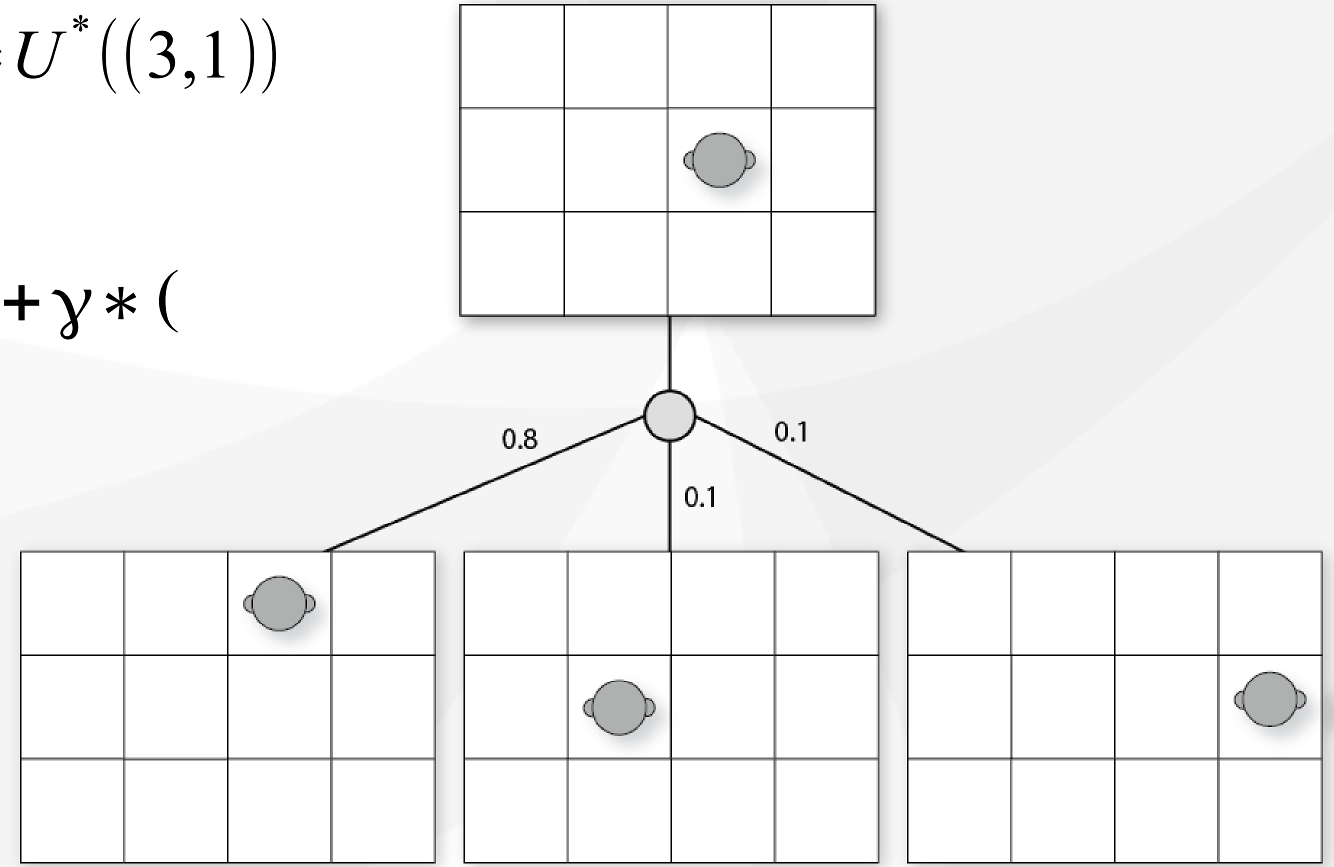
Hence, we can calculate U^* in this way

$$U^*(s) = \max_{a \in A(s)} Q^*(s,a)$$

Calculate Q^* (2)

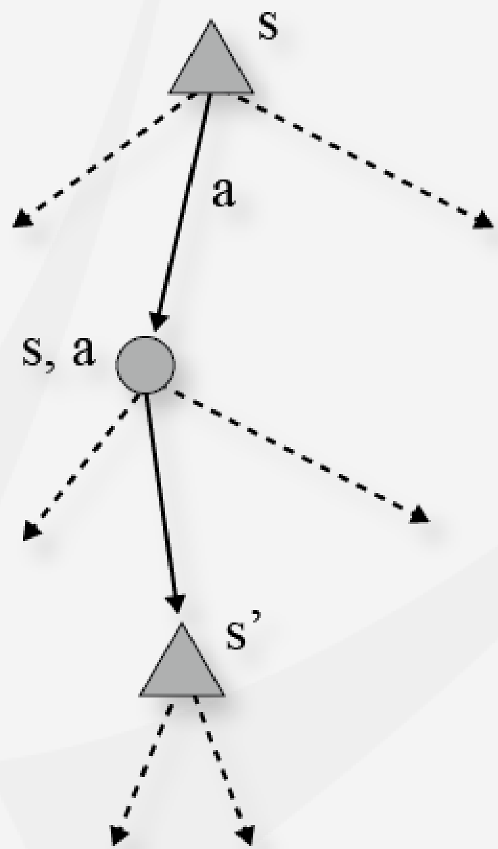
$$Q^*((2,1),N) = R((2,1)) + \gamma * ($$
$$P((2,0)|(2,1),N) * U^*((2,0))$$
$$+P((1,1)|(2,1),N) * U^*((1,1))$$
$$+P((3,1)|(2,1),N) * U^*((3,1))$$
$$)$$

$$Q^*((2,1),N) = -0.04 + \gamma * ($$
$$0.80 * U^*((2,0))$$
$$+0.1 * U^*((1,1))$$
$$+0.1 * U^*((3,1))$$
$$)$$



Calculate U^* *(The Bellman equation)*

- From s we have some possible actions
  - i.e. in the grid world we have NORTH, SOUTH, WEST, EAST
  - to compute the $U^*(s)$ we need the maximum average sum of the utility of the states below
  - we use the discounted rewards
  - the value of a terminal has only the reward $R(s)$



$$U^*(s) = R(s) + \gamma * \max_{a \in A(s)} \sum_{s'} P(s' | s, a) * U^*(s')$$

Extract the policy from U^*

- If we have the optimal utility for all the states:
  - Calculated with:

$$U^*(s) = R(s) + \gamma * \max_{a \in A(s)} \sum_{s'} P(s' | s, a) * U^*(s')$$

- We can calculate the optimal policy in this way:

$$\pi^*(s) = \operatorname*{argmax}_{a \in A(s)} \sum_{s'} P(s' | s, a) * U^*(s')$$

Extract the policy from Q^*

- If we have the q-values:
  - Calculated with:

$$Q^*(s,a)=R(s)+\gamma*\sum_{s'} P(s'|s,a)*U^*(s')$$

  - We know that the state utility can be compute as the following:

$$U^*(s)=\max_{a\in A(s)} Q^*(s,a)$$

  - Hence, we can extract the policy in this way

$$\pi^*(s)=\operatorname{argmax}_{a\in A(s)} Q^*(s,a)$$

Ways to Calculate $U*$

- Calculate the utilities $U*$ from a real tree is too expensive
  - Using the expectimax search
  - A lot of repeated states
  - What should we do with infinite trees?
- If the number of possible states is n, we have n possible Bellman equations.
  - But the equations aren't linear, because there is the “max” operator. The system of equations can't be solved with the linear algebra methods.
- Fortunately, we have two important algorithm:
  - Value Iteration
  - Policy Iteration

Value Iteration

- We start with $\forall s \in S . U_0(s) = 0$
- We update this values until it converges

$$U_{i+1}(s) = R(s) + \gamma * \max_{a \in A(s)} \sum_{s'} P(s' | s, a) * U_i(s')$$

Bellman Update

- There is an unique solution
- The corresponding policy is optimal
- We can see this algorithm as propagating information through the state space by means of local updates
- It's just like a fixed point solution method

Value Iteration (notes)

- Complexity
  - Each iteration needs: $O\left(n^2 * A\right)$
  - It's slow!

- Notes
  - The policy can converge before the values do
  - For each state, the max rarely changes

Policy Iteration

- Alternates two steps:
  - **Policy evaluation**
    - Given the policy π_i we calculate $U_i = U^{\pi_i}$, that is the utility of states where we use the policy π_i

$$U_i(s) = R(s) + \gamma * \sum_{s'} P(s' | s, \pi_i(s)) * U_i(s')$$

these are called "simplified Bellman equations" because use a fixed policy

  - **Policy improvement**
    - Calculate π_{i+1} using the utility of states U_i , π_0 is random

$$\pi_{i+1}(s) = \underset{a \in A(s)}{argmax} \sum_{s'} P(s' | s, a) * U_i(s')$$

Policy Iteration (algorithm)

$repeat$

$U = policyEvaluation(\pi, U)$

$unchanged = true$

$foreach \;\; s \in S:$

$$if \max_{a \in A(s)} \sum_{s'} P(s' | s, a) * U_i(s') > \sum_{s'} P(s' | s, \pi(s)) * U_i(s')$$

$$\pi(s) = \underset{a \in A(s)}{argmax} \sum_{s'} P(s' | s, a) * U_i(s')$$

$unchanged = false$

$until \; unchanged$

i.e. we continue until we reach a fixed point

Policy Evaluation

- At the *i-th* iteration we have π_i and these equations:

$$U_i(s_1) = R(s_1) + \gamma * \sum_{s'} P(s' | s_{1,} \pi_i(a)) * U_i(s')$$

$$U_i(s_2) = R(s_2) + \gamma * \sum_{s'} P(s' | s_{2,} \pi_i(a)) * U_i(s')$$

…

  - These equations are linear (no max operator)
  - Can be solved by a standard linear algebra methods $O(n^3)$

Policy Evaluation (2)

- $O(n^3)$ can be slow for large space states.
- We don't need the exact values
  - We can perform some number of Bellman update

$$U_{i+1}(s) = R(s) + \gamma * \sum_{s'} P(s' | s, \pi_i(a)) * U_i(s')$$

    - Repeated k times to produce the next utility estimation
    - Similar to the value iteration, but simpler

  - This algorithm is called **modified policy iteration**
    - It's often faster than the policy iteration and value iteration.
    - It's still optimal!

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