Seminar 0 - before having the first course
=> geometria de liceu

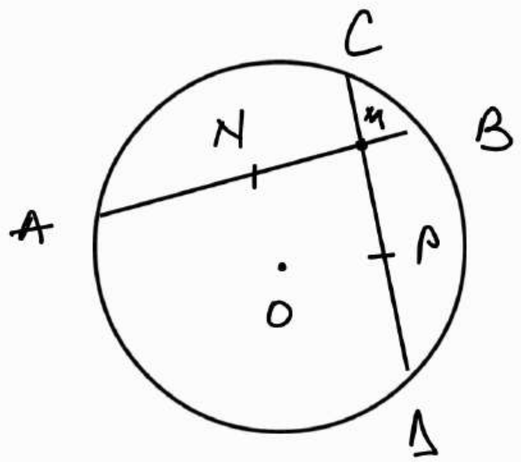
(1) $A, B, C, D \in$ circle of center O

$AB \perp CD$

N mijloc AB $\quad$ M intersection of 2 chords

P mijloc CD

Show that $MNOP$ rectangle



N mijloc $AB \Rightarrow ON \perp AB$

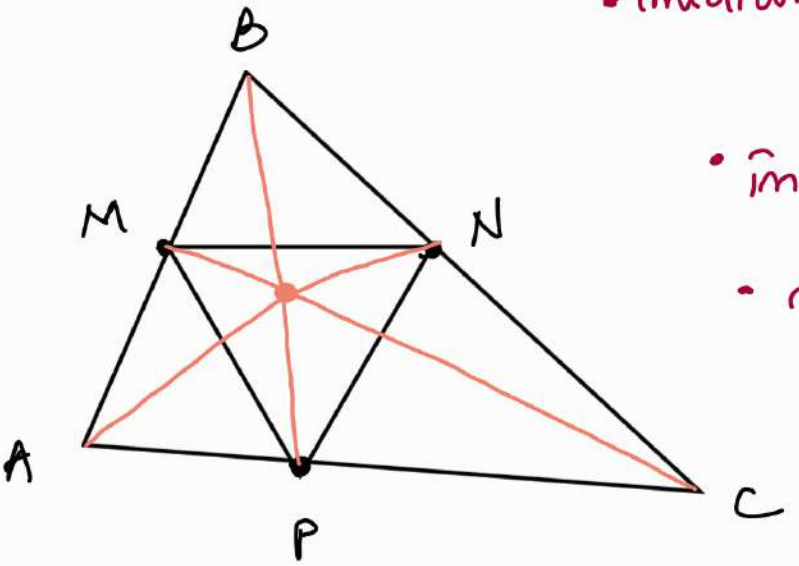
P mijloc $CD \Rightarrow OP \perp CD$

$AB \perp CD$

$\Rightarrow \left.\begin{array}{l} ON \perp OP \\ ON \parallel MP \\ OP \parallel NM \end{array}\right\}$

$\Rightarrow MNOP$ rectangle

② ΔABC



• medianele se intersectează în centrul de greutate
• înălțimile în orthocentru
• mediatoarele în centrul cercului circumscris

Midtriangle:

- ΔABC similar with ΔMNP
- each side of MNP is ∥ to ABC

• înălțimile din ΔABC devin mediatoare în MNP

• medianele din ΔABC sunt medianele din ΔMNP

(4) Area $(ABC) = 4 \cdot$ Area (MNP)

$$\text{Area}(ABC) \leq \frac{1}{3\sqrt{3}} \left(\frac{a+b+c}{2}\right)^2$$

Heron's formula: $\begin{cases} \sqrt{p(p-a)(p-b)(p-c)} \\ p = \dfrac{a+b+c}{2} \end{cases}$

inegalitatea medilor $\sqrt[3]{(p-a)(p-b)(p-c)} \leq \dfrac{(p-a)+(p-b)+(p-c)}{3}$

$$= \frac{3p-(a+b+c)}{3} = \frac{p}{3}$$

combine those two

$$\text{Area} \leq \sqrt{p \cdot \left(\frac{p}{3}\right)^3} = \sqrt{\frac{p^4}{27}} = \frac{p^2}{3\sqrt{3}}$$

$$p = \frac{a+b+c}{2}$$

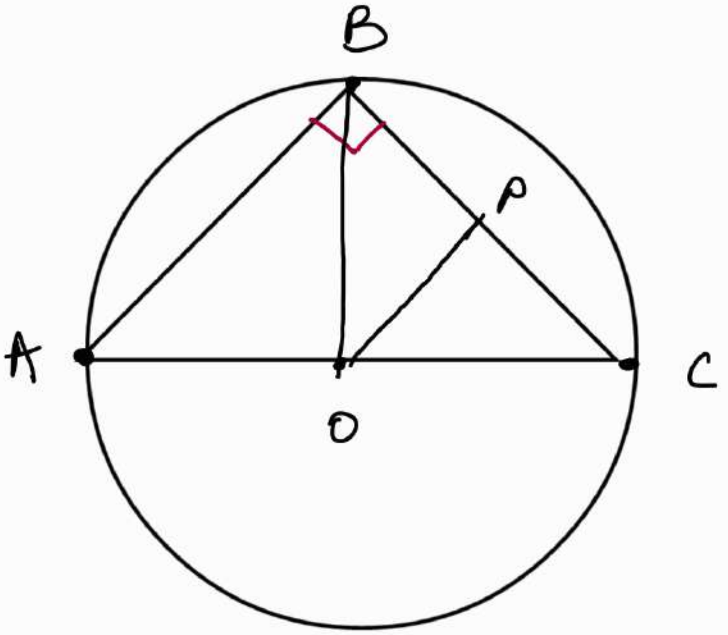
$$\text{Area} \leq \frac{1}{3\sqrt{3}} \left(\frac{a+b+c}{2}\right)^2$$

(5) Thales circle theorem

A, B, C $\in$ circle

[AC] is a diameter iff $\sphericalangle ABC = 90^\circ$

$\boxed{\text{I need to show } \sphericalangle ABC = 90^\circ}$



▌ ΔABC înscris în cerc și una dintre

• laturi este diametrul $\Leftrightarrow$ $\sphericalangle$ opus diametrului este de 90°

O mijloc AC $\Rightarrow$ BO = OC

$\left.\begin{array}{l}\text{P mijloc BC} \\ \text{O mijlo AC}\end{array}\right\}$ OP linie mijlocie

$\Rightarrow$ PO $\parallel$ AB

$PO \perp BC$ | $\Rightarrow AB \perp BC$

⑥ (Central angle theorem)

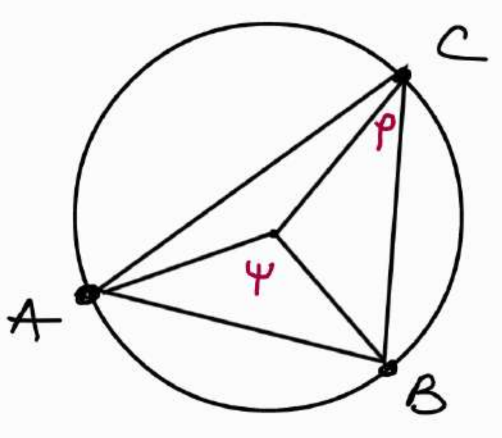
ΔABC triangle circumscris in cerc

$\varphi = \sphericalangle ACB$ subtins de $[AB]$

$\psi = \sphericalangle AOB$ este unghiul la centru

! subtins de coardă = subtended by the chord = unghiul este format din capetele coardei + alt vârf

! unghiul la centru = unghiul format din 2 raze ale cercului



$$\text{Dim}\begin{cases}\varphi = \sphericalangle ACB \text{ subtins de } [AB] \\ \psi = \sphericalangle AOB \text{ este unghiul la centru}\end{cases}$$

$\Rightarrow \psi = 2\varphi$

! unghiul la centru este de 2·(ori care alt unghi înscris pe aceeași arc)

Teorie curs I

angles -> acute $< 90^\circ$

-> right $= 90^\circ$

-> obtuse $> 90^\circ$ $< 180^\circ$

-> straight $= 180^\circ$

-> complementary $\hat{a} + \hat{b} = 90^\circ$

-> supplementary $\hat{a} + \hat{b} = 180^\circ$

midpoint formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

1.1 (A, B) and (C, D) are equidistant $\Leftrightarrow$ $[AB] \equiv [CD]$

1.2 equidistance is an equivalence relation

- reflexivity: $\left.\begin{array}{l}[AB] \equiv [A'B'] \\ [A'B'] \equiv [AB]\end{array}\right\} \Rightarrow [AB] \equiv [AB]$

- symmmetry: if $[AB] \equiv [CD]$ then $[CD] \equiv [AB]$

- transitivity: if $\left.\begin{array}{l}[AB] \equiv [CD] \\ [CD] \equiv [EF]\end{array}\right\} \Rightarrow [AB] \equiv [EF]$

1.5 (A, B) and (C, D) are equidirectional

=> they point in the same direction

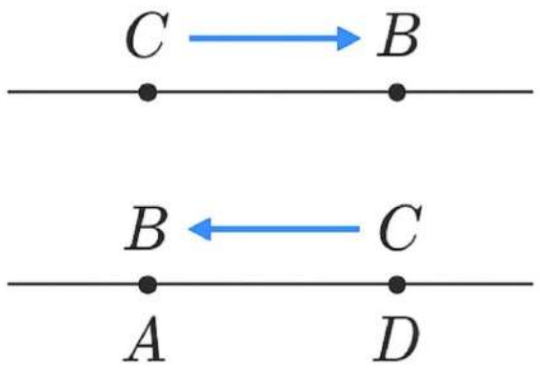
that is true in 3 situations

1. A, B, C, D are points => equidirectional by default

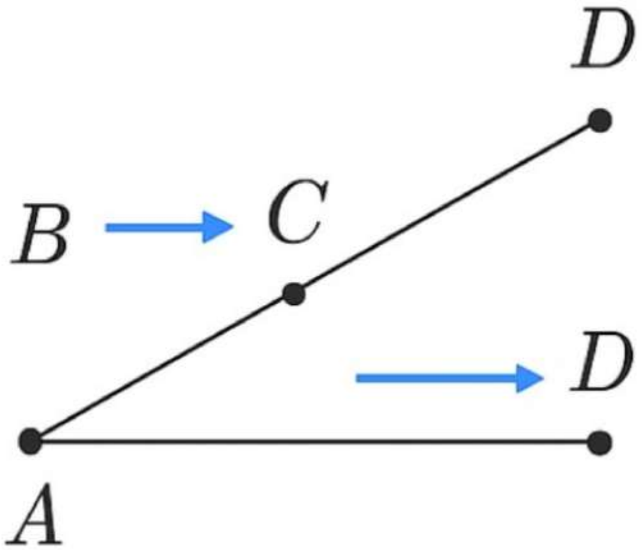
2. $A \neq B$; A, B, C collinear points

if $A = C$ and B, D are on the same side of A

if B, C are on the same side of A ; A, D are on opposite sides of C

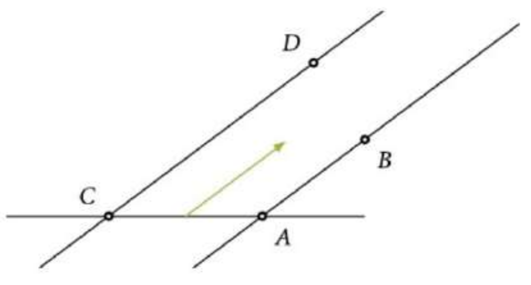


if B, C are on oposite sides of A, and A, D are on the same side of C



3. $A \neq B$, A, B, C not collinear

$AB \parallel CD$ parallel



1.7. equidirectional is an equivalence relation

(A, B) equidirectional (C, D)
(C, D) equidirectional (E, F)

to prove: (A, B) equidirectional (E, F)

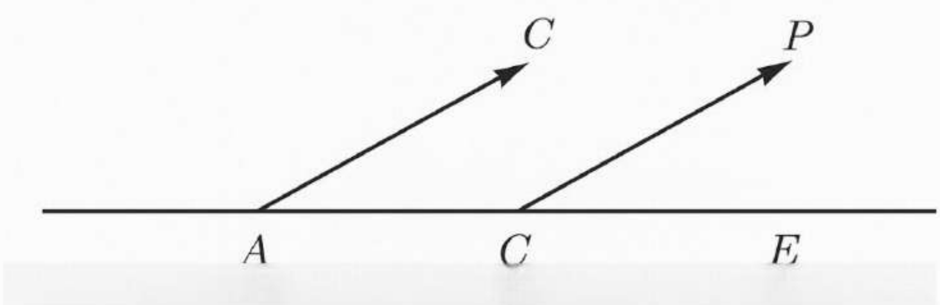
Case 1: $\left.\begin{array}{l} A = B \\ C = D \end{array}\right\}$ just points, no direction

Case 2: all the points are on the same line

for this you can find a point P such that: $\begin{cases} AB = AP \\ CD = CP \\ EF = EP \end{cases}$ => now we compare: (A, P), (C, P), (E, P)

A lies between C and E / E lies between C and A

=> (A, B) Δ (E, F)



Case 3: A, B, C, D collinear

Case 4: the segments have different directions

1.8 equivalence classes = ordered pairs that point in the same direction = DIRECTION

$|AB\rangle$ direction defined by (A,B) => it includes also other pairs equidirectional with (A,B)

simple: $|AB\rangle$ grupuri de point pairs care au aceeași direcție

$-|AB\rangle = |BA\rangle$

1.11 (A,B) & (C,D) = ordered pairs of points

1. $(B,A) \sim (D,C)$
2. $(A,B) \sim (C,D)$

} equipollent (same length, same direction)

3. ABDC parallelogram
4. $|AB\rangle = |CD\rangle$
   $[AB] = [CD]$

} equipollent

5. $|AB\rangle = |CD\rangle$
$|AB| = |CD|$

1.12 equipollence is an equivalence relation
- reflexive
- symmetric
- transitive

(just like the ones before)

1.13 an equivalence class of pairs (A,B) is a VECTOR

$$\vec{AB} = \{(X,Y) \text{ s.t } (X,Y) \sim (A,B)$$

also, (A,B) is a representative of $\vec{AB}$

! 2 vectori sunt egali dacă au aceeași direcție și lungime

$[AB] = |\vec{AB}|$ length of vector

1.18 the set of vectors V with addition is an abelian group

- asociativitate: $\overrightarrow{AB} = a$, $\overrightarrow{BC} = b$, $\overrightarrow{CB} = c$

$(a+b)+c = a(b+c) = \overrightarrow{AB}$

- the zero vector: $\vec{0} + \overrightarrow{AB} = \overrightarrow{AB}$

- inverse: $\overrightarrow{AB} + (-\overrightarrow{AB}) = \overrightarrow{AA} = 0$

- commutativity: $a+b = b+a$

1.19 scalar multiplication

1. $x > 0$ $\vec{a} \neq \vec{0}$ $\Rightarrow$ make it longer/shorter on the same direction

2. $x < 0$ $\vec{a} \neq \vec{0}$ $\Rightarrow$ scale it, but also flipping the direction

3. $x = 0$ / $\vec{a} = \vec{0}$ $\Rightarrow$ the zero vector

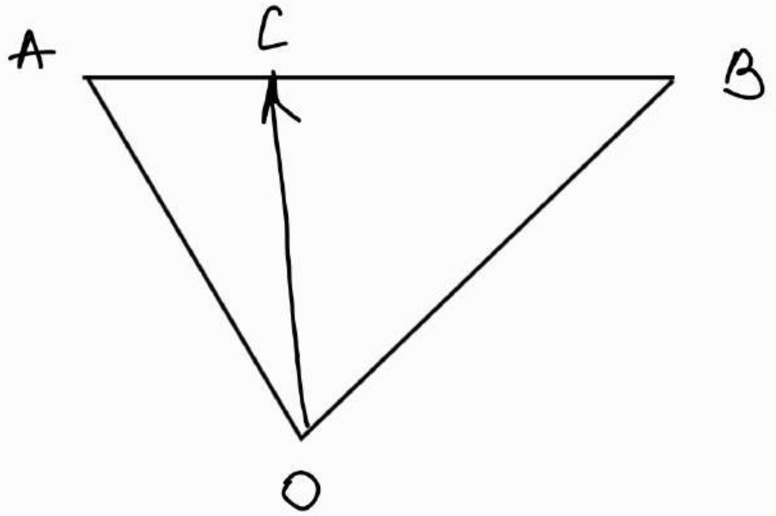
Proposition 1.21. Assume that a unit segment was chosen. For $\mathbf{a}, \mathbf{b} \in \mathbb{V}$ and $x, y \in \mathbb{R}$ we have

1. $0 \cdot \mathbf{a} = \vec{0}$.
2. $1 \cdot \mathbf{a} = \mathbf{a}$
3. $-1 \cdot \mathbf{a} = -\mathbf{a}$
4. $(x+y) \cdot \mathbf{a} = x \cdot \mathbf{a} + y \cdot \mathbf{a}$
5. $x \cdot (y \cdot \mathbf{a}) = (xy) \cdot \mathbf{a}$
6. $x \cdot (\mathbf{a} + \mathbf{b}) = x \cdot \mathbf{a} + x \cdot \mathbf{b}$

Seminar 1

1.1. Let $\lambda = |CA| : |CB|$ be the ratio in which the point $C \in [AB]$ divides the segment $[AB]$. Check that

$$\overrightarrow{OC} = \frac{1}{1+\lambda}\overrightarrow{OA} + \frac{\lambda}{1+\lambda}\overrightarrow{OB}.$$



$|AC| = \lambda\, |BC|$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \lambda\,\overrightarrow{BC} \Rightarrow$$

$$\Rightarrow \overrightarrow{OC} = \overrightarrow{OA} + \lambda(\overrightarrow{CO} + \overrightarrow{OB})$$

$$\overrightarrow{OC} + \lambda\overrightarrow{OC} = \overrightarrow{OA} + \lambda\overrightarrow{OB}$$

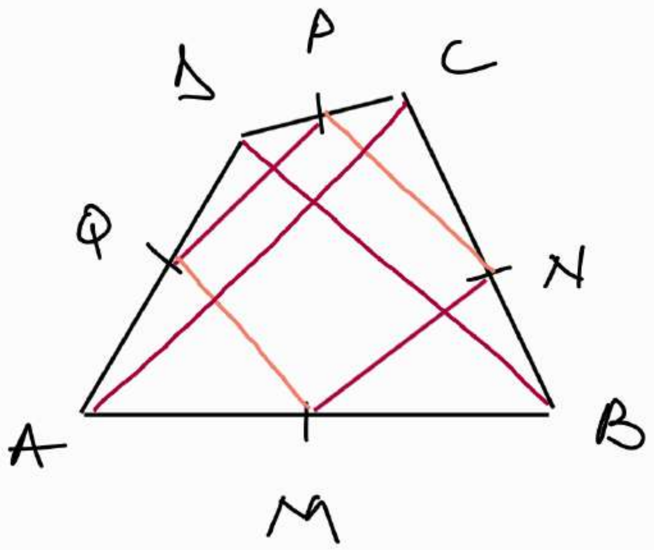
$$\overrightarrow{OC}(1+\lambda) = \overrightarrow{OA} + \lambda\overrightarrow{OB}$$

$$\overrightarrow{OC} = \overrightarrow{OA}\,(1+\lambda) + \frac{\lambda\ \overrightarrow{OB}}{1+\lambda}$$

$$\overrightarrow{OC} = \frac{1}{1+\lambda}\overrightarrow{OA} + \frac{\lambda}{1+\lambda}\overrightarrow{OB}$$

1.2. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN} + \overrightarrow{PQ} = 0.$$



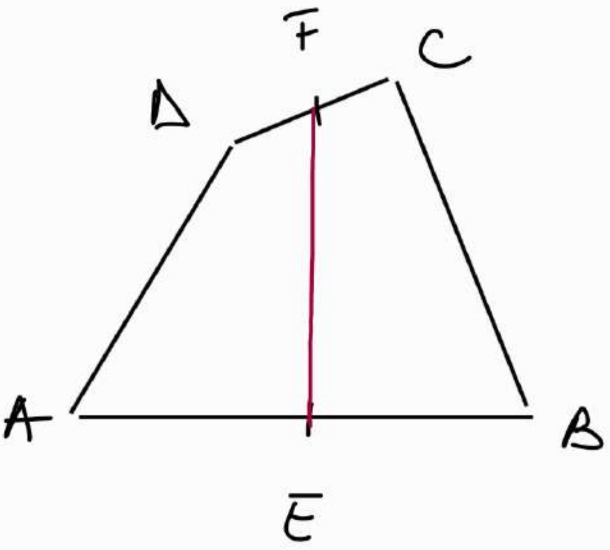
P midpoint DC
Q midpoin AD $\Big|$ => mid segment => QP ∥ AC, $QP = \frac{1}{2} AC$

M midpoint AB
N midpoint BC $\Big|$ => mid segment MN ∥ AC, $MN = \frac{1}{2} AC$

=> MN ∥ QP
MN = QP $\Big|$ => $\overrightarrow{MN} + \overrightarrow{PQ} = 0$

1.3. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC}).$$



E mp $AB \Rightarrow \overrightarrow{AE} = \overrightarrow{EB}$

F mp $CD \Rightarrow \overrightarrow{FD} = \overrightarrow{CF}$

$$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DF}$$
$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CF}$$

(+)

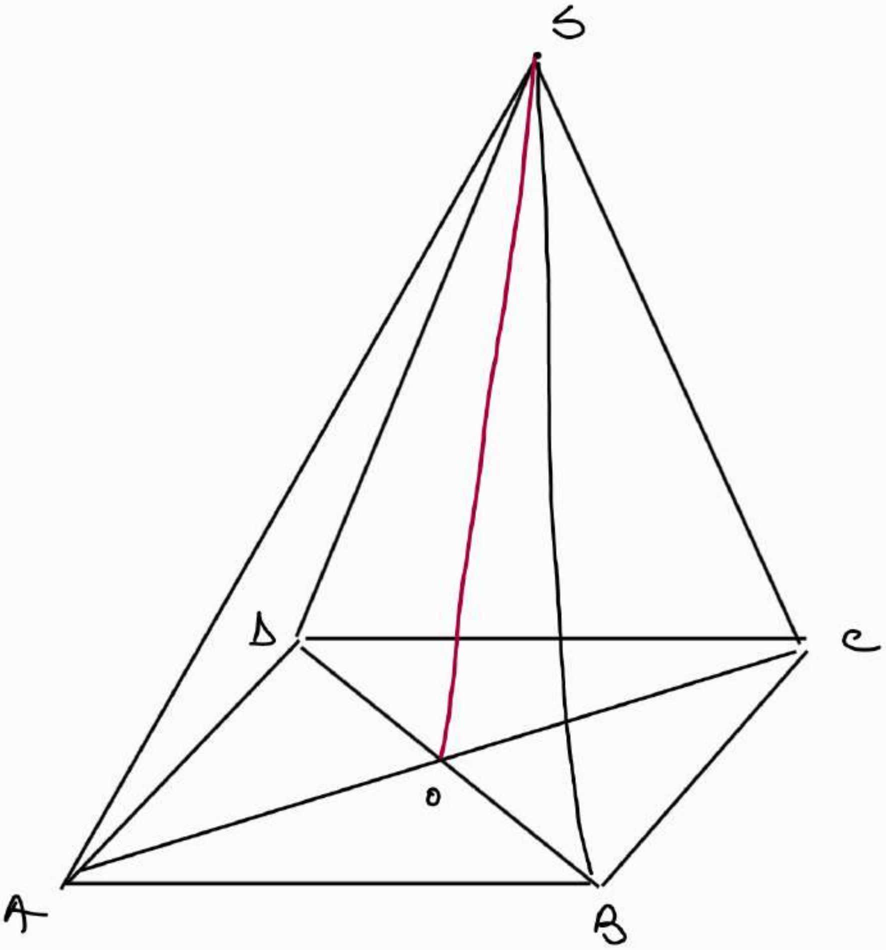
$$2\overrightarrow{EF} = \cancel{\overrightarrow{EA}} + \cancel{\overrightarrow{EB}} + \overrightarrow{AD} + \overrightarrow{BC} + \cancel{\overrightarrow{DF}} + \cancel{\overrightarrow{CF}}$$

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})$$

1.4. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.



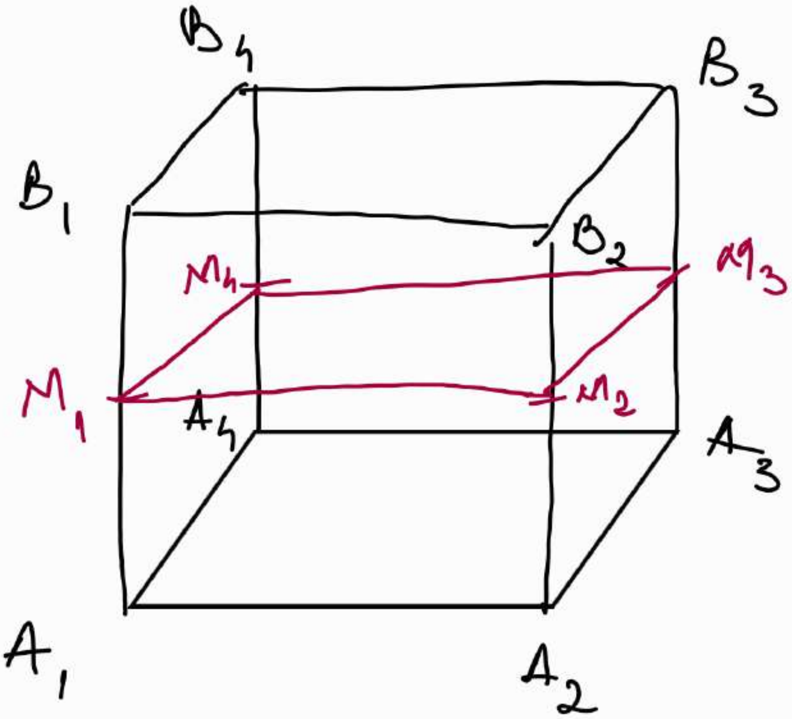
$$\vec{SO} = \vec{SA} + \vec{AO}$$
$$\vec{SO} = \vec{SB} + \vec{BO}$$
$$\vec{SO} = \vec{SC} + \vec{CO}$$
$$\vec{SO} = \vec{SD} + \vec{DO}$$
(+)

$$4\vec{SO} = \vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} + \vec{AO} + \vec{BO} + \vec{CO} + \vec{DO}$$

ABCD parallelogram $\Rightarrow \vec{AO} + \vec{BC} + \vec{CO} + \vec{DO} = 0$

$$\Rightarrow \vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$$

1.5. Consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.



M_1M_4 mid segment

M_2M_3 mid segment

M_3M_4 mid segment

M_1M_2 mid segment

$$M_1M_2 = \frac{1}{2}(\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2})$$

$$M_4M_3 = \frac{1}{2}(\overrightarrow{A_4A_3} + \overrightarrow{B_4B_3})$$

$A_1A_2A_3A_4$, $B_1B_2B_3B_4$ parallelograms $\Rightarrow \overrightarrow{A_1A_2} = \overrightarrow{A_4A_3}$, $\overrightarrow{B_1B_2} = \overrightarrow{B_4B_3}$

$$\Rightarrow \overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$$

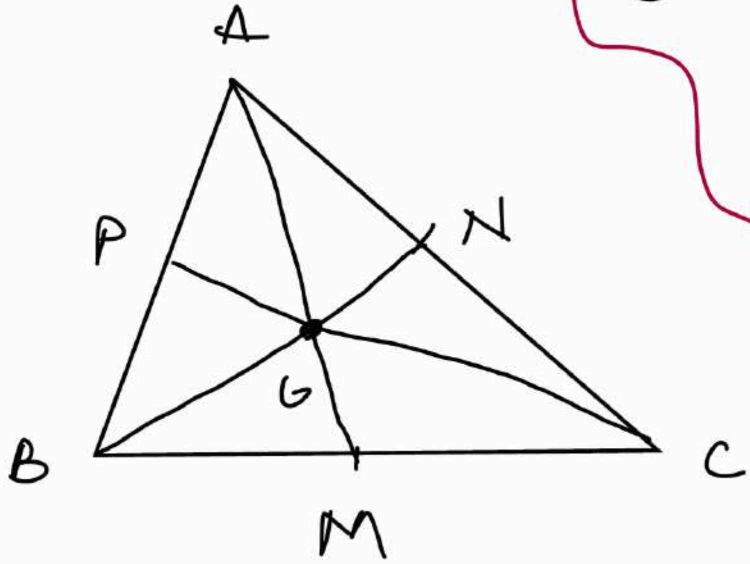
la $\parallel$ și la celelalte $\Rightarrow M_1M_2M_3M_4$ paralelogram

1.6. Starting from Hilbert's Axioms prove that the base angles of an isosceles triangle are congruent.

regulile de congruență

1.7. Using vectors, show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.

ΔABC



G = centru de greutate

$\vec{AG} = \frac{2}{3}\vec{AM}$

$\vec{BG} = \frac{2}{3}\vec{BN}$

M mp $BC \Rightarrow M = \frac{1}{2}(\vec{B}+\vec{C})$

N mp $AC \Rightarrow N = \frac{1}{2}(\vec{A}+\vec{C})$

P mp $AB \Rightarrow P = \frac{1}{2}(\vec{A}+\vec{B})$

$AM = \frac{1}{2}(\vec{B}+\vec{C}) - \vec{A}$

$BN = \frac{1}{2}(\vec{A}+\vec{C}) - \vec{B}$

$CP = \frac{1}{2}(\vec{A}+\vec{B}) - \vec{C}$

$G \in AM \Rightarrow \vec{G} = \vec{A} + \lambda\left(\frac{1}{2}(\vec{B} + \vec{C}) - \vec{A}\right)$

$G \in BN \Rightarrow \vec{G} = \vec{B} + \mu\left(\frac{1}{2}(\vec{A} + \vec{C}) - \vec{B}\right)$

$\vec{G} = (1-\lambda)\vec{A} + \frac{\lambda}{2}(\vec{B} + \vec{C})$

$\vec{G} = (1-\mu)\vec{B} + \frac{\mu}{2}(\vec{A} + \vec{C})$

Teorie curs II

Frame $\Rightarrow K = (O, B)$ where

- O is a point in the plane
- $B = (i, j)$ a basis in the vector space

$\Rightarrow \begin{cases} P \text{ is a point} \\ K \text{ a frame of } (x_P, y_P) \end{cases}$

$\vec{OP} = x_P i + y_P j$

$P_K = \begin{bmatrix} x_P \\ y_P \end{bmatrix}$ the coordinates of point P relative to the frame K

ex: $P = 4.2\, i + 3.8\, j \Rightarrow [P]_K = \begin{bmatrix} 4.2 \\ 3.8 \end{bmatrix}$

Changing frames

$K = (O, i, j) \to K' = (O', i', j')$

$\vec{O'A} = \vec{OA} - \vec{OO'}$

$[\vec{O'A}]_B = [\vec{OA}]_B - [\vec{OO'}]_B$

$[\vec{O'A}]_{B'} = M_{B'B} [\vec{O'A}]_B$

$[A]_{K'} = M_{B'B} \cdot \left([\vec{OA}]_B - [\vec{OO'}]_B \right)$

Example: $[A]_K = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad [O']_K = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$[i']_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\quad [j']_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$M_{B'B} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$

$\Rightarrow$ in the formula above:

$[A]_{K'} = \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$

$= \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Example 2:

Old frame: $K = (O, B)$

New frame: (O', B')

$[P]_K = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

new origin $O' \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$[i']_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad [j']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ new basis vectors

$[P]_B - [O']_B = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$M_{B'B} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = -1 \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$[P]_{K'} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Orientation

f_1, f_2 same side $\Rightarrow$ same orientation

f_1, f_2 opposite side $\Rightarrow$ opposite orientation

Example:

$l_1 = (\lambda, 0)$

$l_2 = (a, b)$

the cangle of basis matrix : $M_{l_1 l_2} = \begin{bmatrix} \lambda & a \\ 0 & b \end{bmatrix}$

$\det(M_{l_1 l_2}) = \lambda \cdot b$

if $\lambda \cdot b > 0$ same orientation

if $\lambda \cdot b < 0$ opposite orientation

left and right orientation

Example: $i = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$ right oriented

$j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example $i = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1$ left oriented

$j = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Now we are movim from $K = (O, B)$ and $B = (i, j)$ to $K = (O, B)$ and $B = (i, j, k)$

$$\vec{OP} = x_p i + y_p j + z_p k$$

$$[P]_K = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

<u>Changing frames in 3D</u> (the same like before

$$[B]_K - [O']_K = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$i' = [-1, -2, 0]$ $\quad j' = [-2, 1, 0]$ $\quad k' = [0, 0, 2]$

$$(M_{K'K})^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

$$[B]_{K'} = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

<u>Orientation</u>

$\det(M(i, j, k)) > 0$ right oriented

$\det(M(i,j,k)) < 0$ left oriented

Seminar 2

2.1. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$

b) $\mathbf{u}(1,2,-1)$, $\mathbf{v}(2,-1,0)$, $\mathbf{w}(1,-3,1)$

a) $u + w - v = 0$

$(7,3) + (-5,5) - (2,8) = (0,0)$ yes

b) $v - u - w =$

$(2,-1,0) - (1,2,-1) - (1,-3,1) = (0,0,0)$ yes

2.2. In each of the following cases, decide if the given points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$

b) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$

a) $\begin{vmatrix} 3 & -5 & 1 \\ -1 & 2 & 1 \\ -5 & 9 & 1 \end{vmatrix} = 0 \Rightarrow$ collinear

b) $\overrightarrow{PQ} = (1,0,-1) - (0,-1,2) = (1,1,-3)$

$\overrightarrow{PR} = (1,0,-1) - (-1,-2,5) = (2,2,-6)$

$2\overrightarrow{PQ} = PR \Rightarrow P, Q, R$ collinear

2.3. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

x goes from 0 to 1

y goes from 0 to 3

z goes from -2 to 0

when $x = 0$
$(0,0,0)$
$(0,0,-2)$
$(0,3,0)$
$(0,3,-2)$

when $x = 1$
$(1,0,0)$
$(1,0,-2)$
$(1,3,0)$
$(1,3,-2)$

2.4. Which of the following sets of vectors form a basis?

a) $\mathbf{v}_1(1,2)$, $\mathbf{v}_2(3,4)$

b) $\mathbf{v}_1(-1,2,1,0)$, $\mathbf{v}_2(2,1,1,0)$, $\mathbf{v}_3(1,0-1,1)$, $\mathbf{v}_4(1,0,0,1)$

a) $a v_1 + b(v_2) = 0$

$a(1,2) + b(3,4) = 0$

$$\begin{cases} a + 3b = 0 \\ 2a + 4b = 0 \end{cases} \Rightarrow a = -3b$$

$-2b = 0 \Rightarrow a = b = 0$

linearly independent

$\Rightarrow$ form a basis

dezvoltăm aici

b) $$\begin{pmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= 1 \cdot (-1)^{1+4} \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} + 1 \cdot (-1)^{4+4} \begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 5 \neq 0 \Rightarrow$ linearly independent

$\Rightarrow$ form a basis

2.5. With respect to the basis $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis and decide if it is left or right oriented.

$$u = i + j = (1, 1, 0)$$

$$v = j + k = (0, 1, 1)$$

$$w = i + k = (1, 0, 1)$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 1 = 2 > 0 \text{ right oriented}$$

$2 \neq 0$ linearly independent $\Rightarrow$ basis

2.6. Let $\mathcal{K} = (O, \mathcal{B})$ and $\mathcal{K}' = (O', \mathcal{B}')$ be two frames in $\mathbb{E}^2$, with $\mathcal{B} = (\mathbf{i}, \mathbf{j})$ and $\mathcal{B}' = (\mathbf{i}', \mathbf{j}')$. Assume that O', $\mathbf{i}'$ and $\mathbf{j}'$ are known relative to $\mathcal{K}$:

$$[O']_{\mathcal{K}} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates with respect to $\mathcal{K}'$ of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

- $[A_K] - [O']_K = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$$(M_{B'B})^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$[A]_{K'} = \frac{1}{-4}\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- $$[B_K] - [O']_K = \begin{bmatrix} 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$(M_{B'B})^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$[B]_{K'} = \frac{1}{-4}\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- $$[C]_{K'} = \frac{1}{-4}\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2.7. Consider the tetrahedron $ABCD$ and the frames

$$\mathcal{K}_A = (A, \mathcal{B}) \text{ with } \mathcal{B} = (\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}),$$

$$\mathcal{K}_B = (B, \mathcal{B}) \text{ with } \mathcal{B} = (\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}),$$

$$\mathcal{K}'_A = (A, \mathcal{B}) \text{ with } \mathcal{B} = (\overrightarrow{BA}, \overrightarrow{BD}, \overrightarrow{BC}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrices from $\mathcal{K}'_A$ to $\mathcal{K}_A$ and from $\mathcal{K}_A$ to $\mathcal{K}_B$,

c) the orientations of the three frames.

a) $B = ((A), \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$ → the standard basis $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

$\downarrow$

origin $\Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$[AB\Gamma\Delta]_{K_A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{B} = (B, \vec{BA}, \vec{B\Gamma}, \vec{B\Delta})$$

$$[AB\Gamma\Delta]_{K_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{B} = (A, \vec{BA}, \vec{B\Delta}, \vec{B\Gamma})$$

$$[AB\Gamma\Delta]_{K'_A} = \begin{matrix} 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$

$$\vec{AB} = -\vec{BA} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{A\Delta} = \vec{AB} + \vec{B\Delta} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{A\Gamma} = \vec{AB} + \vec{B\Gamma} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

b) $K'_A \to K_A$

we want to express $(\vec{BA}, \vec{BD}, \vec{BC})$ in terms of $(\vec{AB}, \vec{AC}, \vec{AD})$

$\vec{BA} = -\vec{AB} = (-1,0,0)$

$\vec{BD} = \vec{BA} + \vec{AD} = (-1,0,0)+(0,0,1) = (-1,0,1)$

$\vec{BC} = \vec{BA} + \vec{AC} = (-1,0,0)+(0,1,0) = (-1,1,0)$

$$M_{K'_A K_A} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{K_B K_A} = \left([\vec{AB}]_{K_B}, [\vec{AC}]_{K_B}, [\vec{AD}]_{K_B} \right)$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $\left.\begin{array}{l} \det M_{K_B K_A} = -1 \\ \det M_{K_A K_A'} = 1 \end{array}\right\} \Rightarrow K_A, K_A'$ same orientation

K_B is diff orientation

2.8 the same ex as 6

Teorie curs 3

Parametric equations

Lines in A^2

ex: we have $Q=(1,2)$ and $P(3,6)$

direction vector $QP = P - Q = (2,4)$

Now, any point on the line can be written as $\boxed{P(t) = Q + t \cdot d_v = (1,2)+t(2,4)}$

and we get the parametric equations

$$\boxed{\begin{cases} x = 1 + 2t \\ y = 2 + 4t \end{cases}}$$

Cartesian equations

we want to eliminate from the equations:

$$\begin{cases} x = 1 + 2t & \Rightarrow t = \frac{x-1}{2} \\ y = 2 + 4t & \Rightarrow t = \frac{y-2}{4} \end{cases}$$

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{4} \Rightarrow 4(x-1) = 2(y-2)$$

$$\Rightarrow 4x - 4 = 2y - 4$$

$$4x - 2y = 0 \Rightarrow 2x - y = 0$$

Intersection of lines

we have the lines

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0 \end{cases}$$

we have 3 possible cases
- lines intersect in one point
- parallel lines
- they are the same line

ex 1: $\begin{cases} x - y + 1 = 0 \\ 2x - 2y + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x - y + 1 = 0 \\ x - y + 1 = 0 \end{cases} \Rightarrow$ the same line

ex 2: $\begin{cases} x - y + 1 = 0 \\ x - y + 3 = 0 \end{cases} \Rightarrow$ parallel lines ex 3: $\begin{cases} x+y = 5 \\ x-y = 1 \end{cases}$ $y=2$ $x=3$

they intersect at some point $(3,2)$

Planes in A3 (cam ac. lucru ca mai sus)

Parametric equations

$Q = (1,1,1)$

$A = (2,3,1)$

$B = (1,2,4)$

$\overrightarrow{QA} = A - Q = (1,2,0) = \vec{v}$

$\overrightarrow{QB} = B - Q = (0,1,3) = \vec{w}$

$$P = Q + s\vec{v} + t\vec{w} = \begin{cases} x = 1+s \\ y = 1+2s+t \\ z = 1+3t \end{cases}$$

Cartesian equations

we want to eliminate s and t

$s = x - 1$

$t = \dfrac{z-1}{3}$

$y = 1 + 2x - 2 + \dfrac{z-1}{3} \Rightarrow 3y = 3 + 6x - 6 + z - 1$

$\Rightarrow \boxed{6x - 3y + z - 4 = 0}$

Relative positions of two planes

$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$

$\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$

3 possibility

1. they intersect in a line

- the sys has infinitely many solutions depending on one parameter

2. they are parallel

- the sys has no solution

3. they coincid (same plane)

- the equations are equivalent and the sys has many solutions depending on two parameters

⇓

we find these by calculating the rank of the matrix of the system

same rank = intersect

rank mismatch = parallel

Lines in A^3 (cam la fel ca lines in A^2 and planes in A^3)

Parametric equations

- a point $Q = (1,2,3)$
- direction vector $\vec{v} = (4,1,-2)$

$$(x,y,z) = (1,2,3) + t(4,1,-2) = \begin{cases} x = 1+4t \\ y = 2+t \\ z = 3-2t \end{cases}$$

Cartesian equations

we want to eliminate t

$$\frac{x-1}{4} = y-2 = \frac{3-z}{2}$$

Relative positions

4 cases ① the direction vectors are scalar multiples $\Rightarrow$ parallel $2\vec{v_1} = \vec{v_2}$

② same line (coincid)

① $l_1: (x, y, z) = (0,0,0) + t(1,2,3)$
$l_2: (x, y, z) = (2,4,6) + s(1,2,3)$
} the same direction vector

② l_2 is $(2,4,6) = 2(1,2,3) \in l_1$

din ① și ② $\Rightarrow$ coincid

③ intersecting lines

$l_1: (x, y, z) = (0,0,0) + t(1,1,1)$

$l_2: (x, y, z) = (1,0,0) + s(-1,1,1)$

dacă facem calcule $t = s = z = \frac{1}{2}$

④ skew lines

$l_1: (x, y, z) = (0,0,0) + t(1,0,0)$

$l_2 : (x, y, z) = (0,1,1) + s(0,1,0)$

$\Rightarrow$ nu are soluții, vectorii nu sunt proporționali $\Rightarrow$ skew lines

Affine subspaces of A^n

a line is a 1D affine subspace

a plane is a 2D ....

a point is a 3D ....

Hyperplanes

$$\left.\begin{array}{l}\text{base point } Q \\ (n-1) \text{ d.v.} \\ \text{a point } P\end{array}\right\} P = Q + t_1 \vec{v_1} + \ldots + t_{n-1} \vec{v}_{n-1}$$

$$\Downarrow$$

$$a_1 x_1 + \ldots + a_n x_n = b$$

Lines

parametric form $(x_1, \ldots x_n) = (q_1 \ldots q_n) + t(v_1 \ldots v_n)$

ex: $(x, y, z) = (1,2,3) + t(4,0,-2)$

$$\frac{x-1}{4} = \frac{y-2}{6} = \frac{z-3}{-2}$$

Relative positions

① parallel
- direction spaces are equal / nested

② equal
- parallel and share a point

③ intersecting
- they may meet in a smaller affine subspace

Changing the Reference Frame

$\Rightarrow$ ce am făcut mai devreme la change of base matrix

Warm up exercises

3.1. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

3.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

[redacted]

3.1 a) $P = A + t(3,-1) = (1,2) + t(3,-1)$

$$\begin{cases} x = 1 + 3t \\ y = 2 - t \end{cases}$$

b) $A(0,0)$ $\quad b(4,5)$

$P = (0,0) + t(4,5)$

$$\begin{cases} x = 4t \\ y = 5t \end{cases}$$

c) $M(1,7)$ parallel to Oy $\quad v = (0,1)$

$P = (1,7) + t(0,1)$

$$\begin{cases} x = 1 \\ y = 7 + t \end{cases}$$

d) the direction vector is

$$\vec{MN} = N - M = (2-2, -5-4) = (0, 9)$$

$$P = M + t(0,9) = (2,4) + t(0,9)$$

$$\begin{cases} x = 2 \\ y = 4 + 9t \end{cases}$$

3.2 a) $\begin{cases} x = 1 + 3t \\ y = 2 - t \end{cases}$ we eliminate t

$$\frac{x-1}{3} = 2 - y \Rightarrow x - 1 = 6 - 3y$$

$$\Rightarrow x + 3y = 7$$

b) $5x - 4y = 0$

c) $x = 1$

d) $x = 2$

3.3. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

a) $P = (1,0,2) + s(3,-1,1) + t(0,3,1)$

$$\begin{cases} x = 1 + 3s \\ y = -s + 3t \\ z = 2 + s + t \end{cases}$$

b) $\vec{AB} = B - A = (2, 1, 2)$

$\vec{AC} = C - A = (3, -1, -2)$

$P = (-2, 1, 1) + s(2, 1, 2) + t(3, -1, -2)$

$$\begin{cases} x = -2 + 2s + 3t \\ y = 1 + s - t \\ z = 1 + 2s - 2t \end{cases}$$

3.4. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u$ with $u, v \in \mathbb{R}$,

b) $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v$ with $u, v \in \mathbb{R}$.

a) $\begin{cases} x = 2 + 3u - 4v \\ y = 4 - v \quad \rightarrow \quad v = y - 4 \end{cases}$

$z = 2 + 3u \Rightarrow u = \dfrac{z-2}{3}$

$x = \cancel{2} + z - \cancel{2} - 4y + 16$

$\boxed{x + 4y - z - 16 = 0}$

b) $\begin{cases} x = u + v \\ y = u - v \\ z = 5 + 6u - 4v \end{cases}$

$$\begin{array}{l} x = u + v \\ \underline{y = u - v} \\ \qquad\qquad (+) \end{array}$$

$x + y = 2u \Rightarrow u = \dfrac{x+y}{2}$

$$\begin{array}{l} x = u + v \\ \underline{y = u - v} \\ \qquad\qquad (-) \end{array}$$

$x - y = 2v \Rightarrow v = \dfrac{x-y}{2}$

$z = 5 + 3(x+y) - 2(x-y)$

$z = 5 + 3x + 3y - 2x + 2y$

$\Rightarrow \quad \boxed{x + 5y - z + 5 = 0}$

3.5. Determine parametric equations for the plane π in the following cases:

a) $\pi : 3x - 6y + z = 0,$

b) $\pi : 2x - y - z - 3 = 0.$

a) choose 2 variables as free parameters

$y = u$

$z = v$

$3x - 6u + v = 0 \quad \Rightarrow \quad 3x = 6u - v$

$x = 2u - \frac{v}{3}$

$x = 2u - \frac{v}{3}$

$y = u$

$z = v$

b) choose 2 variables as free parameters

$y = u$

$z = v$

$2x = u + v + 3 \Rightarrow \quad x = \frac{u + v + 3}{2}$

$$\begin{cases} x = \frac{u + v + 3}{2} \\ y = u \\ z = v \end{cases}$$

3.6. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

$$\vec{AB} = B - A = (-1, 2, 4)$$

$$\vec{AC} = C - A = (-3, 1, 2)$$

$$\vec{AD} = D - A = (3, 2, 4)$$

if $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0 \Rightarrow$ coplanar

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = i \cdot \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + j \begin{vmatrix} 2 & -3 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} \cdot k$$

$$= 18j - 9k = 9(2j - k) = (0, 18, -9)$$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = (-1, 2, 4) \cdot (0, 18, -9)$$

$$= -1 \cdot 0 + 2 \cdot 18 + 4(-9) = 0 + 36 - 36 = 0$$

Seminar 3

3.1. Let $a \in \mathbb{R}$ be a parameter. Consider the lines $\ell_1 : 2x + y = 1$ and $\ell_2 : x + ay = -1$. Discuss the relative position of the two lines in terms of the parameter a. Furthermore, determine the values a for which the origin and the point $P(-2,-2)$ lie on the same side of ℓ_2.

(pag 35)

- $\ell_1 : 2x + y = 1 \quad \vec{v_1}$

  $\ell_2 : x + ay = -1 \quad \vec{v_2}$

  so, if $a = \frac{1}{2} \Rightarrow \vec{v_1} = 2\vec{v_2}$ parallels

  if $a \neq \frac{1}{2} \Rightarrow$ lines intersect

- $O = (0,0)$

  $P = (-2,-2)$

$\ell_2 : f(x,y) = x + ay + 1$

$f(0,0) = 1$

$f(-2,-2) = -2a - 1$

same side : $-2a - 1 > 0$

$-2a > 1 \Rightarrow 2a < -1 \Rightarrow a < -\frac{1}{2}$

3.2. Consider the lines $\ell_1 : x - 1 = \frac{y-1}{2} = z - 3$ and $\ell_2 : \frac{x-3}{3} = \frac{y}{6} = \frac{z+2}{3}$. Are the two lines parallel? If they are, give an equation for the plane containing them. Moreover, check if this plane separates the point $P(1, 1, 1)$ from the origin.

$$l_1 : x - 1 = \frac{y-1}{2} = z - 3$$

$$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$$

(x_0, y_0, z_0) is point on the line

(v_x, y_z, z_x) is direction vector of the line

l_1: $\vec{v_1} = (1, 2, 1)$

$P_1 = (1, 1, 3)$

l_2: $\vec{v_2} = (3, 6, 3)$

$P_2 = (3, 0, -2)$

$\vec{v_2} = 3\vec{v_1} \Rightarrow$ the lines are parallel

$\overrightarrow{P_1P_2} = P_2 - P_1 = (2, -1, -5)$

$$\vec{v_1} \times \overrightarrow{P_1P_2} = \begin{vmatrix} i & j & k \\ 3 & 6 & 3 \\ 2 & -1 & -5 \end{vmatrix} = -9i + 7j - 5k = (-9, 7, -5)$$

plane equation using P_1

$$-9(x-1) + 7(y-1) - 5(z-3) = 0$$

$$\boxed{9x - 7y + 5z - 17}$$

$$\left.\begin{aligned} O(0,0,0) &= 9 \cdot 0 - 7 \cdot 0 + 5 \cdot 0 = 0 \\ P(1,1,1) &= 9 \cdot 1 - 7 \cdot 1 + 5 \cdot 1 = 7 \end{aligned}\right\} \text{they are on}$$

the same side

3.3. What is the relative position of the lines $x = -3t, y = 2+3t, z = 1, t \in \mathbb{R}$ and $x = 1+5s, y = 1+13s, z = 1+10s, s \in \mathbb{R}$?

$$l_1: \begin{cases} x = -3t \\ y = 2 + 3t \\ z = 1 \end{cases} \qquad P_1 = (0,2,1) + t(-3,3,0)$$

$$l_2: \begin{cases} x = 1 + 5s \\ y = 1 + 13s \\ z = 1 + 10s \end{cases} \quad P_2 = (1,1,1) + s(5,13,10)$$

$\Rightarrow$ not parallel, so we need to try to solve them

$$\begin{cases} -3t = 1 + 5s & \Rightarrow t = -\frac{1}{3} \\ 2 + 3t = 1 + 13s \\ 1 = 1 + 10s & \Rightarrow s = 0 \end{cases}$$

true

$\downarrow$

it holds

$\Downarrow$

the lines intersect

3.4. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax + y - 2z + d = 0$.

$P(2, -1, 3)$

$\vec{u}(3, 2, -2)$

$ax + y - 2z + d = 0$

$P: 2a - 1 - 6 + d = 0 \Rightarrow 2a + d = 7$

$\vec{v}: 3a + 2 - 4 = 0 \Rightarrow a = -2$

$d = 11$

3.5. Determine the relative positions of the line $\ell : \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-1}{-4}$ and the plane $\pi : 2x - y + z - 1 = 0$. If the line punctures the plane, determine the intersection point.

$\ell_1: \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-1}{-4}$

$P: (1, 2, 1)$

$\vec{v}: (1, -2, -4)$

$\pi: 2x - y + z - 1 = 0$

$P: (1, 2, 1) + t(1, -2, -4)$

$\begin{cases} x = 1 + t \end{cases}$

$$\begin{cases} y = 2 - 2t \\ z = 1 - 4t \end{cases}$$

$2(1+t) - 2 + 2t + 1 - 4t - 1 = 0$

$2 + 2t - 2 + 2t + 1 - 4t - 1 = 0$

$0 = 0 \Rightarrow$ always true for any t

$\Rightarrow$ entirely contained in the plane

3.6. Determine Cartesian equations for the line ℓ passing through $Q = (1, 1, 2)$ and coplanar to the lines $\ell' : 3x - 5y + z = -1, 2x - 3z = -9$ and $\ell'' : x + 5y = 3, 2x + 2y - 7z = -7$. Furthermore, establish whether ℓ meets or is parallel to ℓ' and ℓ''.

$$\ell' \begin{cases} 3x - 5y + z = -1 \\ 2x - 3z = -9 \end{cases}$$

$$\ell'' \begin{cases} x + 5y = 3 \\ 2x + 2y - 7z = -7 \end{cases}$$

Find points in each line

- in ℓ' $z=0$ take $z = 0 \Rightarrow x = -\frac{9}{2}$

$y = -\frac{5}{2}$

$$P_1 = \left(-\frac{9}{2}, -\frac{5}{2}, 0\right)$$

$-\mathrm{in}\ \ell^{1)}\ y = 0 \Rightarrow$ $x = 3$

$z = \frac{13}{4}$

$$P_2 = \left(3, 0, \frac{13}{4}\right)$$

$$\mathrm{d.v} = \overrightarrow{P_1 P_2} = P_2 - P_1 = \left(\frac{15}{2}, \frac{5}{2}, \frac{13}{4}\right)$$

$$P = (1,1,2) + t\left(\frac{15}{2}, \frac{5}{2}, \frac{13}{4}\right)$$

$$\begin{cases} x = 1 + \frac{15}{2}t \\ y = 1 + \frac{5}{2}t \\ z = 2 + \frac{13}{4}t \end{cases}$$

$$\frac{2(x-1)}{15} = \frac{2(y-1)}{5} = \frac{4(z-2)}{13}$$

3.7. Determine the value k for which the lines $\ell : x = k + t, y = 1 + 2t, z = -1 - kt, t \in \mathbb{R}$ and $\ell' : x = 2 - 2t, y = 3 + 3t, z = 1 + t, t \in \mathbb{R}$ are coplanar. Moreover, determine a Cartesian equation for the plane that contains them and the intersection point $\ell \cap \ell'$ if it exists.

the lines are coplanar if the dv of the lines AND the vector between a point in each line is in the same plane

$$\ell : \begin{cases} x = k + t \\ y = 1 + 2t \\ z = -1 - kt \end{cases} \qquad P = (k, 1, -1) + t\,\overset{\vec{v_1}}{(1, 2, -k)}$$

$$\ell' \begin{cases} x = 2 - 2t \\ y = 3 + 3t \\ z = 1 + t \end{cases} \qquad P = (2, 3, 1) + t\,\overset{\vec{v_2}}{(-2, 3, 1)}$$

$$\overrightarrow{P_1 P_2} = P_2 - P_1 = (2 - k, 2, 2)$$

$$\overrightarrow{P_1 P_2} \cdot (v_1 \times v_2) = 0$$

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -a \\ -2 & 3 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & -a \\ 3 & 1 \end{vmatrix} + j \begin{vmatrix} -a & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} \cdot k$$

$$= (2+3a)\, i + (2a-1)\, j + 7k$$

$$(2-k, 2, 2) \quad (2+3k, 2k-1, 7)$$

$$(2-k)(2+3k) + 4k - 2 + 14 = 0$$

$$4 + \underline{6k} - \underline{2k} - \underline{3k^2} + \underline{4k} - 2 + 14 = 0$$

$$-3k^2 + 8k + 16 = 0$$

$$3k^2 - 8k - 16 = 0$$

$$\Delta = 64 + 48 \cdot 4 = 64 + 192 = 256$$

$$k_1, k_2 = \frac{8 \pm 16}{6} \begin{cases} 4 \\ -\frac{4}{3} \end{cases}$$

3.8. Determine the relative positions of the planes in the following cases

a) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : x + 2y - 3z - 1 = 0$.

b) $\pi_1 : 3x + y + z - 1 = 0$, $\pi_2 : 2x + y + 3z + 2 = 0$, $\pi_3 : -x + 2y + z + 4 = 0$.

c) $\pi_1 : 3x + y + z - 5 = 0$, $\pi_2 : 2x + y + 3z + 2 = 0$, $\pi_3 : 5x + 2y + 4z + 1 = 0$.

a) (pag 34-35)

$$\tilde{M} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & -3 & 1 \end{array}\right)$$

rank M = rank $\tilde{M} = 2 \Rightarrow$ not parallel

b)

$$\tilde{A} = \left(\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 2 & 1 & 3 & -2 \\ -1 & 2 & 1 & -4 \end{array}\right)$$

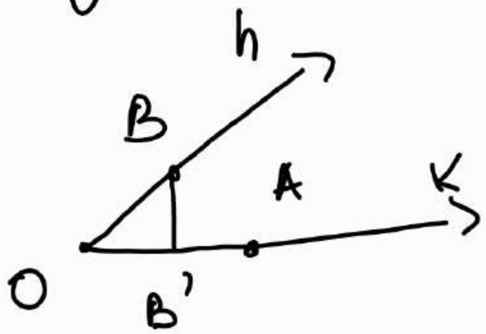
$$\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = -15 \neq 0 \rightarrow \text{rank} = 3$$

rank $\tilde{A} = 3$

$\Rightarrow$ not parallel

Teorie curs 4

Angles



$$\sin \sphericalangle(h,k) = \frac{|AB'|}{|OA|}$$

$$\cos \sphericalangle(h,k) = \pm\left|\frac{OB'}{OA}\right|$$

$$\cos(\vec{a}, \vec{b}) = 0 \Rightarrow \vec{a} \perp \vec{b}$$

Turning

from $(1,0)$ to $(0,1)$ is $+90$

from $(0,1)$ to $(1,0)$ is -90

Scalar product

$$\langle a, b \rangle = \begin{cases} 0 & \text{, if one is zero} \\ |a| \cdot |b| \cdot \cos(\sphericalangle(a,b)) & \text{, both nonzero} \end{cases}$$

if $\vec{a}$ and $\vec{b}$:

- point the same way => product > 0
- perpendicular => product is 0
- opposite direction => product < 0

Properties of the dot product

1. $\langle a + bw, u \rangle = \langle a, u \rangle + b \langle w, u \rangle$

2. $\langle v, w \rangle = \langle w, v \rangle$

3. $\langle v, v \rangle > 0$, if $v \neq 0$

4. $v \perp w \iff \langle v, w \rangle = 0$

   $|v| = 1 \iff \langle v, v \rangle = 1$

$$\cos(\sphericalangle(a,b)) = \frac{\langle a, b \rangle}{|a|\,|b|}$$

Euclidean space

$$d(P, Q) = \sqrt{(p_1 - q_1)^2 + \ldots + (p_n - q_n)^2}$$

ex: $P = (3, 4)$ $\quad Q = (0, 0)$

$$d(P, Q) = \sqrt{(3-0)^2 + (4-0)^2} = 5$$

Normal vector

plane: $2x - y + \frac{1}{3} z + 7 = 0$

$\Rightarrow$ normal vector $\vec{n} = (2, -1, \frac{1}{3})$

Distance from a point to a Hyperplane

$$d(P, H) = \frac{a_1 p_1 + a_2 p_2 + \ldots + a_n p_n - b}{\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}}$$

plane: $3x + 4y = 12$ at point $(0, 0)$

$$d = \frac{|3 \cdot 0 + 4 \cdot 0 - 12|}{\sqrt{3^2 + 4^2}} = \frac{12}{5}$$

Loci of points at Equal Distance

locus = collection of points

$Q = (0,0,0)$ is a point
we want all point of distance 5

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = 25$$

$$x^2 + y^2 + z^2 = 25 \Rightarrow \text{a sphere of radius } 5$$

distance from a point to a plane:

plane: $2x - y + z = 7$
point: $(1, 2, 1)$

$$d = \frac{2\cdot 1 - 1\cdot 2 + 1\cdot 1 - 7}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

Warm-up exercises

In the following exercises, all coordinates and components are given with respect to an orthonormal frame $\mathcal{K} = (O, \mathcal{B})$. For the 2-dimensional cases $\mathcal{B} = (\mathbf{i}, \mathbf{j})$ and for the 3-dimensional cases $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$.

4.1. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

$$\ell : y - \underline{y_0} = \underline{m}(x - \underline{x_0})$$

point on the line

the slope

a) $m = tg(60°) = \sqrt{3}$

the slope can be find with tan apparently

$$y - 3 = \sqrt{3}(x + 2)$$

$$y = \sqrt{3}x + 3 + 2\sqrt{3}$$

b) $\ell : n_x(x - x_0) + n_y(y - y_0) = 0$

$$\ell : 4(x-1) + 3(y-7) = 0$$

$$\ell : 4x + 3y - 25 = 0$$

4.3. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

4.4. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.

4.5. Determine Cartesian equations for the lines situated at distance 4 from the line $12x-5y-15=0$.

4.6. Determine the values k for which the distance from the point $(2,3)$ to the line $8x+15y+k=0$ equals 5.

4.3 a) slope of $BC = \dfrac{y_C - y_B}{x_C - x_B} = \dfrac{6}{6} = 1$

$\Rightarrow y = x + b$
use B

$3 = -4 + b \rightarrow b = 7$

$BC : y = x + 7$

$y - x - 7 = 0 \Rightarrow x - y + 7 = 0$

distance $\dfrac{Ax + By + C}{\sqrt{A^2 + B^2}} = \dfrac{5\sqrt{2}}{2}$

$A = (1, 3)$

4.4 $\ell_1 : 2x - y + 1 = 0$

normal vector $n_1 = (2, -1)$

$\ell_2 : x + y - 2 = 0$

normal vector $n_2 = (1,1)$

$$\cos(\theta) = \frac{\langle n_1, n_2 \rangle}{|n_1|\,|n_2|}$$

$$|n_1| = \sqrt{4+1} = \sqrt{5}$$

$$|n_2| = \sqrt{2}$$

$$\langle n_1, n_2 \rangle = \langle (2,-1),(1,1) \rangle = 2\cdot 1 + (-1)\cdot 1 = 1$$

$$\cos(\theta) = \frac{1}{\sqrt{10}}$$

4.5 $12x - 5y - 15 = 0$

normal vector $\vec{n} = (12, -5)$

$$\frac{|Ax + By + C|}{\sqrt{A^2+B^2}} = 4$$

we want to look for different values of c; if A and B stay the same the lines are parallel

$$\frac{c' - (-15)}{\sqrt{12^2 + (-5)^2}} = 4 \Rightarrow |c' + 15| = 52$$

$$c' = 37 \text{ or } c' = -67$$

$$l_1 : 12x + 5y + 37 = 0$$

$$l_2 : 12x - 5y - 67 = 0$$

4.6 $\quad 8x + 15y + k = 0 \qquad$ point $(2, 3)$

$$d = \frac{8 \cdot 2 + 15 \cdot 3 + k}{\sqrt{64 + 225}} = 5$$

$$\Rightarrow \frac{|61 + k|}{17} = 5 \Rightarrow |61 + k| = 85$$

$$\Rightarrow k_1 = 24$$

$$k_2 = -146$$

Seminar 4

4.1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.

$$\cos \measuredangle(a,b) = \frac{\langle a, b\rangle}{|a||b|}$$

$$\langle a, b\rangle = \langle 2m + 4n, m - n\rangle$$

$$= 2\langle m, m-n\rangle + 4\langle n, m-n\rangle$$

$$= 2\langle m, m\rangle - 2\langle m, n\rangle + 4\langle n, m\rangle - 4\langle n, n\rangle$$

$$= 2|m|^2 - 6\langle m, n\rangle + 4|n|^2 =$$

$$= 6 - 6\langle m, n\rangle \qquad = 6 - 6\cdot\frac{1}{2} = 3$$

$$|a^2| = \langle a, a\rangle = \langle 2m + 4n, 2m + 4n\rangle =$$

$$= \underbrace{4|m|^2}_{1} + 8\underbrace{\langle m, n\rangle}_{-\frac{1}{2}} + 8\underbrace{\langle n, m\rangle}_{-\frac{1}{2}} + \underbrace{16|n|^2}_{1}$$

$$= 12$$

$$|b|^2 = \langle b, b\rangle = \langle m - n, m - n\rangle =$$

$$= |m|^2 - \langle m, n \rangle - \langle n, m \rangle + |n|^2 = 3$$

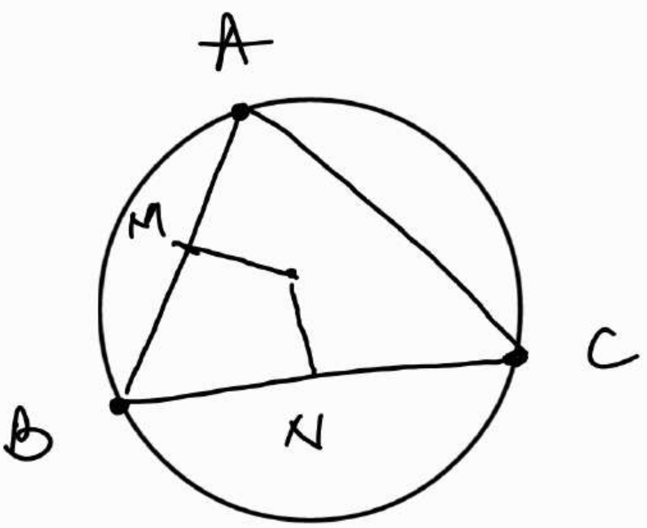
$$\cos \measuredangle(a, b) = \frac{\langle a, b \rangle}{|a| \cdot |b|} = \frac{-3}{\sqrt{12} \cdot \sqrt{3}} = \frac{1}{2}$$

4.2. Consider the points $A(1,2)$, $B(3,-2)$, $C(5,6)$.

a) Calculate the circumcenter and the orthocenter of the triangle ABC.

b) Determine an equation for the angle bisector $\measuredangle ABC$.

a)



M mp $AB \Rightarrow M(2,0)$

$m_{AB} = \frac{-2-2}{3-1} = -2 \Rightarrow$ the perpendicular slope is $m_1 = -\frac{1}{m_{AB}} = \frac{1}{2}$

$$y - 0 = \frac{1}{2}(x-2) \Rightarrow y = \frac{1}{2}x - 1$$

M mp $BC \Rightarrow M(4,2)$

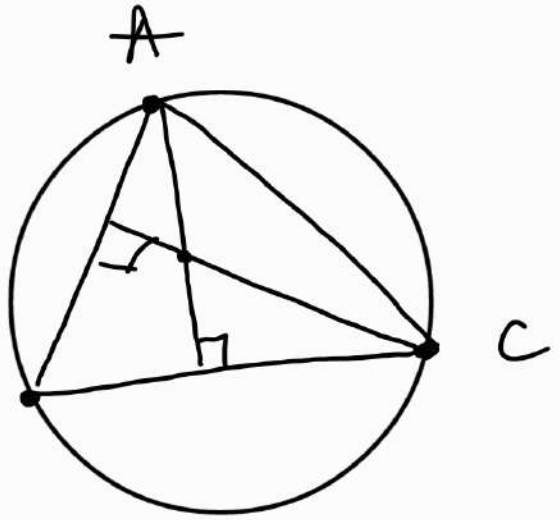
$m_{BC} = \frac{6+2}{5-3} = 4 \Rightarrow$ perp. slope $m_2 = -\frac{1}{4}$

$y - 2 = -\frac{1}{4}(x-4) \Rightarrow y = -\frac{1}{4}x + 3$

Le egalăm $\frac{1}{2}x - 1 = -\frac{1}{4}x + 3 \Rightarrow x = \frac{16}{3}$

$y = \frac{5}{3}$

b)



ne folosim de ce am aflat la a)

doar că schimbăm y_0 și x_0

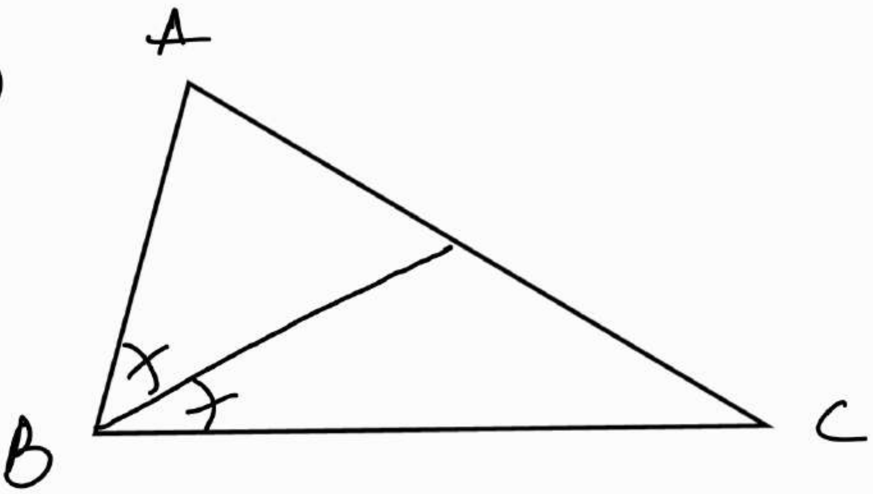
$y - 6 = \frac{1}{2}(x-5) \Rightarrow y = \frac{1}{2}x + \frac{7}{2}$

$$y - 2 = -\frac{1}{4}(x-1) \Rightarrow y = -\frac{1}{4}x + \frac{9}{4}$$

iar le egalăm: $\frac{1}{2}x + \frac{7}{2} = -\frac{1}{4}x + \frac{9}{4}$

$$\Rightarrow x = -\frac{5}{3} \qquad y = \frac{8}{3}$$

c)



$A(x_1, y_1)$

$B(x_2, y_2)$

$C(x_3, y_3)$

$$\frac{\overrightarrow{B\Delta}}{|B\Delta|} = \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} + \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|}$$

$$\overrightarrow{BA} = A - B = (-2, 4)$$

$$\overrightarrow{BC} = (2, 8)$$

$$|\overrightarrow{BA}| = \sqrt{(1-3)^2 + (2+2)^2} = \sqrt{20}$$

$$|\overrightarrow{BC}| = \sqrt{2^2 + 8^2} = \sqrt{68}$$

$$\vec{w} = \left(\frac{-2}{\sqrt{20}}, \frac{4}{\sqrt{20}}\right) + \left(\frac{2}{\sqrt{68}}, \frac{8}{\sqrt{68}}\right)$$

nu știu dacă e bine, dar nici nu mai transf. în ecuație

4.3. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

congruent segment on coordinate axes

=> it cuts the x-axis at $(a,0)$

and the y-axis at $(0,a)$ or $(0,-a)$

$$\frac{x}{a} + \frac{y}{a} = 1 \qquad \frac{x}{a} - \frac{y}{a} = 1$$

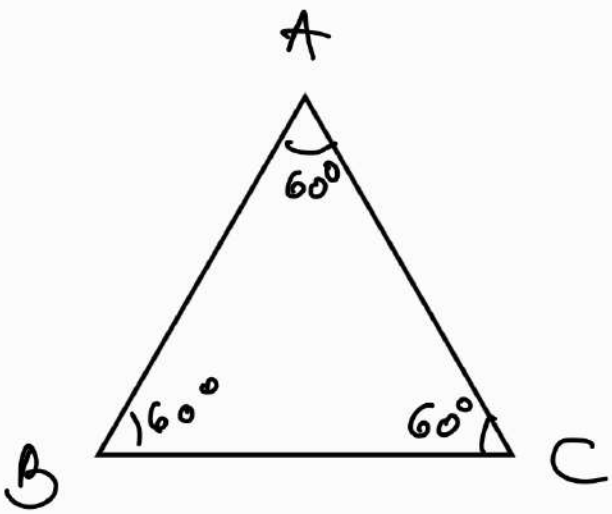
$$x+y=a \qquad x-y=a$$

$A(-2,5)$ $\quad x+y=a \quad x-y=a$

$-2+5=a \Rightarrow$ $\boxed{x-y=3}$

$-2-5=a \Rightarrow$ $\boxed{x-y=-7}$

4.4. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x+y-1=0$. Determine Cartesian equations for the lines containing the other two sides. (AB, AC)



$$BC: \quad x+y-1=0 \Rightarrow y=-x+1$$

$$\Downarrow$$

slope $m = \frac{-1}{1}$

$$\Downarrow$$

direction vector $m = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{m}}{|\vec{v}|\cdot|\vec{m}|}$$

$$\cos(60^\circ) = \frac{1}{2}$$

$$\vec{v}\cdot\vec{m} = a\cdot 1 - b$$

$$|\vec{v}| = \sqrt{2}$$

$$\frac{a-b}{\sqrt{a^2+b^2}\cdot\sqrt{2}} = \frac{1}{2} \Rightarrow 2(a-b) = \sqrt{2}\cdot\sqrt{a^2+b^2}$$

$$4(a-b)^2 = 2(a^2+b^2)$$

$$4a^2 - 8ab + 4b^2 = 2a^2 + 2b^2$$

$$a^2 - 4ab + b^2 = 0 \quad (1) \quad \Big| : b^2$$

$$\frac{a^2}{b^2} - \frac{4a}{b} + 1 = 0 \qquad \text{Note: } \frac{a}{b} = x$$

$$x^2 - 4x + 1 = 0$$

$$\Delta = 16 - 4 = 12$$

$$x_1, x_2 = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

$$a = b(2 \pm \sqrt{3})$$

$$y - y_0 = m(x - x_0)$$

$$m = \frac{1}{2 \pm \sqrt{3}}$$

$A = (2,0)$

$$y - 0 = \frac{1}{2 \pm \sqrt{3}}(x-2)$$

I $\quad y - 0 = \dfrac{1}{2+\sqrt{3}}(x-2) \Rightarrow \boxed{y = (2-\sqrt{3})(x-2)}$

II $\quad y - 0 = \dfrac{1}{2-\sqrt{3}}(x-2) \Rightarrow \boxed{y = (2+\sqrt{3})(x-2)}$

4.5. Consider the vector **v** which is perpendicular on **a**$(4,-2,-3)$ and on **b**$(0,1,3)$. If **v** describes an acute angle with Ox and $|\mathbf{v}| = 26$ determine the components of **v**.

• $v \perp a(4,-2,-3) \Rightarrow \langle v, a\rangle = 0$

$$4x - 2y - 3z = 0 \qquad (1)$$

• $v \perp b(0,1,3) \Rightarrow \langle v, b\rangle = 0$

$$y - 3z = 0 \qquad (2)$$

• $\measuredangle(v, Ox)$ acute $\Rightarrow \cos\measuredangle(v, i) > 0 \Rightarrow$

$$\Rightarrow \frac{\langle v, i\rangle}{|v|\cdot|i|} > 0 \Leftrightarrow \langle v, i\rangle > 0 \qquad (3)$$

• $|v| = 26 \Rightarrow \sqrt{x^2+y^2+z^2} = 26$ (4)

(2) $\Rightarrow y = -3z \Rightarrow z = -\frac{y}{3}$

now substitute in (1)

$$\left.\begin{aligned} 4x - 2y - 3\left(-\frac{y}{3}\right) &\Rightarrow x = \frac{y}{4} \\ z &= -\frac{y}{3} \end{aligned}\right\}$$

$$\Rightarrow \vec{v} = \left(\frac{y}{4}, y, -\frac{y}{3}\right)$$

$$\left(\frac{y}{4}\right)^2 + y^2 + \frac{y^2}{9} = 676$$

$$\left(\frac{1}{16} + 1 + \frac{1}{9}\right) y^2 = 676$$

$$\frac{169}{144} y^2 = 676$$

$$y^2 = \frac{676 \cdot 144}{169} \Rightarrow y = \frac{26 \cdot 12}{13} = \pm 24$$

4.6. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

A is orthogonal projection of the origin on π

$\Rightarrow$ the line conecting O to A is $\perp$ on the plane

$\vec{n} = (1,-1,3)$ is a normal vector

point-normal form of a plane:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$\vec{n} = (a,b,c)$

$A = (x_0, y_0, z_0)$

$$1(x-1)+1(y+1)+3(z-3)=0$$

$$\Rightarrow \boxed{x-y+3z-11=0}$$

4.7. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine equations for the angle bisector of the angle $\measuredangle CAB$.

4.8. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.

4.7 $\vec{AB} = B - A = (1, 0, 0)$

$\vec{AC} = C - A = (2, 2, 2)$

$|\vec{AB}| = 1$

$|\vec{AC}| = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$

$$u_1 = \frac{\vec{AB}}{|\vec{AB}|} = (1, 0, 0)$$

$$u_2 = \frac{\vec{AC}}{|\vec{AC}|} = \left(\frac{2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}\right)$$

$$\vec{v} = u_1 + u_2 = \left(1 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$\measuredangle CAB \Rightarrow P = A + t\,\vec{v}$

$$P = (1, 2, -7) + t\left(1 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\begin{cases} x = 1 + t\left(1 + \frac{1}{\sqrt{3}}\right) \\ y = 2 + \frac{1}{\sqrt{3}} \\ z = -7 + \frac{1}{\sqrt{3}} \end{cases}$$

4.8 $\quad P = (0, 2, 0) \qquad Q = (-1, 0, 0)$

$$\overrightarrow{PQ} = Q - P = (-1, -2, 0)$$

z axis $\Rightarrow \quad \vec{v} = (0, 0, 1)$

$$\cos(60°) = \frac{\underbrace{\vec{n} \cdot \vec{v}}_{= c}}{|\vec{n}| \cdot \underbrace{|\vec{v}|}_{= 1}} = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{2}$$

$$\Rightarrow 2c = \sqrt{a^2 + b^2 + c^2}$$

$$4c^2 = a^2 + b^2 + c^2 \Rightarrow \boxed{a^2 + b^2 = 3c^2}$$

$\vec{n}$ is the normal vector (a, b, c)

$|\vec{n}| = \sqrt{a^2 + b^2 + c^2}$

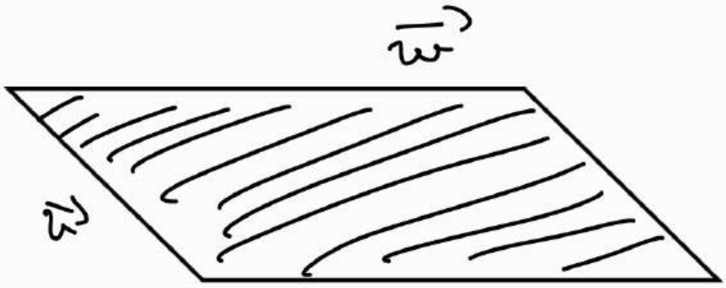
Teorie curs 5

am luat-o direct din seminarul de la Miu

Cross product

$\vec{v}, \vec{w} \in V^3 \to \vec{v} \times \vec{w} = |\vec{v}| \cdot |\vec{w}| \sin(\vec{v}, \vec{w})$

= area (parallogram on $\vec{v}$ and $\vec{w}$)



- $\vec{v} \times \vec{w} \perp \vec{v}$, $\vec{v} \times \vec{w} \perp \vec{w}$
- orientation : right hand rule (screw rule)

Properties of the cross product

- $(\vec{v_1} + \vec{v_2}) \times \vec{w} = \vec{v_1} \times \vec{w} + \vec{v_2} \times \vec{w}$
- $(\alpha v) \times \vec{w} = \alpha (\vec{v} \times \vec{w})$

$\vec{v} \times \vec{w} = - \vec{w} \times \vec{v}$

$\vec{v} \times \vec{v} = 0$

If we assume the reference frame to be right orthonormal

$\vec{v}(x_1, y_1, z_1)$

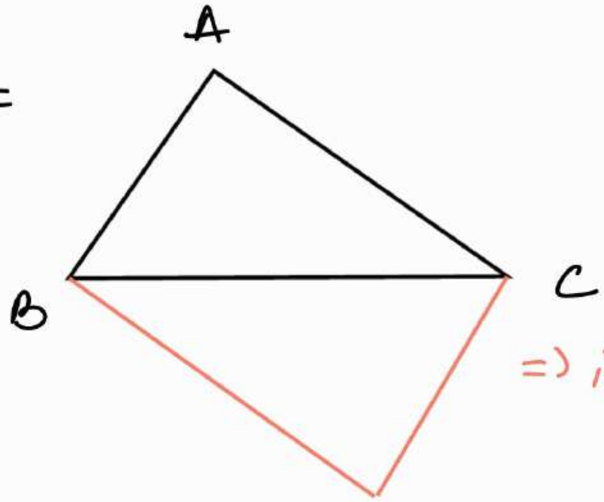
$\vec{w}(x_2, y_2, z_2)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \cdot \vec{i} + \underbrace{\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}}_{} \cdot \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \cdot k$$

we swapped the columns due to "-" coming from the dezvoltare

Applications

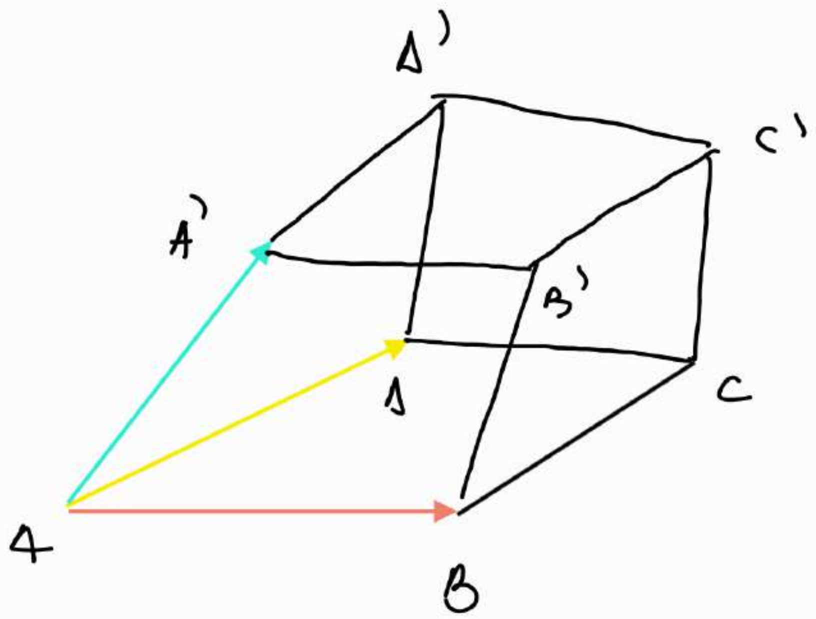
- areas =



=> it is a parallogram now

$$A_{ABC} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

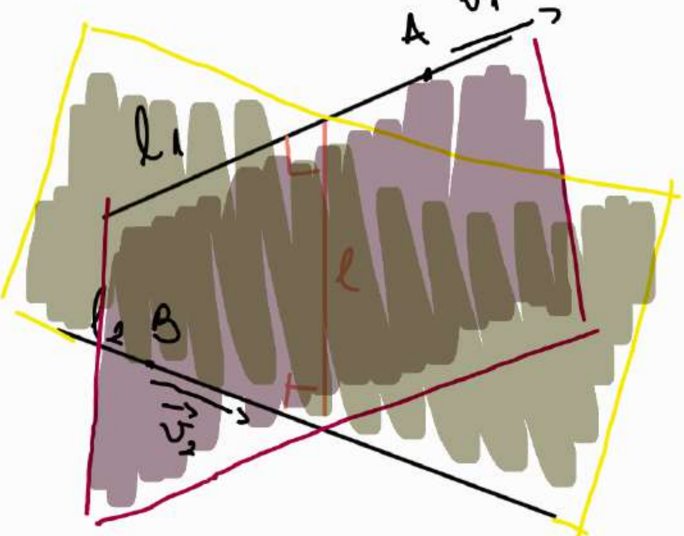
- Volumes



$$vol(ABCDA'B'C'D') = \left[\vec{AB}, \vec{AC}, \vec{AD}\right] = (\vec{AB} \times \vec{AC}) \cdot \vec{AD}$$

$$Vol(ABDA') = \frac{1}{6}\left[\vec{AB}, \vec{AC}, \vec{AA'}\right]$$

The common perpendicular of two skew lines



$A \in l_1$

$B \in l_2$

$\vec{v_1} \in \Delta(l_1)$

$\vec{v_2} \in \Delta(l_2)$

Π_1 = plane given by $A, \vec{v_1}, \vec{v_1} \times \vec{v_2}$

Π_2 = plane given by $B, \vec{v_2}, \vec{v_1} \times \vec{v_2}$

the common perpend is $\ell : \begin{cases} \Pi_1 \\ \Pi_2 \end{cases}$

Warm-up exercises

5.1. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Calculate $\mathbf{a} \times \mathbf{b}$.

5.2. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC}(1.5,2,1)$.

5.3. Consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

5.4. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

5.1

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 7 & 4 & 6 \end{vmatrix} = i \begin{vmatrix} 2 & -2 \\ 4 & 6 \end{vmatrix} + j \begin{vmatrix} -2 & 1 \\ 6 & 7 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 7 & 4 \end{vmatrix} =$$

$$= 20i - 20j - 10k = (20, -20, -10)$$

5.2 cross product gives area of the

parallelogram

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 6 & 0 & 1 \\ 1.5 & 2 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + j \begin{vmatrix} 1 & 6 \\ 1 & 1.5 \end{vmatrix} + k \begin{vmatrix} 6 & 0 \\ 1.5 & 2 \end{vmatrix}$$

$$= -2i - 4.5j + 12k$$

$$\vec{n} = (-2, -4.5, 12)$$

$$\text{Area} = \text{base} \cdot \text{hight} \Rightarrow h = \frac{\text{Area}}{\text{base}}$$

$$= \frac{\|\vec{n}\|}{\|\vec{AB}\|} = \sqrt{\frac{168.25}{37}}$$

$$\|\vec{n}\| = \sqrt{4 + 20.25 + 144} = \sqrt{168.25}$$

$$\|\vec{AB}\| = \sqrt{6^2 + 0^2 + 1^2} = \sqrt{37}$$

5.3 $a(2,-3,1)$ $c(1,2,3)$

$b(-3,1,2)$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = i\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} + j\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ -3 & 1 \end{vmatrix}$$

$$= -7i - 7j - 7k = (-7,-7,-7)$$

$$(a \times b) \times c = \begin{vmatrix} i & j & k \\ -7 & -7 & -7 \\ 1 & 2 & 3 \end{vmatrix} =$$

$$= i\begin{vmatrix} -7 & -7 \\ 2 & 3 \end{vmatrix} + j\begin{vmatrix} -7 & -7 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} -7 & -7 \\ 1 & 2 \end{vmatrix}$$

$$= -7i + 14j - 7k = (-7, 14, -7)$$

5.4 $\overrightarrow{AB} = B - A = (-1,-1,6)$

$\overrightarrow{AC} = C - A = (-2, 0, 2)$

$\vec{AD} = D - A = (1, -1, 4)$

$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0 \Rightarrow$ coplanare

Seminar 5

5.1. Consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

[redacted] $(2\mathbf{a}+\mathbf{b})\times\mathbf{b}, \quad (2\mathbf{a}+\mathbf{b})\times(2\mathbf{a}-\mathbf{b}).$

• $(2a+b)\times b = 2a\times b + \underbrace{b\times b}_{0}$

$$2a\times b = \begin{vmatrix} i & j & k \\ 6 & -2 & -4 \\ 1 & 2 & -1 \end{vmatrix} = \dots$$

• $(2a+b)\times(2a-b)$

$= 4\vec{a}\times\vec{a} + 2(\vec{a}\times\vec{b}) + 2(\vec{b}\times\vec{b}) - \vec{b}\times\vec{b}$

5.2. Consider the points $A(1,2,0)$, $B(3,0,-3)$, $C(5,2,6)$, $D(1,0,1)$. Determine

a) the area of the triangle ABC and the distance from C to AB.

b) the volume of the tetrahedron $ABCD$ and the distance from D to ABC.

c) the common perpendicular line of the lines AB and CD.

a) $A_{\Delta ABC} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} 28 = 14$

$\overrightarrow{AB} = B - A = (2, -2, -3)$

$\overrightarrow{AC} = C - A = (4, 0, 6)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} = (-12, -24, 8)$$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{144 + 256 + 64} = 28$

$d(C, AB) = \frac{2 \cdot A_{\Delta ABC}}{|\overrightarrow{AB}|} = \frac{28}{\sqrt{17}}$

b) $Vol_{ABCD} = \frac{1}{6} |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| =$

$= \frac{1}{6} \cdot 56 = \frac{28}{3}$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = (-12, 24, 8) \cdot (0, -2, 1) = 56$$

$$\vec{AD} = D - A = (0, -2, 1)$$

$$\text{dist}(D, ABC) = \frac{3 \cdot Vol}{A_{\Delta ABC}} = 2$$

c) $\vec{AB} = (2, -2, -3)$

$\vec{CD} = (-4, -2, -5)$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ -4 & -2 & -5 \end{vmatrix} = (4, 22, -12)$$

$$\Pi_1 : (A, \vec{AB}, \vec{AB} \times \vec{CD}) \qquad A(1, 2, 0)$$

$$\Pi_1 : \begin{vmatrix} x-1 & y-2 & z \\ 2 & -2 & -3 \\ -4 & -2 & -5 \end{vmatrix} = 0$$

$$\Rightarrow \Pi_1 : 90x + 12y + 52z - 113 = 0$$

$$\Pi_2 : (C, \overrightarrow{CD}, \overrightarrow{AB} \times CD)$$

$$\Pi_2 : \begin{vmatrix} x-5 & y-2 & z-6 \\ -4 & -2 & -5 \\ 4 & 22 & -12 \end{vmatrix} = 0$$