Extra Homework 1 - Ilie Miruna Andreea

1. Prove that the following statements are true

- any union of open sets is open

Let A, B be open set and $\begin{cases} x_1, y_1 \in A \\ x_2, y_2 \in B \end{cases}$ so $\Rightarrow$

$\Rightarrow A \cup B = (x_1, y_1) \cup (x_2, y_2)$

$\inf(A \cup B) = \{x_1, y_1, y_2, \cancel{y}\ x_2\}$

$\sup(A \cup B) = \{x_1, y_1, y_2, x_2\} / \inf(A \cup B)$

$\left.\begin{array}{l} \text{If } A \text{ is an open set} \Rightarrow \nexists \max(A), \min(A) \\ \text{If } B \text{ is an open set} \Rightarrow \nexists \max(B), \min(B) \end{array}\right\} \Rightarrow$

$\Rightarrow$ so $\nexists \max(A \cup B)$, $\nexists \min(A \cup B)$ $\Rightarrow$

$\Rightarrow A \cup B = (\min(x_1, y_1, x_2, y_2); \max(x_1, y_1, x_2, y_2))$

- any intersection of closed ~~seti~~ sets is closed

$A = [x_1, y_1]$, $B = [x_2, y_2]$

$\begin{cases} \sup(A) = y_1 \\ \max(A) = y_1 \end{cases}$ $\begin{cases} \inf(A) = x_1 \\ \min(A) = x_1 \end{cases}$

$\begin{cases} \sup(B) = \max(B) = y_2 \\ \inf(B) = \min(B) = x_2 \end{cases}$

$X = A \cap B$; $X \subseteq (A \cup B)$

and $\exists\ m \in B$ ~~so tha~~ $\Rightarrow m \in A \cup B$

$$\left.\begin{aligned} \min(X) &\in A \cup B \\ \max(A) &\in A \cup B \end{aligned}\right\} \Rightarrow X \text{ closed}$$

- any finite intersection of open sets is open

$$\left.\begin{aligned} A &= (x_1, y_1) \\ B &= (x_2, y_2) \end{aligned}\right\} \Rightarrow X = A \cap B$$

A, B open $\Rightarrow \nexists \max, \min(A, B)$

$$\left.\begin{aligned} A &= (\inf(A), \sup(A)) \\ B &= (\inf(B), \sup(B)) \end{aligned}\right\} \Rightarrow A \cap B \Rightarrow$$

$\cancel{\Rightarrow A}$ $\Rightarrow A \cap B \ (\max(\inf(A), \inf(B)); \min(\sup(A), \sup(B)))$

- any finite union of closed set is closed

$$\left.\begin{aligned} A &= [x_1, y_1] \\ B &= [x_2, y_2] \end{aligned}\right\} \Rightarrow X = A \cup B$$

$$X = [\min(x_1, y_1, x_2, y_2), \max(x_1, x_2, y_1, y_2)]$$

- $\left.\begin{aligned} A &= (-1, 1) \\ B &= (0, 2) \end{aligned}\right\} \Rightarrow \cancel{A \cap B(} \ A \cap B = (0, 1)$

- $\left.\begin{aligned} A &= [0, 1] \\ B &= [2, 3] \end{aligned}\right\} \Rightarrow A \cup B = [0, 1] \cup [2, 3]$

2. Let α be an irrational number and consider the set $S_\alpha := \{ \{n\alpha\} \mid n \in \mathbb{N}\}$ where $\{\cdot\}$ denotes the fractional part of a number

a) Show that the set S_α is dense in $[0,1]$, meaning that its closure is $[0,1]$

$$S_\alpha = \{ \{n\alpha \mid n \in \mathbb{N}\}$$

$$\{n\alpha\} = n\alpha - [n\alpha]$$

$$0 \leq \{n\alpha\} < 1$$

$\forall \varepsilon > 0$ and $x \in [0,1]$, $\exists n$ so that $|\{n\alpha - x\}| < \varepsilon$

$\Rightarrow S_\alpha$ dense in $[0,1]$

~~b) Then show that the set $\{\{n\alpha\} + m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$~~