Convergence Dynamics in Graphical Replication Games

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***Abstract*—We consider graphical replication games as a model of content placement in distributed systems. The nodes replicate objects, which are accessed by local or remote users. Local users access objects according to a given demand. Accessing an object from a local or remote node has different costs. Each node chooses a set of objects to replicate in her local storage, so as to minimize her own total cost. The underlying communication network limits the interactions among the nodes and is modeled using a social graph. We show the existence of Nash equilibria in graphical replication games and analyze the conditions under which replication games converge to a Nash equilibrium. We provide a bound on the complexity of computing and of converging to a Nash equilibrium.**

I. Introduction

Content replication is widely used in computer architectures and in the Internet to improve system performance. In computer architectures CPU caches decrease memory access latencies [**?**]. In the Internet caches provide faster access to content for local customers and at the same time decrease the amount of network traffic [**?**]. Content replication is also at the core of clean-slate information centric network architectures [**?**].

A distributed replication group [**?**], [**?**], [**?**], consisting of nodes and users located near the nodes, is often used as a model of the problem of content replication. The nodes can replicate objects, which are accessed by local or remote users. The cost incurred by a user accessing an object depends on the object's placement. A commonly studied problem is how to minimize the total cost of users accessing the objects through either optimal node placement [**?**] or through the optimal allocation of objects to nodes [**?**].

Nevertheless, autonomous nodes would not implement the optimal solution in lack of a central authority [**?**], [**?**], instead they would replicate the objects that minimize their own costs, and would update the set of replicated objects as a response to the decisions made by other nodes. Given the autonomous nature of the nodes, an important question is whether there exists an equilibrium state in terms of object placement from which no node has an interest to deviate and whether in a distributed system the nodes would be able to reach an equilibrium state if each node follows a myopic strategy to minimize its own cost. Finally, assuming that the nodes would be able to reach an equilibrium state after some number of updates made by every node, an important question is how many steps it would take to reach equilibrium, and what factors influence the number of steps needed to reach equilibrium.

II. System Model

Consider a set N of nodes and a set $\mathcal{O}$ of objects of unit size. The demand for object $o \in \mathcal{O}$ at node $i \in N$ is given by the rate $w_i^o \in \mathbb{R}_+$. Each node i replicates $K_i \in \mathbb{N}_+$ different objects. We describe the set of objects replicated at node i with the $|\mathcal{O}|$ dimensional vector $r_i = (r_i^1, \ldots, r_i^{|\mathcal{O}|})$, whose component $r_i^o \in \{0,1\}$ is 1 if object o is replicated in node i. We model the neighborhood relationships between the players by a social graph that allows us to capture the limited interactions between nodes imposed by an underlying communication network [**?**]. We denote by $\mathcal{N}(i)$ the set of neighbors of player i. To ease notation we define $\pi_i^o \triangleq \prod_{j\in\mathcal{N}(i)} (1-r_j^o)$. Note that π_i^o is a function of r^o and it is equal to 0 if object o is replicated by at least one of node i's neighbors, 1 otherwise. The marginal cost of serving requests for object o in node i is α_i if the object is replicated in node i, it is β_i if the object is replicated in a node $j \in \mathcal{N}(i)$ neighboring i, and it is γ_i otherwise. It is reasonable to consider that it is not more costly to access an object replicated locally than one replicated at a neighbor, and it is less costly to access an object replicated at a neighbor than retrieving it directly from the common set of objects. Formally $\alpha_i \le \beta_i < \gamma_i$. The cost of node i due to object o is proportional to the demand w_i^o, and is a function of r_i and the replication states r_{-i} of the neighboring nodes

$$C_i^o(r_i^o, r_{-i}^o) = w_i^o\left(\alpha_i r_i^o + (1-r_i^o)\left[\gamma_i\pi_i^o \; + \; \beta_i(1-\pi_i^o)\right]\right),$$

and its total cost is $C_i(r_i, r_{-i}) = \sum_{o\in\mathcal{O}} C_i^o(r_i^o, r_{-i}^o)$

The goal of node i is to choose a replication strategy r_i^* that minimizes its total cost given the strategy profile r_{-i} of the other nodes. It is possible to show that finding the minimum cost is equivalent to maximizing the aggregated cost saving

$$CS_i(r_i, r_{-i}) = \sum_o r_i^o w_i^o\left[\beta_i(1-\pi_i^o) + \gamma_i\pi_i^o - \alpha_i\right].$$

We model this problem of selfish replication as a multiplayer non-cooperative game played on a graph, called a graphical game. The players are the nodes, the set of actions of player i is the set of feasible replication configurations $\mathcal{A}_i = \{r_i | \sum_o r_i^o \le K_i\}$, and the utility function of player i is $U_i(r_i, r_{-i}) = CS_i(r_i, r_{-i})$.

III. Results and Ongoing Work

The concept of best reply is widely used in game theory. A strategy played by a player is a *best reply* when it yields her

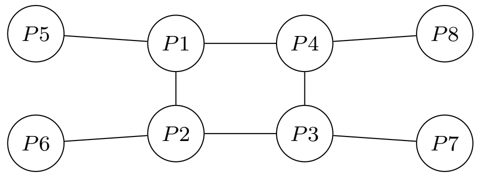

Fig. 1. Social graph that allows a cycle in best replies.

the highest utility given the other players' strategies. A Nash equilibrium is a strategy profile where every player is playing a best reply to the other players' strategies in the profile. In our previous work [?] we address the problem of the existence of Nash equilibria in replication games and of the convergence of the players' strategies to a Nash equilibrium.

Consider the strategy profile that consists of the best replies that the players would play on an edgeless social graph. In this strategy profile every player i replicates the K_i objects with highest demands w_i^o. Starting from this profile and performing a sequence of best replies in a particular order it is always possible to compute a Nash equilibrium.

Given the existence of Nash equilibria in graphical replication games, a natural question is whether it is always possible to reach an equilibrium by letting the players perform best replies one at a time. Considering the social graph in Figure 1, it is possible to find a sequence of best replies that contains a cycle, hence not all the sequences of best replies are finite in graphical replication games. Nevertheless our results show that it is always possible to find an order of strategy updates so as to reach a Nash equilibrium from an arbitrary strategy profile in a bounded number of steps. As a consequence, if the players make their updates in random order, they will eventually reach an equilibrium.

The possibility for the players to cycle arbitrarily long before reaching a Nash equilibrium raises the question of whether it is possible to guarantee the finiteness of every sequence of best replies through enforcing constraints on the game structure. We show that each best reply performed by a player can only move towards objects with higher demand if the social graph is complete. Since the number of objects is finite, the players always reach a Nash equilibrium after a finite number of updates if they are arranged on a complete social graph. In addition we prove that for singleton replication games, best reply cycles cannot exist either when the social graph is a forest.

Another way to ensure convergence to a Nash equilibrium is constraining the utility function. We prove that, if $\beta_i = \alpha_i$ $\forall i \in N$, then every sequence of *lazy improvement steps* is finite. We say that player i performs a *lazy improvement step* if she improves her strategy, making the minimal number of changes that lead to the same utility.

New issues arise when designing the implementation of sequences of best replies or lazy improvement steps in a distributed system. In order to guarantee that the nodes of a distributed system update their strategies one at a time, we would require global synchronization, which can be impractical in large systems. Hence, an important question is whether the players would always reach a Nash equilibrium even if some players would update their strategies simultaneously. We prove that the previous results, concerning sequences of best replies and lazy improvement steps, hold also in the case when every player i makes an improvement step at time t only if no neighboring player $j \in \mathcal{N}(i)$ makes an improvement step at time t. Hence, in order to maximize the convergence speed, it is sufficient to find a minimum vertex coloring of $\mathcal{G}$.

We are also working on assessing the complexity of computing and converging to a Nash equilibrium. The preliminary results show that it is possible to compute a Nash equilibrium of a graphical replication game in at most $\sum_{i\in N}\sum_{j\in\mathcal{N}(i)} K_j$ steps. Furthermore it is possible to show that from an arbitrary strategy profile there exists a sequence of best replies that reaches a Nash equilibrium in at most $\sum_{i\in N} K_i + \sum_{i\in N}\sum_{j\in\mathcal{N}(i)} K_j$ steps.

IV. Summary and Future Work

We considered graphical replication games and we showed the existence of Nash equilibria and analyzed the conditions under which replication games converge to a Nash equilibrium. We provided a bound on the complexity of computing a Nash equilibrium and on the number of steps required to converge to a Nash equilibrium.

Our future work will include the investigation of the properties of social graphs that would allow fast convergence of the players' strategies to an equilibrium. Finding such graphs could ultimately allow the design of network topologies that guarantee the number of updates to reach a stable state to be minimal.

Another interesting extension of our model would include the cost of accessing replicas. In fact, in the current replication model, a player does not incur any costs due to other players accessing her replicas. In a real scenario instead, such accesses could potentially be costly, and the properties of existence and convergence to a Nash equilibrium would be called into question again.