Extra Homework 10

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function having a minimum $x^* \in \mathbb{R}^n$ and consider the gradient descent method

$$x_{k+1} = x_k - s\nabla f(x_k).$$

Assume that there exist $0 < m < M$ such that for any $x \in \mathbb{R}^n$ the Hessian matrix $H(x)$ of f has all its eigenvalues $\lambda \in [m, M]$.

1. Using the Taylor expansion prove that

$$f(x_{k+1}) \leq f(x_k) - s\|\nabla f(x_k)\|^2 + \frac{Ms^2}{2}\|\nabla f(x_k)\|^2. \tag{1}$$

2. Choosing the step $s > 0$ in order to minimize the right-hand side in (1), prove that

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2M}\|\nabla f(x_k)\|^2. \tag{2}$$

3. Using the Taylor expansion prove that

$$f(x^*) \geq f(x_k) - \frac{1}{2m}\|\nabla f(x_k)\|^2. \tag{3}$$

4. Combining (2) and (3), prove that

$$f(x_{k+1}) - f(x^*) \leq \left(1 - \frac{m}{M}\right)\big(f(x_k) - f(x^*)\big). \tag{4}$$

   and hence that $f(x_k) \to f(x^*)$.

These questions are extra. You will get bonus points for solving them.