Coding theory

Seminar 12

(n, k)
- n = # of digits of the code
- k = the message (last k digits)

1. **parity check** = the first digit is the sum of the message (last k digits)

$(3, 2)$ - parity check

110	010	001	111	101	000
✓	✗	✗	✗	✓	✓

repeating code = first two digits should repeat the message (k)

(3,1) - repeating code ♡

111	011	101	010	000	001
✓	×	×	×	✓	×

(2) ! to see if polynomials (a) are code words for another polynomial (b) => împarți „a" la „b" => dacă ai rest atunci „a" $\boxed{\text{nu}}$ e code word, dacă restul $=0$ „a" este code word

$$
\begin{array}{l|l}
x^7+x^6+x^4+x^3+1 & x^4+x^3+x^2+1 \\
\cline{2-2}
x^7+x^6+x^5+x^3 & x^3+x \\
\cline{1-1}
x^5+x^4+1 & \\
x^5+x^4+x^3+x & \\
\cline{1-1}
\end{array}
$$

$x^3+x+1 \neq 0 \Rightarrow x^7+x^6+x^4+x^3+1$

nu este mesaj ptr p

$$
\begin{array}{l|l}
x^6 + x^3 + x^2 + x & x^4 + x^3 + x^2 + 1 \\ \cline{2-2}
x^6 + x^5 + x^4 + x^2 & x^2 + x \\ \cline{1-1}
x^5 + x^4 + x^3 + x & \\
x^5 + x^4 + x^3 + x & \\ \cline{1-1}
\quad 0 &
\end{array}
$$

$\Rightarrow x^6 + x^3 + x^2 + x$ este mesaj pt. p

③ 2^K - code words

(n, K) - n length of the code
- K length of the message

$(6, 3) \Rightarrow 2^3$ code words

the message = 3 digits

8 code words: $\{000, 001, 010, 011, 100, 101, 110, 111\}$

$m \cdot x^{n-K} \,|\, p$

iar restul se adaugă la $m \cdot x^{n-K}$

$$000 = 0\cdot 1 + 0\cdot x^1 + 0\cdot x^2 \Rightarrow m = 0$$

$$\Rightarrow \text{code} = 000\boxed{000}$$

$$001 = x^2 \Rightarrow m = x^2$$

$$m \cdot x^{n-k} = x^2 \cdot x^3 = x^5$$

$$\begin{array}{l|l} x^5 & x^3+x^2+1 \\ \hline x^5+x^4+x^2 & x^2+x+1 \\ \hline x^4+x^2 & \\ x^4+x^3+x & \\ \hline x^3+x^2+1 & \\ x^3+x^2+1 & \\ \hline / & \end{array}$$

$$V = x^5 \Rightarrow 000001$$

??? (vezi seminar postat)

nu înțeleg dacă trb să fac pt. toate 8 code words

update: trb pt. toate

4. <u>generator matrix G</u> $\left[\frac{P}{I_K}\right]$

<u>parity check matrix H</u> $\left[I_{n-K} \,\vdots\, P\right]$

$$H = \left[I_{5-3} \,\vdots\, P\right] = \left[I_2 \,\vdots\, P\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ o \\ d \\ e \\ w \\ o \\ r \\ d \end{pmatrix}$$

$(a, b, c) \in (000, 001, 010, 011, 100, 101, 110, 111)$

code words = 00000, 11001, 1 0

. . . .

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$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$G = \left(\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$m = 9 \quad K = 4$

$$\left[\, I_{m-k} \;\middle|\; P \,\right] = H$$

$$= \left(\begin{array}{ccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right)$$

we add c_2, c_6, c_9

$\Rightarrow d(\mathcal{C}) = 3$ = Hamming distance

- we can detect at most $d(\mathcal{C})-1$ errors
- error correcting $\frac{d(\mathcal{C})-1}{2}$

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$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a, b, c, d \in \{(1,1,0,1), (0,1,1,1), (0,0,0,0), (1,0,0,0)\}$

$1101 \Rightarrow (000101101)$

(7) $(4,1)$

$n=4$ $\qquad$ $p = 1 + x + x^2 + x^3$

$k=1$

2^k - code words $\qquad \{1, 0\}$

$m = 1 \qquad m = 0$

$m = 1 \Rightarrow m \cdot x^3 = x^3$

$$\begin{array}{r|l} x^3 & x^3 + x^2 + x + 1 \\ \hline x^3 + x^2 + x + 1 & 1 \\ \hline x^2 + x + 1 & \end{array}$$

$x^3 + x^2 + x + 1$

$\gamma(m) = (1111)$

$\gamma(m) = (0000)$

8 (7,3)-code

$n = 7$

$k = 3$

$$p = 1 + x^2 + x^3 + x^4$$

$2^3 \Rightarrow m \in \{000, 001, 010, 011, 100, 101, 110, 111\}$

$m = 001 \Rightarrow x^2 \cdot x^4$

$m = 010 \Rightarrow x$

$m = 011 \Rightarrow x + x^2$

Seminar 13

! To choose the coset leader among the vectors corresponding to a syndrome:

1) it has the lowest number of 1's

2) the 1's are clustered together as closely as possible

① $(63, 56)$ - code

$\underset{n}{63}$, $\underset{K}{56}$

i) $K = 56$ digits in the message

ii) $n - k = 63 - 56 = 7$ check digits

iii) Info rate $= \frac{k}{n} = \frac{56}{63} = \frac{8}{9}$

IV) number of syndrome

$$2^{n-k} = 2^{7} = 128$$

② $H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

Synds	000	001	010	011
Coset l	000000	001000	010000	000010

100	101	110	111
100000	000110	000100	000001

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$e = 000000$

$$\begin{array}{r} 101110 \; + \\ 000000 \\ \hline 101110 \end{array}$$

$$\boxed{m = 110}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$e = 000010$

$$\begin{array}{r} 011000 \; + \\ 000010 \\ \hline 011010 \end{array} \Rightarrow \boxed{m = 010}$$

00000 etc

③ $(7,4)$ - code

u_1, u_2, u_3 - check digits

u_4, u_5, u_6, u_7 - message digits

$$\begin{cases} u_1 = u_4 + u_5 + u_7 \\ u_2 = u_4 + u_6 + u_7 \\ u_3 = u_4 + u_5 + u_6 \end{cases}$$

$$\begin{array}{llcccc} & & u_4 & u_5 & u_6 & u_7 \\ e_1 & = & 1 & 0 & 0 & 0 \\ \end{array} \longrightarrow 1111000$$

$$e_2 = 0100 \longrightarrow 1010100$$

$$e_3 = 0010 \longrightarrow 0110010$$

$$e_4 = 0001 \longrightarrow 1100001$$

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$\Downarrow$

syndrome	e
000	0000000
001	0010000
010	0100000
011	0000010
100	1000000
101	0000100
110	0000001
111	0001000

$$\Rightarrow \quad H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad e = 0000000$$

$$\begin{array}{l} 0000000 \\ 0000111 \\ \hline 0000111 \end{array}$$

$$H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow e = 0001000$$

$$\begin{array}{r} 0001111 \; + \\ 0001000 \\ \hline 0000111 \end{array}$$

④ $m = \{01, 10, 11, 00\}$

000	0
001	1
010	1
011	0
100	1
101	0
110	0
111	1

???

(5) $H = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$

coset leaders and syndrome for the (7,4)-code with the parity check matrix H

S	e
000	0000000
001	0010000

010	0100000
011	0000001
100	1000000
101	0000010
110	0000100
111	0001000

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$$G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

S	coetlaclus
00	00000
01	01000
10	10000
11	00010

(8) $(7,3)$ - code

$$p = 1 + x^3 + x^4$$

$$m = 100 \Rightarrow g = 1 \cdot x^4$$

$$\begin{array}{l|l} x^4 & x^4 + x^3 + 1 \\ \hline x^4 + x^3 + 1 & 1 \\ \hline x^3 + 1 & \end{array}$$

$$x^3 + 1 = R \Rightarrow x^4 + x^3 + 1$$

$$\Rightarrow \boxed{1001100}$$

$$m = 010 \Rightarrow g = x \cdot x^4 = x^5$$

$$\begin{array}{l|l} x^5 & x^4 + x^3 + 1 \\ \hline x^5 + x^4 + x & x + 1 \\ \hline x^4 + x & \\ x^4 + x^3 + 1 & \\ \hline \end{array}$$

$x^3 + x + 1 = R \Rightarrow 1 + x + x^3 + x^5$

$\boxed{1101010}$

$m = 001 \Rightarrow g = x^2 \cdot x^4 = x^6$

$$\begin{array}{l|l} x^6 & x^4 + x^3 + 1 \\ \cline{2-2} x^6 + x^5 + x^2 & x^2 + x + 1 \\ \hline x^5 + x^2 & \\ x^5 + x^4 + x & \\ \hline x^4 + x^2 + x & \\ x^4 + x^3 + 1 & \\ \hline x^4 + x^3 + 1 & \end{array}$$

$1 + x + x^2 + x^3 + x^6$

$\boxed{1111001}$

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Sindr.	Cl
0000	00000000
0001	00010000
0010	00100000
0011	0011000
0100	010000
0101	01010000
0110	01100000
0111	1000000
1000	1000000
1001	0000100
1010	1010000
1011	0010100
1100	11000000

1101	00000010
1110	00001001
1111	00000001

7 $(3,1)$ - code

$p = 1 + x + x^2$

$m \in \{0, 1\}$

$m = 1 \Rightarrow 1 \cdot x^{3-1} = x^2$

$$\begin{array}{r|l} x^2 & x^2+x+1 \\ \cline{2-2} x^2+x+1 & 1 \\ \hline x+1 & \end{array}$$

$\Rightarrow x^2 + x + 1$

$\Rightarrow 111$

$G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$H = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Sym	CL
0 0	0 0 0
0 1	0 1 0
1 0	1 0 0
1 1	0 0 1