Homework 3 - Ilie Miruna Andreea

① Find the sum for each of the following series:

a) $\sum\limits_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum\limits_{n\geq 2} \ln\left(\frac{n^2-1}{n^2}\right)$

$= \sum \ln(n-1) + \ln(n+1) - 2\ln n$

$= \sum\limits_{n\geq 2} (\ln(n-1) - \ln(n)) + ((\ln(n+1) - \ln(n)))$

$S_K = \ln 1 + \ln 3 - 2\ln 2 + \ln 2 + \ln 4 - 2\ln 3 + \ln 3 - ..$
$+ \ ... \ \ln(K-1) + \ln(K+1) - 2\ln(K)$

$S_K = \ln 1 - \ln 2 + \ln(K+1) = \ln\left(\frac{K+1}{2}\right)$

$\lim\limits_{K\to\infty} S_K = \lim\limits_{K\to\infty} \left(\ln\left(\frac{K+1}{2}\right)\right) = \infty$

b) $\sum \frac{n+1}{3^n}$

$S_K = \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots + \frac{K+1}{3^K}$

$\sum\limits_{n\geq 1} \frac{n+1}{3^n} = \sum\limits_{n\geq 1} \frac{n}{3^n} + \sum \frac{1}{3^n}$

$$\sum_{n\geq 1} \frac{1}{3^n} = ?$$

$$S_k = \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^k}$$

$$\lim_{k\to\infty} \frac{1-\left(\frac{1}{3}\right)^k}{2} = \frac{1}{2}$$

$$\sum_{n\geq 1} \frac{n}{3^n} = ? \qquad \frac{x_{n+1}}{x_n} = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{n+1}{3n}$$

$$\sum_{n\geq 1} \frac{n}{3^n} = \frac{1}{3}$$

$$\sum_{n\geq 1} \frac{1}{3^n} + \sum_{n\geq 1} \frac{n}{3^n} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

c) $$\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{n}{n^4+2n^2-n^2+1}$$

$$= \sum_{n\geq 1} \frac{n}{(n^2+1)^2-n^2} = \sum_{n\geq 1} \frac{n}{(n^2+n+1)(n^2+n-1)}$$

$$= \sum_{n\geq 1} \frac{\frac{1}{2}}{n^2+1-n} + \sum \frac{-\frac{1}{2}}{n^2+n+1}$$

$$= \sum_{n\geq 1} \frac{1}{2(n^2-n+1)} - \sum_{n\geq 1} \frac{1}{2(n^2+n+1)}$$

$$= \frac{1}{2}\left(\sum_{n\geq 1} \frac{1}{n^2-n+1} - \sum_{n\geq 1} \frac{1}{n^2+n+1}\right)$$

$$\sum_{n\geq 1} \frac{1}{n^2-n+1}$$

$$S_{k_1} = 1 + \frac{1}{3} + \frac{1}{7} + \dots + \frac{1}{k^2-k+1}$$

$$\sum_{n\geq 1} \frac{1}{n^2+n+1}$$

$$S_{k_2} = \frac{1}{3} + \frac{1}{7} + \dots + \frac{1}{k^2+k+1}$$

$$\left.\begin{matrix} S_{k_1} \\ S_{k_2} \end{matrix}\right\} S_{k_1} - S_{k_2} = 1 - \frac{1}{k^2+k+1}$$

$$\Rightarrow \lim_{k\to\infty} 1 - \frac{1}{k^2+k+1} = 1 \Rightarrow \sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \frac{1}{2}$$

2. Study the convergence of the following series

a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x > 0$, $p \in \mathbb{N}$

$\sum_{n\geq 1} x^n$ is a geometric series $\Rightarrow$ convergent

$$\frac{x_{n+1}}{x_n} = \frac{\frac{x^{n+1}}{(n+1)^p}}{\frac{x^n}{n^p}} = x \frac{n^p}{(n+1)^p}$$

$$\lim_{n\to\infty} x \frac{n^p}{(n+1)^p} = x$$

I If $x > 1 \Rightarrow$ the series converges

II If $x < 1 \Rightarrow$ the series diverges

III If $x = 1 \Rightarrow \sum_{n \geq 1} \frac{1}{n^p}$

III 1. $x = 1$, $p > 1$ the series converges
2. $x = 1$, $p \leq 1$ the series diverges

b) $\sum_{n \geq 2} \frac{1}{(\ln n)^{\ln n}}$

$$(\ln n)^{\ln n} > (\ln n)^{\ln 2}$$

$$\Rightarrow \frac{1}{(\ln n)^{\ln n}} \leq \frac{1}{(\ln n)^2}$$

$\sum_{n \geq 2} \frac{1}{(\ln n)^2}$ converges $\Rightarrow \sum_{n \geq 2} \frac{1}{(\ln n)^{\ln n}}$ converges

c) $\sum_{n \geq 1} (\sqrt[n]{n} - 1)$

$$e^x \geq x + 1 \Rightarrow e^{\frac{\ln n}{n}} \geq \frac{\ln n}{n} + 1$$

$$e^{\frac{\ln n}{n}} = n^{\frac{1}{n}} = \sqrt[n]{n}$$

$$\sum_{n \geq 1} \frac{\ln n}{n}$$

Let $y_n = \frac{1}{n} \Rightarrow \frac{x_n}{y_n} = \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \ln n = \infty$

$\Rightarrow \sum_{n \geq 1} \frac{\ln n}{n}$ diverges (1)

5

$$\cancel{\sum_{n\geq 1}\left(\frac{\ln n}{n}\right)}$$

$$\sum_{n\geq 1}(\sqrt[n]{n}-1) \geq \sum_{n\geq 1}\frac{\ln n}{n} \quad (2)$$

Din ① + ② $\Rightarrow \sum_{n\geq 1}(\sqrt[n]{n}-1)$ diverges

d) $$\sum_{n\geq 1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{2n+1} = \sum_{n\geq 1}\frac{\frac{1\cdot 2\cdot 3\cdot\ldots\cdot(2n-1)(2n)}{2\cdot 4\cdot\ldots\cdot 2n}}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{2n+1}$$

$$= \sum_{n\geq 1}\frac{1\cdot 2\cdot 3\cdot\ldots\cdot(2n)}{(2\cdot 4\cdot\ldots\cdot 2n)^2(2n+1)} \Rightarrow \sum_{n\geq 1}\frac{(2n)!}{2^n\cdot n!\,(2n+1)!}$$

$$1\cdot 2\cdot 3\cdot\ldots\cdot 2n = (2n)!$$

$$(2\cdot 4\cdot\ldots\cdot 2n)^2(2n+1) = 2^n\cdot n!\,(2n+1)$$

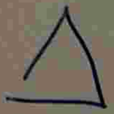
$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \frac{\frac{(2(n+1))!}{2^{n+1}(n+1)!\,(2(n+1)+1)}}{\frac{(2n)!}{2^n\cdot n!\,(2n+1)}}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(2(n+1)(2n+3)(n!\,(n+1)!)}$$

$$= \lim_{n\to\infty}\frac{2n+1}{(2n+3)(n+1)} = \lim_{n\to\infty}\frac{2n+3}{2n+3} - \frac{2}{(2n+3)} = 0 < 1$$

$\Rightarrow$ converges

3. $3 \xrightarrow{\times 4} 12 \xrightarrow{\times 4} 48 \xrightarrow{\times 4} 192$

it 0 initial form — it 1 — it 2 — it 3

$$\left.\begin{array}{l} it\ 0 = 3 \\ it\ 1 = 3 \cdot 4 \\ it\ 2 = 3 \cdot 4^2 \\ it\ 3 = 3 \cdot 4^3 \end{array}\right\} \Rightarrow$$

• at iteration n there will be $3 \cdot 4^n$ sides

initial form = echilateral triangle where s is the length of each segment

it 0 $\quad l = s$

it 1 $\quad l = \frac{s}{3}$

it 2 $\quad l = \frac{\frac{s}{3}}{3} = \frac{s}{9}$

it 3 $\quad l = \frac{\frac{s}{3}}{9} = \frac{s}{27} \Rightarrow$ So $l_n = \frac{s}{3^n}$ length of a side

$$P_n = \underbrace{\text{nr. of sides}}_{= N_n} \cdot l_n = N_n \cdot l_n = 3 \cdot s \left(\frac{4}{3}\right)^n$$

• $\lim\limits_{n \to \infty} P_n = \lim\limits_{n \to \infty} 3 \cdot s \cdot \left(\frac{4}{3}\right)^n = \infty$

Area

- with every iteration comes a new triangle

$\Rightarrow$ the number of triangles at every iteration is $T_n = \frac{3}{4} \cdot 4^n$ and the area of every triangle adds is $\frac{a}{9^n}$ (a = initial area of the first triangle)

- the area added at iteration is

$$T_n \cdot a_n = \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot a_0$$

$$A_n = a_0\left(1 + \frac{3}{4}\sum_{k=1}^{n}\left(\frac{4}{9}\right)^k\right) = a_0\left(1 + \frac{1}{3}\sum_{k=0}^{n-1}\left(\frac{4}{9}\right)^k\right)$$

$$A_n = a_0\left(1 + \frac{3}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)\right) = \frac{a_0}{5}\left(8 - 3\left(\frac{4}{9}\right)^n\right)$$

- $$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{a_0}{5}\left(8 - 3\left(\frac{4}{9}\right)^n\right) = \frac{8}{5} \cdot a_0$$