$\exists X \subset \mathbb{R}$

если $\forall x \in X : x \leq M$, $M$ - верхнее граница $X$ (X огр сверху)

$\exists m \in \mathbb{R} : \forall x \in X \quad m \leq x$ то $m$ - нижнее гр. (X - огр снизу)

$X$ - огр. $\Leftrightarrow X$ огр. сверху и снизу $\Leftrightarrow \exists m, M :$

$: m \leq x \leq M \quad \forall x \in X \Leftrightarrow \exists C : |x| \leq C \quad \forall x \in X$

1) Д-ть огр. посл-ти, т.е. мн-во знач огр.

Найти в. и н. границы

1. $a_n = \frac{1}{n} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\}$

$m = 0$ или $-1$ или $-5000$

$\forall x = \frac{1}{n} : \frac{1}{n} > 0$

$M = 1$ или $\pi$ : $\frac{1}{n} \leq 1$

2. $a_n = \frac{2n^2 - 1}{2 + n^2} \overset{0 \atop n}{=} 2 - \frac{5}{2 + n^2} < 2$

$m = 0$

$M = 2$

3. $a_n = \frac{1 - n}{\sqrt{n^2 + 1}}$

$0 \geq \frac{1 - n}{\sqrt{n^2 + 1}} > \frac{-n}{\sqrt{(n+1)^2}} = -\frac{n}{n+1} > -1$

$$|a_n| = \frac{n-1}{\sqrt{n^2+1}} \leq \frac{n}{\sqrt{n^2}} = \frac{n}{n} = 1$$

$$\Rightarrow \quad m = -1 \quad M = 1$$

4) $a_n = \sqrt{n+1} - \sqrt{n-1}$

$$1 < \sqrt{n+1} - \sqrt{n-1} = \frac{(\sqrt{n+1} - \sqrt{n-1})(\sqrt{n+1} + \sqrt{n-1})}{\sqrt{n+1} + \sqrt{n-1}} =$$

$$= \frac{n+1-(n-1)}{\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{\underbrace{\sqrt{n+1} + \sqrt{n-1}}_{\geq 0}} \leq 2$$

5) $a_n = \sqrt[3]{n^3+1} - \sqrt{n^2-1} \leq \sqrt[3]{n^3} - \sqrt{n^2} = n - n = 0$

$$\sqrt{n^3+1} < n+1 \quad \Big| \quad \sqrt{n^2-1} \geq n-1$$

$$n^3+1 < (n+1)^3 \quad \Big| \quad n^2-1 \geq n^2-2n+1$$

$$\overset{n+1}{\wedge}\!\!\sqrt{n^3+1} - \underset{n-1}{\underbrace{\sqrt{n^2-1}}} < 2 \qquad 2n \geq 2$$

$X$ – неогр. $\Leftrightarrow \forall C \in \mathbb{R} \quad \exists x \in X : |x| > C$

Док-ть неограниченность:

1) $a_n = (-1)^n \cdot n$ $\qquad |a_n| = n, \ \forall C \in \mathbb{R}$

$(-1)^n = \pm 1 \cdot n$ $\qquad N = |\underbrace{[C]}_{\text{целая ч.}}|, \ |a_{N+1}| > C$

2) $a_n = \dfrac{n^2 - n}{n+3} = \dfrac{n-1}{1+\underbrace{\frac{3}{n}}_{\leq 4}} \geqslant \dfrac{n-1}{4}$ неогр сверху

$a_n = \dfrac{n^2 - n}{n+3} = n - \dfrac{4n}{n+3} = n - 4 + \underbrace{\dfrac{12}{n+3}}_{<4} > n$

3) $a_n = 5^n - 4^n = \left(\underbrace{\left(\frac{5}{4}\right)^n - 1}_{\geqslant \frac{1}{4}}\right) \cdot 4^n$ неогр. сверху $\geqslant \frac{1}{4} \cdot \underbrace{4^n}_{>n}$

4) $a_n = \dfrac{2^n}{n^2}$

$\exists\, \exists C \in \mathbb{R}: \forall n \in \mathbb{N} \quad a_n < C \iff \dfrac{2^n}{n^2} < C \iff$

$2^n < C \cdot n^2$ док-м по индукции:

Б: $n = 10$: $2^{10} > 1000$ - верно

Ин. п.: $2^n > n^3$

$n = k$: $2^k > k^3$

: $2^{k+1} > (k+1)^3$

$2^{n+1} > n^3 + 3n^2 + 3n + 1 \Leftarrow 2k^3 > n^3 + 3n^2 + 3n + 1 \iff$

$\iff n^3 > 3n^2 + 3n + 1 \iff \underbrace{n}_{>10} > \underbrace{3 + \frac{3}{n} + \frac{1}{n^2}}_{<5}$

$\sup X :=$ наименьшая верхнее граница } не об. $\in$ мн-во,

$\inf X :=$ наибольшая нижнее граница } но может достигатся

1) $\sup \frac{1}{n} = 1$ $\qquad \inf \frac{1}{n} = 0$

2) $\sup \frac{2n^2-1}{2+n^2} = \sup\left(2 - \underbrace{\frac{5}{2+n^2}}_{\to 0}\right) = 2$

$\inf \frac{2n^2-1}{2+n^2} = \frac{1}{3} = a_1, \quad a_n \uparrow$

3) $\sup \frac{1-n}{\sqrt{n^2+1}} = 0 = a_1, \quad a_n \leq 0$

$\inf \frac{1-n}{\sqrt{n^2+1}} = -1 \iff \sup \frac{n-1}{\sqrt{n^2+1}} = 1$

$\inf = -\sup$

$\frac{n-1}{\sqrt{n^2+1}} > 1 - \varepsilon \quad , \quad \varepsilon > 0$

$\frac{n-1}{\sqrt{n^2+1}} > \frac{n-1}{n+1} = 1 - \frac{2}{n+1} > 1 - \varepsilon$

$\Updownarrow$

$\exists n \to \frac{2}{n+1} < \varepsilon \Rightarrow \inf \frac{1-n}{\sqrt{n^2+1}} = -1$