Homework 4 - Ilie Miruna Andreea

1 a) $\sum_{n\geq 0} \frac{nx^n}{2^n}$, $a_n = \frac{n}{2^n}$, $c = 0$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} =$$

$$= \lim_{n\to\infty} \frac{1}{2}\sqrt[n]{n} = \frac{1}{2} = L \Rightarrow R = \frac{1}{L} = 2$$

if $x = 2 \Rightarrow \sum_{n\geq 0} \frac{n\cdot 2^n}{2^n} = \sum_{n\geq 0} n \Rightarrow$ diverges

if $x = -2 \Rightarrow \sum_{n\geq 0} \frac{n\cdot(-2)^n}{2^n} = \sum_{n\geq 0} (-1)^n \cdot n \Rightarrow$ diverges

$\Rightarrow$ convergence set $(-2,2)$

b) $\sum_{n\geq 1} \frac{x^{2n}}{\sqrt{n}}$, $a_n = \frac{1}{\sqrt{n}}$, $c = 0$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{\sqrt{n}}\right|} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{-\frac{1}{2}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n^{-\frac{1}{2n}} = \lim_{n\to\infty} e^{\ln\left(n^{-\frac{1}{2n}}\right)}$$

$$= \lim_{n\to\infty} e^{-\frac{1}{2n}\ln n} = \lim_{n\to\infty} e^0 = 1 = L$$

$R = \frac{1}{1} = 1$

If $x = 1$ $\sum_{n \geq 1} \frac{1}{\sqrt{n}} = \sum_{n \geq 1} \frac{1}{n^{\frac{1}{2}}}$ $\Rightarrow$ $p = \frac{1}{2} < 1 \Rightarrow$ the series diverges

$\sum_{n \geq 1} \frac{1}{n^p}$

If $x = -1$ $\sum_{n \geq 1} \frac{(-1)^n}{\sqrt{n}}$

Leibniz's Test $\Longrightarrow$ $\sum_{n \geq 1} (-1)^n b_n$; $b_n = \frac{1}{\sqrt{n}}$

$\forall n \geq 1 \Rightarrow b_n > 0$

$b_{n+1} - b_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n^2+n}} < 0$

$\Rightarrow b_n$ is decreasing

$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

$\Rightarrow$ the series converges

convergence set $[-1, 1)$

c) $\sum_{n \geq 1} (-1)^n \frac{(x-1)^n}{n}$; $a_n = \frac{(-1)^n}{n}$, $c = 1$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{(-1)^{n+1}}{n+1}}{\frac{(-1)^n}{n}} = \lim_{n \to \infty} \left| \frac{-1}{n+1} \cdot \frac{n}{1} \right| = \lim_{n \to \infty} \frac{n}{n+1}$

$= 1 = L \Rightarrow R = \frac{1}{1} = 1$

$c = 1 \Rightarrow x = 0$ or $x = 2$

$x=0$ $\quad \sum_{n\geq 1} \frac{(-1)^n (-1)^n}{n} = \sum_{n\geq 1} \frac{1}{n}$ (harmonic series) $\Rightarrow$ diverges

If $x=2$ $\quad \sum_{n\geq 1} \frac{(-1)^n \cdot (1)^n}{n} = \sum_{n\geq 1} \frac{(-1)^n}{n}$

$\underline{\underline{\text{Leibniz's Test}}} \Rightarrow \sum_{n\geq 1} (-1)^n \cdot b_n \; ; \; b_n = \frac{1}{n}$

$$\left.\begin{array}{l} \forall\, n\geq 1 \,,\; b_n \geq 0 \\ b_{n+1} - b_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-n-1}{n(n+1)} = -\frac{1}{n^2+n} < 0 \\ \Rightarrow b_n \text{ decreases} \\ \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\}$$

$\Rightarrow \sum_{n\geq 1} \frac{(-1)^n}{n}$ converges

$\Rightarrow$ convergence set $[0, 2]$

2. Study the convergence and compute the sum for the series $\sum_{n\geq 2} \frac{x^n}{n(n-1)}$

$\sum_{n\geq 2} \frac{x^n}{n(n-1)} \xRightarrow{\text{comp test}} \sum_{n\geq 2} \frac{x^n}{n^2}$

$$\left.\begin{array}{l} \text{If } |x|<1 \Rightarrow \text{the series converges} \\ \text{If } |x|\geq 1 \Rightarrow \text{the series diverges} \end{array}\right\} \Rightarrow$$

$\Rightarrow \sum_{n\geq 2} \frac{x^n}{n(n-1)} \Rightarrow \begin{cases} \text{converges} & , |x|<1 \\ \text{diverges} & , |x|\geq 1 \end{cases}$

$$\sum_{n\geq 2} \frac{x^n}{n(n-1)}$$

$$\sum_{n\geq 2} x^n \frac{1}{n(n-1)} = \sum_{n\geq 2} x^n \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= x^2\left(1-\frac{1}{2}\right) + x^3\left(\frac{1}{2}-\frac{1}{3}\right) + x^4\left(\frac{1}{3}-\frac{1}{4}\right) \dots + x^n\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= x^2 - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^3}{3} + \frac{x^4}{3} - \frac{x^4}{4} + \dots + \frac{x^n}{n-1} - \frac{x^n}{n}$$

$$= x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \dots + \frac{x^n}{n-1}$$

$$= x^2\left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-2}}{n-1}\right)$$

$$= \sum_{n\geq 2} \frac{x^n}{n(n-1)} = x^2(-\ln(1-x)) = -x^2\ln(1-x), \ |x|<1$$

.gitignore   analiza.py

```python

def standard_partial_sum(N):
    #Calculate the partial sum of the alternating series sum((-1)^(n+1) / n) up to N terms.
    total_sum = 0
    for n in range(1, N + 1):
        total_sum += (-1)**(n + 1) / n
    return total_sum

def rearranged_partial_sum(N, p, q):

    #Calculate the partial sum by adding p positive terms and q negative terms repeatedly.

    total_sum = 0.0  # Initialize sum
    n = 1  # Start from term 1
    terms_used = 0  # Track the number of terms used

    while terms_used < N:
        # Add p positive terms
        for _ in range(p):
            if terms_used >= N:
                break
            total_sum =total_sum+ 1 / n
            n =n+ 1
            terms_used =terms_used+ 1

        # Add q negative terms
        for _ in range(q):
            if terms_used >= N:
                break
            total_sum =total_sum- 1 / n
            n =n+ 1
            terms_used =terms_used+ 1
    return total_sum

N = int(input("Enter the number of terms (N): "))
p = int(input("Enter the number of positive terms (p): "))
q = int(input("Enter the number of negative terms (q): "))

# Calculate the standard partial sum up to N terms
approx_ln2_standard = standard_partial_sum(N)
# Calculate the rearranged partial sum (p positive terms, q negative terms)
approx_ln2_rearranged = rearranged_partial_sum(N, p, q)

print(f"Standard partial sum (up to N={N} terms): {approx_ln2_standard}")
print(f"Rearranged partial sum (p={p}, q={q}, up to N={N} terms): {approx_ln2_rearranged}")
```

```
C:\Users\INTEL\PycharmProjects\pb3\.venv\Scripts\python.exe C:\Users\INTEL\PycharmProjects\pb3\analiza.py
Enter the number of terms (N): 6
Enter the number of positive terms (p): 2
Enter the number of negative terms (q): 2
Standard partial sum (up to N=6 terms): 0.6166666666666666
Rearranged partial sum (p=2, q=2, up to N=6 terms): 1.2833333333333334

Process finished with exit code 0
```

```
\Users\INTEL\PycharmProjects\pb3\.venv\Scripts\python.exe C:\Users\INTEL\PycharmProjects\pb3\analiza.py
ter the number of terms (N): 100
ter the number of positive terms (p): 3
ter the number of negative terms (q): 3
andard partial sum (up to N=100 terms): 0.688172179310195
arranged partial sum (p=3, q=3, up to N=100 terms): 1.4453216048473239

ocess finished with exit code 0
```