Extra Homework 2 - Ilie Miruna Andreea

1. Let $f:[a,b]\to[a,b]$ be a contraction, meaning that there exists $\alpha\in(0,1)$ such that
$|f(x)-f(y)|\le\alpha|x-y|$, $\forall\, x,y\in[a,b]$

For an arbitrary $x_1\in[a,b]$ consider the sequence (x_n) given by $x_{n+1}=f(x_n)$, $\forall\, n\in\mathbb{N}$

- So, $f:[a,b]\to[a,b]$

f has a unique fixed point $x^*\in[a,b]$ and $\forall\, x_0\in[a,b]$ we have $f^n(x_0)\xrightarrow{n\to\infty}x^*$

$x_0\in[a,b]$; $x_n=f^n(x_0)$

$$d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1})\le\alpha\cdot d(x_n,x_{n-1})$$
$$=\alpha d(f(x_{n-1}),f(x_{n-2})\le\alpha^2\cdot d(x_{n-1},x_{n-2})\le$$
$$\dots\le\alpha^n\cdot d(x_1,x_0)$$

We have 2 indexes n, m =>

$$\Rightarrow d(x_n,x_m)\le d(x_n,x_{n-1})+\dots d(x_{m+1},x_m)\le$$
$$\le(\alpha^{n-1}+\alpha^{n-2}+\dots+\alpha^m)\cdot d(x_1,x_0)=\alpha^m\cdot\sum_{k=0}^{n-m-1}(x_1,x_0)$$
$$\le\sum_{k=0}^{\infty}\alpha^k=\frac{1}{1-\alpha}\quad\le\frac{\alpha^m}{\alpha}\cdot d(x_1,x_0)$$

- $f(x^*)=f(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}f(x_{n+1})=x^*$

2. Let α (0, 1) and $x_1, x_2 \in R$. Consider the sequence x_n given by $x_{n+2} = \alpha x_{n+1} + (1-\alpha)x_n$, $\forall n \in N$

Prove that (x_n) is convergent and find its limit in terms of α, x_1, x_2

- $x_{n+2} = \alpha x_{n+1} + (1-\alpha)x_n$

$x_n = y^n$

$y^{n+2} = \alpha y^{n+1} + (1-\alpha) y^n$

$y^2 = \alpha y + (1-\alpha)$

$y^2 - \alpha y = 1-\alpha$

$y^2 - \alpha y - 1 + \alpha \Rightarrow y = \frac{\alpha + \sqrt{\alpha^2 + y(1-\alpha)}}{2}$

$\alpha^2 + y(1-\alpha) = (2-\alpha)^2$

$y_1 = 1$

$y_2 = \alpha - 1$

We note X, Y as constants

$x_n = X + Y(\alpha-1)^n$

$\lim_{n\to\infty} x_n = 4 \Rightarrow$ convergent

$x_1 = X + Y(\alpha - 1)$

$x_2 = X + Y(\alpha-1)^2$

$x_1 = y[(\alpha-1)^2 - (\alpha-1)] = y(\alpha-1)(\alpha-2)$

$$B = \frac{x_2 - x_1}{(1-\alpha)(2-\alpha)}$$

$$A = x_1 = B(\alpha - 1) = x_1 + \frac{x_2 - x_1}{2-\alpha}$$

$$\Rightarrow \lim_{n\to\infty} x_n = x_1 + \frac{x_2 - x_1}{2 - \alpha}$$

3. Give an example of a sequence having the set of its limit points equal to $[0,1]$

$$x_n = \sin^2(n)$$

$\sin(n) \in [-1,1]$; $\sin^2(n) \in [0,1]$

$\sin(n)$ is dense on $[-1,1]$

$\sin^2(n)$ is dense on $[0,1]$