Curs 7

- Echelon form of a matrix:

$A \in M_{m,n}(K)$ is in echelon form; $r \geq 1$ nonzero rows $\Longleftrightarrow$ ① rows $1, \dots, r$ are nonzero, rows $r+1, \dots, m$ are zero

② $0 \leq N(1) < N(2) < \dots < N(r)$

$N(i)$ = nr. of 0 elements from the beginning of the row i

(adică sub fiecare pivot se află 0)

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & -2 & 6 \\ -1 & 1 & 1 & 0 \end{pmatrix} \underset{L_3 = L_3 + L_1}{\overset{L_2 \leftarrow L_2 - 3L_1}{\sim}} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 2L_1}{\sim} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

echelon

- $\operatorname{rank}(A) = \dim\langle a_1, \dots, a_m \rangle = \dim\langle a^1, \dots, a^n \rangle$

$\operatorname{rank}(A) = \operatorname{rank}(C) = r$ (! C is a matrix in echelon form)

$\Downarrow$

ptr rank transf. matricea în forma eșalonată $\Rightarrow$ câte rânduri non-zero = rank

$$A = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $\Big\}$ non zero row $\Rightarrow$ $rank(A) = 2$

- $det(A) \neq 0 \Rightarrow$ invertible

ptr a afla A^{-1} adăugăm în dreapta matricii A matricea I_n și facem calcule pentru a transforma partea stângă în matricea I_n și rezultatul va fi în dreapta

$$A \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

Seminarul 8

1) $A = \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$

$\det(A) = -3+0+12-18-0+8$

$= -9+8 = -1 \neq 0 \Rightarrow$ inversabilă

$\underset{L_3 = L_3 - 3L_1}{\overset{L_2 = L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_2 \leftarrow L_2 \cdot (-\frac{1}{5})}{\sim}$

$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$

$\underset{L_3 \leftarrow L_3 + 12L_2}{\overset{L_1 \leftarrow L_1 - 4L_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{11}{5} & \frac{6}{5} & \frac{12}{5} & 1 \end{array}\right)$

$$L_3 \leftarrow L_3 \cdot \frac{5}{11} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{6}{11} & \frac{12}{11} & \frac{5}{11} \end{array}\right)$$

$$\begin{array}{l} L_1 \leftarrow L_1 - \frac{5}{2} L_3 \\ \sim \\ L_2 \leftarrow L_2 - \frac{5}{3} L_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{11} & \frac{2}{11} & -\frac{10}{11} \\ 0 & 1 & 0 & -\frac{1}{11} & -\frac{14}{11} & \frac{15}{11} \\ 0 & 0 & 1 & \frac{6}{11} & \frac{12}{11} & \frac{5}{11} \end{array}\right)$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 9 & 2 & -10 \\ -1 & -14 & 15 \\ 6 & 12 & 5 \end{pmatrix}$$

$$X = A^{-1} B = \frac{1}{11} \begin{pmatrix} 9 & 2 & -10 \\ -1 & -14 & 15 \\ 6 & 12 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} -11 \\ 29 \\ 16 \end{pmatrix}$$

2 a)

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -19 \neq 0 +$$ calcul de rang $\Rightarrow$ compatibil

$\to x_1, x_2, x_3$ necunoscute principale

x_4 necunoscută secundară

$$\begin{cases} x_1 + x_2 + x_3 = 5 - 2\lambda \\ 2x_1 + x_2 - 2x_3 = 1 - \lambda \\ 2x_1 - 3x_2 + x_3 = 3 - 2\lambda \end{cases}$$

$$x_1 = \frac{\Delta x_1}{\Delta}$$

$$\Delta x_1 = \begin{vmatrix} 5+2\lambda & 1 & 1 \\ 1-\lambda & 1 & -2 \\ 3-2\lambda & -3 & 1 \end{vmatrix} = -40$$

$$x_1 = \frac{40}{19}$$

$\vdots$

old way

THE NEW WAY $\Rightarrow$ calc. cu pivot pe diag. în jos $\Rightarrow$ sistem $\Rightarrow$ matrice $\Rightarrow$ calc cu pivot pe diag în sus $\Rightarrow$ sistem = rezultat transf into echelon form

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 2L_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & -5 & -1 & 6 & -7 \end{array}\right) L_3 \leftarrow L_3 - 5L_2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & 0 & 19 & -19 & 38 \end{array}\right) L_3 \leftarrow L_3 \cdot \frac{1}{19}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$\Rightarrow$ switch back to system

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ -x_2 - 4x_3 + 5x_4 = -9 \\ x_3 - x_4 = 2 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \underbrace{\begin{array}{c} L_1 \leftarrow L_1 - L_3 \\ \\ L_2 \leftarrow L_2 + 4L_3 \end{array}}_{} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 3 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$$\underbrace{L_1 = L_1 - L_2}_{} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

back to system

$$\begin{cases} x_1 = 2 \\ -x_2 + x_4 = -1 \\ x_3 - x_4 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 + x \\ x_3 = 2 + x \\ x_4 = x \end{cases}$$

b) $\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right) \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ 2x_4 = 2 \\ 4x_4 = 4 \end{cases}$$

$x_4 = 1 \qquad x_1 = 2x_2 - x_3$

(x_2, x_3 – free parameters)

⑤ a)

$$\begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$$

$$\bar{A} = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

$$\underset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 + L_1}}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right) \underset{L_2 \leftrightarrow L_3}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 4L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

$$\underset{L_2 \leftarrow L_2 + L_3}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_1 \leftarrow L_1 + L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \Rightarrow \begin{cases} x_1 = -7 \\ x_2 = -8 \\ x_3 = 11 \end{cases}$$

b) $\begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim}$$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \begin{array}{l} \underset{\sim}{L_2 \leftarrow L_2 - 2L_1} \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 + L_2} \sim$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \quad z = 0$$

$$\left.\begin{array}{l} x + 2y - z = 3 \\ y = 1 - 3z \end{array}\right\}$$

$$\Rightarrow x = 3 + z - 2(1 - 3z)$$

$$= 3 + z - 2 + 6z = 1 + 7z$$

If I reach a matrix unde in stânga am doar 0 și în stânga valoare $\Rightarrow$ sistemul este incomp

(6) $$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right)$$

$$\underset{L_3 \leftarrow L_1 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda - 2 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda - 5 \end{array}\right)$$

2 cases $\Rightarrow$ 1 $\lambda \neq 5 \Rightarrow$ incompatibility

2 $\lambda = 5 \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \end{array}\right)$

????

Curs 8

- <u>Matrix of a list of vectors</u> in the basis B = the matrix having as its rows the coordinates of the vectors X in the basis B

ex: $B = (e_1, e_2, e_3, e_4)$ - canonical base

$X = (u_1, u_2, u_3)$

$u_1 = (1, 2, 3, 4)$

$u_2 = (5, 6, 7, 8)$

$u_3 = (9, 10, 11, 12)$

$$\Rightarrow [X]_B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

dimension of the generated subspace = rangul matricei (non-zero rows)

ex: $S = \langle (-3,5,-1,1), (1,1,0,1), (1,3 \rangle$

$T = \langle (1,0,2,0), (2,1,-1,2) \rangle$

dimension of $S+T$ = 4

$$\begin{pmatrix} -3 & 5 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -3 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 2 \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 33 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form

$\Downarrow$

4 non-zero rows

basis of $\langle X \rangle$ = list of non-zero rows of the echelon form

ex: basis of $(S+T) = ((1,1,-1,3),$
$(0,-1,3,3),(0,0,5,4),(0,0,0,33))$

! $S+T = S \cup T$
$S \cap T = \dim S + \dim T - \dim(S+T)$

Matrix of a vector in a base

ex: $v = (1,2,3)$
basis = canonical basis $\Big\} \Rightarrow$

$\Rightarrow [v]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Matrix of a linear map

ex: $f(x,y,z,t) = (x+y+z, y+z+t, z+t+x)$

$E = (e_1, e_2, e_3, e_4)$ canoniial bases

$E' = (e_1', e_2', e_3')$

$f(e_1) = (1,0,0,0) = f(x,y,z,t) = (1,0,1)$

$f(e_2) = (0,1,0,0) = (1,1,0)$

$f(e_3) = (0,0,1,0) = (1,1,1)$

$f(e_4) = (0,0,0,1) = (0,1,1)$

$$\Rightarrow [f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

rezultatele de mai sus sunt așezate pe coloane

Rank of a linear map

f linear map $f: V \to V'$

$$\operatorname{rank}(f) = \dim(\operatorname{Im}(f))$$

ex: $f(x,y,z,t) = (x+y,+z, y+z+t, z+t+x)$

$E = (e_1, e_2, e_3, e_4)$

$E' = (e_1', e_2', e_3')$

$$\Rightarrow [f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \text{echelon form}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow \operatorname{rank}([f]_{EE'}) = 3$$

$$\Rightarrow \operatorname{rank}(f) = 3$$

! rank of a linear map = rank of its matrix

Seminar 9

① $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$

$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

$\overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 3$

(2) $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 + 2L_1}{\sim}}$

$$\begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix} \overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 0 & \frac{15}{2} & \frac{5}{2} \end{pmatrix}$$

$\Rightarrow$ rank $= 3$

(3) $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{L_1 \leftarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$

$$\underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - \beta L_1}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3(1-\beta) & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix}$$

$$\underset{\sim}{L_2 \leftarrow \alpha L_3} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$$

<u>If $\alpha = 0$</u> $\Rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \underset{\sim}{L_2 \leftrightarrow L_3}$

$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix} \xRightarrow{\text{echelon form}} \text{rank} = 3$$

If $\alpha \neq 0 \Rightarrow \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$

$\underset{\sim}{L_3 \leftarrow L_3 - \frac{1-\beta\alpha}{\alpha} L_2}$ . . . .

4 Inversa

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\overset{L_2 \leftarrow L_2 \cdot \left(-\frac{1}{3}\right)}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\underset{L_3 \leftarrow L_3 + 6L_1}{\overset{L_1 \leftarrow L_1 - 2L_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right)$$

$$\overset{L_3 \leftarrow L_3 \cdot \frac{1}{9}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\xrightarrow[L_2 \leftarrow L_2 - 2L_3]{L_1 \leftarrow L_1 + 2L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{9} & \frac{4}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{5}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

5 Inversa

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - 3L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow L_2 \cdot \left(-\frac{1}{5}\right)} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 + 12L_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 0 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 \cdot 5} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 0 \end{array}\right)$$

$$L_2 \leftarrow L_2 - \tfrac{3}{5}L_3 \quad L_1 \leftarrow L_1 - 2L_3 \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \quad L_1 \leftarrow L_1 - 4L_2 \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \quad A^{-1}$$

6

$$\begin{cases} v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4 \\ v_2 = 3e_1 - e_2 + 3e_3 - 3e_4 \\ v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4 \end{cases}$$

$$\begin{pmatrix} 3 & 3 & 3 \\ 2 & -1 & 5 \\ -5 & 3 & -13 \\ 4 & -3 & 11 \end{pmatrix} \underset{L_1 \leftarrow \frac{1}{3}L_1}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ -5 & 3 & -13 \\ 4 & -3 & 11 \end{pmatrix} \quad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 + 5L_1 \\ L_4 \leftarrow L_4 - 4L_1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 8 & -8 \\ 0 & -7 & 7 \end{pmatrix} \overset{L_2 \leftarrow L_2 \cdot (-\frac{1}{3})}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 8 & -8 \\ 0 & -7 & 7 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 7L}{\overset{L_3 \leftarrow L_3 - 8L_2}{\sim}}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ echelon form $\Rightarrow$ rank $= 2$

rank $= 2 <$ # of vectors (3)

$\Rightarrow$ linearly dependant

! matrix of a list of vectors is linearly dependent $\Leftrightarrow$ rangul matricii este mai mic decât nr. de vectori

(7) $\dim\langle X\rangle$ = rank of echelon form

basis of $\langle X\rangle$ = non-zero rows of echelon form

! • calc $\dim\langle X\rangle$ / basis => pui

• pe linie vectorii

• representing vectors with respect to a basis => pui pe coloana

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix}$$

$$\underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 2$$

$\dim \langle x \rangle = 2$

basis of $\langle x \rangle = \{(1,0,4), (0,1,-8)\}$

(8) $\begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{pmatrix} \overset{L_2 \leftarrow \frac{1}{2}L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 4 & 6 & 2 \end{pmatrix}$

$\overset{L_3 \leftarrow L_3 - 4L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \dim \langle X \rangle = 2$

basis of $\langle x \rangle = \{(1, 0, 4, 3), (0, 1, 3/2, 1/2)\}$

(9) $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_2}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim}$

$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim = 2$

basis = $\{(1,0,4),(0,1,0)\}$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \underset{L_1 \leftarrow L_1 \cdot \left(-\frac{1}{3}\right)}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & \frac{4}{3} \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix}$$

$$\underset{\substack{L_2 \leftarrow L_2 - 5L_1 \\ L_3 \leftarrow L_3 + 2L_1}}{\sim} \begin{pmatrix} 1 & 2/3 & 4/3 \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & 4/3 & -\frac{32}{3} \end{pmatrix} \underset{L_2 \leftarrow L_2 \cdot \left(-\frac{3}{4}\right)}{\sim}$$

$$\begin{pmatrix} 1 & 2/3 & 4/3 \\ 0 & 1 & -8 \\ 0 & 4/3 & -\frac{32}{3} \end{pmatrix} \underset{L_3 \leftarrow L_3 - \frac{4}{3}L_1}{\sim} \begin{pmatrix} 1 & 2/3 & 4/3 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(T) = 2$

basis of $(T) = \{(1, 2/3, 4/3), (0, 1, -8)\}$

$$S+T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \\ -3 & -2 & 4 \\ 5 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim(S+T) = 2$$

$$\dim(S \cap T) = 2+2-2 = 2$$

Curs 9

Change matrices

- Let V be a vector space over K

$-\ B = (v_1, \dots, v_n)$ bases

$$\left.\begin{array}{l} - B' = (v_1', \dots, v_n') \\ - B'' = (v_1'', \dots, v_n'') \end{array}\right\} \text{ of } V$$

$$\Rightarrow \boxed{T_{BB''} = T_{BB'} \cdot T_{B'B''}}$$

- $$\left.\begin{array}{l} V \text{ V.S.} \\ B, B' \text{ bases of } V \end{array}\right\} \Rightarrow$$

$$\boxed{\begin{array}{l} T_{BB'} \text{ - invertible} \\ (T_{BB'})^{-1} = T_{B'B} \end{array}}$$

- $$\left.\begin{array}{l} B = (v_1, \dots, v_n) \\ B' = (v_1', \dots, v_n') \end{array}\right\} \text{ bases of } V$$

$$\Rightarrow [v]_B = T_{BB'} \cdot [v]_{B'}$$

<u>Example:</u> $E = (e_1, e_2, e_3)$ canonical bases

$B = (v_1, v_2, v_3)$

$v_1 = (0,1,1) \quad v_2 = (1,1,2) \quad v_3 = (1,1,1)$

$$T_{EB} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow T_{BE} = (T_{EB})^{-1}$$

muși pe ciorna

$$T_{BE} = (T_{EB})^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

u is a vector ; $u = (1,2,3)$

$$[u]_B = T_{BE} \cdot [u]_E = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

Alt exemplu

$E = (e_1, e_2, e_3)$; $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$B = (v_1, v_2, v_3)$; $v_1 = (0,1,1)$, $v_2 = (1,1,2)$, $v_3 = (1,1,1)$

$f(x,y,z) = (x+y, y-z, z+x)$

$$\left.\begin{aligned} f(e_1) &= (1,0,0) = (1,0,1) \\ f(e_2) &= (0,1,0) = (1,1,0) \\ f(e_3) &= (0,0,1) = (0,-1,1) \end{aligned}\right\}$$

punem pe coloană

$$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$[f]_B = T_{BE} \cdot [f]_E \cdot T_{EB}$$

$$[f]_B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -3 & -2 \\ 1 & 4 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

Eigenvectors and eigenvalues

Let $f \in End_K(V)$. A non-zero vector $v \in V$ is called an eigenvectors of f

if there exists $\lambda \in K$ such that $f(v) = \lambda \cdot v$
Here λ is called an eigenvalue of f

! $f \in End_K(V)$
$V(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$ = set of zero vector and the eigenvectors of f with eigenvalue λ
$V(\lambda)$ is a subspace of V

=> then $V(\lambda)$ is called the eigenspace (characteristic subspace) of λ with respect to f

! Let V be a v.s. over K, B a basis of V and $f \in End_K(V)$ with the matrix $[f]_B = A = (a_{ij}) \in M_n(K)$. Then $\lambda \in K$ is an eigenvalue of f iff $\det(A - \lambda \cdot I_n) = 0$

Un exemplu simplu ChatGPT:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

$$\left.\begin{matrix} \lambda_1 = 5 \\ \lambda_2 = 2 \end{matrix}\right\} \rightarrow \text{eigenvalues}$$

$$\lambda_1 = 5 \Rightarrow (A - 5I)\cdot v = 0$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow v_1 = (1,1)$$

$$\lambda_2 = 2 \Rightarrow (A - 2I)\cdot v = 0$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow v_2 = (-1,2)$$

eigenvectors

$$\left.\begin{matrix} V(5) = \{(1,1)\} \\ V(2) = \{(-1,2)\} \end{matrix}\right\} \text{eigenspace}$$

Seminar 10

① $E = (e_1, e_2, e_3)$ $f(x,y,z) = (x+y, y-z, 2x+y+z)$

$e_1 = (1,0,0)$; $e_2 = (0,1,0)$; $e_3 = (0,0,1)$

$f(e_1) = (1,0,0) = (1,0,2)$

$f(e_2) = (0,1,0) = (1,1,1)$

$f(e_3) = (0,0,1) = (0,-1,1)$

$$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

② $f(x,y,z) = (y, -x)$

$f(v_1) = f(1,1,0) = (1,-1)$

$f(v_2) = f(0,1,1) = (1,0)$

$f(v_3) = f(1,0,1) = (0,-1)$

$$[f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

- $(1,-1) = a(1,1) + b(1,-2) = (a+b, a-2b)$

$$\begin{cases} a+b = 1 \Rightarrow a = 1-b \\ a-2b = -1 \end{cases} \Rightarrow -3b = 2 \Rightarrow b = \frac{2}{3}$$

$$a = \frac{1}{3}$$

$$[f(v_1)_{B'}] = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

- $(1,0) = a(1,1) + b(1,-2) = (a+b, a-2b)$

$$\begin{cases} a+b = 1 \\ a-2b = 0 \end{cases} \quad a = 1-b \quad \Bigg\} \Rightarrow -3b = -1$$

$$b = \frac{1}{3}$$

$$a = \frac{2}{3}$$

$$[f(v_2)_{B'}] = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

- $(0,-1) = a(1,1) + b(1,-2) = (a+b, a-2b)$

$$\begin{cases} a+b = 0 \\ a-2b = -1 \end{cases} \quad a = -b \Rightarrow -3b = -1$$

$$b = \frac{1}{3}$$

$$a = -\frac{1}{3}$$

$$[f(v_3)_{B'}] = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$[f]_{BB'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

3 a) $f(e_1) = (1,2,3,4)$

$f(e_2) = (4,3,2,1)$

$f(e_3) = (-2,1,4,1)$

$$x_1(1,2,3,4) + x_2(4,3,2,1) + x_3(-2,1,4,1)$$

$$(x_1 + 4x_2 - 3x_3,\ 2x_1 + 3x_2 + x_3,\ 3x_1 + 2x_2 + 4x_3,\ 4x_1 + x_2 + x_3)$$

$$f(v) = \begin{pmatrix} x_1 + 4x_2 - 2x_3 \\ 2x_1 + 3x_2 + x_3 \\ 3x_1 + 2x_2 + 4x_3 \\ 4x_1 + x_2 + x_3 \end{pmatrix}$$

b) $$[f]_E = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix}$$

c) Ker $f \Rightarrow f(v) = 0$

$[f] \cdot v = 0$

$$\begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc} 1 & 4 & -2 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{cccc} 1 & 4 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -15 & 9 & 0 \end{array}\right)$$

$$\overset{L_2 \leftarrow L_2 \cdot \left(-\frac{1}{5}\right)}{\sim} \left(\begin{array}{cccc} 1 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -15 & 9 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 + 15L_2}{\overset{L_3 \leftarrow L_3 + 10L_2}{\sim}}$$

$$\left(\begin{array}{cccc} 1 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \overset{L_3 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc} 1 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 4x_2 - 2x_3 = 0 \\ x_2 - x_3 = 0 \\ -6x_3 = 0 \end{cases} \Rightarrow x_1, x_2, x_3 = 0$$

$\Rightarrow \dim(\operatorname{Ker} f) = 0 \Rightarrow \operatorname{Ker} f = 0$

$$\begin{pmatrix} 1 & 4 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -6 \end{pmatrix} \Rightarrow \text{rank} = 3 \Rightarrow \operatorname{Im} f = 3$$

basis for $\operatorname{Im} f$ $\left((1,4,-2),(0,-1,1),(0,0,-6)\right)$

Seminarul 11

$\mathcal{B} = (v_1, v_2, v_3)$ basis in $\mathbb{R}^3$

$v_1 = (1,0,1)$; $v_2 = (0,1,1)$; $v_3 = (1,1,1)$

$\mathcal{B}' = (v_1', v_2', v_3')$ basis in $\mathbb{R}^3$

$v_1' = (1,1,0)$; $v_2' = (-1,0,0)$; $v_3' = (0,0,1)$

$$a(1,0,1) + b(0,1,1) + c(1,1,1) = (1,1,0)$$

$$\left.\begin{cases} a+c=1 \Rightarrow a = 1-c \\ b+c=1 \Rightarrow b = 1-c \\ a+b+c=0 \end{cases}\right\} \Rightarrow 1-c+1-\cancel{c}+\cancel{c}=0$$

$$\boxed{c=2}$$

$$\boxed{a=b=-1}$$

$$a(1,0,1) + b(0,1,1) + c(1,1,1) = (-1,0,0)$$

$$\left.\begin{cases} a+c=-1 \Rightarrow a = -1-c \\ b+c=0 \Rightarrow b = -c \\ a+b+c=0 \end{cases}\right\} \begin{matrix} -1-c-\cancel{c}+\cancel{c}=0 \\ \Rightarrow \boxed{c=-1} \\ \boxed{a=0} \\ \boxed{b=1} \end{matrix}$$

$$a(1,0,1)+b(0,1,1)+c(1,1,1)=(0,0,1)$$

$$\begin{cases} a+c=0 \Rightarrow a=-c \\ b+c=0 \Rightarrow b=-c \\ a+b+c=1 \end{cases} \Rightarrow -2c+c=1$$

$$-c=1$$

$$\boxed{c=-1}$$

$$\boxed{a=b=1}$$

$$T_{BB'}=\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$$

$$a(1,1,0)+b(-1,0,0)+c(0,0,1)=(1,0,1)$$

$$\left.\begin{array}{l} a-b=1 \\ \boxed{a=0} \\ \boxed{c=1} \end{array}\right\} \Rightarrow \boxed{b=-1}$$

$$a(1,1,0)+b(-1,0,0)+c(0,0,1)=(0,1,1)$$

$$\left.\begin{array}{l} a-b=0 \\ \boxed{a=1} \\ \boxed{c=1} \end{array}\right\} \Rightarrow \boxed{b=1}$$

$$a(1,1,0)+b(-1,0,0)+c(0,0,1)=(1,1,1)$$

$$\left.\begin{array}{l} a-b=1 \\ \boxed{a=1} \\ \boxed{c=1} \end{array}\right\} \Rightarrow \boxed{b=0}$$

$$T_{B'B}=\begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$u=(2,0,-1)$$

$$[u_B]=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\left.\begin{cases} a+c=2 & \Rightarrow a=2-c \\ b+c=0 & \Rightarrow b=-c \\ a+b+c=-1 & \end{cases}\right\} \Rightarrow \begin{array}{l} 2-2c+c=-1 \\ -c=-3 \\ \boxed{c=3} \\ \boxed{a=-1} \\ \boxed{b=-3} \end{array}$$

$$[u]_B=\begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

$$[u]_{B'} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$a - b = 2 \Rightarrow -b = 2 - a \Rightarrow b = -2 + a$

$a = 0$

$b = -2$

$c = -1$

$$[u]_{B'} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

(2) $B = (v_1, v_2)$ $\quad B' = (v_1', v_2')$

$v_1 = (1,2)$

$v_2 = (1,3)$

$v_1' = (1,0)$

$v_2' = (2,1)$

$$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \qquad [g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$2[f]_B = 2\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B$$

$$\boxed{[g]_B = [id]_{B'B} \cdot [g]_{B'} \cdot [id]_{BB'}}$$

$$[id]_{B'B} = \left([v_1']_B \ [v_2']_B \right)$$

$$a(1,2) + b(1,3) = (1,0)$$

$$\begin{cases} a+b=1 & \Rightarrow a = 1-b \\ 2a+3b=0 \end{cases} \quad \Rightarrow \quad 2(1-b)+3b=0$$

$$2-2b+3b=0$$

$$\boxed{b=-2}$$

$$\boxed{a=3}$$

$$[v_1']_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$a(1,2) + b(1,3) = (2,1)$$

$$\begin{cases} a+b=2 & a = 2-b \\ 2a+3b=1 \end{cases} \quad \Rightarrow \quad 2(2-b)+3b=1$$

$$4-2b+3b=1$$

$$4+b=1 \Rightarrow \boxed{b=-3}$$

$$\boxed{a=5}$$

$$[v_2']_{B'} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$[id]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[id]_{BB'} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}^{-1} = \left(\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 \cdot \frac{1}{3}}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{5}{3} & \frac{1}{3} & 0 \\ -2 & -3 & 0 & 1 \end{array}\right) \overset{L_2 \leftarrow L_2 + 2L_1}{\sim} \left(\begin{array}{cc|cc} 1 & 5/3 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right)$$

$$\overset{L_2 \leftarrow 3L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 5/3 & \frac{1}{3} & 0 \\ 0 & 1 & 2 & 3 \end{array}\right) \overset{L_1 \leftarrow L_1 - \frac{5}{3}L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$[id]_{BB'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$\Rightarrow [f+g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B,B'} \cdot [g]_{B',B}$$

④ a) $f(x,y) = (3x+3y, 2x+4y)$

$f(1,0) = (3,2)$

$f(0,1) = (3,4)$

$[f]_E = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

$\det([f]_E - \lambda I) = 0$

$\begin{vmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 6$

$= 12 - 3\lambda - 4\lambda + \lambda^2 - 6$

$= \lambda^2 - 7\lambda + 6 = 0$

$\Delta = 49 - 24 = 25 \qquad \lambda_{1,2} = \frac{7 \pm 5}{2} \begin{cases} 6 \\ 1 \end{cases}$ eigen values

$\lambda_1 = 1 \Rightarrow ([f]_E - 1 I) \cdot v = 0$

$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\begin{cases} 2x + 3y = 0 \\ 2x + 3y = 0 \end{cases} \Rightarrow x = \frac{3y}{2}$$

$$S(\lambda_1) = \left\{ \left(\frac{3y}{2}, y \right) / y \in R \right\} = \langle \left(\frac{3}{2}, 1 \right) \rangle$$

eigenvector$_1$

$$\lambda_2 = 6 \Rightarrow ([f]_E - 6I) \cdot v = 0$$

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} -3x + 3y = 0 \\ 2x - 2y = 0 \end{cases} \quad x = y \quad \Rightarrow \langle (1,1) \rangle$$

eigenvector$_2$

b) $v_1 = \left(-\frac{3}{2}, 1 \right)$

$v_2 = (1, 1)$

$$B = \begin{pmatrix} -\frac{3}{2} & 1 \\ 1 & 1 \end{pmatrix}$$

$$f[B] = B \cdot [f]_E \cdot B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

$[g]_{B'} \rightarrow [g]_B$

schimbarea de bază și nr. de id

$B', B' \rightarrow B, B$

$B, B' \rightarrow B, B$

5. $A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 1 & 0 \\ -4 & -1-\lambda & 0 \\ -4 & -8 & -2-\lambda \end{pmatrix}$

$= -x^3 + 3x - 2 \quad = (x+2)(-x^2 + 2x - 1)$

$\lambda_1 = -2 \quad , \quad \lambda_2 = 1 \Rightarrow$ eigen values

$$\lambda_1 = -2 \Rightarrow \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x + y = 0 \\ -4x - y = 0 \\ -4x - 8y - 2z = 0 \end{cases} \Rightarrow x = y = 0, \quad z \in \mathbb{R}$$

$$\Rightarrow S(\lambda) = \{(0,0,z) \mid z \in \mathbb{R}\}$$

$$= \langle (0,0,1) \rangle$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ -4 & -8 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + y = 0 \\ -4x - 2y = 0 \\ -4x - 8y - 3z = 0 \end{cases} \Rightarrow x = -\frac{y}{2}$$

$$\Rightarrow z = \frac{-8y - 4\left(-\frac{y}{2}\right)}{3}$$

$$\Rightarrow z = -2y$$

$$= \left\{ \left(-\frac{y}{2}, y, -2y \right) / y \in R^2 \right\}$$

$$= \langle \left(-\frac{1}{2}, 1, -2 \right) \rangle$$