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ECCENTRIC ORBIT GENERALIZATION AND SIMULTANEOUS SOLUTION OF BINARY STAR LIGHT AND VELOCITY CURVES

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ABSTRACT

This paper discusses the computation of binary star light and radial velocity curves, including the effects of eccentric orbits and nonsynchronous rotation. Logical relations needed to impose physical constraints (semidetached condition, etc.) are specified. Some of the more important mathematical relations are also given. One can now model semidetached, detached, double-contact, and X-ray binaries for arbitrary rotation rates and orbital eccentricity. Contact binaries can also be modeled, but are restricted to the synchronous, circular orbit case. In all cases the figures of the components and the surface gravity fields are described by surfaces of constant potential energy. An essentially rigorous treatment is given of the solution constraint imposed by X-ray eclipse durations in eccentric orbit cases, assuming photospheric eclipses. Attention is drawn to a previously unrecognized morphological type of close binary, for which the name *double-contact binary* is suggested. This is a binary in which both stars fill their limiting lobes and at least one spins faster than synchronously, so that the components do not touch, even at one point. Some proposed members of this group are β Lyr, V356 Sgr, and perhaps U Cep. As examples of program applications, light curves are computed for SW Lyn and radial velocity curves for HD 77581 (4U 0900−40). One can show that the photometry of SW Lyn does not permit an eccentricity as large as has been reported, and that there is no photometric evidence for $e \neq 0$. Radial velocity distortions for HD 77581 due to proximity effects are small enough to neglect, considering the scatter of the best present observations. Values of X-ray eclipse duration (Θ_e) for two rotation rates and two values of e are given. In order to overlap the range of observational Θ_e values, the rotation should be slow (perhaps half-synchronism) and e should not exceed about 0.10. Several kinds of evidence point to a recent rapid increase in radius of HD 77581. A method is given by which *simultaneous* differential corrections solutions of light and radial velocity curves can be made. Proper weighting schemes are especially crucial to the success of such solutions.

Subject headings: stars: binaries — stars: eclipsing binaries

I. INTRODUCTION

Over the past decade there have been numerous advances in the computation of light and radial velocity curves of binary stars and in the inverse problem of analyzing observed light and velocity curves. Many of these contributions have concerned various special configurations (e.g., contact binaries, detached binaries, X-ray binaries, etc.) with no apparent attempt at generalization.[1] With reasonable planning, however, it is quite possible to make one computer program which can handle a wide variety of systems. In fact, such a program can be envisioned to handle any type of binary after having reached a sufficient level of development. This paper discusses precepts, constraints, and mathematics relevant to any computing scheme with such objectives. It is not intended as a description of a particular program, although the Wilson and Devinney (WD) (1971) program can serve as a concrete example. According to Hutchings (1978), a program has been developed by G. Hill which has many of the objectives of this paper. A description of the Hill program is not yet available, however. Generalizations of the WD work have been published from time to time, usually in papers on specific binaries (e.g., Wilson *et al.* 1972; Wilson and Biermann 1976; Wilson and Wilson 1976; Wilson and Sofia 1976; Leung and Wilson 1977). In this paper, the generalizations to nonzero orbital eccentricity and nonsynchronous rotation are described. Principles involved in the *simultaneous* solution of light and velocity curves by differential corrections are also given. Examples of differential corrections solutions for eccentric cases will be given in a later paper. As before, the figures of the components and the surface gravity fields are described by surfaces of constant potential energy. Since some of the assumptions of Roche geometry are violated for nonzero eccentricity and asynchronous rotation, some small approximations are involved in the potential formulations used here, as is well known and recognized in earlier work. Attention will be called to these assumptions later in the paper, where appropriate.

[1] We are not interested here in models in which the surfaces of the components are simple geometrical figures, such as ellipsoids. Such models cannot be generalized to include the case of contact binaries. However, one such program—that by Wood (1971)—has been used successfully for a number of eccentric systems (e.g., Clausen, Gyldenkerne, and Gronbech 1977).

II. OVERALL LOGISTICS

Eccentric orbit computations are fairly intricate for two main reasons. First, many internal calculations must be done *at each phase* of an eccentric orbit since the figures and surface gravity fields of the components are phase-dependent. These extra calculations become redundant and can be bypassed if the orbital eccentricity e is equal to zero. Second, schemes must be devised to impose constancy on certain conserved, or essentially conserved, quantities. For example, we expect that the stellar volume and bolometric luminosity should be very nearly independent of phase. While it may be that these quantities have some small phase variation in real binaries, this problem will be neglected in the present treatment. Purely orbital calculations connected with the solution of the two-body problem are too well known to require description here. It is presumed that anyone who uses this work to write a computer program can fill in such details.

The surfaces of our component stars are to follow "equipotential surfaces," insofar as that term is meaningful for eccentric orbit cases, and insofar as Coriolis forces due to asynchronism can be neglected. The potential formulation should reduce to that of the Roche model when $e = 0$ and $F = 1$, where F is the ratio of angular rotation rate to the synchronous rate. Note that F is a constant (not phase dependent). In order to have some kind of treatment of the eccentric problem, some workers (van Paradijs *et al.* 1977; Hutchings 1978) have set the rotation rate equal to the instantaneous orbital angular rate, thus having a phase-variable rotation rate. While this procedure is perhaps preferable to having no calculation, it is certainly not correct since the star has no mechanism for temporary storage of rotational angular momentum which can operate on the requisite time scale (much shorter than an orbit). The solution to the asynchronism problem was first given by Plavec (1958), although a detailed derivation and discussion was later published by Limber (1963), apparently independently of Plavec's paper. Another important and extensive discussion is that of Kruszewski (1966).[2] The Plavec-Limber solution accounts for asynchronism by including a factor of F^2 in the centrifugal term of the Roche binary potential equation, the synchronous form of which may be found in numerous references, such as Kuiper (1941) and Kopal (1959). Departures from exact validity in the Plavec-Limber potential arise from the neglect of Coriolis forces due to the flow of stellar fluid through the star's tidal bulge. These forces are small and were discussed thoroughly by Limber (1963). Their effect on the potential could be included by means of numerical techniques, but that does not seem warranted at present. A generalization of the binary potential to include eccentric orbit effects was given recently by Avni (1976). Strictly speaking, such a potential does not exist, because the force field is time-dependent and therefore nonconservative. However, if a binary component can readjust to equilibrium on a time scale which is short compared to that on which forces vary, one can define an effective potential locally at each point of the orbit, without significant inconsistency, as Avni has done. The time scale of principal interest here is that for free nonradial oscillations, which will normally be much shorter than an orbit period. Williams (1977) has derived a potential formulation essentially identical to Avni's, using a different approach. The binary potential generalized to account for nonsynchronism and noncircular orbits, and in the same notation as our earlier work (eq. [1] of Wilson and Devinney 1971) is:

$$\Omega = r^{-1} + q[(D^2 + r^2 - 2r\lambda D)^{-1/2} - r\lambda/D^2] + \tfrac{1}{2}F^2(1 + q)r^2(1 - \nu^2)\,. \tag{1}$$

Here q is the mass ratio (M_2/M_1), D is the instantaneous separation of centers, and λ, ν are direction cosines. The coordinates are r, θ, ϕ (with θ the polar angle), and the unit of length is the semimajor axis of the relative orbit. The origin of coordinates is at the center of star 1, and the value of F pertains to star 1. When treating the secondary component, one should place the origin of coordinates at its center and use an internal (i.e., unseen by the user) potential, Ω', which is related to Ω by

$$\Omega' = \Omega/q + \tfrac{1}{2}(q - 1)/q\,. \tag{2}$$

Actually Ω is a modified potential since a constant term has been subtracted, as in Kopal's (1959) convention. To compute local effective gravity and the orientation of surface elements, one needs Ω-derivatives, the (right-handed) rectangular-coordinate forms of which seem more convenient to use than the polar forms. These are:

$$\frac{d\Omega}{dx} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \frac{q(D - x)}{([D - x]^2 + y^2 + z^2)^{3/2}} + F^2(1 + q)x - q/D^2\,, \tag{3}$$

$$\frac{d\Omega}{dy} = -y\left[\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{q}{([D - x]^2 + y^2 + z^2)^{3/2}} - F^2(1 + q)\right], \tag{4}$$

and

$$\frac{d\Omega}{dz} = -z\left[\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{q}{([D - x]^2 + y^2 + z^2)^{3/2}}\right]. \tag{5}$$

[2] The paper by Lubow (1979) appeared too recently to influence the present work.

If only synchronously rotating, circular orbit models are to be computed, a particularly convenient way to find components of the potential gradient is to use the above forms of the Ω-derivatives directly for both components. Since we have only one potential system for the entire binary, there is no reason not to take advantage of this simplification. The alternative is to introduce a second potential system, with origin of coordinates at the secondary component and with the masses interchanged, which is related to the first system by a suitable transformation. However, for the more general situation with $F \neq 1$ and $e \neq 0$ we have no choice. In a physical sense there exist separate intermeshed and distinct potential systems for components 1 and 2, so the origin of coordinates should be moved to the center of star 2 when computing its gradient components. Furthermore, we would not want to do the calculations in a different way for synchronous and nonsynchronous rotation, so we should have two potential systems for synchronous rotation also. Then synchronism will be a special case of nonsynchronism. It can be shown that the equations for $d\Omega/dy$ and $d\Omega/dz$ are then unchanged, but that for $d\Omega/dx$ becomes

$$\frac{d\Omega_2}{dx} = \frac{q(D-x)}{([D-x]^2 + y^2 + z^2)^{3/2}} - \frac{x}{(x^2+y^2+z^2)^{3/2}} - F^2(1+q)(1-x) + 1/D^2 \,. \tag{6}$$

That is, equation (3) applies for component 1, while (6) applies for component 2. Equations (4) and (5) apply for both components. The coordinates x, y, and z have the identical meanings (i.e., are referenced to the same physical star) in (6) as in (3), (4), and (5) since the proper transformations from the secondary to the primary coordinate system have been applied. The Ω-system is then the same as that for the primary (i.e., for the synchronous circular case, eqs. [3]–[6] give identical derivatives regardless of which component is under consideration).

In view of the fact that binary stars in eccentric orbits will change not only their figures as functions of orbital phase but also, of course, their surface temperature distributions, it becomes particularly important to specify clearly what is meant by *the* "temperature" of such a star. In earlier work it has been common practice to specify the polar effective temperature as an input parameter, and to compute temperatures at other surface points relative to T_{pole}. When $e = 0$, this has only the minor disadvantage that T_{pole} differs from a *mean* surface effective temperature, so that one should apply a small correction when comparing model (i.e., polar) temperatures with spectral type or color index based observational temperatures. However, for $e \neq 0$, there is the further, more troublesome point that T_{pole} is not expected to be constant. Thus one should adopt mean surface temperatures as input parameters. Here the mean surface effective temperature, $\overline{T}_e$, is defined through the bolometric luminosity,

$$L = \sigma S \overline{T}_e^{\,4} \,, \tag{7}$$

where S is the star's surface area. It then follows that $(T_e)_{\text{pole}}$ is related to $\overline{T}_e$ by

$$(T_e)_{\text{pole}} = \overline{T}_e \left\{ S \Big/ \left[\int_{\text{surface}} (\nabla\phi)^g dS \right] \right\}^{0.25} , \tag{8}$$

where $\nabla\phi$ is the local gradient of potential (normalized to the polar value), g is the exponent in the gravity darkening law, as in our notation of earlier papers, and dS is an element of surface area. All quantities within the parentheses of equation (8) are normal by-products of computations which must be done anyway in integrating numerically the brightness of each star, so they require no extra computing time. Equation (8) permits one to use $\overline{T}$ as an input parameter and convert (at each phase) to T_{pole} for internal computations. Thus the convenience of using the pole as a reference point is retained, while allowing the temperature of that reference point to vary with time. Notice that a model component whose *volume* is constant will then have a bolometric luminosity which is virtually (although not exactly) constant. The bolometric luminosity would be exactly constant if the surface area were independent of phase, which will be extremely close to the actual case if we have constant volume.

In the potential formulation devised by Avni, the volume of an equipotential surface of fixed numerical value not only varies with phase (i.e., with instantaneous separation) but varies rather strongly. Since we have adopted this potential function, we are obliged to reconcile this fact with the intuitive and observational[3] notion that real stars do not change their volumes very significantly with orbital phase. However, this apparent difficulty presents no real problem if we realize that the eccentric potential system is effectively defined anew at each orbital phase. With the system for expressing potential energy being phase-variable, our real problem is to determine that potential value which will yield a prespecified volume at each phase. This may be accomplished by a suitable numerical scheme, operating on equation (1). One should interpret the input parameter, Ω, as the potential which applies at *periastron*. One then computes the corresponding periastron volume, which is fixed at that value for all other phases. The corresponding (phase variable) Ω value is then used internally to compute the figure, surface gravity field, and surface element orientations at any arbitrary phase.

[3] Eclipse durations of eccentric systems agree with the assumption of constant component size.

III. MODEL CONSTRAINTS

It is appropriate to apply several types of constraints to solutions of light and velocity curves. Some of these are not particularly related to the eccentric orbit and nonsynchronous problems, such as the constraint that geometric and bolometric quantities should have the same values at all wavelengths of observation. Others, however, must be viewed in a wider context when $e \neq 0$ or $F \neq 1$. For example, we have the constraint that one component of a semidetached system fill its Roche lobe exactly. For reasons which are well known, a star which is transferring mass to its companion by Roche lobe overflow is normally expected to assume the dimensions of the lobe quite accurately. Evidence for this condition is commonly seen in the form of emission-line activity or a secular period change. Such a constraint can be applied by invoking a "semidetached mode" of computation, in which the given surface potential Ω is replaced by the critical value for exact filling of the Roche lobe. Although such a procedure is a mere convenience for the computation of light curves (since one could simply supply the critical potential as input), it is essential for differential corrections solutions in which the mass ratio q is adjusted. This is because Ω_{crit} is functionally dependent on q, and this functional dependence must be accounted for in computing numerical derivatives with respect to q (i.e., $\partial l/\partial q$ or $\partial v_r/\partial q$, where l and v_r are system light and radial velocity, respectively).

Now assume nonsynchronous rotation. There still exists a critical lobe which sets a maximum size for the star, although it would not properly be called a Roche lobe. For faster-than-synchronous rotation, this lobe is smaller than the Roche lobe, while for slower-than-synchronous rotation it is larger (Plavec 1958; Limber 1963). The diminished or increased size of the lobe follows the sliding of the null point of effective gravity along the line of centers, toward or away from a given component. Therefore we apply our semidetached constraint such that the primary or secondary is made to fill its critical lobe exactly, whether that be a Roche lobe (synchronous) or the larger or smaller lobe we find for asynchronism.

Next, allow the orbit to be eccentric. A star which exceeds its critical lobe when not at periastron will certainly exceed it at periastron. Therefore, the effective critical size of the star is that which causes it to fill its critical lobe exactly at periastron. At periastron, the intersection of the star's surface with the line of centers occurs at the null point of effective gravity which is the analog of the inner Lagrangian point of the Roche problem. Our semidetached constraint can now be implemented as follows. For given values of F and periastron distance ($D_p = 1 - e$) we compute the x-coordinate of the null point of effective gravity along the line of centers. This is done by solving numerically for x in equation (3) or (6), as appropriate, with y, z, and $d\Omega/dx$ set to zero. Substitution of $x = x_{\text{null}}$, $y = z = 0$, into equation (1) provides the periastron Ω value, from which follows, by suitable quadrature, the periastron volume. This volume, as for all eccentric orbit cases, is the quantity which remains fixed for all phases, while the instantaneous Ω value is allowed to vary.

The constraint requiring the most logic to impose is the eclipse-duration constraint for X-ray binaries. It is, of course, well known that an observed value of Θ_e, the semiduration of X-ray eclipse, can be used to eliminate one parameter among the set $[i, \Omega, q]$, where i is orbital inclination. It is assumed that the X-ray eclipse limits are set by the photosphere of the optical star, and that the X-ray star has negligible dimensions. A discussion of this problem which compares the available information with the number of parameters to be determined can be found in Wilson and Wilson (1976). For circular orbits, the logic is relatively simple, since one only needs a computing routine which will invert the function

$$\Theta_e = \Theta_e(i, \Omega, q) \tag{9}$$

to solve for one of the three input parameters. Such a circular orbit scheme has been published by Chanan, Middleditch, and Nelson (1976). A similar scheme has been part of the WD program for several years. The WD program finds

$$\Omega = \Omega(i, q, \Theta_e)\,. \tag{10}$$

For $e \neq 0$, the computations are trickier for several reasons:

a) The computation of $\Omega(i, q, \Theta_e)$ must be generalized to apply at the instantaneous separation and to account for asynchronism.

b) Since Ω for *constant volume* varies with phase, there are *two* Ω's to consider—one for ingress and one for egress.

c) The value of Θ_e, which represents a time interval (a difference of mean anomalies), represents equally well a geometrical angle (a difference of true anomalies) when $e = 0$, but does not when $e \neq 0$. The corresponding angle in the orbit plane depends also on the longitude of periastron, ω. The angle is of the nature of a true anomaly, but is not referenced to periastron, but rather to conjunction.

d) Conjunction does not, in general, occur midway in time between ingress and egress of the X-ray star, as it would for a circular orbit.

e) Even when Ω has been determined for ingress and/or egress, one does not have the fundamental quantity needed, because Ω is phase-dependent. One needs the periastron value of Ω corresponding to given $[i, q, \Theta_e, \omega, e, F]$.

The first subproblem to be solved is that of computing $\Omega(i, q, \theta, D, F)$ where θ is the difference of true anomalies at ingress (or egress) and conjunction. It can be shown, with some rather lengthy algebra, that this can be accomplished by solving by iteration for x in the equation

$$\left[(D - x)\left(\tan\theta \sin\theta \sin i + \frac{\cos i}{\cos\theta \tan i}\right) - x\cos\theta \sin i\right] \Big/ [x^2 + (D - x)^2 \tan^2\theta(1 + \csc^2\theta \cot^2 i)]^{3/2}$$
$$+ \frac{q\cos^3\theta \sin^3 i}{(D - x)^2}\left(\cos\theta \sin i + \tan\theta \sin\theta \sin i + \frac{\cos i}{\cos\theta \tan i}\right)$$
$$+ F^2(1 + q)x(\cos\theta \sin i + \tan\theta \sin\theta \sin i)$$
$$- DF^2(1 + q)\tan\theta \sin\theta \sin i - \frac{q}{D^2}\cos\theta \sin i = 0 . \tag{11}$$

In equation (11), x is the line-of-centers coordinate of the point of intersection between the line of sight and the surface of the optical star (tangent point). The y, z coordinates of the tangent point are then

$$y = (x - D)\tan\theta , \tag{12}$$

and

$$z = \frac{x - D}{\cos\theta \tan i} . \tag{13}$$

Having now the x, y, z coordinates of the tangent point, Ω can be found from (the rectangular version of) equation (1).

The second subproblem related to the eccentric duration constraint is to convert the time-interval information represented by the eclipse duration into computed Ω-values for ingress and egress, under the constraint that these two Ω-values produce the same *volume*. This can be done by a two-dimensional Newton-Raphson scheme which imposes the conditions that: (1) the difference in the mean anomalies of ingress and egress is $2\Theta_e$; (2) the volume of the optical star is the same at ingress and egress. To effect this, we set

$$\delta M = \frac{\partial(\delta M)}{\partial\nu_1}\delta\nu_1 + \frac{\partial(\delta M)}{\partial\nu_2}\delta\nu_2 = 0 , \tag{14}$$

$$\delta V = \frac{\partial(\delta V)}{\partial\nu_1}\delta\nu_1 + \frac{\partial(\delta V)}{\partial\nu_2}\delta\nu_2 = 0 , \tag{15}$$

where

$$\delta M = M_2 - M_1 - 2\Theta_e , \tag{16}$$

and

$$\delta V = V_2 - V_1 . \tag{17}$$

Here V refers to stellar volume, M to mean anomaly, ν to true anomaly, with $\theta = \nu - \nu_{\text{conj}}$. Subscripts 1 and 2 refer to ingress and egress, respectively. The δM derivatives are

$$\frac{\partial(\delta M)}{\partial\nu_{1,2}} = \mp\left(\frac{1 - e}{1 + e}\right)^{1/2}(1 - e\cos E_{1,2})\frac{\cos^2(\frac{1}{2}E_{1,2})}{\cos^2(\frac{1}{2}\nu_{1,2})} , \tag{18}$$

where the upper ($-$) sign applies for subscript 1 and E is the eccentric anomaly. The δV derivatives have no simple analytic form, and must be found numerically. The useful output of the above iterative calculation consists of values of the Ω potential at ingress and egress.

The third subproblem is that of finding the *periastron* value of Ω which produces the same volume as we have at ingress and egress. Since a computing routine to find $\Omega(V, F, D, q)$ must exist for us to have gotten this far, we simply substitute $D = 1 - e$ and other appropriate values into that operator to find $\Omega_{\text{periastron}}$.

The treatment of contact binaries with nonzero eccentricity and nonsynchronous rotation presents advanced difficulties and is not considered here. For such cases one could not ignore the dynamical aspects of the problem. Circularization and resynchronization times for such binaries are expected to be extremely short, and at present no observational examples are known.

Perhaps to compensate for the lack of nonsynchronous contact binaries, nature seems to have provided a previously unrecognized morphological type of close binary which occurs only for nonsynchronism, and which might be called a double-contact binary. This is a binary in which both components fill their limiting lobes, yet are not in contact—even at one point. This can occur, for example, when at least one of the components rotates faster than synchronously, so that its limiting lobe is smaller than the Roche lobe. The situation can occur as a natural consequence of mass transfer because mass transfer normally spins-up an accreting star by converting orbital into rotational angular momentum. Thus, near to or after the end of the rapid phase of mass transfer, the mass-losing star would normally be found to fill its Roche lobe (rotating synchronously) while the accreting star (or perhaps only its outer envelope) may have accepted all the angular momentum it can hold, and thus fill its limiting "rotational" lobe. It will detach from the rotational lobe only after some of its angular momentum has transferred back into the orbit via tidal braking. It appears quite likely that β Lyrae and V356 Sgr (Wilson and Caldwell 1978) are such binaries. Other systems which are obviously near the end of the rapid phase, such as U Cep, should be suspected of being double-contact systems. The main evolutionary significance of the finding of double contact is that, since the accreting star cannot accept new high-angular-momentum material, such a binary must find an alternate repository for transferred matter, as first noted by Wilson and Caldwell (1978); this is a probable explanation for the thick circumstellar disks seen in β Lyr and V356 Sgr, and is perhaps related to numerous strange effects seen in U Cep.

In order to treat double-contact binaries, a mode of program operation may be introduced which constrains *both* components to fill their limiting lobes exactly. This is mode 6 of our program. Notice that the existence of double-contact systems introduces a symmetry into the morphology of close binaries, in that stars may fill lobes exactly not only as a result of mass loss, but also as a result of mass gain.

A point of possible confusion is the temperature to be assigned to the X-ray star in X-ray mode operation. Clearly the characteristic X-ray temperature (of, say, 5×10^7 K) has no direct significance for this problem, unless we actually treat the X-ray flux *as X-rays* (with regard to fluorescence, X-ray opacities in upper atmosphere, etc.). Assuming that they are used here only as a source of heating, the significant quantity is the X-ray luminosity. Since the X-ray temperature is not of direct consequence, it is not advisable to "couple together" the temperatures and luminosities of the two components. The best procedure is to treat temperatures and luminosities as *independent* parameters, and set the temperature of the X-ray star to the same value as that of the optical star. This will ensure that no differential bolometric correction is applied in the X-ray heating effect. Since changing the input value of L_x can only affect the amount of heating ("reflection effect") and L_{optical} enters only as a scale factor in the optical light curve, one can interpret the input luminosities as *bolometric* quantities in this mode. Thus in doing a simultaneous fit to n light curves, all of which correspond to essentially the same epoch, one should expect the ratio L_x/L_{optical} to be the same for all light curves, and to ascribe any wavelength-dependent differences to noise in the observations. Since such observations typically are quite noisy, and since the only "handle" which the least squares operator has on the X-ray luminosity is through the reflection effect, one should not be surprised to find fairly large changes in L_x/L_{optical} from light curve to light curve. A reasonable procedure would be simply to average values found for the various passbands.

IV. EXAMPLE: SW LYNCIS

There are few binaries with significantly eccentric orbits *and* large proximity effects, so as to provide a strong test of a program constructed according to the above principles. However, such rare systems are very interesting because (*a*) they will have large rates of periastron advance, thus potentially providing tests of stellar structure models; (*b*) they will often be in rapid evolutionary transition stages; and (*c*) they can provide checks on the theory of tidal damping of eccentricity and rotation (e.g., Zahn 1966*a*, *b*, *c*; Kopal 1968, 1972*a*, *b*; Alexander 1973; Press, Wiita, and Smarr 1975).

The intention here is not to make a thorough study of even one binary—that will be left to subsequent papers—but rather to give an idea of what can be learned in such applications. SW Lyncis seemed particularly interesting in this regard because it has the rather large fractional radii (r/a) of about 0.41 and 0.26, and Vetesnik (1977) has determined spectroscopically an eccentricity of 0.11 ± 0.01 p.e. Furthermore, Gleim's (1967) light curve shows asymmetries suggestive of orbital eccentricity, for which the longitude of periastron would be similar to that found from the radial velocities, allowing for precession at a rate suggested by Gleim (based on a period study). However, several points suggest strongly that there is no significant eccentricity, and the overall weight of evidence leaves little room for doubt that the orbit is circular. Nevertheless, since some of these points resulted from our attempts to model the system quantitatively, and since the light and velocity curves mimic those of an eccentric orbit fairly well, SW Lyn still provides some excellent examples of what can be learned with the aid of a generalized program.

SW Lyncis has components only a little removed from the zero-age main sequence. Preliminary light curve analysis suggests that it is semidetached, with the secondary filling its limiting lobe; but since all such analyses done to date are tentative, the system's evolutionary history (with possible mass ratio reversal) will not be discussed here. An issue of primary importance with regard to the reality of orbital eccentricity is the expected period of apsidal motion. The period calculated from the relation given by (for example) Schwarzschild (1958) is of the order of *1 to 2 years*. Thus Gleim's suggestion of a 75 year period is not tenable. The apsidal period is so short because it depends on the fifth power of the relative radii, which are about a factor of 2 larger for SW Lyn than for typical apsidal motion binaries. Such apsidal motion should be evident in several ways, if the orbit were really eccentric. For example, there should be periodic phase excursions of the eclipses, of the order of

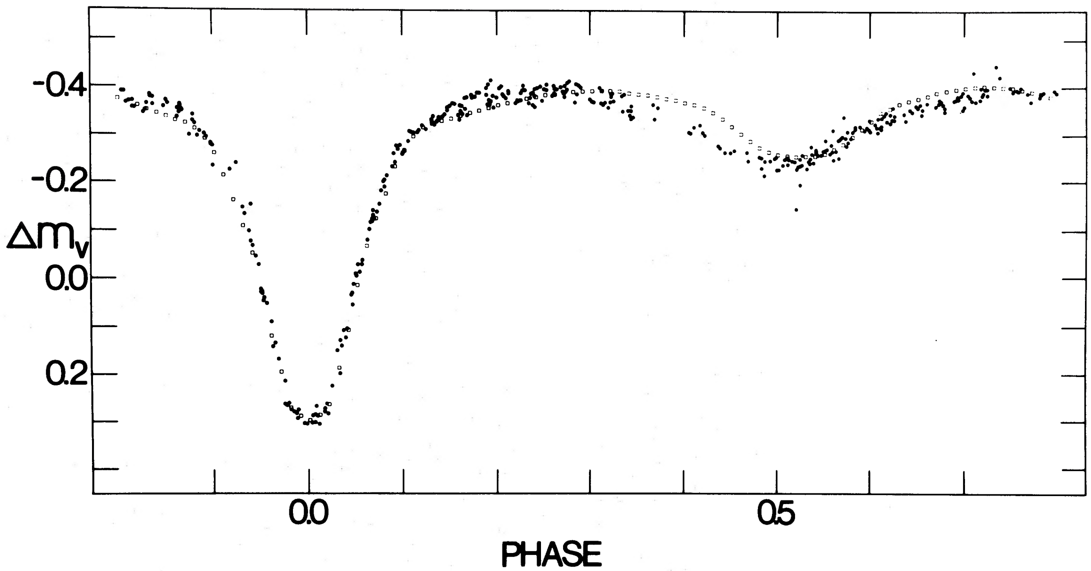

FIG. 1.—Theoretical and observed light curves of SW Lyn, under the hypothesis of an eccentric orbit. While the asymmetries resemble eccentric orbit effects, they are probably due to another circumstance (see text). No combination of e and ω will solve all fitting problems, and this graph is part of the argument that the reported eccentricity is not real.

$\pm 0^{d}02$, which could be noticed in the $O - C$ diagram (Vetesnik 1968) or individual photoelectric minima within a given series. Since the primary and secondary eclipses shift out of phase with one another, the spacing between eclipses doubles the phase-shift effect and should be easily noticed. Also there should be a smearing effect in phase in any light curve compiled over more than a few months. In brief, none of these effects are seen in the various light curves of SW Lyn (Gleim 1967; Vetesnik 1968, 1977; Landolt 1973). Thus it seems virtually certain that the eccentricity measured by Vetesnik is a manifestation of some other effect, such as mass transfer. In view of the short time scale for circularization of the orbit ($\tau \approx 2 \times 10^5$ years [Press, Wiita, and Smarr 1975]) the eccentricity is probably nil.

Attempts at quantitative fitting of the Gleim light curve reveal additional problems. There is no value of ω which will solve all fitting problems. The best results are for ω (primary) near 290°, but a smaller value is needed to obtain the right eclipse spacing and a larger one to represent properly the asymmetries in the maxima and eclipses. One such fit is shown in Figure 1. So that others may check their programs against ours, we list the parameters for the theoretical light curve of SW Lyn in Table 1 and the digital light curve in Table 2. These tables are the most important part of this section. Because of the complexity of such a program, it would be good to see digital examples published for other programs when they are first used.

TABLE 1

PARAMETERS FOR SW LYNCIS THEORETICAL 5500 Å LIGHT CURVE

A. FREE PARAMETERS			
i	84°00	$(T_1)_{\rm mean}$	7230 K
F_1	0.00	$(T_2)_{\rm mean}$	3800 K
F_2	1.00	M_2/M_1	0.3800
e	0.08	Ω_1	2.8500
ω_1	295°	A_1	1.00
g_1	1.00	A_2	1.00
g_2	1.00	x_1	0.55
l_3	0.0000	x_2	0.80
B. AUXILIARY QUANTITIES			
Ω_2	2.8802	$L_1/(L_1 + L_2)_V$	0.9848
$(r_1/a)_{\rm pole}$	0.4042	$(r_2/a)_{\rm pole}$	0.2496
$(r_1/a)_{\rm point}$	0.4419	$(r_2/a)_{\rm point}$	0.3035
$(r_1/a)_{\rm side}$	0.4046	$(r_2/a)_{\rm side}$	0.2571
$(r_1/a)_{\rm back}$	0.4204	$(r_2/a)_{\rm back}$	0.2807
$(r_1/a)_{\rm mean}$	0.4112	$(r_2/a)_{\rm mean}$	0.2640

TABLE 2

COMPUTED SW LYNCIS 5500 Å LIGHT CURVE

Phase[a]	D/a	l	Phase	D/a	l
0.5808......	0.920	0.9798	0.1008......	1.080	0.9576
0.6208......	0.923	1.0357	0.1408......	1.075	1.0122
0.6608......	0.932	1.0615	0.1808......	1.067	1.0325
0.7008......	0.945	1.0777	0.2208......	1.055	1.0512
0.7408......	0.962	1.0795	0.2608......	1.039	1.0651
0.7808......	0.981	1.0650	0.3008......	1.021	1.0706
0.8208......	1.001	1.0402	0.3408......	1.001	1.0668
0.8608......	1.021	1.0131	0.3808......	0.981	1.0549
0.9008......	1.039	0.9498	0.4208......	0.962	1.0371
0.9408......	1.055	0.7764	0.4608......	0.945	0.9851
0.9808......	1.067	0.5808	0.5008......	0.932	0.9475
0.0208......	1.075	0.5856	0.5408......	0.923	0.9455
0.0608......	1.080	0.7864	0.5808......	0.920	0.9798

[a] Periastron is at phase 0.5808, primary conjunction at phase 0.0000.

Should suspicion arise that the problems with SW Lyn could lie in the theoretical light curves, it should be mentioned that light curves of V1647 Sgr, a normal eccentric orbit binary, were computed, and these agree extremely well with those observed by Clausen, Gyldenkerne, and Nelson (1977). The periastron effect was reproduced quite accurately with no difficulty.

A curious feature of the Vetesnik (1977) radial velocity observations is the absence of a noticeable rotational eclipse disturbance (Rossiter effect). One should perhaps wait for more observations to accumulate before trying to explain this strange result of a star 85% as large as its critical lobe, with no apparent rotation. However, mainly as a demonstration of the computation of a light curve for nonsynchronous rotation, F_1 was set to zero for our theoretical light curves of SW Lyn. It has been verified that the fit is not materially improved by adoption of synchronous rotation.

Some examples of what can be learned, even by trial-and-error fitting are:

a) The photometry of SW Lyn does not permit an eccentricity as large as that ($e = 0.11 \pm 0.01$ p.e.) reported spectroscopically because the theoretical ellipsoidal variation is then too small. That happens because the stars can fill their limiting lobes only at periastron, while over most of the orbit they are smaller than the (instantaneous) lobes and thus show relatively little tidal distortion. Actually one could compensate for this effect by increasing the mass ratio to something like 0.45 (from Vetesnik's suggested 0.32), but then the absolute masses are much too small to make any sense.

b) The very difficulty one finds in satisfying all features of the light and velocity curves with *one* set of parameters renders it quite clear that a system for which this can be accomplished will provide very strong values of e, ω, and other parameters needed for comparison of theoretical and observed rates of periastron advance.

c) It is clear that one needs the method of least squares more than ever, when e and ω (and perhaps F_1, F_2) are added to the usual list of parameters. There are then just too many numbers for subjective adjustment, if one is to retain much confidence in the detailed results. It is also clear that one cannot afford to disregard any constraints on the elements, such as the requirement that photometric and spectroscopic values be the same.

Since the reported eccentricity of SW Lyn probably is not real, it becomes natural to ask what binaries with real eccentricities would be logical test objects of a program which accurately treats both eccentricity and proximity effects. Important in this respect would be systems with measurable apsidal motion (Batten 1973 and references therein). Since the ratio of orbital to apsidal periods depends on the fifth power of the relative radii, it is crucial to determine the radii as accurately as possible. Because of parameter correlations, shortcomings in the model physics and geometry can lead to errors in the radii which are important when one attempts to predict apsidal periods. Only when such effects are quite small can we make strong tests of stellar structure models through observations of apsidal motion.

V. EXAMPLE: HD 77581

To illustrate computed radial velocity curves for our general model, parameters similar to those of the X-ray binary 4U 0900−40 (HD 77581) have been chosen. The velocities of the (theoretical) X-ray star (component 1 here) are, of course, the same as those of a point source, but those of the optical star show the various proximity effects as departures from a normal velocity curve. It seems highly likely that the orbit of HD 77581 is significantly noncircular because the e and ω (longitude of periastron) found from optical and X-ray observations are in good agreement (van Paradijs *et al.* 1977). Also, a fairly large eccentricity would not be unprecedented among such systems (cf. Rappaport *et al.* 1978). However, this is not a closed issue since Hutchings (1978) finds the light curve compatible with negligible eccentricity, while Milgrom and Avni (1976) have postulated an effect which could simulate orbital eccentricity. The rotation is probably slower than synchronous (Wickramasinghe *et al.*

TABLE 3

PARAMETERS[a] FOR "HD 77581" 4350 Å VELOCITY CURVE

A. FREE PARAMETERS

i	90°	T_2	25,000 K
F_2	0.50	M_2/M_1	12.5
e	0.10	A_2	1.00
ω_x	175°	L_1/L_2 (BOLO)	0.0008
V_γ	-7.2 km s^{-1}	x_2	0.43
Θ_e	35°.39	a	4.04×10^7 km
g_2	1.00	P	7.748×10^5 s

B. AUXILIARY QUANTITIES

Ω_2	17.2283	$(r_2/a)_{\rm side}$	0.581
$(r_2/a)_{\rm pole}$	0.567	$(r_2/a)_{\rm back}$	0.596
$(r_2/a)_{\rm point}$	0.646	$(r_2/a)_{\rm mean}$	0.583

[a] Most quantities are defined in the text. A_2 and x_2 are the bolometric albedo and limb darkening coefficient, as used in earlier papers. The radii pertain to periastron.

1974), so F_2 was set to 0.5. Tables 3 and 4 give the chosen input parameters and resulting computed velocity curves. The inclination is not an important fitting parameter unless its value drops below 80°. Here it was set to 90°. It should be emphasized that no detailed fitting to radial velocities of HD 77581 was done.

The effect of orbital eccentricity is to diminish distortions of the velocity curves which are due to the rotational-tidal effect. This is so because a component can be nearly as large as its limiting lobe (and thus have maximum tidal distortion) only near periastron. Thus, although eccentricity is a complicating factor from some points of view, it simplifies velocity curves by restricting the effect predicted by Sterne (1941) and treated quantitatively by Wilson and Sofia (1976) and more recently by many others to a fairly small phase range. Hutchings (1978) has found the same result and made similar comments. Figure 2 shows the velocity curve of Table 4 and shows how small the rotational-tidal effect is for $e = 0.10$ and $F_2 = 0.50$. Slower than synchronous rotation strongly reduces the predicted effect. Since the velocity distortion is due to a combination of tidal deformation and rotation, a reduction in either of these will reduce the distortion. Slow rotation leads to a relatively large limiting lobe so that, for a given radius, a star finds itself then relatively undeformed. For synchronous rotation, the distortion effect seen in Figure 2 would be about 3 times larger, which is still rather small. The result for HD 77581 is that the velocity amplitude is changed very little from that of the purely orbital motion. Van Paradijs *et al.* assumed that the tidal-rotational effect could be neglected in their solution for the masses of the components. In that solution, of course, the mass of the X-ray component is of particular interest. It appears that the effect is indeed negligible for this binary, considering the scatter in the available observations. However, note that the effect will be larger when the orbit has precessed to $\omega_x \approx 270°$, since maximum tidal distortion will then coincide with transit conjunction. Incidentally, for all the computations of this section, the X-ray luminosity was assumed to be 0.0008 of the optical luminosity. An X-ray luminosity 10 times larger would give virtually the same results.

TABLE 4

COMPUTED "HD 77581" RADIAL VELOCITY CURVE

Phase[a]	D/a	$(V_2)_{\rm orb}$ (km s^{-1})	$(V_2)_{\rm total}$ (km s^{-1})	Phase	D/a	$(V_2)_{\rm orb}$ (km s^{-1})	$(V_2)_{\rm total}$ (km s^{-1})
0.00	0.900	+19.50	+17.86	0.52	1.099	−29.12	−28.63
0.04	0.904	19.01	18.06	0.56	1.094	−28.53	−28.04
0.08	0.915	16.33	16.98	0.60	1.084	−26.86	−26.53
0.12	0.932	11.91	13.58	0.64	1.069	−24.15	−24.05
0.16	0.954	6.31	7.89	0.68	1.050	−20.42	−20.45
0.20	0.978	0.20	0.95	0.72	1.028	−15.90	−15.80
0.24	1.004	−5.99	−6.15	0.76	1.004	−10.31	−10.24
0.28	1.028	−11.78	−12.63	0.80	0.979	−4.02	−3.99
0.32	1.050	−16.45	−18.03	0.84	0.954	+2.39	+2.52
0.36	1.069	−21.30	−22.25	0.88	0.932	8.57	8.70
0.40	1.084	−24.73	−25.32	0.92	0.915	13.91	13.81
0.44	1.094	−27.22	−27.35	0.96	0.904	17.70	17.05
0.48	1.099	−28.69	−28.43	0.00	0.900	19.50	17.86

[a] Phases reckoned from periastron.

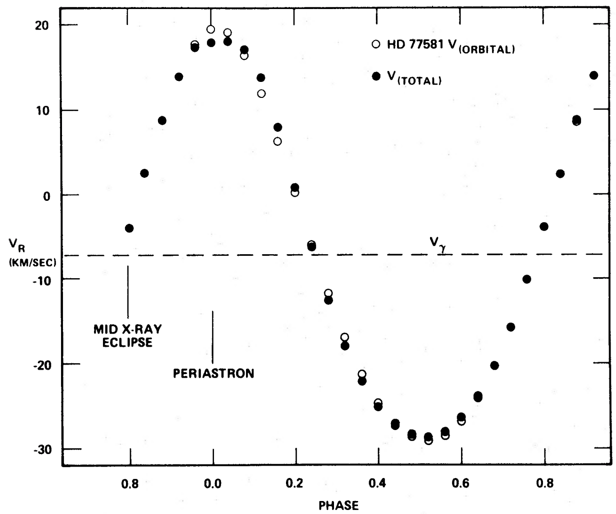

FIG. 2.—Theoretical radial velocity curves of HD 77581, including (*filled circles*) and without (*open circles*) proximity corrections

It is somewhat difficult to produce X-ray eclipse durations quite as large as those observed for 4U 0900−40. This suggests either that one or more of the assumed parameters is in error or that the eclipse may be produced at a level higher than the photosphere. However, with a combination of relatively slow rotation and an eccentricity of 0.10, there is agreement with the smaller eclipse durations reported to date. Observed values range between about 34° and 40° (Forman *et al.* 1973; Watson and Griffiths 1977; Charles *et al.* 1976), while Table 5 summarizes the largest possible theoretical values (star fills lobe at periastron) for two values of e and two rotation rates. Unless we restrict e to about 0.1 or less and adopt slower-than-synchronous rotation, there is little overlap between the theoretical and observational Θ_e values. The calculations were done with M_2/M_1, i, and ω_x equal to 12.5, 90°, and 175°, respectively (van Paradijs *et al.* 1977).

To summarize the developing picture of HD 77581 (4U 0900−40), it seems likely that the orbit is eccentric and that the rotation is slower than synchronous. This fits in with arguments derived from the observed rapid spin-up of the neutron star, and from theoretical equilibrium spin rates, to the effect that there has been a recent rapid increase in the radius (Savonije and van den Heuvel 1978). That is, the spin-up would be due to the turning-on of a stellar wind due to rapid expansion, while the persistence of eccentricity and asynchronism would be explained by the short time that tidal dissipation has had to act at the present relatively large radius. Changing rotation and reduction of orbital eccentricity would have been much less efficient at the smaller radius the optical star must have had until recently.

VI. SIMULTANEOUS LIGHT AND VELOCITY SOLUTIONS

Several difficulties are inherent in the treatment of light curves and radial velocity curves as separate and unconnected sources of information. For example, a parameter such as the mass ratio, which can be extracted from both types of observations, will in general be found to have two different values. Although some kind of weighted mean of these values might be adopted, both the spectroscopic and photometric solutions will be inconsistent with that mean value. A second difficulty is that minor sources of information, such as the rotational-tidal distortion of velocity curves, actually take on the role of *nuisances*. Thus we have the ironic situation that an observable effect, which should aid the solution, in practice degrades the solution either because there is no provision for computing the effect or because the requisite parameter values come mainly from the *other* type of solution. A third problem is that the computed error estimates (probable errors or standard deviations) of the various parameters fail to show the effects of correlations with parameters as they occur in the *other* type of solution. Thus results will appear to be more accurate than they are.

TABLE 5

THEORETICAL LIMITING X-RAY ECLIPSE SEMIDURATIONS Θ_e

F_2/e	0.1360	0.1000
0.50	33°5	35°4
1.00	32°3	33°9

The above difficulties can be overcome by fitting light and velocity curves simultaneously, much in the way that Wilson and Devinney (1971) introduced the practice of fitting multicolor light curves simultaneously. The total number of parameters in such an adjustment will be greater than that in an ordinary light curve solution by only two, since most parameters are common to light curve and radial velocity cases. The two additional parameters could be $a \sin i$ and the systemic velocity, V_γ. The individual semimajor axes, a_1 and a_2, would not be separate parameters because the mass ratio q is a parameter and, along with a, determines a_1 and a_2. However, should the additional two parameters significantly impede solution convergence due to enhanced correlation problems, the method of multiple subsets (Wilson and Biermann 1976) can always be used to ensure convergence. For the present we assume that the period P will have been determined separately from the other elements in the preliminary analysis of the data, although in the future it may be found profitable to include P in the adjustment.

The first simultaneous least squares fits of multicolor light curves appeared in an addendum to the paper by Wilson and Devinney (1971) on MR Cyg. Such solutions require that parameters which are not different for different light curves (the geometric and bolometric parameters and the mass ratio) have only one value each, while curve-dependent parameters have n values for n curves. Simultaneous multicolor light and velocity curve solutions can be done by the same mathematics as simultaneous light curves (Wilson and Devinney 1971). As before, one has a general equation of condition of the form

$$f_{\text{obs}} - f_{\text{comp}} = \Delta f = \frac{\partial f}{\partial p_1}\,\Delta p_1 + \frac{\partial f}{\partial p_2}\,\Delta p_2 + \cdots + \frac{\partial f}{\partial p_n}\,\Delta p_n\,, \tag{19}$$

containing parameters $p_1, p_2, \ldots, p_n$ and a function, f, of these parameters. The function f is understood to be a light value (i.e., relative flux) for photometric observations, and may be, for example, in the U, B, or V (etc.) system. It is understood to be a radial velocity for spectroscopic observations and might be for component 1 or component 2. Thus we combine n light curves in n-passbands and one or two velocity curves, which should also be assigned some mean wavelength of observation based on the wavelengths of measured spectral lines. Of course, Δp_1, Δp_2, etc., are corrections to parameter values of the previous iteration, and $f(p_1, p_2, \ldots, p_n, \text{phase})$ is to be computed from our computer model of the earlier sections or a similar model. The derivatives $\partial f/\partial p_n$ are to be found by differencing, either by

$$\frac{\partial f}{\partial p_n} = \frac{f(p_n + \delta p_n) - f(p_n)}{\delta p_n} \tag{20}$$

(asymmetric "derivatives") as in Wilson and Devinney (1971) or by

$$\frac{\partial f}{\partial p_n} = \frac{f(p_n + \tfrac{1}{2}\delta p) - f(p_n - \tfrac{1}{2}\delta p)}{\delta p_n} \tag{21}$$

(symmetric "derivatives"), an alternative suggested by Wilson and Biermann (1976). In computing such derivatives, naturally one must apply all invoked constraints (such as the semidetached condition, etc.) to the incremented as well as the base values. Simultaneous light and velocity solutions may be obtained by including *one* term in equation (19) for each curve-independent parameter, but $(n + m)$ terms for each curve-dependent parameter for light curves in n-passbands and velocity curves of m components ($m = 0$, 1, or 2). Then for the ith observation with respect to the kth parameter,

$$\frac{\partial f_i}{\partial p_k} = 0$$

if parameter k has no influence on an observation of the type (i.e., light or velocity) or passband (i.e., U, B, V, etc.) or component (1 or 2) represented by i. For example, derivatives of light with respect to a (orbital semimajor axis) may be set to zero, rather than actually computed, since a does not affect light values. Similarly, derivatives of light in the B-passband with respect to limb darkening in the V-passband may be set to zero. An interesting situation arises for certain derivatives which are very nearly zero, but not exactly so, such as a velocity differentiated with respect to a component's temperature. We expect some small dependence here in a binary with strong reflection, because the measured velocity depends on the amount of "reflection" heating, which, in turn, depends on the temperatures of both stars. This problem is easily handled by suppressing the adjustment of that parameter for that component, except in cases where there is some reasonable expectation of finding a meaningful answer (i.e., for which at least some derivatives in the appropriate column are significantly different from zero).

In our current DC program, based on the above rules and ideas, the complete parameter list is summarized in Table 6.

One must be especially careful about weighting the observations in simultaneous light and velocity solutions. If only light *or* velocity observations are processed, good results can be obtained without giving much thought to weighting problems, provided that all subsets of observations (i.e., B, V, etc.) have approximately the same precision, and all are affected by essentially the same sources of accidental error (i.e., shot noise, sky transparency

TABLE 6

SUMMARY OF ADJUSTABLE PARAMETERS IN WD PROGRAM

Curve-independent Parameters	
$a \sin i$........	Orbital semimajor axis X sin (inclination)
V_γ..........	Systemic velocity
e............	Orbital eccentricity
ω............	Longitude of periastron, star 1
F_1, F_2........	Ratio of surface rotation rate to synchronous rate for stars 1, 2
ϕ_s...........	Phase shift parameter; the phase of conjunction
i............	Orbital inclination to plane of sky
g_1, g_2........	Gravity darkening exponents for stars 1, 2
T_1, T_2........	Mean surface effective temperatures for stars 1, 2
A_1, A_2.......	Bolometric albedos for stars 1, 2
Ω_1, Ω_2.......	Modified surface potentials for stars 1, 2
q............	Mass ratio M_2/M_1, where star 1 is farther from the observer at primary conjunction
Curve-dependent Parameters	
L_1, L_2........	Relative monochromatic luminosities of stars 1, 2 *over* 4π *steradians*
x_1, x_2........	Limb darkening coefficients for stars 1, 2 in linear cosine law, $I/I_0 = 1 - x + x \cos \gamma$
l_3............	Third light in same units as relative flux from the binary system. This is *not* third luminosity, and is not directly comparable to L_1, L_2.

fluctuations, etc.). Since these conditions are satisfied in most work, one can often be less than exacting in specifying input quantities used by a program to compute weights. However, light and velocity observations will almost never satisfy the above conditions when mixed together, so it is essential to enter reasonably correct values for the weighting quantities. Three considerations (types of weights) enter into the overall weighting scheme. These are:

A) *Intrinsic weights.*—These would normally be assigned to the individual observations, so that the input data stream contains triplets of [phase; relative flux or radial velocity; weight]. The intrinsic weights would most commonly be based on the number of individual observations per normal point or the amount of time over which the signal was averaged, but might also be partly or entirely subjective. Since there is a school with nonnegligible influence which advocates that subjectivity is *good* in such solutions, it is important to note that the introduction of subjective weights provides *controllable continuity* between wholly objective and subjective solutions. One can thus introduce an arbitrarily small subjective element by modifying the weights of only a few observations by factors differing very little from unity.

B) "*Curve*"*-dependent weight factor.*—In simultaneous solutions in several passbands, or with mixed light and velocity observations, the various light and velocity curves will usually have unequal precision. If this fact were ignored, the least squares operator would pay most attention to the curve with the most scatter, which is certainly not what is wanted. Therefore, one should enter an estimate of the probable error or standard deviation of a single observation in each curve, at some arbitrarily chosen reference phase. It is important that this estimate pertain to the observations scaled as they actually are scaled in the input data. Many persons may be satisfied to use rough estimates for these p.e.'s or s.d.'s (σ's), but it is worthwhile to take the small trouble of making an accurate estimate, especially since this need be done only once for a given data collection. A weighting factor proportional to σ^{-2} should be computed for each observation, where σ varies from curve to curve.

C) *Level-dependent weights.*—The precision (a measure of the weight) of a photometric observation usually depends on the brightness of the star observed, so it will be different at the various levels of a light curve. The dependence on level is governed by the source of noise. For noise proportional to light level l, such as variable sky transparency or scintillation, the weight is proportional to l^{-2}, while for noise proportional to the square root of the light level, such as photoelectron shot noise, the weight is proportional to l^{-1}. Some types of noise are independent of light level (weight proportional to l^0), such as detector noise in photoconductive devices. Radial velocity scatter does not vary with level in a way subject to such simple modeling, so one should use a constant level weight factor for velocities, unless willing to attempt more sophisticated weighting models.

Examples of simultaneous light and velocity curve differential correction solutions will be given in a future paper.

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Note added in proof.—It should be emphasized, although it is made clear by eq. (8), that the mean surface temperatures defined here are *those which would exist in the absence of the reflection effect*. Local temperatures in irradiated regions must be increased, as described in Wilson *et al.* (1972).

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