Homework 7 - Ilie Miruna Andreea - 914

1. Compute the following limits using a Riemann sum:

a) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}}{n^2}$

$\frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}}{n^2} = \sum\limits_{k=1}^{n} k \cdot e^{\frac{k}{n}}$

$\sqrt[n]{e} = k \cdot e^{\frac{k}{n}}$

$2\sqrt[n]{e^2} = k \cdot e^{\frac{k}{n}}$

$\Rightarrow \frac{\sum\limits_{k=1}^{n} k \cdot e^{\frac{k}{n}}}{n^2} = \frac{1}{n^2} \sum\limits_{k=1}^{n} k \cdot e^{\frac{k}{n}} = \frac{1}{n} \cdot \frac{1}{n} \sum\limits_{k=1}^{1} k \cdot e^{\frac{k}{n}}$

$x_k = \frac{k}{n}$; $x_k \in [0,1]$

$\frac{1}{n^2} \sum\limits_{k=1}^{n} k e^{\frac{k}{n}} = \frac{1}{n} \sum\limits_{k=1}^{n} x_k e^{x_k} = \int_0^1 x e^x \, dx$

$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x \Big|_0^1 = 1$

b) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \cdots \sin\frac{(n-1)\pi}{2n}}$

$P_n = \prod\limits_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)$

$$\sqrt[n]{P_n} = \left(\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{n}}$$

$$\ln \sqrt[n]{P_n} = \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\sin\left(\frac{k\pi}{2n}\right)\right)$$

$$\sin \frac{k\pi}{2n} \text{ "like" } \frac{k\pi}{2n}$$

$$\ln\left(\sin\left(\frac{k\pi}{2n}\right)\right) \text{ "like" } \ln\left(\frac{k\pi}{2n}\right) \rightarrow \ln(k) + \ln\left(\frac{\pi}{2}\right) - \ln(n)$$

$$\ln\left(\sqrt[n]{P_n}\right) \rightarrow \frac{1}{n} \sum_{k=1}^{n-1} \left(\ln(k) + \ln\left(\frac{\pi}{2}\right) - \ln(n)\right)$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n-1} \ln(k) + \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\frac{\pi}{2}\right) - \frac{1}{n} \sum_{k=1}^{n-1} \ln(n)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_0^1 \ln(x) \qquad \ln\left(\frac{\pi}{2}\right) \qquad \ln(n)$$

$$\downarrow$$

$$-1$$

$$\Rightarrow \lim_{n \to \infty} \sqrt[n]{P_n} = -1 + \ln\left(\frac{\pi}{2}\right) - \ln(n) = \frac{\pi}{2e}$$

2. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \alpha > 0$

$$F(\alpha+1) = \int_0^\infty x^\alpha e^{-x} \, dx$$

$$F(\alpha+1) = -x^\alpha e^{-x} \Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} \, dx$$

$$(\alpha+1)=\alpha\int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

$$\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha)$$

$$\Gamma(m)=(m-1)! \qquad m\in\mathbb{N}^*$$

- $\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha)$

$$\Gamma(m)=(m-1)\,\Gamma(m-1)$$

$$\Gamma(m-1)=(m-2)\,\Gamma(m-2)$$

$$\Gamma(m)=(m-1)(m-2)(m-3)\ldots\Gamma(1)$$

$$\Gamma(1)=\int_0^\infty x^{1-1}e^{-x}\,dx=\int_0^\infty e^{-x}\,dx=-e^{-x}\Big|_0^\infty=1$$

$$\Gamma(1)=1$$

$$\Gamma(m)=(m-1)(m-2)(m-3)\ldots\Gamma(1)=(m-1)!$$

```python
import numpy as np
import matplotlib.pyplot as plt

# Function to calculate e^(-x^2)
def f(x):
    return np.exp(-x**2)

# Numerical integration using the trapezoidal rule
def trapezoidal_rule(f, a, b, n):
    x = np.linspace(a, b, n + 1)
    y = f(x)
    h = (b - a) / n
    return h * (0.5 * y[0] + 0.5 * y[-1] + np.sum(y[1:-1]))

# Values of 'a' and number of intervals 'n'
a_values = np.linspace( start: 1,  stop: 10,  num: 10)  # Increasing values of 'a'
n = 1000  # Number of subintervals for trapezoidal rule

# Calculate the integral for different values of 'a'
results = []
for a in a_values:
    integral = trapezoidal_rule(f, -a, a, n)
    results.append((a, integral))

# Print the results
print("a (Interval Bound) | Integral Value")
print("------------------------------------")
for a, integral in results:
    print(f"{a:.2f}                | {integral:.6f}")


plt.plot( *args: a_values, [r[1] for r in results], marker='o', label='Numerical Integration')
plt.axhline(np.sqrt(np.pi), color='red', linestyle='--', label='√π')
plt.xlabel('a (Interval Bound)')
plt.ylabel('Integral Value')
plt.title('Convergence of Numerical Integration to √π')
plt.legend()
plt.grid()
plt.show()
```

```
C:\Users\INTEL\PycharmProjects\pb3\.venv\Scripts\python.exe C:\Users\INTEL\PycharmProjects\pb3\.venv\
a (Interval Bound) | Integral Value
-----------------------------------
1.00               | 1.493648
2.00               | 1.764163
3.00               | 1.772415
4.00               | 1.772454
5.00               | 1.772454
6.00               | 1.772454
7.00               | 1.772454
8.00               | 1.772454
9.00               | 1.772454
10.00              | 1.772454

Process finished with exit code 0
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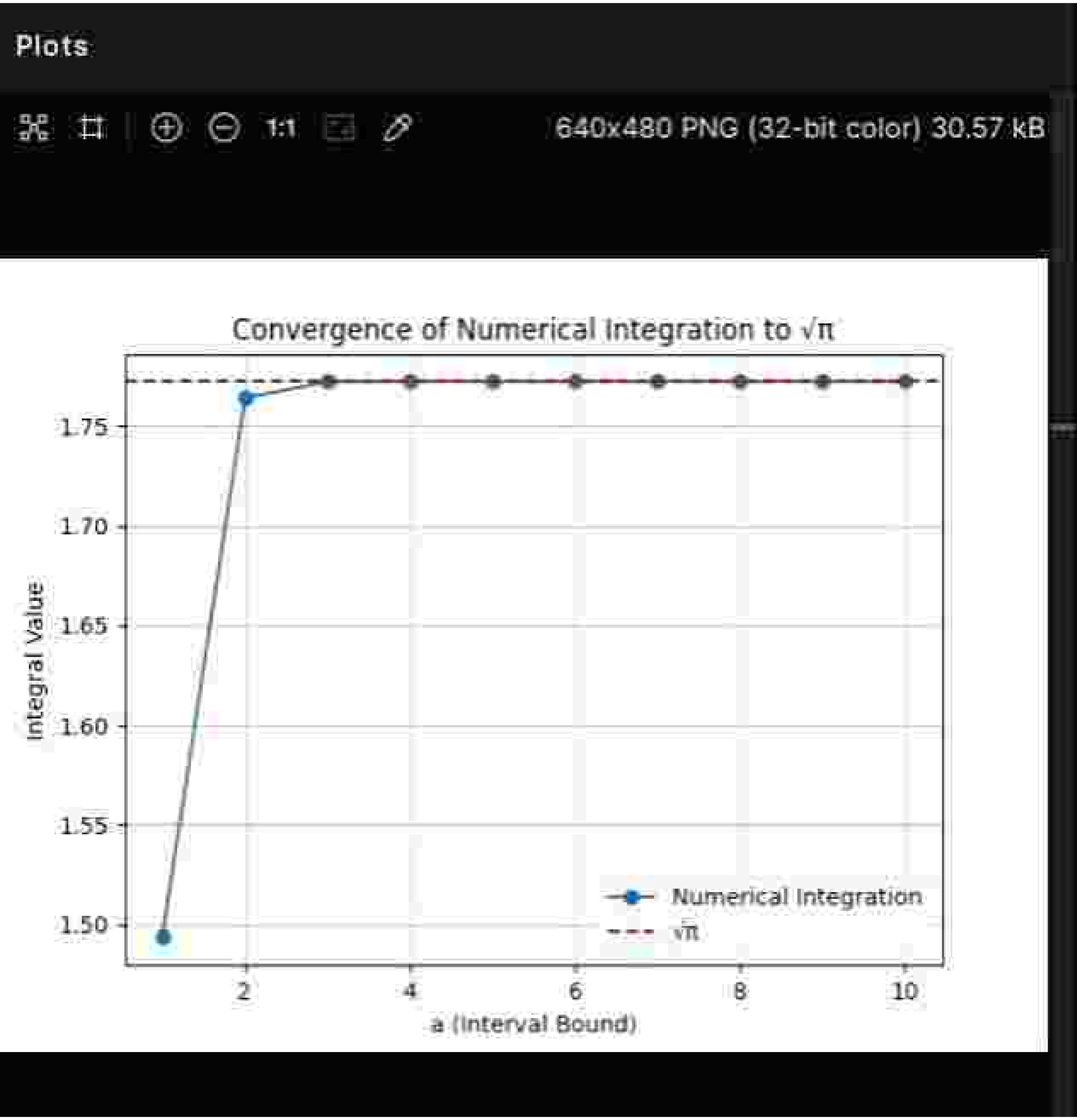
Plots
640x480 PNG (32-bit color) 30.57 kB
1:1
Convergence of Numerical Integration to √π
Integral Value
1.75
1.70
1.65
1.60
1.55
1.50
2
4
6
8
10
a (Interval Bound)
Numerical Integration
√π