Homework 6 - Ilie Miruna Andreea

1) Using Taylor series, compute the following limits:

a) $\lim_{x \to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$

$\sin x - x + \frac{x^3}{6} = \left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(7)\right) - x + \frac{x^3}{6}$

$= \frac{x^5}{120} + O(x^7)$

$\Rightarrow \dfrac{\sin x - x + \frac{x^3}{6}}{x^5} = \dfrac{\frac{x^5}{120} + O(7)}{x^5} = \frac{1}{120} + O(x^7)$

$\Rightarrow \lim_{x \to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}$

b) $\lim_{x \to 0} \dfrac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4}$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + O(x^6)$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$

$\Rightarrow e^{x^2} - \cos x - \frac{3x^2}{2} = \left(1 + x^2 + \frac{x^4}{2} + O(x^6)\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right) - \frac{3x^2}{2}$

$$= x^2 + \frac{x^2}{2} - \frac{3x^2}{2} + \frac{x^4}{2} - \frac{x^4}{24} + O(x^6)$$

$$= \frac{x^4}{3} + O(x^6)$$

$$\Rightarrow \frac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4} = \frac{\frac{x^4}{3} + O(x^6)}{x^4} = \frac{1}{3} + O(x^2)$$

$$\Rightarrow \lim_{x \to 0} \frac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4} = \frac{1}{3}$$

2. Prove that the Taylor series of $\ln(1+x)$ around 0 is

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{x+1}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$\vdots$$

$$f^n(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}$$

For $x = 0$

$$f^n(0) = (-1)^{n+1} (n-1)!$$

The taylor series is

$$\ln(1+x) = \sum_{n \geq 1} \frac{f^n(0)}{n!} x^n = \sum_{n \geq 1} \frac{(-1)^{n+1} (n-1)!}{n!} \cdot x^n = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$$

```python
import numpy as np


# Define the function and its exact derivative
def f(x):  2 usages
    return np.sin(x)  # Example function


def f_prime_exact(x):  1 usage
    return np.cos(x)  # Exact derivative


# Define the forward and centered difference approximations
def forward_difference(f, x, h):  1 usage
    return (f(x + h) - f(x)) / h


def centered_difference(f, x, h):  1 usage
    return (f(x + h) - f(x - h)) / (2 * h)


# Point of interest and range of h values
x = np.pi / 4  # Example point (45 degrees)
h_values = np.logspace(-8, -1, num: 10)  # Reduced range of h values for readability

# Calculate errors for forward and centered differences
print(f"{'h':<12}{'Forward Error':<20}{'Centered Error':<20}")
print("=" * 52)

for h in h_values:
    forward_approx = forward_difference(f, x, h)
    centered_approx = centered_difference(f, x, h)
    exact = f_prime_exact(x)

    forward_error = abs(forward_approx - exact)
    centered_error = abs(centered_approx - exact)

    # Print in standard scientific notation
    print(f"{h:<12.2e}{forward_error:<20.2e}{centered_error:<20.2e}")
```

```
C:\Users\INTEL\PycharmProjects\pb3\.venv\Scripts\python.exe C:\Users\INTEL\PycharmProjects\pb3\bla.py
h           Forward Error       Centered Error
==================================================
1.00e-08    8.05e-09            3.05e-09
5.99e-08    2.17e-08            5.28e-10
3.59e-07    1.27e-07            1.92e-10
2.15e-06    7.62e-07            2.04e-11
1.29e-05    4.57e-06            1.40e-11
7.74e-05    2.74e-05            7.07e-10
4.64e-04    1.64e-04            2.54e-08
2.78e-03    9.85e-04            9.12e-07
1.67e-02    5.93e-03            3.28e-05
1.00e-01    3.65e-02            1.18e-03

Process finished with exit code 0
```