Homework 1 - Ilie Mirună Andreea

1) Let $a, b \in \mathbb{R}$ with $a>0$. If $S \subset \mathbb{R}$ is nonempty and bounded above, prove that $\sup\limits_{x\in S}(ax+b) = a\sup(S)+b$

S is nonempty and bounded above => will have a sup(S)

=> $X = \sup(S)$

$\forall x \in S$, $x \le X \quad | \cdot a$

$ax \le aX \quad | + b$

$ax + b \le aX + b \Rightarrow aX + b \in ub(ax+b)$

$\forall \varepsilon > 0, \exists x \in S$

$ax + b > aX + b - \varepsilon$

$$\left.\begin{array}{l} \Rightarrow aX + b \ge ax + b > aX + b - \varepsilon \\ A = ax + b \end{array}\right\} \Rightarrow \exists y \in A \quad y > a\sup(S) + b - \varepsilon$$

$\Rightarrow \sup(A) = a \cdot \sup(S) + b$

$\Rightarrow \sup(ax+b) = a \cdot \sup(S) + b$

2) Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ such that $U \cap V = \emptyset$

So $U \in \mathcal{V}(a)$

$V \in \mathcal{V}(b)$

$U \in \mathcal{V}(a)$ only if only $\exists\, \varepsilon$ so that $[a-\varepsilon, a+\varepsilon] \subset U$

$V \in \mathcal{V}(b)$ only if only $\exists\, \varepsilon$ so that $[b-\varepsilon, b+\varepsilon] \subset V$

So let's say $d = \frac{|a-b|}{2}$

$U = (a-d, a+d)$
$V = (b-d, b+d)$

$$\left.\begin{aligned} a+d &= \frac{a+b}{2} \\ b-d &= \frac{a+b}{2} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow a+b = b-d \Rightarrow U \cap V = \emptyset$

3. Let $A = (0,1) \cap \mathbb{Q}$. Show rigorously (using the definitions)

- $\forall x \in A$, $x \geq 0 \Rightarrow \inf(A) = \max(lb(A))$

$\Rightarrow lb(1) = (-\infty, 0] \Rightarrow \boxed{\inf(A) = 0}$

- $\forall x \in A$, $x \leq \sup(A)$

  $\sup(A) = \min(ub(A)) \Rightarrow ub(A) = [1, +\infty) \Rightarrow \boxed{\sup(A) = 1}$

- $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$

  $\nexists (a,b)$ so that $\forall x \in (a,b) \in A \Rightarrow \cancel{A \neq} \boxed{\operatorname{int}(A) = \emptyset}$

- $\forall x \in A$, $\exists \varepsilon \in A$ so that $\begin{cases} x - \varepsilon \in A \\ x + \varepsilon \in A \end{cases}$ $\Rightarrow$ this means

that ~~cl~~ $\boxed{\operatorname{cl} A = [0,1]}$