Extra homework 7 - Ilie Miruna Andreea

space $L^2([-\pi,\pi])$ of square-integrable functions defined on $[-\pi,\pi]$:

$$L^2([-\pi,\pi]) := \{ f:[-\pi,\pi] \to R : \|f\|^2 := \int_{-\pi}^{\pi} f^2(x)\,dx < \infty \}$$

On $L^2([-\pi,\pi])$ we can define the inner product (or scalar product) between any $f,g \in L^2([-\pi,\pi])$

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x)\,dx$$

A set of functions $\{\phi_k(x)\}_{k\in N}$ is called orthogonal if $\langle \phi_m, \phi_n \rangle = 0$ if $m \neq n$

A set of orthogonal functions $\{\phi_k(x)\}_{k\in N}$ is called orthogonal if $\|\phi_k\| - \langle \phi_k, \phi_k \rangle^{1/2} = 1$

a) prove that the following functions are orthogonal in $L^2([-\pi,\pi])$

$$\{ 1, \cos x, \sin x, \cos 2x, \sin 2x \}$$

For ortogonality, the inner product of two functions $=0$ $\quad \langle f,g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\,dx$

$$\left.\begin{aligned} f(x) &= \cos(mx) \\ g(x) &= \cos(nx) \\ m &\neq n \end{aligned}\right\} \Rightarrow \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0$$

$$\Rightarrow \cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m+n)x) + \cos((m-n)x)\right]$$

where ~~is~~ the integral of $\cos(hx)$ ~~over the integral~~ over the symmetrical interval $[-\pi, \pi]$ is 0 for $h \neq 0$

Similary, $f(x) = \sin x \, (mx)$

$g(x) = \sin(mx)$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(mx) \, dx = 0$$

$\Rightarrow \sin(mx) \sin(mx) = \frac{1}{2} [\cos(m-m)x - \cos(m+m)x]$

$$\int_{-\pi}^{\pi} \cos(mx) \sin(mx) \, dx = 0$$

Orthogonality of 1 with the others works bcs

$$\int_{-\pi}^{\pi} \cos(mx) \, dx = \int_{\pi} \sin(mx) \, dx = 0, \ \forall m \geq 1$$

$\Rightarrow$ the set $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ is orthogonal

b) Prove that the following functions are orthonormal in $L^2([-\pi, \pi])$

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x \dots \right\}$$

$\Rightarrow$ orthogonality, normalization $\Big\} \Rightarrow \| f_n \| = \sqrt{\langle f_n, f_n \rangle} = 1$

$$\langle f_n, f_n \rangle = \int_{-\pi}^{\pi} \left(\frac{f(x)}{\sqrt{\pi}} \right)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx$$

For $f(x) = \cos(mx)$ or $f(x) = \sin(mx)$

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \int_{-\pi}^{\pi} \sin^2(mx)\,dx$$

$$\| f_n \| = \sqrt{\frac{\pi}{\pi}} = \sqrt{1} = 1$$

$\Rightarrow$ the functions are orthonormal

c) assume that $f \in L^2([-\pi, \pi])$ can be expanded in this orthonormal basis as a Fourier series

$$f(x) = a_0 + \sum_{m=1}^{\infty} [a_m \cos(mx) + b_m (\sin(mx))]$$

Calc. the fourier coeff

a_0 $\langle f, 1 \rangle = \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{2\pi}}\,dx$ (inner product $\frac{1}{\sqrt{2\pi}}$)

a_m inner prod $\frac{\cos(mx)}{\sqrt{\pi}}$

$$\langle f, \cos(mx) \rangle = \int_{-\pi}^{\pi} f(x) \cos(mx)\,dx$$

b_m inner prod $\frac{\sin(mx)}{\sqrt{\pi}}$

$$\langle f, \sin(mx) \rangle = \int_{-\pi}^{\pi} f(x) \sin(mx)\,dx$$

d) Prove Parseval's theorem (also known as Parseval's identity)

$$\|f\|^2 = \int_{-\pi}^{\pi} f^2(x)\,dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

The norm of $f(x)$ $\Rightarrow$ $\|f\|^2 = \int_{-\pi}^{\pi} f^2(x)\,dx$

Fourier expansion: $f^2(x) = a_0^2 + \sum_{n\geq 1} (a_n^2 + b_n^2)$

$\Rightarrow$ Parseval's identity is proven