Неопределенный интеграл

1. $\int \frac{dx}{x+5} = \ln|x+5| + C$

2. $\int \frac{dx}{3x-5} = \frac{1}{3}\int \frac{dx}{x-\frac{5}{3}} = \frac{1}{3}\ln\left|x-\frac{5}{3}\right| + C$

3. $\int \frac{dx}{3x^2+5} = \frac{1}{3}\cdot\int \frac{dx}{x^2+\sqrt{\frac{5}{3}}^2} \ominus$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C}$$

$\ominus\ \frac{1}{3}\cdot\frac{1}{\sqrt{5/3}}\cdot\operatorname{arctg}\frac{x}{\sqrt{5/3}} + C$

4. $\int \frac{dx}{3x^2-5} = \frac{1}{3}\int \frac{dx}{x^2-\frac{5}{3}} \ominus$

$$\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| + C}$$

$\ominus\ \frac{1}{3}\cdot\frac{1}{2\cdot\sqrt{5/3}}\cdot\ln\left|\frac{x-\sqrt{5/3}}{x+\sqrt{5/3}}\right| + C$

5. $\int \frac{x\,dx}{3x+5} = \int \frac{3x+5-5\,dx}{3(3x+5)} = \int \frac{3x+5}{3(3x+5)} - \frac{5}{3(3x+5)}\,dx = \int \frac{1}{3} - \frac{5}{3(3x+5)}\,dx = \frac{x}{3} - \frac{5}{3}\cdot\int\frac{1}{3x+5}\,dx \ominus$

$\ominus\ \frac{x}{3} - \frac{5}{9}\cdot\ln\left|x+\frac{5}{3}\right| + C$

6. $\int \frac{x^2dx}{3x+5} = \int \frac{9x^2-25+25}{9(3x+5)}\,dx = \frac{1}{9}\int\frac{(3x-5)(3x+5)+25}{(3x+5)}\,dx = \frac{1}{9}\int (3x-5)+\frac{25}{3x+5}\,dx \ominus$

$\ominus\ \frac{1}{9}\left(\int 3x-5\,dx + 25\int\frac{1}{3x+5}\,dx\right) = \frac{1}{9}\left(\frac{3}{2}x^2-5x+\frac{25}{3}\ln\left|x+\frac{5}{3}\right|\right) + C$

7. $\int\sqrt{3x+5}\,dx = \frac{1}{3}\int\sqrt{3x+5}\,d(3x+5) = \frac{1}{3}\frac{(3x+5)^{\frac{3}{2}}}{\frac{3}{2}} + C$

8. $\int\frac{dx}{\sqrt{3x^2+5}} \ominus$

$$\boxed{\int\frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C}$$

$\ominus\ \int\frac{dx}{\sqrt{3}\left(\sqrt{x^2+\frac{5}{\sqrt{3}}}\right)} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2+\frac{5}{\sqrt{3}}}} = \frac{1}{\sqrt{3}}\ln\left|x+\sqrt{x^2+\frac{5}{3}}\right| + C$

9. $\int \frac{x\,dx}{\sqrt{3x^2+5}} = \int \frac{d(3x^2+5)}{6\sqrt{3x^2+5}} = \frac{1}{6}\int \frac{d(3x^2+5)}{\sqrt{3x^2+5}} \ominus$

$d(3x^2+5) = 6dx$

$\ominus$ $t = 3x^2+5$ $= \frac{1}{6}\int \frac{dt}{\sqrt{t}} = \frac{1}{6}\frac{\sqrt{t}}{\frac{1}{2}} = \frac{1}{3}\cdot\sqrt{3x^2+5} + C$

10. $\int \cos(1-6x)\,dx = \int \cos(1-6x)\,d(1-6x) = \frac{1}{6}\cdot\sin(1-6x) + C$

11. $\int tg\,x\,dx = \int \frac{\sin x}{\cos x}dx \ominus d\cos x = -\sin x\,dx \ominus \int \frac{-1}{\cos x} d\cos x = -\ln|\cos x| + C$

12. $\int tg^2 x\,dx = \int \frac{(1-\cos^2 x)}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - \int 1\,dx \ominus$

$\int \frac{dx}{\cos^2 x} = tg\,x + C$

$\ominus\ tg\,x - x + C$

13. $\int e^{\cos x}\cdot \sin x\,dx = -\int e^{\cos x} d\cos x = -e^{\cos x} + C$

$d\cos x = -\sin dx$

13. $\int \frac{arctg\,x}{1+x^2}dx = \int arctg\,x\,d\,arctg\,x = \frac{arctg^2 x}{2} + C$

$d(arctg\,x) = \frac{dx}{1+x^2}$

14. $\int \frac{\sqrt{\arcsin x}}{1-x^2}dx = \int \sqrt{\arcsin x}\,d(\arcsin x) = \frac{\arcsin^{\frac{3}{2}} x}{3/2} + C$

15. $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2+x+\frac{1}{4}-\frac{5}{4}} = \int \frac{dx}{(x+\frac{1}{2})^2-\frac{5}{4}} = \int \frac{d(x+\frac{1}{2})}{(x+\frac{1}{2})^2-\frac{5}{4}} \ominus$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

$\ominus\ \frac{1}{2\frac{\sqrt{5}}{2}}\ln\left|\frac{x+\frac{1}{2}-\frac{\sqrt{5}}{2}}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right|$

16. $\int \underbrace{x}_{u}\,\underbrace{\cos 2x\,dx}_{dv}$

$u = x \quad du = dx$

$dv = \cos 2x\,dx$

$v = \int \cos 2x\,dx = \frac{\sin(2x)}{2}$

$$\int x\cos(2x)\,dx = x\cdot\frac{\sin(2x)}{2} - \int\frac{\sin(2x)}{2}dx = x\cdot\frac{\sin(2x)}{2} + \frac{1}{2}\cdot\frac{\cos(2x)}{2} + C$$

17. $\int (3x-2)\cdot e^{-x}\,dx \ominus$

$u = 3x-2 \qquad du = 3\,dx$

$dv = e^{-x}dx$

$v = \int e^{-x} = -e^{-x}$

$\ominus\ (3x-2)(-e^{-x}) - \int -3\cdot e^{-x}\,dx = (3x-2)(-e^{-x}) - 3e^{-x} + C$

1. $\int \sin^2 x\,dx \ominus$

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}}$$

$\ominus \int \frac{1-\cos 2x}{2}dx = \frac{1}{4}\int 1-\cos 2x\ d(2x) = \frac{1}{4}(2x) - \frac{1}{4}\int \cos 2x\ d(2x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $\int \sin^3 x\,dx = \int \sin^2 x\cdot \sin x\ dx = -\int \sin^2 x\ d(\cos x) = -\int 1-\cos^2 x\ d(\cos x) = -\left(\cos x - \int \cos^2 x\ d\cos x\right) =$

$= -\cos x + \frac{\cos^3 x}{3} + C$

3. $\int \sin^4 x\ dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}x - \frac{1}{4}\int 2\cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx =$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{4}\int\frac{1+\cos 4x}{2}dx = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$

4. $\int \sin^5 x\ dx = \int \sin^4 x\cdot\sin x\ dx = \int(\sin^2 x)^2\cdot\sin x\,dx = -\int (1-t^2)\ dt$ . . .

5. $\int \sin^2 x\cdot\cos^3 x\ dx = \int \sin^2 x\cdot\cos^3 x$

6. $\int\frac{x\,dx}{(x+1)(x+2)(x-3)} =$

$$\frac{x}{(x+1)(x+2)(x-3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}\ \Big|\ (x+1)(x+2)(x+3)$$

$x = A(x+2)(x-3) + B(x+1)(x-3) + C(x+1)(x+2)$

$x = A(x^2-x-6) + B(x^2-2x-3) + C(x^2+3x+2)$

$$\begin{aligned} &\text{при } x^2: \\ &\text{при } x^1: \\ &\text{при } x^0: \end{aligned} \begin{cases} A+B+C=0 \\ -A-2B+3C=1 \\ -6A-3B+2C=0 \end{cases}$$

2 Способ:

$x=-1: \quad -1=-4A \quad A=\frac{1}{4}$

$x=-2: \quad -2=5B$

$x=3: \quad 3=20C$

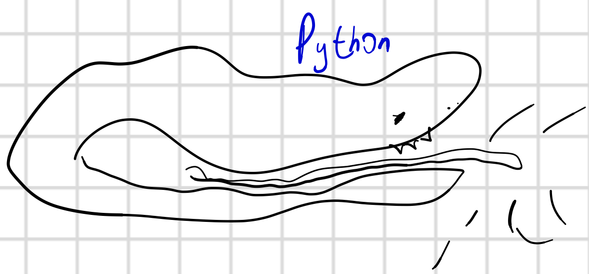
$$\int \frac{x\,dx}{(x+1)(x+2)(x-3)} = \frac{1}{4}\ln|x+1| - \frac{2}{5}\ln|x+2| + \frac{3}{20}\ln|x-3| + C$$

7. $\displaystyle\int \frac{3x^4+3x^3+2}{x^3+x^2-x-1}\,dx$

$$\begin{array}{r|l} 3x^4+3x^3+2 & x^3+x^2-x-1 \\ 3x^4+3x^3-3x^2-3x & \\ \hline & 3x \\ 3x^2+3x+2 & \end{array}$$

$$\frac{3x^2+3x+2}{x^3+x^2-x-1} = \frac{3x^2+3x+2}{(x^2-1)(x+1)} = \frac{3x^2+3x+2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$3x^2+3x+2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$



$x=-1: \quad 2=-2C$

$x=1: \quad 8=4A$

$x=0: \quad 2=A-B-C \Rightarrow B=1$

$$\ominus \int\left(3x+\frac{2}{x-1}+\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx=\frac{3}{2}x^2+2\ln|x-1|+\ln|x+1|+\frac{1}{x+1}+C$$

8. $$\int\frac{dx}{x^3+1}=\int\frac{dx}{(x+1)(x^2-x+1)}$$

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$1=A(x^2-x+1)+(Bx+C)(x+1)$$

$x=-1: \quad 1=3A, \; A=\frac{1}{3}$

$x=0: \quad 1=A+C \Rightarrow C=\frac{2}{3}$

при x^2: $\quad 0=A+B \Rightarrow B=-\frac{1}{3}$

$$\int\frac{dx}{(x+1)(x^2-x+1)}=\int\left(\frac{1/3}{x+1}+\frac{1}{3}\frac{-x+2}{x^2-x+1}\right)dx=\frac{1}{3}\ln|x+1|+\frac{1}{3}\int\frac{-x+2}{x^2-x+1}dx \ominus$$

$$\ominus \frac{1}{3}\ln|x+1|-\frac{1}{3}\int\frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx=\frac{1}{3}\ln|x+1|-\frac{1}{3}\int\frac{x-\frac{1}{2}-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx \ominus$$

$$\ominus \frac{1}{3}\ln|x+1|-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2(x-\frac{1}{2})}{\sqrt{3}}$$

9. $$\int\frac{dx}{\sqrt[3]{x}+\sqrt{x}}=\int\frac{6t^5dt}{t^2+t^3}=6\int\frac{t^3dt}{1+t}=6\int\frac{t^3-1+1}{1+t}dt=6\int(t^2-t+1)\left(\frac{1}{1+t}\right)dt \ominus$$

$\exists\, x=t^6$

$$\ominus\; 2t^3-3t^2+6t-6\ln|1+t|+C=$$

$$\int\frac{x\,dx}{(x^2+3x+1)^2}\longrightarrow\int\frac{t+b}{(t^2-a^2)^2}dt$$

$$\int\frac{dt}{(t^2-a^2)^2}=\frac{1}{a^2}\int\frac{a^2-t^2-t^2}{(t^2-a^2)^2}d-$$

1. $$\int\frac{2x^3+x^2+5x+1}{(x^2+3)(x^2-x+1)}dx$$

$$\frac{2x^3+x^2+5x+1}{(x^2+3)(x^2-x+1)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2-x+1} \quad / \cdot (x^2+3)(x^2-x+1)$$

$$2x^3+x^2+5x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+3)$$

$$2x^3+x^2+5x+1 = Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+3Cx+Dx^2+3D$$

$$\begin{cases} A+C=2 \\ B+D-A=1 \\ A-B+3C=5 \\ 3D+B=1 \end{cases} \quad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 3 & 0 & 5 \\ 0 & 1 & 0 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 6 \\ 0 & 0 & -1 & 2 & -2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 6 \\ 0 & 0 & 0 & 7 & 0 \end{array}\right) \quad \begin{array}{l} D=0 \\ C=2 \\ B=1 \\ A=0 \end{array}$$

$$\int \frac{2x^3+x^2+5x+1}{(x^2+3)(x^2-x+1)}\,dx = \int \frac{dx}{x^2+3} + 2\int \frac{x}{x^2-x+1}\,dx = \frac{1}{\sqrt{3}} \operatorname{arc\,tg} \frac{x}{\sqrt{3}} + * \; \textcircled{=}$$

$$* = 2\int \frac{x}{x^2-x+1}\,dx = \int \frac{x}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx = \int \frac{x+\frac{1}{2}-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx = \int \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx + \int \frac{\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx \, ;$$

$$\int \frac{t\,dt}{t^2+\frac{3}{4}} = \frac{1}{2}\int \frac{du}{u} \, , \; u = t^2+\frac{3}{4}$$

$$\textcircled{=} \; \frac{1}{\sqrt{3}} \operatorname{arc\,tg} \frac{x}{\sqrt{3}} + \ln|x^2-x+1| + \frac{2}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x-1}{\sqrt{3}} + C$$

2. $\int \frac{dx}{5+\cos^2 x} \; \textcircled{=}$

$$\text{J}\; t = \operatorname{tg} x \qquad dt = \frac{1}{\cos^2 x} \qquad dx = \cos^2 x\, dt$$

$$\operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$$

$$\cos^2 x = \frac{1}{tg^2x+1}$$

$$\ominus \int \frac{\cos^2 x}{5+\cos^2 x}\,dt = \int \frac{dt}{(t^2+1)\left(5+\frac{1}{t^2+1}\right)} = \int \frac{dt}{5t^2+6} = \frac{1}{5}\int \frac{dt}{t^2+6/5} = \frac{1}{5}\cdot\sqrt{\frac{5}{6}}\cdot arctg\,\frac{tg\,x\sqrt{5}}{\sqrt{6}}$$

3. $$\int \frac{\cos x\,dx}{\sin^3 x+\cos^3 x} \;\Big|:\cos^3 x\; = \int \frac{d\,tg\,x}{tg^3x+1}$$ ← сумма кубов + простейшие

4. $$\int \frac{dx}{3\sin x+4\cos x-5} \ominus$$

$\exists\; t = tg\,\frac{x}{2}$ ун-ая. тр-ая. по-ка

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\ominus \int \frac{\frac{2\,dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)+\frac{5(1+t^2)}{1+t^2}} = 2\int \frac{dt}{6t+4-4t^2+5+5t^2} = 2\int\frac{dt}{t^2+6t+9} = 2\int\frac{dt}{(t+3)^2} =$$

$$= 2\int \frac{d(t+3)}{(t+3)^2} = -\frac{2}{t+3}+C$$

5. $$\int \frac{\sqrt{x^2+1}}{x^2+2}\,dx \ominus \exists x = tg\,t\;;\; dt = \frac{1}{\cos^2 x}\,dx$$

$\sqrt{a^2-x^2}$ $\exists x = a\cdot\sin t$

$\sqrt{a^2+x^2}$ $\exists x = a\cdot tg\,t$

$\sqrt{x^2-a^2}$ $\exists\; x = \frac{a}{\cos t}$ или $x = a\cdot sh\,t$

$$tg^2x+1 = \frac{1}{\cos^2 x}$$

$$\ominus \int \frac{\sqrt{tg^2t+1}}{(tg^2t+2)\cos^2 t}\,dt = \int \frac{dt}{\cos t\cdot(\sin^2 t+2\cos^2 t)} = \int \frac{\cos t\,dt}{\cos^2 t\,(\sin^2 t+2\cos^2 t)} \ominus \quad \exists\, u = \sin t$$

$$\ominus \int \frac{du}{(1-u^2)(u^2+2-2u^2)} = \int \frac{du}{(1-u^2)(2-u^2)} = \int \frac{(2-u^2)-(1-u^2)}{(1-u^2)(2-u^2)}\,du = \int \ldots$$

$$6.\ \int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos^2 t}\right)^{3/2}} = \int \cos t\, dt = \sin(\operatorname{arctg} x) + C$$

dx $\quad tg^2 t+1$

$x = tg\, t$

2-ой способ:

Подстановка Абеля:

$$t = \left(\sqrt{x^2+1}\right)' = \frac{x}{\sqrt{x^2+1}}$$

$$t^2 = \frac{x^2}{x^2+1} \Rightarrow x^2 = \frac{t^2}{1-t^2} \qquad x = \frac{t}{\sqrt{1-t^2}}$$

$$dx = \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{1-t^2}\, dt = \frac{dt}{(1-t^2)^{3/2}}$$

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dt}{(1-t^2)^{3/2}} (1-t^2)^{3/2} = t + C = \frac{x}{\sqrt{x^2+1}} + C$$

$$7.\ \int \frac{\sqrt{x^2+1}}{x^2}\, dx = \int \sqrt{x^2+1}\, d\left(-\frac{1}{x}\right) = -\frac{1}{x}\sqrt{x^2+1} + \int \frac{x\, dx}{x\sqrt{x^2+1}} = -\frac{1}{x}\sqrt{x^2+1} + \ln\left|x+\sqrt{x^2+1}\right| + C$$

$$8.\ \int \varphi(x)\, dx, \text{ где } \varphi(x) = \int_0^x e^{-t^2} dt \Leftrightarrow \varphi'(x) = e^{-x^2}$$

$$\int \varphi(x)\, dx = x \cdot \varphi(x) - \int x \cdot d(\varphi(x)) = x \cdot \varphi(x) - \int x \cdot e^{-x^2} dx = x \cdot \varphi(x) + \frac{1}{2} \cdot e^{-x^2} + C$$

$d(-x^2) = -2x\, dx$