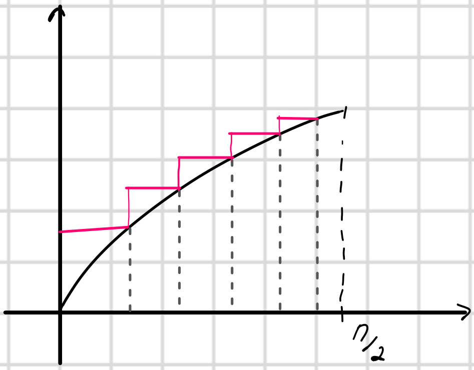
Приколы

1) $I = \int\limits_0^{\pi/2} \sin x \, dx$ $\qquad \tau: \; x_k = \frac{\pi k}{2n}, \; k = 0, \dots, n \qquad \Delta x_k = \frac{\pi}{2n} = \lambda$



$$S_\tau = \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k = \frac{\pi}{2n} \sum_{k=1}^{n} \sin \frac{\pi k}{2n} =$$

$$\sum_{k=1}^{n} \sin(kx) = \frac{1}{\sin\frac{x}{2}} \sum_{k=1}^{n} \sin(kx) \cdot \sin\frac{x}{2} =$$

$$= \frac{\sum_{k=1}^{n} \left(\cos(k-\frac{1}{2})x - \cos(k+\frac{1}{2})x\right)}{2\sin\frac{x}{2}} = \frac{\cos\frac{x}{2} - \cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}} =$$

$$= \frac{\sin\left(\frac{n+1}{2}x\right) \cdot \sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{\pi(n+1)}{4n} \cdot \sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}}$$

$$\lim_{n\to\infty} S = \lim_{n\to\infty} \frac{\pi}{2n} \cdot \frac{\sin\frac{\pi}{4} \cdot \sin\frac{\pi}{4}}{\frac{\pi}{4n}} = 2 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

$$s_\tau = \sum f(x_{k-1}) \Delta x_k = \frac{\pi}{2n} \sum_{k=1}^{n} \sin\frac{\pi(k-1)}{2n} = \dots \to 1$$

$$\forall \; S_\tau - s_\tau = \dots \xrightarrow[n\to\infty]{} 0$$

$$I^* = \inf S_\tau = \inf\left(\frac{\pi}{2n} \cdot \frac{\sin\frac{\pi(n+1)}{4n} \cdot \sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}}\right) = 1$$

$$I_* = \dots$$

2) $\lim\limits_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) = \lim \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{1+\frac{n}{n}}\right) =$

$$\sigma_\tau(f, \xi) = \sum f(\xi_i) \Delta x_i = \frac{1}{n} \sum_{i=1}^{n} f(\xi_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}}$$

$$f(x) = \frac{1}{1+x}$$

$\xi_i = \frac{i}{n}, \; i = 1 \dots n$ $\qquad = \int\limits_0^1 \frac{dx}{1+x} = \ln|1+x| \Big|_0^1 = \ln 2 - \ln 1 = \ln 2$

3) Кто больше?

a. $\int_0^{\pi/2} \frac{\sin x}{x} dx < \int_0^{\pi} \frac{\sin x}{x} dx$

b. $\int_0^{\pi} \frac{\sin x}{x} dx > \int_0^{2\pi} \frac{\sin x}{x} dx$

c. $\int_2^3 \frac{dx}{\log_2 x} < \int_2^3 \frac{dx}{\log_3 x}$

$$\log_2 x = \frac{\ln x}{\ln 2} > \frac{\ln x}{\ln 3} = \log_3 x$$

$$\Rightarrow \frac{1}{\log_2 x} < \frac{1}{\log_3 x}$$

4) Док-ть

a. $0 < \int_0^{\pi} \frac{\sin x}{\sqrt[5]{x^2+2}} dx < \frac{\pi}{\sqrt[5]{2}}$

$\rhd$ $\frac{\sin x}{\sqrt[5]{x^2+2}} \leq \frac{1}{\sqrt[5]{x^2+2}} \leq \frac{1}{\sqrt[5]{2}}$

$f \leq g \Rightarrow \int f \leq \int g$ $\blacktriangleleft$

b. $\frac{\sqrt{2}}{3} < \int_{\pi/4}^{3\pi/4} \frac{\sin x}{x} dx < \ln 3$

$\exists$ $f \geq 0$, $f(x_0) > 0$ и f - непр. в $(\cdot)\ x_0$

$$\Rightarrow \int_a^b f dx > 0$$

$\rhd$ $\frac{\sqrt{2}}{2x} \leq \frac{\sin x}{x}$

$$\frac{\sqrt{2}}{2} \int_{\pi/4}^{3\pi/4} \frac{dx}{x} = \frac{\sqrt{2}}{2} \ln x \Big|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} \cdot \ln 3$$ $\blacktriangleleft$

5) Решить

a. $\frac{d}{dx} \int_a^b \sin(x^2)\, dx = 0$

b. $\frac{d}{dx} \int_a^x \sin(t^2)\, dt = \sin(x^2)$ по т. Барроу

c. $\frac{d}{dx} \int_a^{x^2} \sin(t^2)\, dt \ =$

$$\varphi(u) = \int_a^u \sin(t^2)\,dt$$

$$= \frac{d}{dx}\varphi(x^2) = \varphi'(x^2)\cdot(x^2)' = 2x\sin(x^4)$$

6. $$\lim_{x\to 0} \frac{1}{x}\int_0^x \cos(t^2)\,dt = \text{лопиталь} = \lim_{x\to 0}\frac{\left(\int_0^x \cos t^2\,dt\right)'}{x'} = \lim_{x\to 0}\frac{\cos x^2}{1} = 1$$

2-ой способ: $$\frac{1}{x}\int_0^x \cos(t^2)\,dt = \frac{1}{x}\cdot\cos(\xi^2)\cdot x = \cos(\xi^2) \to 1, \quad \xi\in[0,x]$$

7. $$\int_0^{100\pi}\frac{dx}{3+\cos x} =$$

$t = \operatorname{tg}\frac{x}{2}$ $\qquad dx = \frac{2\,dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$$\int\frac{2\,dt}{(1+t^2)\left(3+\frac{1-t^2}{1+t^2}\right)} = \int\frac{dt}{2+t^2} = \frac{1}{\sqrt2}\operatorname{arctg}\frac{t}{\sqrt2}$$

$$= \frac{1}{\sqrt2}\operatorname{arctg}\left(\frac{1}{\sqrt2}\operatorname{tg}\frac{x}{2}\right)\Big|_0^{100\pi} = 0-0 = \cancel{0}$$