Предел функции.

1. Док-ть сходимость: $X_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \dots + \frac{\sin n}{2^n}$

Критерий Коши: X_n - сходится $\Leftrightarrow$ X_n - фундаментальна $\Leftrightarrow$

$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n \geq n_0 \quad \forall p \in \mathbb{N} : |X_{n+p} - X_n| < \varepsilon$

$$\triangleleft \; |X_{n+p} - X_n| = \left| \frac{\sin(n+1)}{2^{n+1}} + \dots + \frac{\sin(n+p)}{2^{n+p}} \right| \leq \frac{1}{2^{n+1}} + \dots +$$

$$+ \frac{1}{2^{n+p}} \leq \frac{1}{2^{n+1}} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots \frac{1}{2^{p-1}} \right) =$$

$$= \frac{1}{2^{n+1}} \cdot \frac{1 - \frac{1}{2^p}}{1 - \frac{1}{2}} < \frac{1}{2^n} < \varepsilon \quad \text{чтд.}$$

X_n - не фунд $\Leftrightarrow$ $\exists \varepsilon > 0 \quad \forall n_0 \quad \exists n \geq n_0 : \exists p \in \mathbb{N}:$
$|X_{n+p} - X_n| \geq \varepsilon$

2. $\triangleleft \; X_{n+p} - X_n = \frac{n+1}{(n+2)^2} + \frac{n+2}{(n+3)^2} \dots + \frac{n+p}{(n+p+1)^2} >$

$$\frac{n}{(n+p+1)^2} \cdot p \quad \exists p = n \Rightarrow \frac{n^2}{(2n+1)^2} > \frac{1}{6} \geq \varepsilon \quad \text{чтд.}$$

$4n^2 + 4n + 1$

$$\frac{n^2}{4n^2 + 4n + 1}$$

3. Док-ть расходимость

$X_n = \sin n$

$\exists \lim_{n \to \infty} \sin n = A$

$$X_{n+2} - X_n = \underbrace{\sin(n+2)}_{\to A} - \underbrace{\sin(n)}_{\to A} = 2\sin\frac{n+2-n}{2}\cos\frac{n+2+n}{2} =$$

$$= 2\underbrace{\sin 1}_{\neq 0}\cos(n+1) \longrightarrow 0$$

$$\Rightarrow \lim_{n\to\infty}\cos n = 0$$

$$\triangleleft\ \underbrace{\sin(2n)}_{\to A} = 2\underbrace{\sin n}_{\to A}\cos n \Rightarrow A = 0$$

но $\underbrace{\sin^2 n}_{\to 0} + \underbrace{\cos^2 n}_{\to 0} = 1$

противоречие

4. Док-ть по опр по Коши

$$\lim_{x\to 1}\frac{5x^2-4x-1}{x-1} = 6 \iff$$

$$\lim_{x\to x_0} f(x) = A \iff \forall\varepsilon>0\ \exists\delta>0:\ \forall x\in\mathbb{R}: \underbrace{0<|x-x_0|<\delta}_{x_0-\delta<x<x_0+\delta} \Rightarrow$$

$$\Rightarrow |f(x)-A|<\varepsilon$$

$$\left|\frac{5x^2-4x-1}{x-1}-6\right| = \left|\frac{(x-1)(5x+1)}{x-1}-6\right| = 5x-5 =$$

$$\begin{array}{r|l} 5x^2-4x-1 & x-1 \\ \hline -\ 5x^2-5x & 5x+1 \\ \hline x-1 & \\ -\ x-1 & \\ \hline 0 & \end{array}$$

$$5|x-1|<\varepsilon$$

$$|x-1|<\delta \quad \exists\ \delta = \frac{\varepsilon}{5}$$

$$0<|x-1|<\frac{\varepsilon}{5}\ \Big|\cdot 5$$

$$0<5|x-1|<\varepsilon$$

5. $\lim\limits_{x\to 9} \sqrt{x} = 3$

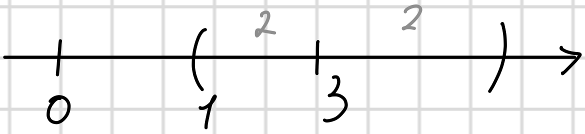
$$|\sqrt{x} - 3| = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(\sqrt{x}+3)} = \left|\frac{x-9}{\sqrt{x}+3}\right| < |x-9| < \varepsilon$$

$$\exists\, \delta = \varepsilon \qquad |x-9| < \delta \Rightarrow |x-9| < \varepsilon$$

6. $\lim\limits_{x\to -3} \frac{2x}{x+3} = \infty$

$$\left|\frac{2x}{x+3}\right| > \frac{1}{\varepsilon} \qquad \left|\frac{2x}{x+3}\right| >$$

$$0 < |x+3| < \delta$$



$$\exists\, \delta < 2 \Rightarrow |x+3| < 2$$

$$-5 < x < -1$$

$$1 < |x| < 5 \Leftrightarrow$$

$$\Leftrightarrow |x| > 1$$

$$\left|\frac{2}{x+3}\right| > \frac{2}{\delta}$$

$$\exists\, \delta = 2\varepsilon \quad \left|\frac{2}{x+3}\right| > \frac{1}{\varepsilon}$$

$$\varepsilon = \min\{2\varepsilon;\ 1.9\}$$

7. Д-ть $\nexists$ в $\bar{R}$ $\lim\limits_{x\to 0} 2^{\frac{1}{x}}$

$$\lim_{x\to 0} 2^{1/x} = 2^{-\infty} = 0$$

$$\neq$$

$$\lim_{x\to 0} 2^{1/x} = 2^{+\infty} = +\infty$$

предела нет из за отсутствия равенства

$x_n^{(1)} = \frac{1}{n}$ $\qquad 2^{1/x_n} = 2^n \xrightarrow[n\to\infty]{} +\infty$

$x_n^{(2)} = -\frac{1}{n}$ $\qquad 2^{1/-\frac{1}{n}} = 2^{-n} \xrightarrow[n\to\infty]{} 0$

3. $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$

Д-ть $\nexists \lim\limits_{x\to x_0} D(x)$

$x_n^{(1)} \in \mathbb{Q}$ $\qquad x_n^1 \to x_0$

$x_n^1 \neq x_0$

$D(x_n^1) = 1$

$x_n^{(2)} \in \mathbb{I}$ $\qquad x_n^2 \to x_0$

$x_2^2 \neq x_0$

$D(x_n^2) = 0$

противоречие

5. $\lim\limits_{x\to 2} \frac{2x^3-5x^2-4x+12}{3x^3-14x^2+10x-8} = \lim\limits_{n\to 2} \frac{(x-2)(2x^2-x-6)}{(x-2)(3x^2-8x+4)} =$

$= \frac{2x^2-x-6}{3x^2-8x+4} = \frac{(x-2)(2x+3)}{(x-2)(3x-2)} = \frac{2x+3}{3x-2} = \frac{7}{4}$

10. $$\lim_{x\to 5} \frac{\sqrt{x+11} - 2\sqrt{x-1}}{x^2-26} = \frac{0}{-1} = 0$$

11. $$\lim_{x\to 5} \frac{\sqrt{x+11} - 2\sqrt{x-1}}{x^2-25} = \frac{x+11-4x+4}{(x^2-25)\left(\sqrt{x+11} + 2\sqrt{x-1}\right)} =$$

$$= \frac{-3x+15}{(x-5)(x+5)\left(\sqrt{x+11} + 2\sqrt{x-1}\right)} = \frac{-3(x-5)}{(x+5)(x-5)\left(\sqrt{x+11} + 2\sqrt{x-1}\right.} =$$

$$= \frac{-3}{\sqrt{x+11} + 2\sqrt{x-1}} = \frac{-3}{10 \cdot 8} = -\frac{3}{80}$$