1. $A^TA=\begin{pmatrix}35 & 44\\ 44 & 56\end{pmatrix}$ $A^Tb=\begin{pmatrix}9\\ 12\end{pmatrix}$

$$\Longrightarrow \quad x=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

2. 解 $A^TAx=A^Tb$

即 $$\begin{pmatrix}6 & 3 & 1 & 1\\ 3 & 9 & 3 & 3\\ 1 & 3 & 1 & 1\\ 1 & 3 & 1 & 1\end{pmatrix}x=\begin{pmatrix}4\\ 3\\ 1\\ 1\end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix}1 & 3 & 1 & 1\\ 0 & 15 & 5 & 5\end{pmatrix}x=\begin{pmatrix}1\\ 2\end{pmatrix}$$

2个独立变量

$$\begin{pmatrix}1 & 3\\ 0 & 15\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}1-x_3-x_4\\ 2-5x_3-5x_4\end{pmatrix}$$

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}\frac{3}{5}\\ \frac{2}{15}-\frac{1}{3}x_3-\frac{1}{3}x_4\end{pmatrix}$$

$$x = \left\{ \begin{pmatrix} \frac{3}{5} \\ \frac{2}{15} - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_3, x_4 \in \mathbb{R} \right\}$$

3. $\|x\|_2 = \|Hx\|_2 \,,\; \alpha > 0 \Rightarrow \alpha = 5$

$$w = \frac{x - Hx}{\|x - Hx\|_2} = \frac{1}{5\sqrt{2}}(0, -5, 0, 0, 3, 4)^T$$

$$H = I - 2ww^T$$

$$= \frac{1}{25}\begin{pmatrix} 25 & & & & & \\ & 0 & & & 15 & 20 \\ & & 25 & & & \\ & & & 25 & & \\ & 15 & & & 16 & -12 \\ & 20 & & & -12 & 9 \end{pmatrix}$$

4. $\alpha = \pm \frac{13}{\sqrt{2}}$

$$\Rightarrow \begin{pmatrix} c \\ s \end{pmatrix} = \frac{\sqrt{2}}{26}\begin{pmatrix} 17 \\ 7 \end{pmatrix} \text{ 或 } -\frac{\sqrt{2}}{26}\begin{pmatrix} 17 \\ 7 \end{pmatrix}$$

5 记 $\|x\|_2=\sqrt{x^H x}=\sqrt{|x_1|^2+|x_2|^2}$

$$Q^H Q = I$$

$y = Qx = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$ 则 $|y_1| = \|x\|_2$

$$x = Q^H y = \begin{pmatrix} c & -\bar{s} \\ s & c \end{pmatrix}\begin{pmatrix} y_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c y_1 \\ s y_1 \end{pmatrix}$$

$x_1 = c y_1$ 取 $c = \dfrac{|x_1|}{\|x\|_2}$

$s y_1 = x_2$

则 $s = \dfrac{x_2 \overline{y_1}}{|y_1|^2} = \dfrac{x_2 \cdot \frac{1}{c}\overline{x_1}}{\|x\|_2^2}$

$$= \frac{x_2 \overline{x_1}}{|x_1| \, \|x\|_2}$$

6. 分别寻找 P、Q，$Px = P_n P_{n-1} \cdots P_2 x = e_1$

$Ry = R_n R_{n-1} \cdots R_2 y = e_1$

其中 P_i, R_i 为 Givens 变换

取 $Q = R^T P$ 即可

7. 令 $\alpha = \dfrac{\|x\|_2}{\|y\|_2}$

当 $\dfrac{x}{\|x\|_2} = \dfrac{y}{\|y\|_2}$，取 $H = I$

否则，取 $\omega = \dfrac{\dfrac{x}{\|x\|_2} - \dfrac{y}{\|y\|_2}}{\left\| \dfrac{x}{\|x\|_2} - \dfrac{y}{\|y\|_2} \right\|_2}$

$H = I - 2\omega\omega^T$ 即可

8. $L^{(0)} = L = (l_{ij}^{(0)})_{m\times n}$ $l_{ij}^{(0)} = 0$ $(i<j)$

1) 找 H_1 使

$$H_1\begin{pmatrix} 0 \\ \vdots \\ 0 \\ l_{nn}^{(0)} \\ \vdots \\ l_{mn}^{(0)} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} n-1 \\ \\ \\ m-n \end{array}$$

只改变 $n\sim m$ 行

记 $L^{(1)} = H_1 L^{(0)}$

k) 找 H_k 使

$$H_k\begin{pmatrix} 0 \\ \vdots \\ 0 \\ l_{n-k+1,n-k+1}^{(k-1)} \\ \vdots \\ l_{m,n-k+1}^{(k-1)} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ l_{n-k+2,n-k+1}^{(k-1)} \\ \vdots \\ l_{n,n-k+1}^{(k-1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} n-k \\ \\ \\ \\ \\ m-n \end{array}$$

只改变 $n-k+1$ 行和 $n+1\sim m$ 行

这时不会改变第 $n-k+2\sim n$ 行产生的 0 $(k\geq 2)$

此时 $H_n \cdots H_1 L$ 满足条件

9 由8. 找正交矩阵 H, 使得 $HL = \begin{array}{cc} \begin{pmatrix} L_1 \\ 0 \end{pmatrix} & \begin{array}{c} n \\ m-n \end{array} \\ n & \end{array}$

$$\|Ly - Pb\|_2 = \|HLy - HPb\|_2$$

$$= \left\| \begin{pmatrix} L_1 \\ 0 \end{pmatrix} y - HPb \right\|_2$$

记 $HPb = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{array}{c} n \\ m-n \end{array}$

则 $\|Ly - Pb\|_2^2 = \|L_1 y - b_1\|_2^2 + \|b_2\|_2^2$

解 $L_1 z = b_1$ 即得上述LS问题的解

而 $\|Ax - b\|_2 = \|LUx - Pb\|_2$

$$= \|Lz - Pb\|_2$$

10. 由题设 Xb 极小化 $\|Ax-b\|_2$ $\forall b\in\mathbb{R}^m$

故 $A^TAXb = A^Tb$ $\forall b\in\mathbb{R}^m$

可得 $A^TAX = A^T$

对A作奇异值分解 $A=U\Sigma V^T$，其中U,V分别为m,n阶正交阵，$\Sigma\in\mathbb{R}^{m\times n}$

$$\Sigma = \begin{array}{cc} \left(\begin{matrix} D & 0 \\ 0 & 0 \end{matrix}\right) & \begin{matrix} k \\ m-k \end{matrix} \\ \begin{matrix} k \quad & \quad n-k \end{matrix} & \end{array}, \quad D=\mathrm{diag}(\sigma_1,\cdots,\sigma_k), \quad \sigma_1>\sigma_2>\cdots>\sigma_k>0$$

则 $A^TAX=A^T$

$$\Rightarrow V\Sigma^TU^TU\Sigma V^TX = V\Sigma^TU^T$$

$$\Rightarrow \Sigma^T\Sigma V^TXU = \Sigma^T$$

记 $$V^TXU = \begin{array}{cc} \left(\begin{matrix} P_1 & P_2 \\ P_3 & P_4 \end{matrix}\right) & \begin{matrix} k \\ n-k \end{matrix} \\ \begin{matrix} k \quad & \quad m-k \end{matrix} & \end{array}$$

则可得 $P_1=D^{-1}$，$P_2=0$

于是 $X = V\begin{pmatrix} D^{-1} & 0 \\ P_3 & P_4 \end{pmatrix} U^T$

有 $AXA = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^T V \begin{pmatrix} D^{-1} & 0 \\ P_3 & P_4 \end{pmatrix} U^T U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^T$

$= U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^T = A$

以及

$AX = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^T V \begin{pmatrix} D^{-1} & 0 \\ P_3 & P_4 \end{pmatrix} U^T$

$= U\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} U^T$

故 $(AX)^T = AX$

11. ① 将A变成形如 $\begin{pmatrix} * & \ddots & * \\ * & \ddots & \ddots \\ 0 & \ddots & * \quad * \end{pmatrix}$

方法：第k步找 $\tilde{G}_k \in \mathbb{R}^{2\times 2}$ 使得

$$\tilde{G}_k \begin{pmatrix} \beta_{n-k} \\ \sqrt{\beta_n^2+\cdots+\beta_{n-k+1}^2} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_n^2+\cdots+\beta_{n-k}^2} \\ 0 \end{pmatrix}$$

$$G_k = \begin{pmatrix} I_{n-k-1} & & \\ & \tilde{G}_k & \\ & & I_{k-1} \end{pmatrix}$$

$$A^{(k)} = G_k A^{(k-1)}$$

② $A^{(n-1)} = \begin{pmatrix} * & \ddots & * \\ * & \ddots & \ddots \\ 0 & \ddots & * \quad * \end{pmatrix} = B^{(0)} = (b_{ij}^{(0)})_{n\times n}$

自上而下消去次对角线

第k步找 $\hat{H}_k \begin{pmatrix} b_{kk}^{(k-1)} \\ b_{k+1,k}^{(k-1)} \end{pmatrix} = \begin{pmatrix} b_{kk}^{(k)} \\ 0 \end{pmatrix}$

$$H_k = \begin{pmatrix} I_{k-1} & & \\ & \hat{H}_k & \\ & & I_{n-k-1} \end{pmatrix}$$

则 $H_{n-1}\cdots H_1 G_{n-1}\cdots G_1 A = R$ 上三角

$$Q = (H_{n-1}\cdots H_1 G_{n-1}\cdots G_1)^T$$

12. $x \in \mathcal{X}_{LS}$

$$\Rightarrow \frac{d}{d\alpha} \| A(x+\alpha w) - b \|_2^2 \Big|_{\alpha=0} = 0$$

$$\overset{代入等式}{\Longrightarrow} \quad 2w^T A^T (Ax - b) = 0$$

由 w 任意性则 $A^T(Ax-b) = 0$