Homework 2 - Ilie Miruna Andreea

1) Prove using the ε-definition

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{n(1+\overbrace{\frac{1}{n}}^{\to 0})}{n(2+\underbrace{\frac{3}{n}}_{\to 0})} = \frac{1}{2}$$

Let $\varepsilon > \frac{1}{2}$, $\exists N \in \mathbb{N}$ such that $|x_n - \frac{1}{2}| < \varepsilon$

$$\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon \Rightarrow \left|\frac{2n+2-2n-3}{2(2n+3)}\right| < \varepsilon$$

$$\cancel{\left|-\frac{1}{2}\right| < \varepsilon} \quad \cancel{\Rightarrow} \quad \cancel{\frac{1}{2} < \varepsilon}$$

$$\left|-\frac{1}{2(2n+3)}\right| < \varepsilon \qquad \frac{1}{2(2n+3)} < \varepsilon$$

$$2(2n+3) > \frac{1}{\varepsilon} \Rightarrow 4n+6 > \frac{1}{\varepsilon}$$

$$4n > \frac{1}{\varepsilon} - 6 \Rightarrow n > \frac{1}{4\varepsilon} - \frac{6}{4} \Rightarrow$$

$$\Rightarrow N = \left[\frac{1}{4\varepsilon} - \frac{6}{4}\right] + 1, \quad n \geq N$$

~~2) a) limit~~

2) a) $\lim_{n\to\infty} (1+2+\dots+n)^{\frac{1}{n}} =$

$$= \lim_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^2+n}{2}\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln\left(\frac{n^2+n}{2}\right)^{\frac{1}{n}} = e^{\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{n^2+n}{2}\right)}$$

$$e^{\lim_{n\to\infty} \ln\left(\frac{n^2+n}{2}\right)^{\frac{1}{n}}} = e^{\lim_{n\to\infty} \frac{\ln\left(\frac{n^2+n}{2}\right)}{n}} = e^{\lim_{n\to\infty} \frac{1}{\frac{n^2+n}{2}} \cdot \left(\frac{n^2+n}{2}\right)'}$$

$$= e^{\lim\limits_{n\to\infty} \frac{2}{n^2+n} \cdot \frac{2n+1}{2}} = e^{\lim\limits_{n\to\infty} \frac{2n+1}{n^2+n}}$$

$$\cancel{= e^{\lim\limits_{n\to\infty} \frac{n^2\left(\frac{2}{n}+\frac{1}{n^2}\right)}{n\left(1+\frac{1}{n}\right)}}} = e^{\lim\limits_{n\to\infty} \frac{2}{2n+1}} = e^{0} = 1$$

c) $$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} = \lim_{n\to\infty} \frac{n^n}{n^n\left(1+\frac{1}{n^n}\left(1+2^2+3^3+\dots+(n-1)^{(n-1)}\right)\right)}$$

$$= \lim_{n\to\infty} \frac{n^n}{n^n} = 1$$

b) $$\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln n}\right)^n = ?$$

$$\ln\left(\frac{\ln(n+1)}{\ln n}\right)^n = n \ln\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$\ln\left(\frac{(n+1)}{\ln n}\right) = \ln(\ln(n+1)) - \ln(\ln(n))$$

$$\left(\frac{\ln(n+1)}{\ln n}\right)^n = n\left(\ln(\ln(n+1)) - \ln(\ln n)\right)$$

$$a_n = \ln(\ln(n+1)) - \ln(\ln(n))$$

$$x_n = n \Rightarrow \lim_{n\to\infty} x_n = \infty \Rightarrow \text{incresing}$$

$$\lim_{n\to\infty} \frac{a_n}{x_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{x_{n+1}-x_n}$$

$$a_{n+1}-a_n = [\ln(\ln(n+2)) - \ln(\ln(n+1)) - [\ln(\ln(n+1)) - \ln(\ln(n))$$

$$= \ln(\ln(n+2)) - 2\ln(\ln(n+1)) + \ln(\ln n)$$

$_{n+1} - x_n = 1$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{x_{n+1} - x_n} = \lim_{n\to\infty} [\ln(\ln(n+2)) - 2\ln(\ln(n+1)) + \ln(\ln(n))]$$

$$\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln n}\right)^n = e^0 = 1$$

3. For $x_n = \frac{\sin(n)}{n}$ study if (x_n) is bounded, monotone, and convergent. Find its limit

$$\lim_{x\to\infty} \frac{\sin x}{x} = 1 \qquad \sin n \in [-1, 1]$$

$$-1 \le \sin(n) \le 1$$

$$-1/n \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty} -\frac{1}{n} \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le \lim_{n\to\infty} \frac{1}{n} \quad \text{(squeeze theorem)}$$

$$0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

4. $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$

$$x_{n+1} - x_n = \left(1 + \frac{1}{2} + \dots \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right)$$

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln n \le 0$$

$$\cancel{= \ln(n) - \ln(n+1) + \frac{1}{n+1} \le 0}$$

$$\Rightarrow x_{n+1} - x_n \le 0 \quad \Rightarrow x_{n+1} \le x_n$$

$$\ln(n) - \ln(n+1) \le -\frac{1}{n+1}$$

$$-\ln\left(1+\frac{1}{n}\right) \le -\frac{1}{n+1}$$

$\ln\left(1+\frac{1}{n}\right) \ge \frac{1}{n+1}$ True $\Rightarrow$ x_n is decreasing

x_n - harmonic sum

$$1+\frac{1}{2}+\dots+\frac{1}{n} \le 1+\ln n$$

$\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) - \ln n \ge 1$ $\Rightarrow$ bounded