1. $(x-x_*)^T A(x-x_*) = x^T A x + x_*^T A x_* - 2x_*^T A x$

$$\overset{Ax^*=b}{=\!=\!=} x^T A x + x_*^T A x_* - 2b^T x$$

$$= \varphi(x) + x_*^T A x_*$$

2. $\varphi(x_k) = \varphi(x_{k-1}) - \dfrac{(r_{k-1}^T r_{k-1})^2}{r_{k-1}^T A r_{k-1}}$

由于 $r_{k-1}^T A^{-1} r_{k-1} = (b - Ax_{k-1})^T A^{-1} (b - Ax_{k-1})$

$$= b^T A^{-1} b + \varphi(x_{k-1}) \geq \varphi(x_{k-1})$$

则 $\varphi(x) \leq \varphi(x_{k-1}) - \dfrac{(r_{k-1}^T r_{k-1})^2}{r_{k-1}^T A r_{k-1}} \cdot \dfrac{\varphi(x_{k-1})}{r_{k-1}^T A^{-1} r_{k-1}} \leq \varphi(x_{k-1}) \left(1 - \dfrac{1}{\|A\|_2 \|A^{-1}\|_2}\right)$

3. $x_* = x_k + \dfrac{r_k^T r_k}{r_k^T A r_k} \cdot r_k$

$\Rightarrow b = Ax_* = Ax_k + Ar_k \cdot \dfrac{r_k^T r_k}{r_k^T A r_k}$

$\Rightarrow Ar_k = r_k \cdot \dfrac{r_k^T A r_k}{r_k^T r_k}$

4. $B = \begin{pmatrix} r_k^T A r_k & r_k^T A p_{k-1} \\ r_k^T A p_{k-1} & p_{k-1}^T A p_{k-1} \end{pmatrix}$

B正定 $\Leftrightarrow (x r_k + y p_{k-1})^T A (x r_k + y p_{k-1}) = 0$

当且仅当 $x = y = 0$

由 A 正定和 r_k 与 p_{k-1} 显然不共线可得
B 正定成立，于是特别地 B 可逆。

5. 令 $\sum_{i=1}^{k} \lambda_i p_i = 0 \quad \lambda_i \in \mathbb{R}$

$$\Rightarrow \quad 0 = \left(\sum_{i=1}^{k} \lambda_i p_i\right)^T A \left(\sum_{i=1}^{k} \lambda_i p_i\right)$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j \, p_i^T A p_j$$
$$= \sum_{i=1}^{k} \lambda_i^2 \, p_i^T A p_i$$

$\Rightarrow \quad \lambda_i = 0 \quad 1 \le i \le k$

$\Rightarrow \quad p_1, \cdots, p_k$ 线性无关

6. $\varphi(x) = x^T A x - 2b^T x$

$$\frac{d\varphi(y_{i-1} + te_i)}{dt} = 2(y_{i-1} + te_i)^T A e_i - 2b^T e_i = 0$$

$$\Rightarrow t = \frac{1}{a_{ii}} (b - A y_{i-1})^T e_i$$

$$(y_i)_i - (y_{i-1})_i = \frac{1}{a_{ii}} (b - A y_{i-1})^T e_i$$

对于 G-S 迭代 $x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)}\right)$

$$\Rightarrow x_i^{(k+1)} - x_i^{(k)} = \frac{1}{a_{ii}} (b - A x_i^{(k)})^T e_i$$

故一次迭代即对应 G-S 迭代中一个元素的计算

7. A实对称 $\Longrightarrow$ A可对角化 $\Rightarrow d_A(\lambda)$ 无重根

$\Rightarrow d_A(\lambda) = \prod_{i=1}^{k}(\lambda-\lambda_k)$

$d_A(A)=0 \Rightarrow A^k = \sum_{i<k} c_i A^i$

由归纳法易证 $A^l r \in \text{span}\{r, Ar, \cdots, A^{k-1}r\}$

$(k \le l \le n-1)$

于是 $\text{span}\{r, Ar, \cdots A^{n-1}r\} \subset \text{span}\{r, Ar, \cdots, A^{k-1}r\}$

维数至多为 k

8 由定理 5.2.2 和习题 7 易得

9. 由定理 5.3.2，

$$\|x_k - x^*\|_A \le 2\left(\frac{\sqrt{\kappa_2}-1}{\sqrt{\kappa_2}+1}\right)^k \|x_0 - x^*\|_A$$

需估计 $\|\cdot\|_2$ 与 $\|\cdot\|_A$ 关系

$A = P\Sigma P^T$，P正交

$\Sigma = \text{diag}(\lambda_1 \cdots \lambda_n)$，$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$

$\forall y \in \mathbb{R}^n$，$x = P^T y$

$$y^T A y = x^T \Sigma x \begin{cases} \leq \lambda_1 x^T x = \lambda_1 y^T y \\ \geq \lambda_n x^T x = \lambda_n y^T y \end{cases}$$

$$\Longrightarrow \sqrt{\lambda_n}\,\|y\|_2 \leq \|y\|_A \leq \sqrt{\lambda_1}\,\|y\|_2$$

而 $\|A\|_2 = \sqrt{\lambda_1}$，$\|A^{-1}\|_2 = \dfrac{1}{\sqrt{\lambda_n}}$

代入即得结论

11、x_k 在 $\mathcal{X}$ 中极小化 $\|x - A^{-1}b\|_A$

$$\Longleftrightarrow \forall \omega \in \mathcal{X} \quad \frac{d}{d\alpha}\|x_k + \alpha\omega - A^{-1}b\|_A^2\Big|_{\alpha=0} = 0$$

$$\Longleftrightarrow \quad \omega^T A(x_k - A^{-1}b) = 0$$

$$\Longleftrightarrow \quad Ax_k - b \perp \mathcal{X}$$

12 $r_0 = A^T(b-Ax) = p_0$

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A^T A p_k} \xrightarrow{\underline{\tilde{p}_k = A p_k}} \frac{r_k^T r_k}{\tilde{p}_k^T \tilde{p}_k}$$

$$\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$$

$$r_{k+1} = r_k - \alpha_k A^T A p_k$$
$$= r_k - \alpha_k A^T \tilde{p}_k$$

大致算法:

x_0 = 初值, $r_0 = A^T(b-Ax)$, $\hat{r}_0 = Ar_0$, $k=0$

while $r_k \neq 0$

$k = k+1$

if $k=1$

$p_0 = r_0$, $\tilde{p}_0 = Ap_0$

else

$\beta_{k-2} = r_{k-1}^T r_{k-1} / r_{k-2}^T r_{k-2}$

$p_{k-1} = r_{k-1} + \beta_{k-2} p_{k-2}$, $\tilde{p}_{k-1} = Ap_{k-1}$

end

$$\alpha_{k-1} = r_{k-1}^T r_{k-1} / \tilde{p}_{k-1}^T \tilde{p}_{k-1}$$

$$x_k = x_{k-1} + \alpha_{k-1} p_{k-1}$$

$$r_k = r_{k-1} - \alpha_{k-1} A^T \tilde{p}_{k-1}$$

end

$x = x_k$