~~E~~xtrahomework 4 - Ilie Miruna Andreea

1. Rearrange the terms in the alternating harmonic series such that its sum is $s \in \overline{\mathbb{R}}$

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$$

S converges to $s = \ln 2$

=> Rieman series - we can arrange its terms to converge to real or complex numbers

$$S = (\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \dots +}_{A}) - (\underbrace{\frac{1}{2} + \frac{1}{4} + \dots}_{B})$$

A, B diverge to $+\infty$, but A-B converges

Let $A_n = \sum_{i=1}^{n} \frac{1}{2i-1}$ $\qquad A_n > R$

Let $B_m = \sum_{j=1}^{m} \frac{1}{2j}$ $\qquad A_n - B_m < R$

2. Let (F_n) be the Fibonacci sequence with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Study the convergence and find the value of the series $\sum_{n=0}^{\infty} F_n x^n$

$F_0 = 1 \quad ; \quad F_1 = 1 \quad ; \quad F_n = F_{n-1} + F_{n-2} \quad , n \geq 2$

$F_n \simeq \frac{\varphi^n}{\sqrt{5}}$

$\varphi = \text{ratio} = \frac{1+\sqrt{5}}{2}$

$F_n = a r^n$

$F_{n+2} = a r^{n+2}$

$F_{n+2} = F_{n+1} + F_n = a r^{n+1} + a r^n$

$\Rightarrow a r^{n+2} = a r^{n+1} + a r^n \rightarrow r^2 = r+1$

$\Rightarrow r = \frac{1+\sqrt{5}}{2}$

$F_n x^n \simeq \frac{\varphi^n}{\sqrt{5}} x^n$

$F_n x^n \rightarrow 0 \iff |x| < \frac{1}{\varphi}$

$R = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}} \simeq 0,61$ $\Rightarrow$ the series converges for $|x| < \frac{1}{\varphi}$

Let $F(x) = \sum_{n \geq 0} F_n x^n$

$F_{n+1} = F_n + F_{n-1}$

$F_n = F_{n-1} + F_{n-2}$

$F_{n-1} = F_{n-2} + F_{n-3}$

$\vdots$

$F_1 = 1$

$F_0 = 1$

$$\frac{F(x)-x}{x} = F(x) + xF(x)$$

$$F(x) - x = xF(x) + x^2 F(x)$$

$$F(x) - xF(x) - x^2 F(x) = x$$

$$F(x)(1-x-x^2) = x$$

$$\Rightarrow F(x) = \frac{x}{1-x-x^2} = \sum_{n\geq 0} F_n x^n$$ $\Rightarrow$ converges for $|x| < \frac{1}{\varphi}$

3. Let C_n be the number of full binary trees with $n+1$ leaves (catalan numbers)

a) Find a recurrence relation for C_n

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{(n+1)\, n!}$$

$$C_{n+1} = \frac{1}{(n+1)+1}\binom{2(n+1)}{n+1} = \frac{(2(n+1))!}{(n+2)!\,(n+1)!} = \frac{(2n+2)!}{(n+2)!\,(n+1)!}$$

$$= \frac{2n!\,(2n+1)(2n+2)}{(n+1)!\,n!\,(n+1)(n+2)}$$

$$\Rightarrow C_{n+1} = C_n \cdot \frac{(2n+1)\,2(n+1)}{(n+1)\cdot(n+2)}$$

$$C_{n+1} = C_n \frac{2(2n+1)}{n+2} = \sum_{i=0}^{n-1} C_{n-i-1}$$

$$C_n = \sum_{h=0}^{n-1} C_h C_{n-1-h}$$

h) Considering the generating function $f(x) = \sum_{n=0}^{\infty} C_n x^n$, prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$

Let $C(x) = \sum_{n\geq 0} C_n x^n = \sum_{n\geq 0} \sum_{h=0}^{n-1} C_h C_{n-1-h} x^n$

$= 1 + \sum_{n\geq 1} \sum_{h=0}^{n-1} C_h C_{n-1-h} x^n = 1 + x \sum_{n\geq 0} \sum_{k=0}^{h} C_h C_{n-1-h} x^n$

$= 1 + x\left(\sum_{n\geq 0} C_n x^n\right)^2 = 1 + x\,C(x)^2$

$\Rightarrow x\,C(x)^2 - C(x) + 1 = 0$

$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$

$C_0 = 1 \Rightarrow \lim_{x\to 0} C(x) = \lim_{x\to 0} \sum_{n\geq 0} C_n x^n = C_0 = 1$

$\Rightarrow C(x) = \frac{1-\sqrt{1-4x}}{2x}$

$(1-4x)^{\frac{1}{2}} = 1 + \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-4x)^n$

$= 1 + \sum_{n\geq 1} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} (-4x)^n$

$= 1 + \sum_{n\geq 1} (-1)^{n-1} \frac{(2n-3)!!}{2^n n!} (-4x)^n$

$= 1 - \sum_{n\geq 1} \frac{2^n (2n-3)!!}{n!} x^n$

$$-2 \sum_{n \geq 1} \frac{2^{n-1} \prod_{k=1}^{n-1} (2k-1)}{n (n-1)!^2} x^n$$

$$= 1 - 2 \sum_{n \geq 1} \frac{\prod_{k=1} 2k \cdot \prod_{k=1}^{n-1} (2k-1)}{n (n-1)!^2} x^n$$

$$= 1 - 2 \sum_{n \geq 1} \frac{(2n-2)!}{n (n-1)!^2} x^n$$

$$= 1 - 2 \sum_{n \geq 1} \frac{1}{n} \binom{2(n-1)}{n-1} x^n$$

$$\Rightarrow C(x) = \frac{1}{2x} \cdot 2 \sum_{n \geq 1} \frac{1}{n} \binom{2(n-1)}{n-1} x^n$$

$$C(x) = \sum_{n \geq 1} \frac{1}{n} \binom{2(n-1)}{n-1} x^{n+1}$$

$$C(x) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n \Rightarrow C_n = \frac{1}{n+1} \binom{2n}{n}$$