Curs 8

I The Euclidean space

- A function $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ is called a norm if

$$\begin{cases} \|x\| = 0 \text{ iff } x = 0 \\ \|\alpha x\| = |\alpha| \, \|x\| \\ \|x+y\| \leq \|x\| + \|y\| \quad (\text{triangle inequality}) \end{cases}$$

- $\langle \cdot , \cdot \rangle$ = scalar product if
  - $\langle x, y \rangle = \langle y, x \rangle$
  - $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
  - $\langle x, x \rangle > 0$

- $|\langle x, y \rangle| \leq \|x\| \, \|y\|$

- $\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \ldots + x_n^2} \Rightarrow$ the length of the vector x

- a function d = a distance (metric)

if $\begin{cases} d(x,y)=0 \text{ iff } x=y \\ d(x,y)=d(y,x) \\ d(x,z) \le d(x,y)+d(y,z) \end{cases}$

- $d(x,y) = \|x-y\|$

- $d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$

II Neighborhoods. Interior. Closure. Boundary

- $A \subseteq \mathbb{R}^n$ is <u>bounded</u> if there exists $r>0$ such that $\|x\| \le r$, $\forall x \in A$

- $x_0 \in \mathbb{R}^n$, $r>0$;- the open ball of centre x_0 and radius r is given by

$$B(x_0, r) := \{ x \in \mathbb{R}^n \mid \|x-x_0\| < r \}$$

- the closed ball of centre x_0 and radius r is given by

$$\overline{B}(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$$

- $V \subseteq \mathbb{R}^n$ is a neighborhood (vecinity) of $x \in \mathbb{R}^n$ if $\exists\, r > 0$ such that $B(x, r) \subseteq V$

- $A \subseteq \mathbb{R}^n$

(interior) $\operatorname{int}(A) := \{x \in \mathbb{R}^n \mid \exists\, V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

(closure) $\operatorname{cl}(A) := \{x \in \mathbb{R}^n \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

(boundary) $\operatorname{bd}(A) := \{x \in \mathbb{R}^n \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset\}$

example: $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

$$\begin{cases} \operatorname{int}(A) = A \\ \operatorname{cl}(A) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \\ \operatorname{bd}(A) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \end{cases}$$

- $A = \operatorname{int}(A) \Rightarrow A = \text{open}$
- $A = \operatorname{cl}(A) \Rightarrow A = \text{closed}$

III Sequences

- a sequence (x^k) converges if $\lim_{k\to\infty} \|x^k - x\| = 0 \Rightarrow \lim_{k\to\infty} x^k = x$

example: $x^k = \left(\frac{1}{k}, \frac{k}{k+1}\right) \Rightarrow \lim_{k\to\infty} x^k = (0,1)$

- a sequence (x^k) converges iff $\lim_{k\to\infty} x_i^k = x_i$

Seminar 8

1. Prove that $\forall\, x, y \in \mathbb{R}^n$

a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\|x\| = \sqrt{x_1^2 + x_2^2} = \sqrt{\langle x, x\rangle}$

$d(x-y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = \sqrt{\langle x-y, x-y\rangle} =$
$= \|x-y\|$

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \underline{\langle x, x+y\rangle} + \langle y, x+y\rangle + \underline{\langle x, x-y\rangle} + \langle -y, x-y\rangle$$

$$= \underline{\langle x, x\rangle} + \underline{\langle x, y)} + \langle y, x\rangle + \langle y, y\rangle +$$

$$+ \underline{\langle x, x\rangle} + \underline{\langle x, -y\rangle} + \langle -y, x\rangle + \langle -y, -y\rangle$$

$$= \langle x, x\rangle + \cancel{\langle x, y)} + \cancel{\langle y, x\rangle} + \langle y, y\rangle + \langle x, x\rangle$$

$$- \cancel{\langle x, y\rangle} - \cancel{\langle y, x\rangle} + \langle y, y\rangle$$

$$= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 +$$

$$+ 2\|y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

b) $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2$

$$= \frac{1}{4}(\langle x+y, x+y) - \langle x-y, x-y\rangle)$$

$$= \frac{1}{4}(\langle x, x+y\rangle + \langle y, x+y) - \langle x, x-y\rangle - (\langle -y, x-y\rangle)$$

$$= \frac{1}{4}(\langle x, x\rangle + \underline{\langle x, y\rangle} + \underline{\langle y, y\rangle} + \underline{\langle y, x\rangle} - \langle x, x\rangle$$

$$- \underline{\langle x, -y)} - \underline{\langle -y, -y\rangle} - \underline{\langle -y, x\rangle}$$

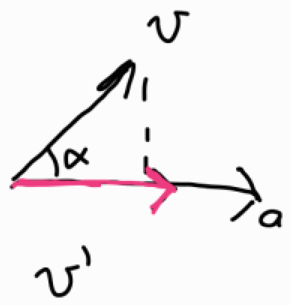
$$= \frac{1}{4}\left(\cancel{\|x\|^2} + 2\langle x, y\rangle - \cancel{\|y\|^2} + 2\langle y, x\rangle - \right.$$

$$\left. - \cancel{\|x\|^2} - \cancel{\|y\|^2} \right)$$

$$= \frac{1}{4} \cdot 4 \langle x, y\rangle = \langle x, y\rangle$$

② Find the orthogonal projection of a vector $v \in R^2$ onto a vector $a \in R^2$

$x \perp y \iff \langle x, y\rangle \; 0$



$$v' = \frac{a}{\|a\|} \cdot \|v'\| \qquad (1)$$

$$\langle v, a\rangle = \|v\| \cdot \|a\| \cdot \cos\alpha = \|v\| \cdot \|a\| \cdot \frac{\|v'\|}{\|v\|} =$$

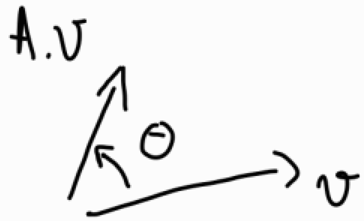
$$= \|a\| \cdot \|v'\| \Rightarrow \|v'\| = \frac{\langle v, a\rangle}{\|a\|} \qquad (2)$$

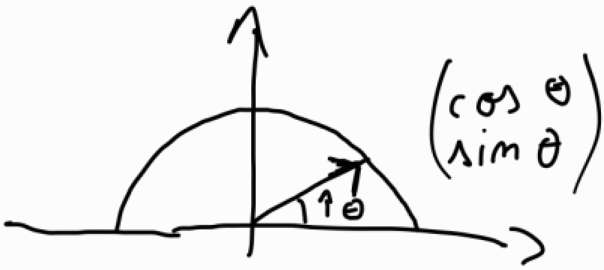
$$\xrightarrow{(1)\,(2)} v' = \frac{a}{\|a\|} \cdot \frac{\langle v, a\rangle}{\|a\|} = \frac{a \langle v, a\rangle}{\|a\|^2}$$

$$= \frac{a \langle v, a\rangle}{\langle a, a\rangle}$$

! $\|a\|^2 = \langle x, x \rangle$

③ Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation matrix with angle θ in R^2





$\|v_1\| = 1$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}_{v_1}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

5. Find the interior, the closure and the boundary for each of the following sets:

a) $[0,1) \times (1,2]$

$\text{int}(A) = (0,1) \times (1,2)$

$\text{cl}(A) = [0,1] \times [1,2]$

$\text{bd}(A) = \{0,1\} \times [1,2] \cup (0,1) \times \{1,2\}$

b) $\{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$

$\text{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$

$\text{cl}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq |y|\}$

$\text{bd}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| = |y|\}$

c) $\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$:

$\text{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$

$\text{cl}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$

$\text{bd}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$

d) $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1,\ x \leq 1\}$

$\text{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1,\ x < 1\}$

$$cl(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \le 1, x \le 1\}$$

$$bd(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1, x \le 1\}$$

⑥ Draw the level sets $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

a) $f(x,y) = x + y$; $c \in \{0, \pm 1\}$

$x + y = 0 \Rightarrow y = -x$

$x + y = 1 \Rightarrow y = 1 - x$

$x + y = -1 \Rightarrow y = -1 - x$

Curs 9 I Partial derivatives and differentiability

Derivatele parțiale => derivate calculate față de o sg. variabilă, celelalte rămânând constante

ex: $f(x,y) = x^2 + y^2$

$$\begin{cases} \text{față de } x: \dfrac{df}{dx} = 2x \\ \text{față de } y: \dfrac{df}{dy} = 2y \end{cases}$$

Gradientul => vector care conține toate derivatele parțiale ale unei funcții

ex: $f(x,y) = x^2+y^2$

$$\nabla f(x,y) = \left(\frac{df}{dx}, \frac{df}{dy}\right) = (2x, 2y)$$

=> la pct $(1,1)$ gradientul este $\nabla f(1,1) = (2,2)$

Diferențiabilitatea

O funcție este diferențiabilă dacă poate fi aproximată liniar în jurul unui punct

$$f(x) \simeq f(x_0) + \underbrace{f'(x_0)(x - y_0)}_{\text{panta liniei tangente la graficul funcției în } x_0}$$

ex: $f(x) = x^2$ la $x_0 = 1$

$f'(x) = 2x \Rightarrow f'(x_0) = 2$

$f(x) \approx f(1) + f'(1) \cdot (x-1) = 1 + 2(x-1)$

$f(x) \simeq 2x - 1$

Jacobian matrix

= o matrice care conține toate derivatele parțiale ale unei funcții

ex: $f(x,y) = (x^2, xy)$

$f_1(x,y) = x^2$

$f_2(x,y) = xy$

$\frac{df_1}{dx} = 2x \qquad \frac{df_1}{dy} = 0$

$\frac{df_2}{dx} = y \qquad \frac{df_2}{dy} = x$

$$J_f(x,y) = \begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} \\ \frac{df_2}{dx} & \frac{df_2}{dy} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix}$$

$$J_f(1,2) = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

Chain rule

ex: $z = f(x,y)$ $\qquad x = g(t)$

$\qquad\qquad\qquad\qquad y = h(t)$

$$\boxed{\frac{dz}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}}$$

$$z = f(x,y) = x^2 + y^2$$

$$\begin{cases} x = g(t) = t^2 \\ y = h(t) = t+1 \end{cases}$$

$$\frac{df}{dx} = 2x \qquad\qquad \frac{df}{dy} = 2y$$

$$\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x\frac{d(t^2)}{dt} + 2y\frac{d(t+1)}{dt}$$

$$= (2x)\cdot(2t) + (2y)\cdot(1) \xRightarrow[\text{cu val date}]{\text{substituim}}$$

$$2t^2 \cdot 2t + 2(t+1)(1) = 4t^3 + 2t + 2$$

$$\Rightarrow \boxed{\frac{dz}{dt} = 4t^3 + 2t + 2}$$

II Functions of several variables. Limits and continuity

- $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ are limita l la x_0 dacă $\forall \varepsilon > 0, \exists \delta > 0$ a.î. $\|x - x_0\| < \delta \Rightarrow$ $\Rightarrow |f(x) - l| < \varepsilon$

<u>Teoremă</u> dacă $\lim_{x \to x_0} f(x) = l \Rightarrow$ ptr. $\forall$ secv $x_k \to x_0$ trebuie să avem $\lim_{k \to \infty} f(x_k) = l$

exemple: a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$0 \leq \frac{x^2}{x^2+y^2} \leq \frac{x^2}{x^2} = 1$ (squeeze theorem)

am $x^2+y^2 \to 0$ când $(x,y)\to(0,0)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = 0$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin x^2+y^2}{x^2+y^2}$

$t = x^2+y^2 \Rightarrow \lim\limits_{t\to 0} \frac{\sin t}{t} = 1$

- $f(x)$ continuă la x_0 dacă

$\lim\limits_{x\to x_0} f(x) = f(x_0)$

Seminar 9

① Study the limits of the following functions when $(x,y) \to (0,0)$

a) $\frac{x^2-y^2}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-0}{x^2+0} = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{0-y^2}{0+y^2} = -1$$

$1 \neq -1$

$\Rightarrow$ nu există $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

b) $\frac{x+y}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \to \infty$$

$$x=0 \Rightarrow \lim_{y\to 0} \frac{y}{y^2} = \lim_{y\to 0} \frac{1}{y} \to \infty$$

dc $x=y \Rightarrow \lim\limits_{x\to 0} \frac{2x}{2x^2} = \frac{1}{x} \to \infty$

$\Rightarrow$ nu există $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$

c) $\frac{x^3+y^3}{x^2+y^2}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^3+y^3}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x^3}{x^2} = x$

$x=0 \Rightarrow \lim\limits_{y\to 0} \frac{y^3}{y^2} = y$

$\Rightarrow$ funcția depinde de calea urmată

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ nu există

2 a) $f(x,y) = e^{-(x^2+y^2)}$

$\frac{d}{dx} = e^{g(x)} \cdot g'(x)$ – derivare fct exponențială

- $\frac{df}{dx} = \frac{d}{dx}\left(e^{-(x^2+y^2)}\right)$

$g(x) = -(x^2+y^2) \Rightarrow \frac{d}{dx}(-x^2-y^2) =$

$= -\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = -2x - 0 = -2x$

$\Rightarrow \boxed{\frac{df}{dx} = e^{-(x^2+y^2)} \cdot (-2x)}$

- $\frac{df}{dy} = \frac{d}{dy} e^{-(x^2+y^2)}$

$g(x) = -(x^2+y^2) \Rightarrow \frac{d}{dy}(-x^2-y^2)$

$= -\frac{d}{dy}(x^2) - \frac{d}{dy}(y^2) = -0 - 2y = -2y$

$\boxed{\frac{df}{dy} = e^{-(x^2+y^2)} \cdot -2y}$

b) $f(x,y) = \cos x \cdot \cos y - \sin x \sin y$

$$\frac{df}{dx} = \frac{d}{dx}(\cos x \cdot \cos y) - \frac{d}{dx}(\sin x \sin y)$$

$$= \boxed{-\sin x \cos y - \cos x \sin y}$$

$$\frac{df}{dy} = \frac{d}{dy}(\cos x \cdot \cos y) - \frac{d}{dy}(\sin x \sin y)$$

$$= \boxed{-\cos x \sin y - \sin x \cos y}$$

c) $f(x,y) = \|x,y\| = \sqrt{x^2+y^2}$

$$\frac{d}{dx} = \frac{(x^2+y^2)'}{2\sqrt{x^2+y^2}}$$

$$\frac{df}{dx} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{df}{dy} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

d) $f(x,y,z) = x^2yz + ye^z$

$$\frac{df}{dx} = 2xyz + ye^z$$

$$\frac{df}{dy} = x^2z + e^z$$

$$\frac{df}{dz} = x^2y + ye^z$$

4) $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{0 \cdot x}{\sqrt{x^2+0}} = 0$$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{0 \cdot y}{\sqrt{y^2+0}} = 0$$

$$\Rightarrow \left.\begin{array}{l} \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 \\ \\ f(0,0) = 0 \end{array}\right\} \Rightarrow \text{continuă în origine}$$

$$\frac{df}{dx}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{df}{dx}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} =$$

$$= \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\frac{df}{dy}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0$$

$\Rightarrow$ f has partial derivatives at $(0,0)$

⑤ Find the gradient of f at point a

a) $f(x,y) = e^{-x} \sin(x+2y)$, $a = (0, \pi/4)$

$$\frac{df}{dx}(x,y) = \frac{d}{dx}(e^{-x}) \sin(x+2y) + e^{-x} \frac{d}{dx}(\sin(x+2y))$$

$$= -e^{-x} \sin(x+2y) + e^{-x} \cos(x+2y)$$

$$\frac{df}{dx}(x,y) = 0 \cdot \sin(x+2y) + e^{-x} \cos(x+2y)$$

$$= e^{-x} \cdot \cos(x+2y)$$

$$a = (0, \tfrac{\pi}{4}) \Rightarrow \frac{df}{dx} = -1 \sin\left(\frac{\pi}{2}\right) + 1 \cdot \cos\left(\frac{\pi}{2}\right) = -1$$

$$\frac{df}{dy} = 1 \cdot \cos\frac{\pi}{2} = 0$$

$$\nabla f(0, \tfrac{\pi}{4}) = (-1, 0)$$

b) $f(x,y) = \arctan\left(\frac{y}{x}\right)$, $a = (1,1)$

- $\frac{df}{dx} = \frac{\left(\frac{y}{x}\right)'}{1+\left(\frac{y}{x}\right)^2} = \frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \boxed{\frac{-y}{x^2+y^2}}$

$\left(\frac{y}{x}\right)' = \frac{-y \cdot x'}{x^2} = \frac{-y}{x^2} = -\frac{y}{x^2}$

- $\frac{df}{dy} = \frac{\left(\frac{y}{x}\right)'}{1+\left(\frac{y}{x}\right)^2} = \frac{1}{x\left(1+\left(\frac{y}{x}\right)^2\right)} = \boxed{\frac{x}{x^2+y^2}}$

$\left(\frac{y}{x}\right)' = \frac{y' \cdot x}{x^2} = \frac{x}{x^2} = \frac{1}{x}$

$\nabla f(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

c) $f(x,y,z) = e^{xyz}$ $a = (0,0,0)$

$(e^{xyz})' = (xyz)' \cdot e^{xyz}$

$\frac{df}{dx} = e^{xyz} \cdot yz$

$$\frac{df}{dy} = e^{xyz} \cdot xz$$

$$\frac{df}{dz} = e^{xyz} \cdot xy$$

$$\nabla f(0,0,0) = (0,0,0)$$

d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$$\sqrt{x^2+y^2+z^2} = \frac{(x^2+y^2+z^2)'}{2\sqrt{x^2+y^2+z^2}}$$

$$\frac{df}{dx} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{df}{dy} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{df}{dz} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Curs 10

Directional derivatives

= arată cum o funcție $f(x)$ se schimbă în direcția unui vector

$$D_v f(x) = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$$

ex: $f(x,y) = x^2 + y^2$, cu punctul $(1,1)$ și direcția $v = (1,0)$

$$\Rightarrow D_v f(x,y) = \lim_{h \to 0} \frac{f(x+hv_x, y+hv_y) - f(x,y)}{h}$$

I variantă cu calcule explicată

$$\frac{df}{dx} = 2x \; ; \quad \frac{df}{dy} = 2y \; ;$$

$$\nabla f(x,y) = \left(\frac{df}{dx}, \frac{df}{dy}\right) = (2x, 2y)$$

$v = (1,0) \Rightarrow$ indică deplasarea numai pe axa x

$$D_v f(x,y) = \nabla f(x,y) \cdot v = (2x, 2y) \cdot (1,0)$$

$$= 2x \cdot 1 + 2y \cdot 0 = 2x$$

$$(x,y) = (1,1) \Rightarrow D_v f(1,1) = 2 \cdot 1 = 2$$

II varianta cu formula direct

$$D_v f(1,1) = \lim_{h \to 0} \frac{f(1+h,1) - f(1,1)}{h} \Longrightarrow$$

$$f(1+h,1) = (1+h)^2 + 1^2 = 2 + 2h + h^2$$

$$f(1,1) = 1^2 + 1^2 = 2$$

$$\Longrightarrow \lim_{h \to 0} \frac{(2+2h+h^2) - 2}{h} = \lim_{h \to 0} 2 + h = 2$$

$$\Rightarrow \boxed{D_v f(1,1) = 2}$$

Gradient

$$\nabla f(x) = \left(\frac{df}{dx_1}, \frac{df}{dx_2}, \dots \right)$$

Gradient descendent

= metodă de a găsi minimul funcției

$$\boxed{x_{k+1} = x_k - s_k \nabla f(x_k)}$$ x_k - punct curent

ex: $f(x,y) = x^2 + 3y^2$ și se alege punctul inițial $x_0 = (3,1)$

$$\nabla f(x,y) = \left(\frac{df}{dx}, \frac{df}{dy}\right) = (2x, 6y)$$

$$\nabla f(3,1) = (2\cdot 3, 6\cdot 1) = (6,6)$$

avem pasul $s = 0{,}5$

$$x_1 = (3,1) - 0{,}5\,(6,6) = (3,1) - (3,3)$$

$= (0,-2)$ $\Rightarrow$ $\boxed{x_1 = (0,-2)}$

$$\nabla f(0,-2) = (2\cdot 0, 6\cdot(-2)) = (0,-12)$$

$$x_2 = (0,-2) - 0{,}5\,(0,-12) = (0,-2+6) = (0,4)$$

. . . . . .

Taylor and Hessian expansion

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0)$$

$\boxed{H(x_0) = \text{matricea hessiană}}$

$\Downarrow$

matricea a derivatelor parțiale de ordin 2

$$H = \begin{bmatrix} \frac{d^2 f}{dx_1^2} & \frac{d^2 f}{dx_1 dx_2} \\ \frac{d^2 f}{dx_2 dx_1} & \frac{d^2 f}{dx_2^2} \end{bmatrix}$$

ex: $f(x, y) = x^2 + 3xy + y^2$

$$\frac{df}{dx} = 2x + 3y \quad ; \quad \frac{df}{dy} = 2y + 3x$$

$$\frac{d^2 f}{dx^2} = 2 \quad ; \quad \frac{d^2 f}{dy^2} = 2$$ => derivatele de ordin 2

$$\frac{d^2 f}{dx dy} = 3 \quad ; \quad \frac{d^2 f}{dy dx} = 3$$ => derivate mixte

$$\Rightarrow H = \begin{bmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx\,dy} \\ \frac{d^2 f}{dy\,dx} & \frac{d^2 f}{dy^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Taylor expansion aproximează o funcție $f(x,y)$ în jurul unui punct (x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + \nabla f(x_0,y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}^T H(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

ex: $f(x,y) = x^2 + xy + y^2 \qquad (x_0, y_0) = (1, 1)$

$f(1,1) = 3$

$$\nabla f(x,y) = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} \Rightarrow \nabla f(1,1) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$H(x,y) = \begin{bmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx\,dy} \\ \frac{d^2 f}{dy\,dx} & \frac{d^2 f}{dy^2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\frac{d^2f}{dx^2} = 2 \quad ; \quad \frac{d^2f}{dy^2} = 2$$

$$\frac{d^2f}{dxdy} = 1 \quad ; \quad \frac{d^2f}{dydx} = 1$$

$$f(x,y) \approx \underbrace{f(1,1)}_{3} + \underbrace{\nabla f(1,1) \cdot \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}}_{} + \underbrace{\frac{1}{2} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}^T H(1,1) \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}}_{}$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} \qquad \frac{1}{2} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}^T \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}}$$

$$\Downarrow$$

$$3(x-1) + 3(y-1) \qquad \begin{bmatrix} 2(x-1) + 1(y-1) \\ (x-1) + 2(y-1) \end{bmatrix}$$

$$(x-1)\left[2(x-1) + (y-1)\right] + (y-1)\left[(x-1) + 2(y-1)\right] =$$

$$= \frac{1}{2}\left[2(x-1)^2 + 2(x-1)(y-1) + 2(y-1)^2\right]$$

$\Rightarrow$ rezultat final $\Rightarrow$

$\Rightarrow f(x,y) \simeq 3 + 3(x-1) + 3(y-1) + (x-1)^2 +$
$+ (x-1)(y-1) + (y-1)^2$

Determinarea minimului și maximului

- gradientul $\nabla f(x,y) = 0$
- calculăm matricea Hessiană
- analizăm val. proprii a matricei
  - $\Rightarrow$ dacă val. propr $> 0 \Rightarrow (x_0, y_0) =$ minim local
  - $\Rightarrow$ dacă val. propr $< 0 \Rightarrow (x_0, y_0) =$ maxim local
  - $\Rightarrow val_1 < 0$ și $val_2 > 0 \Rightarrow (x_0, y_0) =$ saddle point

ex: $f(x,y) = x^2 + xy + y^2$

1 $\nabla f(x,y) = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix}$

$\begin{cases} 2x + y = 0 \\ x + 2y = 0 \end{cases}$ $y = -2x$ $\Rightarrow x - 4x = 0 \Rightarrow x = y = 0$

$(x_0, y_0) = (0, 0)$

② $H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (din calcul anterior)

$\det(H - \lambda I) = 0 \Rightarrow \underbrace{\underset{>0}{\lambda_1 = 3} \wedge \underset{>0}{\lambda_2 = 1}}$

(x_0, y_0) minim local

Seminar 10

direction of steepest ascent $\nabla f(x)$
direction of steepest descent $-\nabla f(x)$

① Compute $\frac{df}{dt}$ directly and using the chain rule:

a) $f(x,y) = \ln(x^2+y^2)$
$x = t$
$y = t^2$

$\frac{df}{dx} = \frac{2x}{(x^2+y^2)}$ $\qquad$ $\frac{df}{dy} = \frac{2y}{(x^2+y^2)}$

$$\frac{d}{dt} = \frac{2x}{(x^2+y^2)}\frac{dx}{dt} + \frac{2y}{(x^2+y^2)}\frac{dy}{dt}$$

$$= \frac{2x}{(x^2+y^2)}\frac{d(t)}{dt} + \frac{2y}{(x^2+y^2)}\frac{d(t^2)}{dt}$$

$$= \frac{2x}{(x^2+y^2)} \cdot 1 + \frac{2y}{(x^2+y^2)} \cdot 2t$$

$$\Rightarrow \frac{2t}{(t^2+t^4)} + \frac{4t^3}{(t^2+t^4)} = \frac{2t+4t^3}{t^2+t^4}$$

$$= \frac{t(2+4t^2)}{t(t+t^3)} = \frac{2(1+2t^2)}{t(1+t^2)}$$

b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$x = \cos t$

$y = \sin t$

$z = t$

$$\frac{df}{dx} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{df}{dz} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{df}{dy} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{d}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{dz}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} \cos t + \frac{y}{\sqrt{x^2+y^2+z^2}} \sin t$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2}} t \quad \Rightarrow$$

$$\Rightarrow \frac{\cos t^2 + \sin t^2 + t^2}{\sqrt{\cos t^2 + \sin t^2 + t^2}} = \sqrt{\cos t^2 + \sin t^2 + t^2}$$

② $f(u, v, w) = u^2 + v^2 - w$

$u(x, y, z) = x^2 y$

$v(x, y, z) = y^2$

$w(x, y, z) = e^{-xz}$

în mod direct:

$f(x, y, z) = x^4 y^2 + y^4 - e^{-xz}$

$$\frac{df}{dx} = 4x^3 y^2 + z e^{-xz}$$

$$\frac{df}{dy} = 2x^4y + 4y^3$$

$$\frac{df}{dz} = xe^{-xz}$$

chain rule:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} + \frac{df}{dv} \cdot \frac{dv}{dx} + \frac{df}{dw} \cdot \frac{dw}{dx}$$

$$f(u,v,w) = u^2 + v^2 - w$$

$$\frac{df}{du} = 2u$$

$$\frac{df}{dv} = 2v$$

$$\frac{df}{dw} = -1$$

$$\Rightarrow 2u \cdot 2xy + 2v \cdot 0 - 1 \cdot e^{-xz} =$$

$$= 2x^2y \cdot 2xy + z \cdot e^{-xz}$$

$$= 4x^3y^2 + z \cdot e^{-xz}$$

③ $f(x,y) = x^2 + xy$

a) $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x+y, x)$

$\nabla f(1,0) = (2,1)$

$-\nabla f(1,0) = (-2,-1) \Rightarrow$ steepest descent

b) $D_v (1,0) = \nabla f(1,0) \cdot v$ $\qquad v = (1,1)$

$\nabla f(x,y) = (2x+y, x) = (2,1)$

$D_v f(1,0) = (2,1) \cdot (1,1) = 2 \cdot 1 + 1 \cdot 1 = 3$

④ $\frac{1}{2} \|x\|^2$

I $\|x\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$

$f(x) = \frac{1}{2} (x_1^2 + x_2^2 + \dots + x_n^2)$

$$\nabla f(x) = (x_1, x_2 \dots, x_m) = x$$

$$\frac{df}{dx_k} = \frac{1}{2} \cdot 2 x_k = x_k$$

$$D_v f(x) = \langle \nabla f(x), v \rangle = \langle x, v \rangle$$

II $D_v f(x) = \lim_{h \to 0} \frac{f(x - hv) - f(x)}{h} =$

$$= \frac{1}{2} \|x + hv\|^2 - \frac{1}{2} \|x\|^2 =$$

$$\langle a+b, a+b \rangle = \|a\|^2 + \|b\| + 2\langle a, b \rangle$$

$$= \frac{1}{2} \frac{(\cancel{\|x\|^2} + \|hv\|^2 + 2\langle x, hv \rangle - \cancel{\|x\|^2})}{h}$$

$$= \frac{1}{2h} (|h|^2 \|v\|^2 + 2h - \langle x, v \rangle =$$

$$= \frac{1}{2h} (h \|v\|^2 + 2 \langle x, v \rangle)$$

⑥ Find the equation of the tangent line to the elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at an arbitrary point (x_0, y_0)

elipsa = curbă de nivel = $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\nabla f(x,y) = \left(\frac{df}{dx}, \frac{df}{dy}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right)$$

tangenta la curbă este $\perp$ pe gradient

$$\Rightarrow \nabla f(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{2x_0}{a^2} \\ \frac{2y_0}{b^2} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = 0$$

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) = 0$$

$$\frac{2x_0}{a^2} \cdot x - \frac{2x_0^2}{a^2} + \frac{2y_0}{b_2} \cdot y - \frac{2y_0^2}{b^2} = 0$$

$$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2}$$

din cerință $\frac{x_0^2}{a^2} + \frac{y_0^2}{y_0} = 1$

$$\Rightarrow \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} = 2 \Rightarrow \boxed{\frac{x_0}{a^2}x + \frac{y_0}{b^2}y = 1}$$

Curs 11 Optimization with constraints

Lagrangrange multipliers

optimization with constraints = maximizarea/minimizarea unei funcții $f(x)$ cu o restricție specifică

$\Downarrow$

optimizează $f(x)$ sub constrângerea $g(x) = c$

Lagrange method

$L(x,\lambda) = f(x) - \lambda(g(x) - c)$, unde:

- $f(x)$ -> funcția pe care o optimizăm
- $g(x) = c$ -> constrângerea
- λ = multiplicatorul (se calc. în proces)

Rezolvare: căutăm punctele critice $L(x,\lambda) = $

rezolvăm simultan
$$\begin{cases} 1.\ \nabla_x L = 0 & \text{(derivata parțială în fct. de x)} \\ 2.\ \dfrac{dL}{d\lambda} = 0 & \text{(respectarea constrângerii)} \end{cases}$$

ex: Se dorește maximizarea $f(x,y) = x + y$ cu restricția $x^2 + y^2 = 1$

1. $L(x,y,\lambda) = x + y + \lambda(x^2 + y^2 - 1)$

2. $\dfrac{dL}{dx} = 1 + 2x\lambda = 0$

rezolvăm =

$$\frac{\partial L}{\partial y} = 1 + 2y\lambda$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

$\Rightarrow$ 3. $x = -\frac{1}{2\lambda}$; $y = -\frac{1}{2\lambda}$ $\Rightarrow x = y$

$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$\Rightarrow x = y = \pm\frac{\sqrt{2}}{2}$

$\Rightarrow$ punctele critice $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

4. $f(x,y) = x + y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$f(x,y) = x + y = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

$f(x,y) = \sqrt{2}$ $\Rightarrow$ maxim

$f(x,y) = -\sqrt{2}$ $\Rightarrow$ minim

Seminar 11 (are legătură cu cursul 10)

① Find the second order Taylor polynomial for the function

$$T_2(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0) + \right.$$
$$\left. + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2\right]$$

a) $f(x,y) = \sin(x+2y)$ la $(0,0)$

$$f(x_0,y_0) = f(0,0) = \sin 0 = 0$$

$$\frac{\partial f}{\partial x} = \cos(x+2y) \qquad \frac{\partial f}{\partial x}(0,0) = \cos 0 = 1$$

$$\frac{\partial f}{\partial y} = 2\cos(x+2y) \qquad \frac{\partial f}{\partial x} = 2$$

$$\frac{d^2 f}{dx^2} = -\sin(x+2y) \qquad \frac{df}{dx}(0,0) = 0$$

$$\frac{d^2 f}{dy^2} = -4\sin(x+2y) \qquad \frac{df}{dx}(0,0) = 0$$

$$\frac{d^2 f}{dxdy} = -2\sin(x+2y) \qquad \frac{d^2 f}{dxdy} = 0$$

$$T_2(x,y) = 0 + 1\cdot x + 2y + \frac{1}{2}\left[0\cdot x^2 + 2\cdot 0\cdot xy + 0\cdot y^2\right]$$

$$T_2(x,y) = x + 2y$$

b) $f(x,y) = e^{x+y}$ f_0 $(0,0)$

$$f(0,0) = e^0 = 1$$

$$\frac{df}{dx} = e^{x+y} \qquad \frac{df}{dx} = 1$$

$$\frac{df}{dy} = e^{x+y} \qquad \frac{df}{dy} = 1$$

$$\frac{d^2 f}{dx^2} = e^{x+y} \qquad \frac{df}{dx} = 1$$

$$\frac{d^2 f}{dy^2} = e^{x+y} \qquad \frac{df}{dx} = 1$$

$$\frac{d^2 f}{dx\,dy} = e^{x+y} \qquad \frac{d^2 f}{dx\,dy} = 1$$

$$T_2(x,y) = 1 + 1\cdot x + 1\cdot y + \frac{1}{2}\left[1\cdot x^2 + 2\cdot 1\cdot xy + 1\cdot y^2\right]$$

$$= 1 + x + y + \frac{1}{2}(x+y)^2$$

c) $f(x,y) = \sin x \cdot \sin y$ at $(\pi/2, \pi/2)$

$$f(\pi/2, \pi/2) = 1\cdot 1 = 1$$

$$\frac{df}{dx} = \cos x \cdot \sin y \qquad \frac{df}{dx}(\pi/2, \pi/2) = 0\cdot 1 = 0$$

$$\frac{df}{dy} = \sin x \cdot \cos y \qquad \frac{df}{dy}(\pi/2, \pi/2) = 1\cdot 0 = 0$$

$$\frac{d^2 f}{dx^2} = -\sin x \cdot \sin y \qquad \frac{d^2 f}{dx^2}(\pi/2, \pi/2) = -1 \cdot 1 = -1$$

$$\frac{d^2 f}{dy^2} = \sin x(-\sin y) \qquad \frac{d^2 f}{dy^2}(\pi/2, \pi/2) = 1 \cdot (-1) = -1$$

$$\frac{d^2 f}{dx\,dy} = \cos x \cdot \cos y \qquad \frac{d^2 f}{dx\,dy}(\pi/2, \pi/2) = 0$$

$$T_2(x,y) = 1 + 0 \cdot (x - \pi/2) + 0 \cdot (y - \pi/2) + \frac{1}{2}\left[-1 \cdot (x - \pi/2)^2 + \right.$$
$$\left. + 2 \cdot 0 \cdot (x - \pi/2)(y - \pi/2) - 1\,(y - \pi/2)^2\right]$$
$$= 1 - \frac{1}{2}\left[(x - \pi/2)^2 + (y - \pi/2)^2\right]$$

d) $f(x,y) = e^{-(x^2+y^2)}$ at $(0,0)$

$$f(0,0) = 1$$

$$\frac{df}{dx} = -2x\, e^{-(x^2+y^2)} \qquad \frac{df}{dx}(0,0) = 0$$

$$\frac{df}{dy} = -2y\, e^{-(x^2+y^2)} \qquad \frac{df}{dy}(0,0)=0$$

$$\frac{d^2f}{dx^2} = (4x^2-2)e^{-(x^2+y^2)} \qquad \frac{d^2f}{dx^2}(0,0)=-2$$

$$\frac{d^2f}{dy^2} = (4y^2-2)\, e^{-(x^2+y^2)} \qquad \frac{d^2f}{dy^2}(0,0)=-2$$

$$\frac{d^2f}{dx\,dy} = 4xy\, e^{-(x^2+y^2)} \qquad \frac{d^2f}{dx\,dy}(0,0)=0$$

$$T_2(x,y) = 1+0\cdot x+0\cdot y+\frac{1}{2}\left[-2\cdot x^2+2\cdot 0\cdot xy-2y^2\right]$$
$$= 1-x^2-y^2$$

② Compute the Hessian matrix and its eigenvalues

!

$$H = \begin{bmatrix} \frac{d^2f}{dx^2} & \frac{d^2f}{dx\,dy} \\ \frac{d^2f}{dy\,dx} & \frac{d^2f}{dy^2} \end{bmatrix}$$

a) $f(x,y) = (y-1)e^x + (x-1)e^y$ la $(0,0)$

$$\frac{df}{dx} = (y-1)e^x + e^y$$

$$\frac{d^2f}{dx^2} = (y-1)e^x \quad \Rightarrow -1$$

$$\frac{df}{dy} = e^x + (x-1)e^y$$

$$\frac{d^2f}{dy^2} = (x-1)e^y \quad \Rightarrow -1$$

$$\frac{d^2f}{dx\,dy} = \frac{d}{dy}\left((y-1)e^x + e^y\right) = e^x \quad \Rightarrow 1$$

$$\frac{d^2f}{dy\,dx} = \frac{d}{dx}\left((x-1)e^y + e^x\right) = e^y \quad \Rightarrow 1$$

$$H = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

eigenvalues $\det(H - \lambda I_2) = \begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}$

$= (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 =$

$= -2\lambda + \lambda^2 = \lambda(\lambda - 2) = 0$

$\lambda_1 = 0 \quad \lambda_2 = -2$

b) $f(x,y) = \sin x \cos y$ at $(\pi/2, 0)$

$\frac{df}{dx} = \cos x \cdot \cos y$

$\frac{d^2f}{dx^2} = -\sin x \cdot \cos y \quad \Rightarrow -1$

$\frac{df}{dy} = \sin x\,(-\sin y)$

$\frac{d^2f}{dy^2} = -\sin x \cos y \quad \Rightarrow -1$

Atenție la calcul ptr. derivata mixtă

$\frac{d^2 f}{dx\,dy}$ = calculăm $\frac{df}{dy}$ în funcție de x

$\frac{d^2 f}{dy\,dx}$ = calculăm $\frac{df}{dx}$ în funcție de y

$$\frac{d^2 f}{dx\,dy} = \cos x\,(-\sin y) = -\cos x \sin y$$

$\Rightarrow 0$

$$\frac{d^2 f}{dy\,dx} = \cos x\,(-\sin y) = -\cos x \sin y$$

$\Rightarrow 0$

$$H = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

eigenvalues $\det(H - \lambda I_2) = \begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 \quad \Rightarrow \quad \lambda = -1$

③ Find and classify the critical points:

a) $f(x,y) = x^3 - 3x + y^2$

$\nabla f(x,y) = (3x^2 - 3, 2y)$

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y = 0 \end{cases} \Rightarrow y = 0 \Rightarrow x = \pm 1$$

$$\left.\begin{aligned} (x_0, y_0) &= (1, 0) \\ (x_0, y_0) &= (-1, 0) \end{aligned}\right\} \text{critical points}$$

$$H(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \qquad H(x,y) = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$$

$$H(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$ $\Rightarrow$ Hessiană diagonala

Hessiana diagonală { - pe diagonala principală

elem. nenule
- pe diagonala secundară doar 0

! când avem o Hessiană diagonală nu
• tb. calculate eigenvalues luăm direct valorile de pe diagonala principală

deci $\lambda_1 = 6 > 0$ $\quad \lambda_2 = 2 > 0$ $\Rightarrow$ (1, 0) minim local

$$H(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_1 = -6 < 0 \quad \lambda_2 = 2 > 0$$

(-1, 0) saddle point

c) $f(x,y) = x^4 + y^4 - 4(x-y)^2$

$$\frac{df}{dx} = 4x^3 - 8(x-y)'(x-y) = 4x^3 - 8(x-y)$$

$$\frac{df}{dy} = 4y^3 - 8(y-x)$$

$$\left.\begin{cases} 4x^3 - 8x + 8y = 0 \\ 4y^3 - 8y + 8x = 0 \end{cases}\right\} \Rightarrow x = y = 0$$

$$\frac{d^2 f}{dx^2} = 12x^2 - 8 \qquad \Rightarrow -8$$

$$\frac{d^2 f}{dy^2} = 12y^2 - 8 \qquad \Rightarrow -8$$

$$\frac{d^2 f}{dx\,dy} = 8$$

$$\frac{d^2 f}{dy\,dx} = 8$$

$$H(0,0) = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \qquad \det(H(0,0) - \lambda I_2)$$

$$= (-8-\lambda)^2 - 64 = (8+\lambda)^2 - 64$$

$$= 64 + 16\lambda + \lambda^2 - 64 = 16\lambda + \lambda^2 = 0$$

$\Rightarrow \lambda(16+\lambda)=0 \Rightarrow \left.\begin{array}{l}\lambda=0\\ \lambda=-16\end{array}\right\}$ saddle point

b) $f(x,y)=x^3+y^3-6xy$

$\nabla f(x,y)=(3x^2-6y,\ 3y^2-6x)$

$$\left\{\begin{array}{l}3x^2-6y=0\\ 3y^2-6x=0\end{array}\right\} \Rightarrow (0,0)\ ;\ (2,2)$$

$$H=\begin{bmatrix}6x & -6\\ -6 & 6y\end{bmatrix}$$

$$H(0,0)=\begin{bmatrix}0 & -6\\ -6 & 0\end{bmatrix} \Rightarrow \begin{vmatrix}-\lambda & -6\\ -6 & -\lambda\end{vmatrix}=\lambda^2-36$$

$\Rightarrow \lambda=\pm 6 \Rightarrow (0,0)$ saddle point

$$H(\quad =\begin{bmatrix}12 & -6\\ -6 & 12\end{bmatrix} \Rightarrow \begin{vmatrix}12-\lambda & -6\\ -6 & 12-\lambda\end{vmatrix}=$$

$$= (12-\lambda)^2 - 36 = 144 - 24\lambda + \lambda^2 - 36$$

$$= \lambda^2 - 24\lambda + 108$$

$$\lambda = \frac{24 \pm \sqrt{576 - 432}}{2} \quad \begin{cases} 18 > 0 \\ 6 > 0 \end{cases} \left.\right\} \begin{matrix} (2,2) \\ \text{minim} \\ \text{local} \end{matrix}$$

d) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$

$$\nabla f(x,y,z) = \left(2x - y + 1,\ 2y - x,\ 2z - 2\right)$$

$$\begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \Rightarrow x = 2y \quad \Rightarrow \\ 2z - 2 = 0 \Rightarrow z = 1 \end{cases}$$

$$\Rightarrow \quad 4y - y + 1 \Rightarrow 3y = -1 \Rightarrow y = -\frac{1}{3}$$

$$x = -\frac{2}{3}$$

$$\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$$

Matricea Hessiană de 3x3

$$H(x,y,z) = \begin{bmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx\,dy} & \frac{d^2 f}{dx\,dz} \\ \frac{d^2 f}{dy\,dz} & \frac{d^2 f}{dy^2} & \frac{d^2 f}{dy\,dz} \\ \frac{d^2 f}{dz\,dx} & \frac{d^2 f}{dz\,dy} & \frac{d^2 f}{dz^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow \det(H - \lambda I) = (2-\lambda)^3 - 1 =$$

$$= (2-\lambda)(\lambda - 3)(\lambda - 1)$$

$$\left.\begin{matrix} \lambda_1 = 2 & \lambda_2 = 3 & \lambda_1 = 1 \\ > 0 & > 0 & > 0 \end{matrix}\right\} \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$$

minim local

Curs 12

Double integrals

$$\iint_A f(x,y)\,dx\,dy \quad \text{, unde} \begin{cases} A: \text{domeniu pe care integrăm} \\ dxdy: \text{elementele mici de arie} \end{cases}$$

Fubini Theorem – ordinea integrării nu contează atât timp cât domeniul este „simplu"

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

! Dacă funcția integrată este <u>separabilă</u> ($f(x,y) = g(x)h(y)$), integrala devine mai ușoară

$$\iint_A f(x,y)\,dx\,dy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

ex: $$\iint_{[0,1]\times[0,2]} (x+y)\,dx\,dy =$$

$$= \iint x\,dx\,dy + \iint y\,dx\,dy$$

Schimbare de variabile

1. $(x,y) \longrightarrow (u,v)$

2. Calculează Jacobian matrix (vezi curs 9)

$$|\det(J)| = \left|\frac{d(x,y)}{d(u,v)}\right|$$

3. Rescrie integrala

$$\iint_{D} f(x,y)\,dx\,dy = \iint_{D^*} f(u,v)\,|\det(J)|\,du\,dv$$

Seminar 12

are legătură cu cursul 11+12

① Use Lagrange multipliers to find the extrema of the following functions subject to constraints

a) $f(x,y) = x^2+y^2$ cu restricția $x-y+1=0$

$$L(x,y,\lambda) = x^2+y^2+\lambda(x-y+1)$$

$$\frac{dL}{dx} = 2x+\lambda = 0$$

$$\frac{dL}{dy} = 2y-\lambda = 0$$

$$\frac{dL}{d\lambda} = x-y+1 = 0$$

$$\begin{cases} 2x+\lambda = 0 \\ 2y-\lambda = 0 \\ x-y+1 = 0 \end{cases} \Rightarrow \begin{cases} 2x+2y = 0 \\ x-y+1 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x+y=0 \\ x-y=-1 \end{cases} \Rightarrow 2y=1 \Rightarrow y=\frac{1}{2}$$

$$\Rightarrow x=-1+y \Rightarrow x=-1+\frac{1}{2}$$

$$\Rightarrow \frac{-2+1}{2}=-\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$f(x,y)=x^2+y^2=\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

b) $f(x,y)=(x+y)^2$ cu restricția $x^2+y^2=1$

$$L(x,y,z)=x^2+2xy+y^2+\lambda(x^2+y^2-1)$$

$$\frac{dL}{dx}=2x+2y+2x\lambda$$

$$\frac{dL}{dy}=2x+2y+2y\lambda$$

$$\frac{dL}{d\lambda}=x^2+y^2-1$$

$$\begin{cases} 2x+2y+2x\lambda = 0 \\ 2x+2y+2y\lambda = 0 \\ x^2+y^2-1 = 0 \end{cases} \Rightarrow \left.\begin{matrix} 2(x+y)+2x\lambda = 0 \\ 2(x+y)+2y\lambda = 0 \end{matrix}\right\}$$

$$\Rightarrow \begin{cases} 2x\lambda - 2y\lambda = 0 \\ x^2+y^2-1 = 0 \end{cases} \Rightarrow \quad x=y \quad \text{sau } \lambda = 0$$

$$\Downarrow$$

nu bun

$$x = y = \pm \frac{1}{\sqrt{2}}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 4$$

c) $f(x,y,z) = x+2y+3z$ cu const. $x^2+y^2+z^2=1$

$$L(x,y,z,\lambda) = x+2y+3z+\lambda(x^2+y^2+z^2-1)$$

$$\frac{dL}{dx} = 1 + 2x\lambda \qquad \frac{dL}{dz} = 3 + 2z\lambda$$

$$\frac{dL}{dy} = 2 + 2x\lambda \qquad \frac{dL}{d\lambda} = x^2+y^2+z^2-1$$

$$\left.\begin{cases} 1+2x\lambda = 0 \Rightarrow \lambda = \frac{1}{2x} \\ 2+2y\lambda = 0 \Rightarrow \lambda = -\frac{1}{y} \\ 3+2z\lambda = 0 \Rightarrow \lambda = -\frac{3}{2z} \\ x^2+y^2+z^2-1=0 \end{cases}\right\} \Rightarrow$$

$$\Rightarrow x = y \quad \text{și} \quad z = 3x \quad \Rightarrow$$

$$\Rightarrow x = y = \pm\frac{1}{\sqrt{11}}, \quad z = \pm\frac{3}{\sqrt{11}}$$

$$\left(\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}\right) \qquad \left(-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f(x,y,z) = \frac{12}{\sqrt{11}} \qquad f(x,y,z) = -\frac{12}{\sqrt{11}}$$

d) $f(x,y,z) = 2x^2 + y^2 + 3z^2$ $\qquad x^2 + y^2 + z^2 = 1$

$$L(x,y,z,\lambda) = 2x^2 + y^2 + 3z^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{dL}{dx} = 4x + 2x\lambda \qquad \frac{dL}{dz} = 6z + 2z\lambda$$

$$\frac{dL}{dy} = 2y + 2y\lambda \qquad \frac{dL}{d\lambda} = x^2 + y^2 + z^2 - 1$$

$$\begin{cases} 4x + 2x\lambda = 0 \\ 2y + 2y\lambda = 0 \\ 6z + 2z\lambda = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases}$$

etc. => nu mai stau să calculez atât

② Find the minimum value of $\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ subject to the following constraints

a) $x_1 + x_2 + x_3 = 3$

$$L(x_1, x_2, x_3, \lambda) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{dL}{dx_1} = x_1 + \lambda \qquad \frac{dL}{dx_3} = x_3 + \lambda$$

$$\frac{dL}{dx_2} = x_2 + \lambda \qquad \frac{dL}{d\lambda} = x_1 + x_2 + x_3 - 3$$

$$\begin{cases} x_1 + \lambda = 0 \\ x_2 + \lambda = 0 \\ x_3 + \lambda = 0 \\ x_1 + x_2 + x_3 = 3 \end{cases} \Rightarrow -3\lambda = 3 \Rightarrow \lambda = -1$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = 1$$

$$f(1,1,1) = \frac{1}{2}(1+1+1) = \frac{3}{2} \Rightarrow$$

$\Rightarrow \frac{3}{2}$ = val. min la pct. $(1,1,1)$

b) $x_1 + x_2 + x_3 = 3$

$x_1 + 2x_2 + 3x_3 = 12$

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = \frac{1}{2}(x_1 + x_2 + x_3) + \lambda_1(x_1 + x_2 + x_3 - 3) + \lambda_2(x_1 + 2x_2 + 3x_3 - 12)$$

$$\frac{dL}{dx} = x_1 + \lambda_1 + \lambda_2 \qquad \frac{dL}{dx_3} = x_3 + \lambda_1 + \lambda_3$$

$$\frac{dL}{dx_2} = x_2 + \lambda_1 + 2\lambda_2 \qquad \frac{dL}{d\lambda_1} = x_1 + x_2 + x_3 - 3$$

$$\frac{dL}{d\lambda_2} = x_1 + 2x_2 + 3x_3 - 12$$

$$\begin{cases} x_1 + \lambda_1 + \lambda_2 = 0 \\ x_2 + \lambda_1 + 2\lambda_2 = 0 \\ x_3 + \lambda_1 + \lambda_3 = 0 \\ x_1 + x_2 + x_3 = 3 \\ x_1 + 2x_2 + 3x_3 = 12 \end{cases}$$

etc. again nu mai calculez atât

③ Compute the double integrals

a) $\iint_R \cos x \cdot \sin y \, dx\, dy \quad R = [0, \pi/2] \times [0, 1]$

$$\left.\int_0^{\pi/2}\int_0^{\pi/2} \underbrace{\cos(x)\cdot\sin(y)}_{\text{este separabilă}}\, dx\, dy \right\} \Rightarrow$$

$$\Rightarrow \int_0^{\pi/2} \cos x\, dx \cdot \int_0^{\pi/2} \sin y\, dy$$

$$= \sin x \Big|_0^{\pi/2} \cdot (-\cos y)\Big|_0^{\pi/2} = 1\cdot 1 = 1$$

b) $\bullet \iint_R \frac{1}{(x+y)^2}\, dx\, dy \qquad R = [1,2]\times[0,1]$

$\Downarrow$

între $[1,2]$ variază x

între $[0,1]$ variază y

• integrăm mai întâi după x (y rămâne fix)

$$\int_1^2 \frac{1}{(x+y)^2}\, dx \qquad \begin{array}{l} u = x+y \\ du = dx \end{array} \qquad \begin{array}{l} x=1 \Rightarrow u = 1+y \\ x=2 \Rightarrow u = 2+y \end{array}$$

$$\int_{1+y}^{2+y} \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_{1+y}^{2+y}$$

$$= -\frac{1}{2+y} + \frac{1}{1+y} = \frac{1}{1+y} - \frac{1}{2+y}$$

• acum integrăm după y

$$\int_0^1 \left(\frac{1}{1+y} - \frac{1}{2+y} \right) dy$$

$$= \int_0^1 \frac{1}{1+y}\, dy - \int_0^1 \frac{1}{2+y}\, dy = \ln(1+y) \Big|_0^1 -$$

$$- \ln(2+y) \Big|_0^1 = \ln(2) - \ln(1) - \ln(3) + \ln(2)$$

$$= 2\ln(2) - \ln(3)$$

• $\iint_R y e^{xy}$ $\qquad R = [1,2] \times [0,1]$

$$\int_0^1 \int_1^2 y e^{xy}\, dx\, dy$$

integrăm mai întâi după x (y rămâne fix)

$$\int_1^2 \underbrace{y}_{\text{const}} e^{xy}\,dx \Rightarrow y\int_1^2 e^{xy}\,dx \Rightarrow$$

$u = xy$ $\qquad x=1 \Rightarrow u = y$

$dx = \frac{dy}{y}$ $\qquad x=2 \Rightarrow u = 2y$

$$\Rightarrow y\cdot\frac{1}{y}\int_y^{2y} e^u\,du = e^{2y} - e^y$$

$$\Rightarrow \int_1^2 y e^{xy}\,dx = y\left(e^{2y} - e^y\right)$$

acum integrăm după y

$$\int_0^1 y\left(e^{2y} - e^y\right)dy$$

$$\underbrace{\int_0^1 y e^{2y}\,dy}_{\mathrm{I}} - \underbrace{\int_0^1 y e^{y}\,dy}_{\mathrm{II}}$$

I

$u = 2y$ $\qquad y = 0 \Rightarrow u = 0$

$du = 2dy \Rightarrow dy = \frac{du}{2}$ $\qquad y = 1 \Rightarrow u = 2$

$$\int_0^1 ye^{2y}\,dy = \int_0^2 \frac{u}{2} e^u \cdot \frac{1}{2}\,du = \frac{1}{4}\int_0^2 ue^u\,du$$

$$\xrightarrow{\text{integrare prin părți}} \frac{1}{4}\left(ue^u \Big|_0^2 - \int_0^2 e^u\,du \right)$$

$$= \frac{1}{4}\left(ue^u - e^u\right)\Big|_0^2 = \frac{1}{4}(e^2+1)$$

II

$u = y$

$du = dy$

$$\int_0^1 ye^y \xrightarrow[\text{părți}]{\text{integrare prin}} ye^y\Big|_0^1 - \int_0^1 e^y\,dy$$

$$= ye^y - e^y\Big|_0^1 = e - 2$$

$$\int_0^1 ye^{2y} - ye^{y} = \frac{1}{4}(e^2+1) - (e-2)$$

$$= \frac{e^2}{4} + \frac{1}{4} - e + 2$$

(5) a) $$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$$

outer integral: y goes from 0 to 1
inner integral: x goes from y to 1

$\Rightarrow$ $0 \le y \le 1$
$y \le x \le 1$

după cerință $\Rightarrow$ reverse the order

$\Rightarrow$ x goes from 0 to 1
y goes from 0 to x
$\Rightarrow$

$$\Rightarrow \int_0^1 \int_0^x \sin(x^2)\, dy\, dx$$

$$\int_0^x \sin(x^2)\, dy = y \sin(x^2)\Big|_0^x = x \sin x^2$$

$$\Rightarrow \int_0^1 x \sin(x^2)\, dx$$

$$\left.\begin{aligned} u &= x^2 \\ du &= 2x\, dx \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 0 \quad u = 0 \\ x &= 1 \quad u = 1 \end{aligned}$$

$$\Rightarrow \frac{1}{2}\int_0^1 \sin(u)\, du = \frac{1}{2}(-\cos(u))\Big|_0^1$$

$$= \frac{1}{2}(1 - \cos(1))$$

b) $\displaystyle\int_0^1\int_x^1 e^{y^2}\, dy\, dx$

outer integral : x goes from 0 to 1

inner integral : y goes from x to 1

$$\left.\begin{aligned} 0 &\le x \le 1 \\ x &\le y \le 1 \end{aligned}\right\}$$ din cerință $\Rightarrow$ reverse this order

$\Rightarrow$ y goes from 0 to 1
x goes from 0 to y $\}$

$$\Rightarrow \int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

$$\int_0^y e^{y^2}\, dy \qquad \begin{aligned} u &= y^2 \\ du &= 2y\, dy \end{aligned} \qquad \begin{aligned} y=0 \quad u=0 \\ y=1 \quad u=1 \end{aligned}$$

$$\frac{1}{2}\int_0^1 e^u\, du = \frac{1}{2} e^u \Big/_0^1 = \frac{1}{2}(e-1)$$

6 Compute the following integrals by a change of variables

a) $\displaystyle\iint_\Delta e^{\frac{x-y}{x+y}}\, dx\, dy$

$\Delta = \{(x,y) \in \mathbb{R}^2 / x \geq 0, y \geq 0, x+y \leq 1\}$

use : $u = x - y$, $v = x + y$

Steps

1. transform Δ
2. compute the Jacobian matrix
3. rewrite the integral

1. $x \geq 0$

$y \geq 0$

$x + y \leq 1$

$u = x - y \rightarrow x = \frac{u+v}{2}, \ y = \frac{v-u}{2}$

$x + y \leq 1 \rightarrow v \leq 1$

$x \geq 0 \rightarrow u + v \geq 0 \rightarrow u \geq -v$

$y \geq 0 \rightarrow v - u \geq 0 \Rightarrow u \leq v$

$\Rightarrow \Delta : \{ -v \leq u \leq v, \ 0 \leq v \leq 1 \}$

Jacobian matrix

$$J = \begin{bmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{bmatrix}$$

$$x = \frac{u+v}{2} \quad , \quad y = \frac{v-u}{2}$$

$$\frac{dx}{du} = \frac{1}{2} \qquad \frac{dy}{du} = -\frac{1}{2}$$

$$\frac{dx}{dv} = \frac{1}{2} \qquad \frac{dy}{dv} = \frac{1}{2}$$

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(J) = \frac{1}{2} \cdot \frac{1}{2} - \left(-\frac{1}{2}\right)\frac{1}{2} = \frac{1}{2}$$

$$\iint_{\Delta} \frac{x-y}{e^{x+y}}\, dx\, dy = \iint_{\Delta_{uv}} \frac{u}{e^{v}} \cdot |\det(J)|\, du\, dv$$

$$= \iint_{\Delta_{uv}} \frac{u}{e^{v}} \cdot \frac{1}{2}\, du\, dv = \frac{1}{2} \iint_{\Delta_{uv}} \frac{u}{e^{v}}\, du\, dv$$

$$\Rightarrow \int_{-v}^{v} u\, du = \frac{u^2}{2} \Big|_{-v}^{v} = 0$$