Tests for Heterogeneous Treatment Effect

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Abstract

Recent advances in causal machine learning have facilitated reliable estimators of the average treatment effect (ATE) with valid statistical inference. However, applying similar techniques to conditional average treatment effects (CATE) poses significant inferential challenges. In this paper, we propose three hypothesis tests to detect the existence of heterogeneous treatment effects. These tests inform researchers about whether the treatment effect is a constant across subpopulations defined by the covariates, thereby bridging the gap between inference on the ATE and the more ambitious task of fully characterizing those heterogeneities. Our tests build on three causal parameters: the projection of CATE on covariates, the variance of the CATE, and the variance difference between the potential outcomes. The test statistics are derived from the influence functions of the proposed parameters and are illustrated through Monte Carlo simulations and two empirical applications.

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1 Introduction

In the literature of empirical treatment effect analysis, most of the papers focus on the estimation and inference of the average treatment effect (ATE) identified under the unconfoundedness. Nonparametric estimators with valid statistical inference have been developed to reduce reliance on model assumptions(e.g. Van Der Laan and Rubin, 2006; Chernozhukov et al., 2018). These practices evaluate the treatment effect by its average over the whole population but may overlook the heterogeneity in the treatment effect. Understanding heterogeneous treatment effect is crucial for two main objectives. First, it provides insights in the key drivers of the treatment effect and informs mechanism analysis. Second, it helps policy makers to find the optimal treatment assignment rule by identifying whether an individual would be better off under with or without the treatment.

To complement the established methods for ATE inference and advance our understanding of heterogeneous treatment effects, we propose three hypothesis tests for detecting the existence of heterogeneity. We focus on the null hypothesis that the conditional average treatment effect (CATE) is a constant across subpopulations defined by the covariates, against the alternative hypothesis of its negation. These tests streamline the empirical workflow by helping researchers determine when more detailed CATE analysis would be beneficial.

Our hypothesis tests are built on three causal parameters that summarize the heterogeneity of treatment effect: the projection of CATE on covariates, the variance of the CATE, and the variance difference between the potential outcomes. These parameters offer a significant advantage over the infinite-dimensional CATE function, as their low-dimensional nature allows us to leverage recent advances in causal machine learning for statistical inference. In developing our approach, we appeal to the influence functions of these parameters to construct $\sqrt{n}$-consistent and asymptotically normal estimators. The resulting estimators are equivalent to those obtained from double/debiased machine learning methods (Chernozhukov et al., 2018). For inference, we provide a set of conditions that unifies the classical Donsker condition on the complexity of nonparametric estimators with the modern approach of cross-fitting.

There has been an emerging literature on nonparametric tests for heterogeneous treatment effect, (e.g. Crump et al., 2008; Chang et al., 2015; Hsu, 2017; Sant'Anna, 2021; Dai et al., 2023), under different definitions of heterogeneity and hypotheses. Our approach distinguishes from prior work primarily through its integration of machine learning methods to address high-dimensionality and nonlinearity of the data.

Our first test, the CATE Projection Test (CPT), examines the joint significance of projection coefficients of the CATE on a set of covariates. The idea of direct examination of the CATE function is similar to Crump et al. (2008), who proposed a test based on series estimation of the regression functions of the treated and untreated potential outcomes. Their method was later generalized by Sant'Anna (2021) to test for heterogeneity in duration outcome. The CPT also shares the theoretical connections with machine learning estimation of the CATE function. CATE estimation is challenging because canonical machine learning methods are designed for outcome prediction by minimizing the mean squared error loss. The literature has developed two main strategies for CATE estimation. Athey and Imbens (2016) and Wager and Athey (2018) modifies regression trees to optimize for the loss of CATE estimation. Another strand decomposes CATE estimation into a sequence of sub-regression problems and solve them using off-the-shelf machine learning methods (Zimmert and Lechner, 2019; Nie and Wager, 2021; Semenova and Chernozhukov, 2021; Fan et al., 2022). Our approach aligns with the latter strategy but diverges from these methods in its primary objective. Rather than pursuing accurate CATE function estimates by series or kernel estimators, we project the CATE onto a set of the covariates to detect heterogeneity, which substantially simplifies the estimation and inference procedure.

Our second and third tests introduce variance-based parameters to detecting treatment effect heterogeneity. The CATE Variance Test (CVT) examines the variance of the conditional effects, offering a novel measure of treatment effect variation across subpopulations. The Potential Outcome Variance Test (POVT) stems from the insight that the heterogeneity of treatment effect manifests in the variance change of potential outcomes. This fundamental idea was previously explored by Ding et al. (2016), who proposed a randomization test based on the variance ratio of potential outcomes. These variance-based parameters have received limited attention in the literature, perhaps because they do not characterize the specific function of the CATE. However, they prove especially valuable for hypothesis testing. An additional advantage of these two parameters is that they are identified without strong assumptions on the joint distribution of potential outcomes (Heckman et al., 1997). We contribute to the literature by developing identification strategies and machine learning estimators for these causal variance parameters.

The rest of the paper is organized as follows. Section 2 reviews the identification of CATE and presents the hypotheses of interest. In Section 3, we develop 3 tests for hetergeneous treatment effect. For each test, we systematically present: the definition of the causal parameter, its identification, the derivation of the corresponding influence function, and the construction of the test statistic. Section 4 gives Monte Carlo simulation results of the performance of our tests. In Section 5, we apply our proposed tests to the survey data from the Chinese Family Panel Studies (CFPS) regarding the effect of being the only child on the mental health of the only children, and the survey data of the 401(k) plan regarding the effect of 401(k) participation on net financial assets of the participants. Some concluding remarks are given in Section 6. All proofs are collected in Appendix A.

2 Identification of CATE and Hypotheses

Suppose we observe n i.i.d data $O_i = (Y_i, D_i, X_i)$ with $i = 1, ..., n$, distribution according to an unknown distribution P_0. D_i is a dummy variable indicating the status of treatment in a population of interest with $D_i = 1$ if individual i receives treatment and $D_i = 0$ otherwise. X_i is a vector of covariates that is potentially high-dimensional. Following the potential outcome framework or Rubin causal model by Rubin (1974), we define $Y_i(1)$ as the potential outcome of individual i with treatment and $Y_i(0)$ as the corresponding potential outcome without treatment. The observed outcome Y_i can be written as

$$Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0).$$

The CATE is defined as

$$\tau(x) = \mathbb{E}[Y_i(1) - Y_i(0) | X_i = x].$$

Further define the conditional expectations of the potential outcomes for the treated and control group by $\mu_0(d, x) = \mathbb{E}[Y_i|D_i = d, X_i = x]$ with $d = 0, 1$. To achieve identification of the CATE, we maintain the following two assumptions.

Assumption 1 (Unconfoundedness). $D_i \perp (Y_i(1), Y_i(0))|X_i$.

Assumption 2 (Overlap).

$$\exists\ \xi > 0,\ \ s.t.\ \xi \leq e_0(x) \leq 1 - \xi$$

where $e_0(x) = P(D_i = 1|X_i = x)$ *is the propensity score.*

Apply the AIPW (Robins et al., 1994) transformation and define the pseudo-outcome $\psi(o)$ as

$$\psi(O_i) = D_i \frac{Y_i - \mu_0(1, X_i)}{e_0(X_i)} - (1 - D_i) \frac{Y_i - \mu_0(0, X_i)}{1 - e_0(X_i)} + \mu_0(1, X_i) - \mu_0(0, X_i). \tag{1}$$

Under Assumption 1 and Assumption 2, standard results (e.g. Abrevaya et al., 2015; Semenova and Chernozhukov, 2021) ensures that the CATE is identified by the conditional expectation of the pseudo-outcome, namely,

$$\tau(x) = \mathbb{E}[\psi(O_i)|X_i = x]. \tag{2}$$

In this paper, we focus on the question whether the CATE is identical for all subpopulations defined by the covariates and consider the following hypotheses for the CATE:

$$\begin{aligned} &H_0 : \tau(x) \text{ is constant for all } x, \\ &H_a : \tau(x) \text{ is not constant for some } x. \end{aligned} \tag{3}$$

Under the null hypothesis, the average treatment effect is constant for all values of covariates. Throuhgout the paper, we refer to H_a in Equation (3) as treatment effect heterogeneity, which is particularly relevant for applications in policy evaluation and optimal treatment assignment. Hetergeneity may exist beyond the conditional average effect, and we will come back to this point in Section 3.3. As mentioned in

Section 1, direct tests on $\tau(x)$ are difficult to implement and therefore, we will instead develop parameters that summarizes the heterogeneity of the CATE and construct corresponding test statistics.

3 Tests for Heterogeneous Treatment Effect

3.1 CATE Projection Test (CPT)

Our first test is based on the projection coefficients of the CATE on the covariates. Let $b(X_i)$ be some function of the covariates which includes a constant. We will discuss the restrictions on $b(X_i)$[1] and some potentially useful specifications later.

Definition 1 (CATE Projection Coefficients). *The projection coefficients of the CATE on $b(X_i)$ is*

$$\dot{\beta}_0 = \mathbb{E}[b(X_i)'b(X_i)]^{-1}\mathbb{E}[b(X_i)'(Y_i(1) - Y_i(0))]$$

Throughout the paper, we use parameters with dots above them to represent causal parameters, which are defined by potential outcomes. The projection of the CATE provides a linear approximation of the true CATE function, and projection coefficients are general targets of inference. A similar parameter is also considered by Semenova and Chernozhukov (2021), where $b(x)$ is set to be a vector of basis functions of x and they propose a series estimator of the CATE by minimizing the the squared approximation error $\mathbb{E}[(\tau(X_i) - b(X_i)\dot{\beta}_0)^2]$.

In this paper, we focus on testing the existence of heterogeneity instead of accurately estimating the CATE. The change of the goalpost gives flexibility in specifying $b(x)$ and also largely reduces the complexity of the inference procedure. In the case when X_i is high-dimensional, for $\dot{\beta}_0$ to exist, we require $b(X_i)$ to be low-dimensional, which means that $b(X_i)$ needs to be a small subset of X_i. This imposes the assumption that

[1] In this paper X represents a $n \times q$ matrix where n is the number of observations and q is the number of covariates, while X_i represents the i-th row of X which is a p-dimensional row vector. The same layout applies to O and O_i.

the treatment effect is identified by a high-dimensional vector of covariates while the heterogeneity is only driven by a small proportion of the the covariates.

Proposition 1 (Identification of CATE Projection Coefficients)**.** *Under Assumption 1 and Assumption 2 and assume that* $\mathbb{E}[b(X_i)'b(X_i)]$ *is positive definite, the projection coefficients* $\dot{\beta}_0$ *is identified by*

$$\beta_0 = \mathbb{E}[b(X_i)'b(X_i)]^{-1}\mathbb{E}[b(X_i)'\psi(O_i)] = \dot{\beta}_0 \tag{4}$$

Given the identification result of $\dot{\beta}_0$ in Proposition 1, we now focus on the estimation and inference of β_0. We partition β_0 as $(\beta_c, \beta_x')'$ where β_c is the intercept and β_x is the rest of the projection coefficients, and then the hypotheses in Equation (3) can be translated into the following hypotheses.

$$\begin{aligned} H_0 &: \beta_x = 0, \\ H_a &: \beta_x \neq 0. \end{aligned} \tag{5}$$

We note that $\beta_x = 0$ is only a necessary conditon for the null hypothesis of no heterogeneity but not sufficient, which means that rejecting $H_0 : \beta_x = 0$ implies the existence of heterogeneity, however, accepting H_0 does not imply there is no heterogeneous treatment effect. There exists special cases when β_0 fails to capture treatment effect heterogeneity. For example, when the CATE function is parabolic in $b(x)$, leading to zero projection coefficient. Consequently, the test based on β_0 will lose power against certain directions in the alternative space. In such cases, it suffices to include polynomials or discretized X_i in $b(X_i)$. Moreover, we will introduce another two tests that do not suffer from this issue in Section 3.2 and Section 3.3.

If a consistent and asymptotically normal estimator of β_0 is available, the null hypothesis in Equation (5) can be eassily tested by a Wald test. In order to use machine learning methods to deal with potentially high-dimensional X_i and nonlinear functions of e_0 and μ_0, we appeal to recent developments in influence-function-based semiparametric estimation.

Proposition 2 (Influence Function of CATE Projection Coefficients)**.** *The influence* *function (IF) of β_0 is given by*

$$IF_\beta(O_i) = \mathbb{E}[b(X_i)'b(X_i)]^{-1}b(X_i)'\left(\psi(O_i) - b(X_i)\beta_0\right). \tag{6}$$

The IF in Equation (6) is the key to construct an estimator and perform hypothesis tests on β_0. There are several semiparametric methods replying on the IF, such as one-step estimation (e.g. Pfanzagl and Wefelmeyer, 1985; Bickel et al., 1993), estimating equation method (e.g. Laan and Robins, 2003; Chernozhukov et al., 2018)[2], and targeted maximum likelihood estimation (Van Der Laan and Rubin, 2006). In this paper, we use the estimating equation method and develop conditions under which the estimator is $\sqrt{n}$-consistent.

Let $\hat{\mu}$ and $\hat{e}$ denote consistent estimators of μ_0 and e_0, and

$$\hat{\psi}(O_i) = D_i\frac{Y_i - \hat{\mu}(1, X_i)}{\hat{e}(X_i)} - (1 - D_i)\frac{Y_i - \hat{\mu}(0, X_i)}{1 - \hat{e}(X_i)} + \hat{\mu}(1, X_i) - \hat{\mu}(0, X_i). \tag{7}$$

The estimating equation estimator of β_0 is given by solving $n^{-1}\sum_{i=1}^n IF_\beta(O_i) = 0$, namely,

$$\hat{\beta} = \left(\sum_{i=1}^n b(X_i)'b(X_i)\right)^{-1}\left(\sum_{i=1}^n b(X_i)'\hat{\psi}(O_i)\right) \tag{8}$$

To develop the asymptotic properties of $\hat{\beta}$, we adopt the following notations. Let $\|\cdot\|$ be the L_2 norm of a vector, $\|\cdot\|_P = \mathbb{E}[(\cdot)^2]^{1/2}$ be the $L_2(P_0)$ norm of a function.

Proposition 3 (Asymptotic Distribution of $\hat{\beta}$)**.** *Assume that the matrices $\mathbb{E}[b(X_i)'b(X_i)]$ and $\sum_{i=1}^n b(X_i)'b(X_i)$ are positive definite, there exists constants $\xi, K \in (0, \infty)$ such that $\|b(X_i)\| < K$, $\mathbb{E}[(Y_i - \hat{\mu})^2|X_i] < K$, $\hat{e}(X_i) \in (\xi, 1 - \xi)$ almost surely, and $\|e_0 - \hat{e}\|_P\|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$. Suppose one of the two conditions holds:*

(i) Donsker condition: the quantities $\hat{\mu}(D_i, X_i)$, $D_i(Y_i - \hat{\mu}(1, X_i))/\hat{e}(X_i)$, and $(1 - D_i)(Y_i - \hat{\mu}(0, X_i))/(1 - \hat{e}(X_i))$ fall within a P-Donsker class with probability approaching 1.

[2] The estimating equation method is called double/debiased machine learning in Chernozhukov et al. (2018), where IF is recognized as the Neyman orthogonal score.

(ii) Cross-fitting: The sample used to estimate $\hat{e}(x)$ and $\hat{\mu}(x)$ is independent of the sample used to construct $\hat{\beta}$.

Then $\hat{\beta}$ is a regular asymptotically linear estimator of β_0 with IF in Equation (6). Hence,

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} N(0, V_\beta)$$

with $V_\beta = \mathbb{E}[IF_\beta(O_i) IF_\beta(O_i)']$.

Proposition 3 provides a set of conditions under which $\hat{\beta}$ is asymptotically normal. The Donsder condition requires not to fit overly complex models and is usually assumed in the traditional nonparametric literature, such as Robins et al. (1994). However, the Donsker condition is not gauranteed to hold when modern machine learning methods are used. Alternative, the cross-fitting condition proposed by Chernozhukov et al. (2018) is able to incorporate a large class of machine learning methods like lasso, random forest, and neural networks, allowing us to handle high-dimensional X_i. The condition $\|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$ is standard in the debiased machine learning estimator (Chernozhukov et al., 2018) or the augmented inverse propensity estimator of the ATE. This condition is also recognized as a double robustness property since it implies that $\hat{\beta}$ is $\sqrt{n}$-consistent if either $\hat{\mu}$ or $\hat{e}$ converges sufficiently fast. It is not surprising to see that the CATE projection coefficients, which is a linear transformation of the ATE, inherits the double robustness property.

A straightforward way to satisfy the cross-fitting condition is to first random split the data into two halves, then obtain $\hat{\mu}$ and $\hat{e}$ by fitting the machine learning methods on one half and calculate $\hat{\beta}$ on the other half. This typically results in the loss of efficiency, as the sample size used to estimate $\hat{\beta}$ is halved. A more efficient algorithm is provided in Chernozhukov et al. (2018), which works as follows. First, randomly partition the n observations into K equal folds indexed by $(I_k)_{k=1}^K$. Each fold has observations $n_K = n/K$. Let I_k^c denote the observations without fold I_k. Using each I_k^c, we estimate the nuisance parameter μ_0 and e_0 by some machine learning method to obtain $\hat{\mu}_k$ and $\hat{e}_k$. Second, we construct $\hat{\psi}_k$ by plugging $\hat{\mu}_k$ and $\hat{e}_k$ into Equation (7) and calculate

$$\hat{\beta}_k = \left(\sum_{i \in I_k} b(X_i)' b(X_i) \right)^{-1} \left(\sum_{i \in I_k} b(X_i)' \hat{\psi}_k \right), \quad \hat{\beta} = \frac{1}{K} \sum_{k=1}^{K} \hat{\beta}_k. \tag{9}$$

Theorem 1 (CATE Projection Test)**.** *If Assumption 1, Assumption 2, and the conditions in Proposition 3 hold, under $H_0 : \beta_x = 0$, the Wald statistic*

$$T = \hat{\beta}_x'(\hat{V}_{\beta_x}/n)^{-1}\hat{\beta}_x \xrightarrow{d} \chi^2_{dim(\beta_x)}$$

where $\hat{\beta}_x$ is the estimator in Equation (8) or Equation (9) without intercept, and $\hat{V}_{\beta_x}$ is a consistent estimator of the submatrix of V_β corresponding to β_x.

The CPT in Theorem 1 is essentially the joint test on the significance of CATE projection coefficients. It is convenient to implement by contructing $\hat{\psi}(O_i)$ and then regressing on $b(X_i)$ as if $\hat{\psi}(O_i)$ is observed. Given the formula of $\hat{\beta}$ and V_β, the test statistic is readily available in the standard output of statistical softwares and no adjustment is needed for plugging in the machine learning estimators. For the α-level test, we reject the null hypothesis when T is greater than or equal to the $(1-\alpha)$-quantile of the $\chi^2_{\mathrm{dim}(\beta_x)}$ distribution. Another benefit of the CPT is that when Assumption 1 does not hold but a binary instrument is available, extension to heterogneity in the local average treatment effect is simple.

3.1.1 Extension to Binary IV

Suppose we observe data $O_i = (Y_i, D_i, X_i, Z_i)$ where Z_i is a binary instrument for the treatment D_i. Define the pseudo-outcome for some random variable R_i as

$$\psi^R(O_i) = Z_i\frac{R_i - r_0(1, X_i)}{q_0(X_i)} - (1 - Z_i)\frac{R_i - r_0(0, X_i)}{1 - q_0(X_i)} + r_0(1, X_i) - r_0(0, X_i)$$

where $r_0(z, x) = \mathbb{E}[R_i|Z_i = z, X_i = x]$ and $q_0(x) = \mathbb{E}[Z_i|X_i = x]$. Under the conditional local average treatment effect (CLATE) set up (e.g. Abadie, 2002), the CLATE $\tau_{late}(x)$ is identified by

$$\tau_{late}(x) = \frac{\mathbb{E}[\psi^Y(O_i)|X_i = x]}{\mathbb{E}[\psi^D(O_i)|X_i = x]}.$$

We focus on testing the null hypothesis of constant CLATE $H_0 : \tau_{late}(x)$ is constant for all x. Let $\alpha_0 = (\alpha_c, \alpha_x')'$ denote the projection coefficients of $\psi^Y(O_i)$ on $b(X_i)$, and $\gamma_0 = (\gamma_c, \gamma_x')'$ denote the projection coefficients of $\psi^D(O_i)$ on $b(X_i)$. The null hypothesis can be translated into

$$H_0 : \alpha_x = \frac{\alpha_c}{\gamma_c} \cdot \gamma_x. \tag{10}$$

Consider the regression of stacked $\psi^Y(O_i)$ and $\psi^D(O_i)$

$$\begin{bmatrix} \psi^Y(O_i) \\ \psi^D(O_i) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & b(X) & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & b(X) \end{bmatrix} \begin{bmatrix} \alpha_c \\ \gamma_c \\ \alpha_x \\ \gamma_x \end{bmatrix} + \begin{bmatrix} \epsilon^Y \\ \epsilon^D \end{bmatrix} = \tilde{b}(X)\beta_0 + \epsilon$$

where ϵ is the projection error of the stacked pseudo-outcome. We see that the coefficient in the above regression takes the same form as in Equation (4). Therefore, Proposition 3 directly applies. We can construct the design matrix $\tilde{b}(X)$ and run the stacked regression, then the null hypothesis in Equation (10) can be tested by combining Theorem 1 with the delta method. Specifically, the Wald statistic under the null is given by

$$T_{late} = l(\hat{\beta})'(\nabla l(\hat{\beta})' \cdot (\hat{V}/n) \cdot \nabla l(\hat{\beta}))^{-1} l(\hat{\beta}) \xrightarrow{d} \chi^2_{\dim(\alpha_x)} \tag{11}$$

where $l(\hat{\beta}) = \hat{\alpha}_x - \frac{\hat{\alpha}_c}{\hat{\gamma}_c} \cdot \hat{\gamma}_x$, $\nabla l(\hat{\beta})$ is the gradient of $l(\hat{\beta})$, and $\hat{V}$ is a clustered consistent estimator of the covariance matrix of $\hat{\beta}$ with clusters defined by whether the stacked pseudo-outcome is $\psi^Y(O_i)$ or $\psi^D(O_i)$.

3.2 CATE Variance Test (CVT)

The CPT is an intuitive and practical test for detecting the presence of treatment effect heterogeneity. However, as discussed in Section 3.1, the test may lose power against certain directions in the alternative space and requires low-dimensional heterogeneity. To address these issues, we propose a variance test for the CATE.

Definition 2 (CATE Variance). *The variance of the CATE is defined by*

$$\dot{\theta}_0 = Var(\tau(X_i)).$$

Assuming $\|\tau(X_i)\| < \infty$, $\dot{\theta}_0$ is well-defined. The variance of the CATE is a natural parameter for measuring the heterogeneity, which informs the extent to which the heterogeneity of treatment effect is explained by covariates. The benefit of considering $\dot{\theta}_0$ in hypoethsis testing is that, it summarizes the potentially high-dimensional heterogeneity into a scalar parameter, which avoids the issue of high-dimensional inference as we discussed in CPT.

Proposition 4 (Identification of the CATE Variance)**.** *Let $\nu_0(x) = \mu_0(1,x) - \mu_0(0,x)$. Under Assumption 1 and 2, the variance of the CATE $\dot{\theta}_0$ is identified by*

$$\theta_0 = \mathbb{E}\left[(\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])^2\right] = \dot{\theta}_0.$$

A sufficient and necessary condition for heterogeneous CATE is its variance not equal to zero. Then the hypotheses in Equation (3) can be translated into:

$$\begin{aligned} H_0 &: \theta_0 = 0, \\ H_a &: \theta_0 > 0. \end{aligned} \tag{12}$$

Now we proceed to develop consistent and asymptotically normal estimators of θ_0 based on the IF.

Proposition 5 (Influence Function for the CATE Variance)**.** *The IF of θ_0 is given by*

$$IF_\theta(O_i) = (\psi(O_i) - \tau_0)^2 - (\psi(O_i) - \nu_0(X_i))^2 - \theta_0 \tag{13}$$

where $\tau_0 = \mathbb{E}[\nu_0(X_i)]$ is the ATE.

Based on $IF_\theta(O_i)$, the estimating equation estimator of θ_0 is given by

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left((\hat{\psi}(O_i) - \hat{\tau})^2 - (\hat{\psi}(O_i) - \hat{\nu}(X_i))^2\right) \tag{14}$$

where $\hat{\tau}$ is a consistent estimator of τ_0. One choice of $\hat{\tau}$ is the augmented inverse propensity weighting estimatior $n^{-1}\sum_{i=1}^{n}\hat{\psi}(O_i)$ (Chernozhukov et al., 2018).

Equation (14) also gives the intuition of the CATE variance. Since $\tau(X_i)$ works as the conditional expectation of $\psi(O_i)$, we can estimate $Var(\tau(X_i))$ by subtracting the variance of the "error" term $\psi(O_i) - \nu_0(X_i)$ from $Var(\psi(O_i))$.

Proposition 6 (Asymptotic Distribution of $\hat{\theta}$)**.** *Assume that there exists constants $\xi, K \in (0, \infty)$ such that $Var(\psi(O_i)|X_i) < K$, $\mathbb{E}[(Y_i - \hat{\mu})^2|X_i] < K$, $(\hat{\nu}(X_i) - \hat{\tau})^2 < K$, $\hat{e}(X_i) \in (\xi, 1-\xi)$ almost surely. Also assume that $\|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$, $\|\tau_0 - \hat{\tau}\|_P$ and $\|\nu_0 - \hat{\nu}\|_P$ are both $o_P(n^{-1/4})$. Suppose one of the two conditions holds:*

(i) Donsker condition: the quantities $(\hat{\psi}(O_i) - \hat{\nu}(X_i))^2$, $(\hat{\psi}(O_i) - \hat{\tau})^2$, and $\hat{\psi}(O_i)(\hat{\nu}(X_i) - \hat{\tau})$ fall within a P-Donsker class with probability approaching 1.

(ii) Cross-fitting: The sample used to estimate $\hat{e}(x)$, $\hat{\mu}(x)$, $\hat{\nu}(x)$, and $\hat{\tau}$ is independent of the sample used to construct $\hat{\theta}$.

Then $\hat{\theta}$ is a regular asymptotically linear estimator of θ_0 with IF in Equation (13). Hence,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V_\theta)$$

with $V_\theta = \mathbb{E}[IF_\theta(O_i)^2]$.

Compared to Proposition 3, an extra convergence rate condition is imposed by $\|\nu_0 - \hat{\nu}\|_P = o_P(n^{-1/4})$. By definition, $\hat{\nu}(x)$ can be constructed by $\hat{\mu}(1,x) - \hat{\mu}(0,x)$. The condition is satisfied if $\|\mu_0 - \hat{\mu}\|_P = o_P(n^{-\kappa})$ with $\kappa \geq 1/4$. Then $\|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$ implies that $\|e_0 - \hat{e}\|_P = o_P(n^{-1/2+\kappa})$. It suggests that $\hat{e}$ is allowed to converge at a slower rate but $\hat{\mu}$ has to converge fast. A better way to construct $\hat{\nu}$ is to use $\hat{\psi}$, such that only the condition $\|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$ binds and the double robustness property is preserved.

The construction of a cross-fitting estimator of $\hat{\theta}$ follows a similar procedure to that described in Section 3.1, which we omit for brevity. Given the result in Proposition 6, it is straightforward to construct a Z-test for the null hypothesis in Equation (12).

Theorem 2 (CATE Variance Test)**.** *If Assumption 1, Assumption 2, and the conditions in Proposition 6 hold, under $H_0 : \theta_0 = 0$, the Z-statistic*

$$Z_\theta = \frac{\hat{\theta}}{\sqrt{\hat{V}_\theta / n}} \xrightarrow{d} N(0, 1)$$

where $\hat{V}_\theta$ is a consistent estimator of V_θ.

Since the variance cannot be negative, one-sided test should be used. The null hypothesis is rejected at significance level α if $Z_\theta > z_{1-\alpha}$, where $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

3.3 Potential Outcome Variance Test (POVT)

In this section, we propose a test based on the difference between the variances of the potential outcomes.

Definition 3 (Variance Difference of Potential Outcomes)**.** *The variance difference of the potential outcomes is defined by*

$$\dot{\lambda}_0 = Var(Y_i(1)) - Var(Y_i(0)).$$

The causal parameter $\dot{\lambda}_0$ can be interpreted as the treatment effect on the variance of the potential outcomes. To see how the variance difference of the potential outcomes can be used to detect heterogeneity, let $\tau_i = Y_i(1) - Y_i(0)$ denote the individual treatment effect. We have

$$Y_i(1) = \tau_i + Y_i(0). \tag{15}$$

If the individual treatment effect is constant, then the variances of the potential outcomes are the same, i.e., $Var(Y_i(1)) = Var(Y_i(0))$. The equality fails to hold in the presence of heterogeneity, thus making $\dot{\lambda}_0$ a measure of treatment effect heterogeneity.

We propose an identification stategy by $Var(Y_i(d)) = \mathbb{E}[Y_i(d)^2] - \mathbb{E}[Y_i(d)]^2$. Formally, let

$$\begin{aligned}
\psi^1(O_i) &= \frac{D_i(Y_i - \mu_0(1, X_i))}{e_0(X_i)} + \mu_0(1, X_i) \\
\psi^0(O_i) &= \frac{(1 - D_i)(Y_i - \mu_0(0, X_i))}{1 - e_0(X_i)} + \mu_0(0, X_i) \\
\phi^1(O_i) &= \frac{D_i(Y_i^2 - \mu_0^2(1, X_i))}{e_0(X_i)} + \mu_0^2(1, X_i) \\
\phi^0(O_i) &= \frac{(1 - D_i)(Y_i^2 - \mu_0^2(0, X_i))}{1 - e_0(X_i)} + \mu_0^2(0, X_i)
\end{aligned}$$

where $\mu_0^2(d,x) = \mathbb{E}[Y_i^2(d)|X_i = x]$ for $d = 0,1$. Note that $\psi(O_i) = \psi^1(O_i) - \psi^0(O_i)$. Then the identification of $\dot{\lambda}_0$ is given by the following proposition.

Proposition 7 (Identification of the Variance Difference of Potential Outcomes)**.** *Under Assumption 1 and 2, the variance difference of the potential outcomes is identified by*

$$\lambda_0 = \left(\mathbb{E}[\phi^1(O_i)] - \mathbb{E}[\psi^1(O_i)]^2\right) - \left(\mathbb{E}[\phi^0(O_i)] - \mathbb{E}[\psi^0(O_i)]^2\right) = \dot{\lambda}_0.$$

And the hypothesis in Equation (3) can be translated into

$$\begin{aligned} H_0 &: \lambda_0 = 0, \\ H_a &: \lambda_0 \neq 0. \end{aligned} \tag{16}$$

It is important to note that $\lambda_0 = 0$ serves as a sufficient but not necessary condition for the null hypothesis that $\tau(x)$ is constant for all x. This distinction arises because λ_0 captures broader distributional treatment effects beyond the heterogeneity of conditional average effects $\tau(x)$. There may exist cases where these distributional effects aggregate to a constant for each value of x, resulting in $\lambda_0 \neq 0$ despite the null hypothesis of constant $\tau(x)$ holding true. Essentially, $\lambda_0 = 0$ is equivalent to the null hypothesis that τ_i is constant for all i. While the test in Equation (16) can be used to investigate distributional effects, it needs to be treated with caution for examining heterogeneity in conditional average effects.

Proposition 8 (Influence Function for the Variance Difference of Potential Outcomes)**.** *The IF of λ_0 is given by*

$$\begin{aligned} IF_\lambda(O_i) = \phi^1(O_i) - \mathbb{E}[\phi^1(O_i)] - 2\mathbb{E}[\psi^1(O_i)]\left(\psi^1(O_i) - \mathbb{E}[\psi^1(O_i)]\right) \\ - \left(\phi^0(O_i) - \mathbb{E}[\phi^0(O_i)] - 2\mathbb{E}[\psi^0(O_i)]\left(\psi^0(O_i) - \mathbb{E}[\psi^0(O_i)]\right)\right) \end{aligned} \tag{17}$$

Based on the IF, the estimating equation estimator of λ_0 is given by

$$\begin{aligned} \hat{\lambda} = \frac{1}{n}\sum_{i=1}^n \left(\hat{\phi}^1(O_i) + \left(\frac{1}{n}\sum_{i=1}^n \hat{\psi}^1(O_i)\right)^2 - \hat{\psi}^1(O_i) \cdot \frac{2}{n}\sum_{i=1}^n \hat{\psi}^1(O_i)\right) \\ - \frac{1}{n}\sum_{i=1}^n \left(\hat{\phi}^0(O_i) + \left(\frac{1}{n}\sum_{i=1}^n \hat{\psi}^0(O_i)\right)^2 - \hat{\psi}^0(O_i) \cdot \frac{2}{n}\sum_{i=1}^n \hat{\psi}^0(O_i)\right) \end{aligned} \tag{18}$$

where $\hat{\phi}^d$ and $\hat{\psi}^d$ are estimators of ϕ^d and ψ^d, analogous to the plug-in estimator $\hat{\psi}$ in Equation (7).

Based on the IF, we can derive the asymptotic distribution of $\hat{\lambda}$ and construct a Z-test similar to Proposition 6 and Theorem 2.

Proposition 9 (Asymptotic Distribution of $\hat{\lambda}$)**.** *Assume that there exists constants $\xi, K \in (0, \infty)$ such that $\hat{e}(X_i) \in (\xi, 1-\xi)$ almost surely, $\mathbb{E}[(Y_i - \hat{\mu})^2|X_i] < K$, $\mathbb{E}[(Y_i^2 - \hat{\mu}^2)^2|X_i] < K$, $\|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P = o_P(n^{-1/2})$, and $\|e_0 - \hat{e}\|_P \|\mu_0^2 - \hat{\mu}^2\|_P = o_P(n^{-1/2})$. Suppose one of the two conditions holds:*

(i) Donsker condition: the quantities $\hat{\phi}^d(O_i)$, $\hat{\psi}^d(O_i)$, and $\hat{\psi}^{d,2}(O_i)$ for $d = 0, 1$ fall within a P-Donsker class with probability approaching 1.

(ii) Cross-fitting: The sample used to estimate $\hat{e}(x)$, $\hat{\mu}(x)$, $\hat{\mu}^2(x)$ is independent of the sample used to construct $\hat{\lambda}$.

Then $\hat{\lambda}$ is a regular asymptotically linear estimator of λ_0 with IF in Equation (17). Hence,

$$\sqrt{n}(\hat{\lambda} - \lambda_0) \xrightarrow{d} N(0, V_\lambda)$$

with $V_\lambda = \mathbb{E}[IF_\lambda(O_i)^2]$.

Theorem 3 (Potential Outcome Variance Test)**.** *If Assumption 1, Assumption 2, and the conditions in Proposition 9 hold, under $H_0 : \lambda_0 = 0$, the Z-statistic*

$$Z_\lambda = \frac{\hat{\lambda}}{\sqrt{\hat{V}_\lambda / n}} \xrightarrow{d} N(0, 1)$$

where $\hat{V}_\lambda$ is a consistent estimator of V_λ.

Unlike the one-sided CVT, the POVT is a two-sided test. The reason is as follows. By decomposing $Var(Y_i(1))$ according to Equation (15), we have

$$Var(Y_i(1)) = Var(Y_i(0)) + Var(\tau_i) + 2Cov(\tau_i, Y_i(0)).$$

This decomposition yields a variance difference of $\dot{\lambda}_0 = Var(\tau_i) + 2Cov(\tau_i, Y_i(0))$.

Without additional assumptions on $Cov(\tau_i, Y_i(0))$, $\dot{\lambda}_0$ can take both positive and negative values under H_a.

An alternative quantity for testing the equality of variances is the ratio

$$Var(Y_i(1))/Var(Y_i(0)),$$

which was previously considered by Ding et al. (2016) in experiments. Proposition 7 suggests that this quantity is identified using our strategy, allowing the extension to observational data under Assumption 1. However, the F-test for the variance ratio requires the marginal distribution of potential outcomes to be normal, an assumption that is difficult to verify and impractical in empirical applications. Based on Equation (15), another possible way to test heterogeneity is by the variance of individual treatment effect $Var(Y_i(1) - Y_i(0))$. This quantity cannot be identified without imposing assumptions on the joint distribution of $(Y_i(1), Y_i(0))$. Therefore, we focus on $\dot{\lambda}_0$ as it is identified under standard assumptions in the treatment effect literature.

A final note is that, since both CPT and POVT are two-sided tests of equality, they can be combined to form a joint Wald test. According to Propostion 3 and 9, each estimator of β and λ is asymptotically normal based on the IFs. One can stack IF_β and IF_λ to obtain the variance-covariance matrix of $\hat{\beta}$ and $\hat{\lambda}$, and the construction of the joint Wald test statistic follows naturally. This joint testing approach enhances statistical power compared to individual tests. As discussed previously, the CATE projection test may fail to reject the null hypothesis in certain cases, combining it with POVT can address this limitation. On the other hand, combining the CPT benefits POVT, as variance, being a second-order statistic, typically requires larger samples to achieve asymptotic properties. Thus, a joint test will leverage the complementary strengths of each test to provide more robust detection of treatment effect heterogeneity. This idea of using joint test to improve statistical power was previously considered by Kleibergen (2005) in the context of GMM identification tests.

4 Simulation

In this section, we conduct Monte Carlo simulations to evaluate the size and power of the proposed tests. For each simulation exercise, we generate 1000 datasets with sample size $n = 500, 1000, 2000, 4000$, and report the rejection proportions at 5% significance level. The data generating process is specified as follows:

- Low-dimensional ($p = 10$) correlated covariates, nonsparse $\mu(\cdot, x)$, $e(x)$ and $\tau(x)$

  1. Constant CATE model:

  $$\begin{aligned} &X_i \sim N(0, \Sigma), \quad \Sigma_{ii} = 1,\ \Sigma_{ij,i\neq j} \sim \text{Uniform}(0.1, 0.3) \\ &D_i|X_i = x \sim \text{Bern}(e(x)), \quad e(x) = \frac{1}{1+\exp(-x'\alpha)},\ \alpha = (0.1, \ldots, 0.1)_{10\times 1} \\ &Y_i|D_i = d, X_i = x \sim N(x'\beta + 2d, 1/2), \quad \beta = (0.5, \ldots, 0.5)_{10\times 1} \end{aligned}$$

  2. Linear outcome model: treatment and covariates are distributed as above. The outcome is defined by

  $$\begin{aligned} Y_i|D_i = d, X_i = x \sim N(x'(\beta + d\cdot\gamma_1 + (1-d)\cdot\gamma_0), 1/2), \\ -\gamma_0 = \gamma_1 = (1, \ldots, 1)_{10\times 1} \end{aligned}$$

  3. Kinked outcome model: treatment and covariates are distributed as above. The outcome is defined by

  $$\begin{aligned} Y_i|D_i = d, X_i = x \sim N((1-d)\cdot x'\gamma + d\cdot \text{diag}(\mathcal{I}(x>0))\gamma, 1/2), \\ \gamma = (2, \ldots, 2)_{10\times 1} \end{aligned}$$

  4. Nonlinear outcome model: treatment and covariates are distributed as above. The outcome is defined by

  $$\begin{aligned} Y_i|D_i = d, X_i = x \sim N(\exp(x'\beta) + d\cdot x'\gamma, 1/2), \\ \gamma = (2, \ldots, 2)_{10\times 1} \end{aligned}$$

- High-dimensional ($p = 100$) uncorrelated covariates, sparse $\mu(\cdot, x)$, $e(x)$ and

$\tau(x)$

1. Constant CATE model: Models similar to the low-dimensional case, but we create sparsity by setting the parameters differently:

$$\Sigma \text{ is identity matrix,}$$
$$\alpha = \left((0.2, \ldots, 0.2)_{10\times 1}, (0, \ldots, 0)_{90\times 1}\right),$$
$$\beta = \left((2, \ldots, 2)_{20\times 1}, (0, \ldots, 0)_{80\times 1}\right)$$

2. Linear outcome model: Models similar to the low-dimensional case, with parameters defined as above, except that:

$$-\gamma_0 = \gamma_1 = \left((5, \ldots, 5)_{50\times 1}, (0, \ldots, 0)_{50\times 1}\right)$$

3. Kinked outcome model: Models similar to the low-dimensional case, with parameters defined as in constant CATE model, except that:

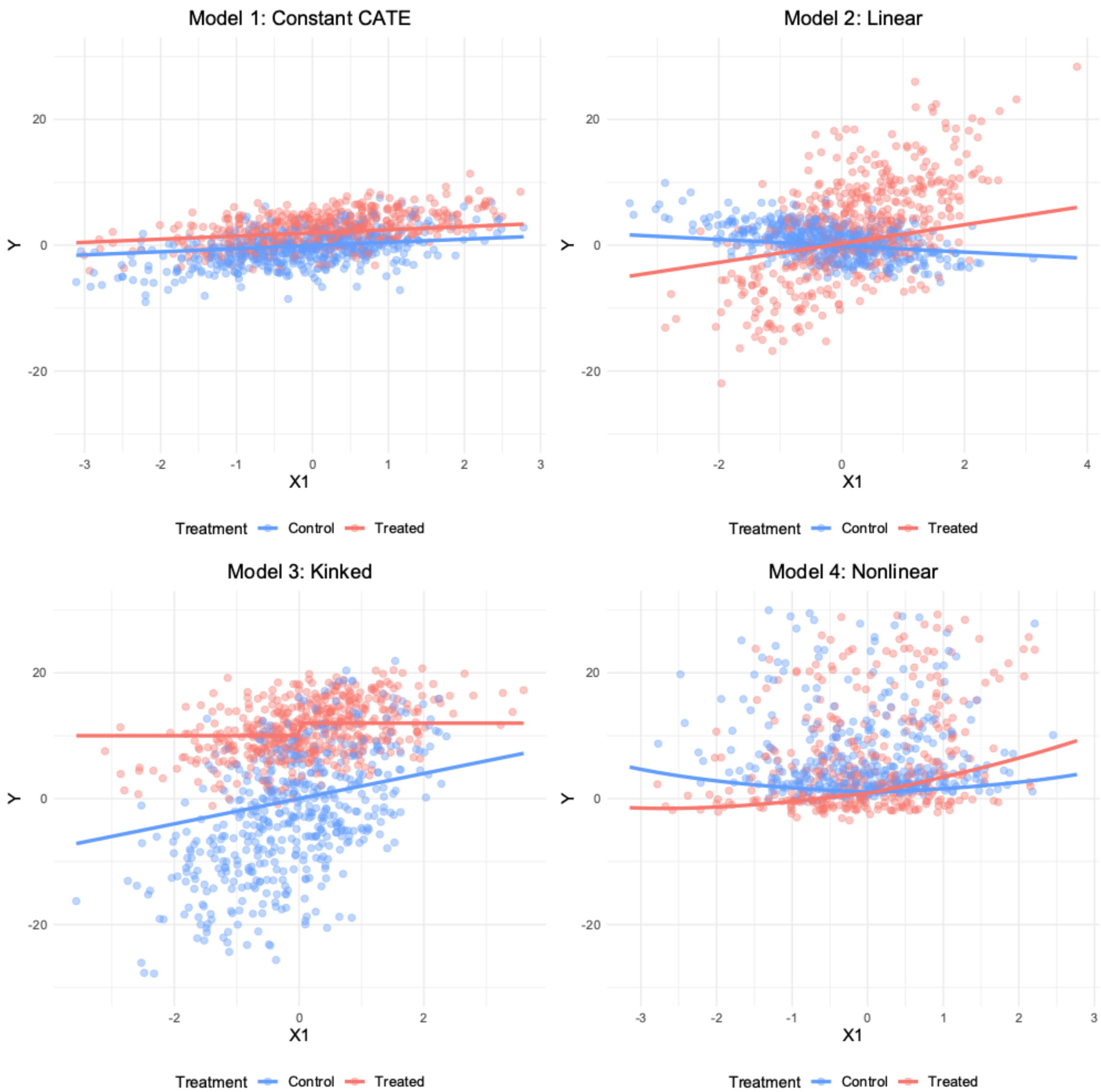
$$\gamma = \left((10, \ldots, 10)_{50\times 1}, (0, \ldots, 0)_{50\times 1}\right)$$

4. Nonlinear outcome model: Models similar to the low-dimensional case, with parameters defined as in constant CATE model, except that:

$$\beta = \left((1, \ldots, 1)_{20\times 1}, (0, \ldots, 0)_{80\times 1}\right),$$
$$\gamma = \left((5, \ldots, 5)_{50\times 1}, (0, \ldots, 0)_{50\times 1}\right)$$

Figure 1 provides visualizations of the data generating processes through the scatter plots of Y_i against X_{1i}, alongside the true relationships between potential outcomes and X_{1i}. The models are designed to reflect different patterns of treatment effects. The constant CATE model falls under the null of Equation (3), while the other three models fall under the alternative. The linear and nonlinear models are specifically designed to have a zero ATE, where the conventionalATE-targeted approaches such as OLS or IPW would fail to detect the existence of treatment effects.

Figure 1: Sketches of the Data Generating Processes



We consider two machine learning algorithms for estimating the nuisance parameters: Lasso and XGBoost[3]. The cross-fitting estimators analogous to Equation (9) are used to estimate β_0, θ_0 and λ_0, and we implement the tests proposed in Theorems 1, 2 and 3.

[3]We use the `glmnet` R package for Lasso and the `xgboost` package for XGBoost. For both algorithms, 5-fold cross-validation is used to select the optimal tuning parameters.

Table 1: Simulated Rejection Rate

	n	Lasso CPT	Lasso CVT	Lasso POVT	XGBoost CPT	XGBoost CVT	XGBoost POVT
Low Dimension							
Model 1	500	4.8	5.9	3.2	4.7	5.9	3.7
	1000	4.9	5.6	3.4	4.8	5.7	3.7
	2000	5.0	5.6	3.6	5.0	5.6	3.9
	4000	5.0	5.5	3.7	5.0	5.6	4.0
Model 2	500	93.3	87.4	71.5	91.4	89.1	80.1
	1000	100	96.2	87.5	100	100	98.4
	2000	100	100	99.9	100	100	100
	4000	100	100	100	100	100	100
Model 3	500	85.4	81.9	64.4	84.8	82.7	78.7
	1000	92.2	89.7	82.3	91.7	90.5	82.3
	2000	100	100	90.3	100	100	90.8
	4000	100	100	100	100	100	100
Model 4	500	91.9	88.7	61.2	92.2	91.9	74.9
	1000	96.7	93.9	88.3	99.4	97.8	88.9
	2000	100	100	95.3	100	100	94.6
	4000	100	100	100	100	100	100
High Dimension							
Model 1	500	4.5	5.6	3.0	4.5	5.6	3.0
	1000	4.6	5.5	3.2	4.6	5.5	3.2
	2000	4.6	5.3	3.3	4.6	5.3	3.3
	4000	4.7	5.2	3.5	4.7	5.2	3.5
Model 2	500	88.6	82.7	67.5	89.4	87.0	67.9
	1000	98.3	94.3	86.1	98.7	95.2	85.9
	2000	100	100	92.5	100	100	92.4
	4000	100	100	100	100	100	100
Model 3	500	77.2	79.9	62.6	84.3	86.8	61.3
	1000	89.6	92.5	80.0	94.9	93.3	81.9
	2000	99.9	100	88.6	100	100	89.7
	4000	100	100	98.6	100	100	97.8
Model 4	500	80.1	83.2	50.4	87.5	89.5	60.2
	1000	93.2	92.8	87.9	96.7	95.1	88.3
	2000	100	100	94.6	100	100	94.3
	4000	100	100	99.5	100	100	100

Empirical rejection proportions, in percentage points, at 5% significance level based on 1000 simulations.

The empirical rejection proportions of the tests are presented in Table 1. CPT and

CVT have good size properties, with the rejection proportions close to the nominal level 5%, while size of POVT is below the nominal level. In terms of power, CPT and CVT reach statisfactory rejection rate close to 100% at $n = 1000$ across all four models, whereas POVT requires larger sample sizes to attain comparable power. While all tests perform marginally better in the low-dimensional setting, this difference is not substantial. Finally, we observe that the performance of the tests also depends on the choice of machine learning algorithms. For the kinked and nonlinear models, which exhibit nonlinearity in the outcome regression, XGBoost-based tests outperform Lasso-based tests, with the improvement being more significant in the high-dimensional setting. This can be attributed to XGBoost's capability in capturing nonlinear patterns in the outcome regression. Overall, these simulation results demonstrate the robustness of our proposed tests across diverse data generating processes.

5 Application

In this section, we demonstrate the application of the proposed tests through the NSW job training program. In this program, participants were randomly assigned to either a job training program or a control group, and the treatment effect on future earnings can be estimated by directly comparing outcomes of the treated and control group. In order to evaluate the validity of econometric estimators of treatment effects, LaLonde (1986) compared the treated individuals from the experiment to control groups drawn from two survey datasets: the Panel Study of Income Dynamics (PSID) and Current Population Survey (CPS). The resulting datasets have been extensively analyzed in the influential works by Dehejia and Wahba (1999); Smith and Todd (2005); Angrist and Pischke (2009); Słoczyński (2022), among others. In the context of CATE hypothesis testing, the dataset was analyzed by Hsu (2017) and Dai et al. (2023), who focused specifically on the heterogeneity with respect to the age of the individuals. Using the proposed tests, we will examine the heterogeneity with respect to all available covariates.

The dataset we use is NSW-CPS, which contains 185 treated units from the experiment and 15992 control units from the CPS. The outcome Y_i is the earnings in 1978, and the treatment D_i is a binary indicator of whether the individual received the job training. We consider the set of covariates same as those in column 4 of Table 3.3.3 in Angrist and Pischke (2009), which includes age, age squared, education, dummy variables for black and Hispanic, marriage status, dummy indicator for high school degree, and pre-treatment earnings in 1974 and 1975. For this set of covariates X_i, we test for $H_0 : \tau(x) = 0$ for all covariate values x. For nuisance parameter estimation, we employ XGBoost, as well as a parametric approach with OLS for outcome regression and logistic regression for propensity score. Since OLS and logistic regression falls in the P-Donsker class trivially, we implement the estimators in Equation (8), (14) and (18) for CPT, CVT and POVT, respectively.

The test statistics and their corresponding p-values are presented in Table 2. All tests reject the null of constant CATE at 1% significance level, providing strong evidence for the presence of heterogeneous treatment effects. Table 3 presents parameter estimates from both parametric and XGBoost approaches. While there are some differences in the magnitude of the estimates between the parametric and XGBoost approaches, the signs of the estimates remain consistent. These estimates can offer additional insights into the heterogeneity. For example, parametric estimates of $\hat{\beta}$ suggests that marriage status and earnings in 1975 are potential drivers of the heterogeneity; the negative estimate of $\hat{\lambda}$ indicates that the treatment group exhibits smaller variance of earnings compared to the control group, suggesting that the job training program is effective in reducing the individual earning gaps. These findings complement the conventional ATE-focused analyses.

Table 2: Test Statistics and P-values

	Parametric		XGBoost	
Test	Stat.	p-value	Stat.	p-value
CPT	7722.17	< 0.01	4806.72	< 0.01
CVT	9.55	< 0.01	10.78	< 0.01
POVT	-2.91	< 0.01	-10.82	< 0.01

Table 3: Parameter Estimates and Standard Errors

		Parametric		XGBoost	
Parameter		Est.	S.E.	Est.	S.E.
	age	0.01	0.12	0.07	0.31
	age^2	0.00	0.00	-0.00	0.00
	educ	0.18	0.11	-0.15	0.31
β	*black*	1.18	2.34	5.24	7.00
	hispanic	-0.48	0.82	-0.16	2.45
	married	1.57	0.66	0.70	1.61
	nodegree	-1.82	0.60	-0.97	1.35
	$re74$	-0.09	0.18	-0.05	0.55
	$re75$	-0.50	0.16	-0.66	0.46
θ		23.70	2.48	51.91	4.82
λ		-9.72	3.34	-68.18	6.30

6 Conclusion

This paper develops three hypothesis tests for detecting heterogeneous treatment effects in observational studies, bridging the gap between average treatment effect inference and heterogeneity analysis. Our tests build on distinct but complementary measures: the CATE Projection Test examines linear projections of treatment effects on covariates, the CATE Variance Test examines variation in conditional effects, and the Potential Outcome Variance Test investigates variance change in the potential outcomes. By developing influence functions for these parameters, we enable valid inference under both classical nonparametric and modern machine learning frameworks, with Monte Carlo simulations demonstrating good size control and power against various data generating processes, including high-dimensional settings.

The empirical application to the NSW job training program reveals significant heterogeneity in treatment effects, highlighting the importance of moving beyond average effects in policy evaluation. Our work contributes to the growing literature on heterogeneous treatment effects by providing practitioners with practical tools to determine when more detailed analysis of effect heterogeneity is warranted. Since the building block of our tests is the influence function of the CATE, it is possible to extend the tests to other causal parameters, such as heterogeneity in treatment effect in the treated or quantile treatment effects, and we refer to Chernozhukov et al. (2018) and Firpo (2007) for the corresponding influence functions. We leave this for future research.

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A Appendix: Proofs

Proof of Proposition 1. By the law of iterated expectation, we have $\mathbb{E}[b(X_i)'\psi(O_i)] = \mathbb{E}[b(X_i)'\mathbb{E}[\psi(O_i)|X_i]]$. Under Assumption 1 and 2, $\mathbb{E}[\psi(O_i)|X_i] = \mathbb{E}[Y_i(1) - Y_i(0)|X_i]$, which follows from standard identification of augmented inverse propensity weighting estimators. $\square$

Proof of Proposition 2. We follow a simple strategy provided by Hampel (1974) to derive the influence function. Consider β_0 as a statistical functional $\beta(P_0)$ which maps the data distribution to the parameter space. Let f_t denote the probability density function of distribution P_t and $\mathbb{I}_{\tilde{o}}(o)$ denote the Dirac delta function with respect to $\tilde{o}$, that is, the density which is a point mass at o and zero elsewhere. Consider a perturbed density of f_0 in the direction of a single observation $\tilde{o}$,

$$f_t(o) = t\mathbb{I}_{\tilde{o}}(o) + (1-t)f_0(o)$$

with $t \in [0,1]$. The IF of $\beta(P_0)$ given by

$$IF_\beta(O_i) = \left.\frac{d\beta(P_t)}{dt}\right|_{t=0}$$

We begin with deriving the IF of $\mathbb{E}[Y_i(1)]$. First, notice that

$$\frac{d}{dt}f_t(o) = \mathbb{I}_{\tilde{o}}(o) - f_0(o), \tag{A.1}$$

which implies the IF of expectation $\mathbb{E}[O_i]$ is $O_i - \mathbb{E}[O_i]$. Under Assumption 1 and Assumption 2,

$$\mathbb{E}[Y_i(1)] = \mathbb{E}\left[D_i\frac{Y_i - \mu_0(1,X_i)}{e_0(X_i)} + \mu_0(1,X_i)\right] = \mathbb{E}[\mathbb{E}[Y_i|D_i = 1, X_i]].$$

Denote $\mathbb{E}_{P_t}[\cdot]$ as the expectation evaluated at P_t and $\mathbb{E}[\cdot] = \mathbb{E}_{P_0}[\cdot]$. The IF of $\mathbb{E}[Y_i(1)]$

is then

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}_{P_t}[\mathbb{E}_{P_t}[Y_i|D_i=1,X_i]]\bigg|_{t=0} \\
=&\frac{d}{dt}\int\int y\frac{f_t(y,1,x)f_t(x)}{f_t(1,x)}dydx\bigg|_{t=0} \qquad\qquad (A.2)\\
=&\int\int y\left(\frac{f(x)}{f(1,x)}\frac{d}{dt}f_t(y,1,x)\bigg|_{t=0}+\frac{f(y,1,x)}{f(1,x)}\frac{d}{dt}f_t(x)\bigg|_{t=0}-\frac{f(y,1,x)f(x)}{f(1,x)^2}\frac{d}{dt}f_t(1,x)\bigg|_{t=0}\right)dydx\\
=&\int\int y\frac{f(y,1,x)f(x)}{f(1,x)}\left(\frac{\mathbb{I}_{\tilde{y},\tilde{d},\tilde{x}}(y,1,x)}{f(y,1,x)}+\frac{\mathbb{I}_{\tilde{x}}(x)}{f(x)}-\frac{\mathbb{I}_{\tilde{d},\tilde{x}}(1,x)}{f(1,x)}-1\right)dydx\\
=&\frac{\mathbb{I}_{\tilde{d}}(1)}{e_0(\tilde{x})}(\tilde{y}-\mu_0(1,\tilde{x}))+\mu_0(1,\tilde{x})-\mathbb{E}[Y_i(1)].
\end{aligned}
$$

Similarly, we can replace $d=1$ with $d=0$ and $e_0(x)$ with $1-e_0(x)$ to obtain the IF of $\mathbb{E}[Y_i(0)]$. The IF of $\mathbb{E}[Y_i(1)-Y_i(0)]$ is then the difference of the two IFs, which is $\psi(O_i)-\mathbb{E}[\tau(X_i)]$. Then we have

$$
\begin{aligned}
IF_\beta(O_i)=&\frac{d\beta(P_t)}{dt}\bigg|_{t=0}\\
=&\frac{d}{dt}\left(\mathbb{E}_{P_t}[b(X_i)'b(X_i)]^{-1}\mathbb{E}_{P_t}[b(X_i)'\psi(O_i)]\right)\bigg|_{t=0}\\
=&\frac{d}{dt}\left(\mathbb{E}_{P_t}[b(X_i)'b(X_i)]^{-1}\right)\bigg|_{t=0}\mathbb{E}_{P_t}[b(X_i)'\psi(O_i)]+\mathbb{E}_{P_t}[b(X_i)'b(X_i)]^{-1}\frac{d}{dt}\mathbb{E}_{P_t}[b(X_i)'\psi(O_i)]\bigg|_{t=0}\\
=&\mathbb{E}[b(X_i)'b(X_i)]^{-1}\left(b(X_i)'\psi(O_i)-\mathbb{E}[b(X_i)'\psi(O_i)]\right)\\
&-\mathbb{E}[b(X_i)b(X_i)']^{-1}\left(b(X_i)'b(X_i)-\mathbb{E}[b(X_i)'b(X_i)]\right)\mathbb{E}[b(X_i)b(X_i)']^{-1}\mathbb{E}[b(X_i)'\psi(O_i)]\\
=&\mathbb{E}[b(X_i)'b(X_i)]^{-1}b(X_i)'\left(\psi(O_i)-b(X_i)\beta_0\right)
\end{aligned}
$$

where the fourth equation follows from the chain rule and the fact that the IF of $\mathbb{E}[b(X_i)'\psi(O_i)]$ is $b(X_i)'\psi(O_i)-\mathbb{E}[b(X_i)'\psi(O_i)]$. □

Proof of Proposition 4.

$$Var(\tau(X_i))=Var(\mathbb{E}[Y_i(1)-Y_i(0)|X_i])=Var(\mu_0(1,X_i)-\mu_0(0,X_i))$$

where the second equation holds under Assumption 1 and 2. □

Proof of Proposition 5. Let $\mu_t(d,x) = \mathbb{E}_{P_t}[Y_i|D_i = d, X_i = x]$ be the conditional expectation function evaluated at P_t. Accordingly, $\nu_t(x) = \mu_t(1,x) - \mu_t(0,x)$, and $e_t(x) = \mathbb{E}_{P_t}[D_i|X_i = x]$. Then

$$
\begin{aligned}
IF_\theta(O_i) &= \frac{d}{dt}\mathbb{E}_{P_t}[(\nu_t(X_i) - \mathbb{E}_{P_t}[\nu_t(X_i)])^2]\big|_{t=0} \\
&= \mathbb{E}\left[2(\nu_t(X_i) - \mathbb{E}[\nu_t(X_i)])\frac{d}{dt}(\nu_t(X_i) - \mathbb{E}_{P_t}[\nu_t(X_i)])\big|_{t=0}\right] \\
&\quad + \frac{d}{dt}\mathbb{E}_{P_t}[(\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])^2]\big|_{t=0} \\
&= \mathbb{E}\left[2(\nu_t(X_i) - \mathbb{E}[\nu_t(X_i)])\mathbb{E}\left[\frac{d}{dt}(\nu_t(X_i))\right]\Bigg|_{t=0}\right] \\
&\quad + (\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])^2 - \theta_0
\end{aligned}
$$

where the last equation follows from the influence function of expectations, i.e., $\frac{d}{dt}\mathbb{E}_{P_t}[A_i]|_{t=0} = A_i - \mathbb{E}[A_i]$ for any random variable A_i. Now we continue with the first term, which can be written as an integration form

$$
\frac{d}{dt}\int\int 2(\nu_0(x) - \mathbb{E}[\nu_0(X_i)])\cdot\left(y\frac{f_t(y,1,x)f_t(x)}{f_t(1,x)} - y\frac{f_t(y,0,x)f_t(x)}{f_t(0,x)}\right)dydx\Bigg|_{t=0}.
$$

Note that this term is similar to Equation (A.2). Following the same steps, we have

$$
2(\nu_0(\tilde{x}) - \mathbb{E}[\nu_0(X_i)])\left(\frac{\mathbb{I}_{\tilde{d}}(1)}{e_0(\tilde{x})}(\tilde{y} - \mu_0(1,\tilde{x})) - \frac{\mathbb{I}_{\tilde{d}}(0)}{1 - e_0(\tilde{x})}(\tilde{y} - \mu_0(0,\tilde{x}))\right).
$$

Replacing variables with tildes with generic forms,

$$
\begin{aligned}
&2(\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])\left(\frac{D_i}{e_0(X_i)}(Y_i - \mu_0(1,X_i)) - \frac{(1-D_i)}{1 - e_0(X_i)}(Y_i - \mu_0(0,X_i))\right) \\
&= 2(\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])(\psi(O_i) - \nu_0(X_i))
\end{aligned}
$$

The IF of θ_0 is then given by

$$
\begin{aligned}
IF_\theta(O_i) &= 2(\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])(\psi(O_i) - \nu_0(X_i)) + (\nu_0(X_i) - \mathbb{E}[\nu_0(X_i)])^2 - \theta_0 \\
&= (\psi(O_i) - \tau_0)^2 - (\psi(O_i) - \nu_0(X_i))^2 - \theta_0
\end{aligned}
$$

□

Proof of Proposition 7. Under Assumption 1 and 2, $\mathbb{E}[Y_i(d)^2] = \mathbb{E}[\phi^d(O_i)]$, which follows the same identification procedure as $\mathbb{E}[Y_i(d)]$. Therefore,

$$\dot{\lambda}_0 = \left(\mathbb{E}[\phi^1(O_i)] - \mathbb{E}[\psi^1(O_i)]^2\right) - \left(\mathbb{E}[\phi^0(O_i)] - \mathbb{E}[\psi^0(O_i)]^2\right).$$

□

Proof of Proposition 8. Let $\bar{\phi}^d = \mathbb{E}[\phi^d(O_i)]$ and $\bar{\psi}^d = \mathbb{E}[\psi^d(O_i)]$. In this proof, we use the IFs of $\bar{\psi}^d$ and $\bar{\phi}^d$ as building blocks and derive the IF of $\dot{\lambda}_0$ by applying the chain rule of influence function. First note that the IF of $\bar{\psi}^d$ is $\psi^d(O_i) - \bar{\psi}^d$, which has been proved in the literature of augmented inverse propensity weighting estimators. Following a similar procedure, it is easy to show that the IF of $\bar{\phi}^d$ is $\phi^d(O_i) - \bar{\phi}^d$.

Consider the mapping $g(\bar{\psi}^1, \bar{\phi}^1) = \bar{\phi}^1 - \bar{\psi}^{1,2}$. We have

$$\begin{aligned} IF_g(\bar{\psi}^1, \bar{\phi}^1) &= \nabla g \cdot (IF_{\bar{\psi}^1}(O_i), IF_{\bar{\phi}^1}(O_i))' \\ &= \phi^1(O_i) - \bar{\phi}^1 - 2\bar{\psi}^1(\psi^1(O_i) - \bar{\psi}^1) \end{aligned}$$

where the first equation follows from the chain rule of influence function. The second equation holds by plugging in $\nabla g = (-2\bar{\psi}^1, 1)$ and the IFs of $\bar{\psi}^1$ and $\bar{\phi}^1$. Similarly, we can obtain the IF when $d = 0$. Combining the results, we have

$$\begin{aligned} IF_\lambda(O_i) = \phi^1(O_i) &- \mathbb{E}[\phi^1(O_i)] - 2\mathbb{E}[\psi^1(O_i)](\psi^1(O_i) - \mathbb{E}[\psi^1(O_i)]) \\ &- \left(\phi^0(O_i) - \mathbb{E}[\phi^0(O_i)] - 2\mathbb{E}[\psi^0(O_i)](\psi^0(O_i) - \mathbb{E}[\psi^0(O_i)])\right) \end{aligned}$$

□

Proof of Propositions 3, 6, and 9. Here we provide a general proof for the asymptotic properties of $\hat{\beta}$, $\hat{\Phi}$, and $\hat{\Omega}$ using a commonempirical process notation. Let P_0 and P_n denote linear operators such that for some function $f(O_i)$, $P_0(f) = \mathbb{E}[f(O_i)] = \int f(O_i) dP_0$ and $P_n(f) = P_n(f(O_i)) = n^{-1}\sum_{i=1}^n f(O_i)$. Let $\hat{P}$ denote the plug-in estimator, e.g., $\psi(O_i, \hat{P}) = \hat{\psi}(O_i)$, $\nu(X_i, \hat{P}) = \hat{\mu}(1, X_i) - \hat{\mu}(0, X_i)$.

For arbitrary statistical functional $\varphi(P_0)$ with estimating equation estimator $\hat{\varphi}$, we have

$$\begin{aligned}
\hat{\varphi} - \varphi_0 &= \varphi(\hat{P}) - \varphi(P_0) + P_n(IF_\varphi(O_i, \hat{P})) \\
&= (P_n - P_0)IF_\varphi(O_i, \hat{P}) + R_n \\
&= (P_n - P_0)IF_\varphi(O_i, P_0) + (P_n - P_0)(IF_\varphi(O_i, \hat{P}) - IF_\varphi(O_i, P_0)) + R_n \\
&= (P_n - P_0)IF_\varphi(O_i, P_0) + E_n + R_n
\end{aligned}$$

where

$$\begin{aligned}
E_n &= (P_n - P_0)(IF_\varphi(O_i, \hat{P}) - IF_\varphi(O_i, P_0)) \\
R_n &= \varphi(\hat{P}) - \varphi(P_0) + P_0(IF_\varphi(O_i, \hat{P}))
\end{aligned}$$

The first term $(P_n - P_0)IF_\varphi(O_i, P_0)$ is a sample average of a fixed function, which behaves like a normally distributed random varible by central limit theorem, up to an error $o_P(n^{-1/2})$. The task is to show that the empirical process term E_n and the remainder term R_n are $o_P(n^{-1/2})$.

The empirical process term E_n:

The task is to prove

$$\|IF_\varphi(O_i, \hat{P}) - IF_\varphi(O_i, P_0)\|_P = o_P(1). \tag{A.3}$$

Then under Donsker condition and Lemma 19.24 of Van der Vaart (1998), or under cross-fittingcondition and Lemma 2 of Kennedy et al. (2020), the above condition implies that E_n is $o_P(n^{-1/2})$.

For CPT:

$$
\begin{aligned}
E_n &= (P_n - P_0)(IF_\beta(O_i, \hat{P}) - IF_\beta(O_i, P_0)) \\
&= \mathbb{E}[b(X_i)'b(X_i)]^{-1}(P_n - P_0)(b(X_i)'b(X_i)(\beta(\hat{P}) - \beta(P_0))) \qquad \text{(A.4)} \\
&\quad + \mathbb{E}[b(X_i)'b(X_i)]^{-1}(P_n - P_0)\Bigg(b(X_i)'[\\
&\quad + \left(\left(1 - \frac{D_i}{e_0(X_i)}\right)(\hat{\mu}(1, X_i) - \mu_0(1, X_i))\right) \qquad \text{(A.5)} \\
&\quad + \frac{D_i(Y_i - \hat{\mu}(1, X_i))}{\hat{e}(X_i)e_0(X_i)}(e_0(X_i) - \hat{e}(X_i)) \qquad \text{(A.6)} \\
&\quad - \left(\left(1 - \frac{1 - D_i}{1 - e_0(X_i)}\right)(\hat{\mu}(0, X_i) - \mu_0(0, X_i))\right) \qquad \text{(A.7)} \\
&\quad - \frac{(1 - D_i)(Y_i - \hat{\mu}(0, X_i))}{(1 - \hat{e}(X_i))(1 - e_0(X_i))}(\hat{e}(X_i) - e_0(X_i))]\Bigg) \qquad \text{(A.8)}
\end{aligned}
$$

The first term (A.4) is equal to $(\beta(\hat{P}) - \beta(P_0))\mathbb{E}[b(X_i)'b(X_i)]^{-1}(P_n - P_0)(b(X_i)'b(X_i)) = o_p(n^{-1/2})$ by central limit theorem. Under the assumptions in Proposition 3, we also have the following conditions:

$$
\begin{aligned}
\|(\text{A.5})\|_P &\leq \left(1 + \frac{1}{\xi}\right)\|\mu_0(1, X_i) - \hat{\mu}(1, X_i)\|_P = o_P(1) \\
\|(\text{A.6})\|_P &\leq \frac{\sqrt{K}}{\xi^2}\|e_0(X_i) - \hat{e}(X_i)\|_P = o_P(1) \\
\|(\text{A.7})\|_P &\leq \left(1 + \frac{1}{\xi}\right)\|\mu_0(0, X_i) - \hat{\mu}(0, X_i)\|_P = o_P(1) \\
\|(\text{A.8})\|_P &\leq \frac{\sqrt{K}}{\xi^2}\|e_0(X_i) - \hat{e}(X_i)\|_P = o_P(1).
\end{aligned}
$$

Combined with the condition that $\|b(X_i)\|$ is bounded, the above conditions imply that Equation (A.3) holds, and thus $E_n = o_P(n^{-1/2})$.

For CVT:

$$
\begin{aligned}
E_n &= (P_n - P_0)(IF_\theta(O_i, \hat{P}) - IF_\theta(O_i, P_0)) \\
&= (P_n - P_0)(\theta_0 - \theta(\hat{P})) && \text{(A.9)} \\
&\quad + 2(P_n - P_0)((\hat{\psi} - \psi)(\hat{\nu} - \hat{\tau})) && \text{(A.10)} \\
&\quad + (P_n - P_0)((\psi - \hat{\tau})^2 - (\psi - \tau_0)^2) && \text{(A.11)} \\
&\quad - (P_n - P_0)((\psi - \hat{\nu})^2 - (\psi - \nu_0)^2) && \text{(A.12)}
\end{aligned}
$$

Here, for simplicity, we slightly abuse the notations to use $\hat{\tau}$ to denote $\tau(\hat{P})$, instead of the estimating equation estimator of τ_0. Note that the first term (A.9) is zero since $(P_n - P_0)(\theta_0 - \theta(\hat{P})) = (\theta_0 - \theta(\hat{P}))(P_n - P_0)[1] = 0$. Consider the quantity $\mathbb{E}[(\text{A.10})^2] = \mathbb{E}[(\hat{\psi} - \psi)^2(\hat{\nu} - \hat{\tau})^2] < K\mathbb{E}[(\hat{\psi} - \psi)^2]$. The second term $\mathbb{E}[(\hat{\psi} - \psi)^2]$ also appears in the derivation of ATE estimators in e.g., Theorem 5.1 of Chernozhukov et al. (2018), which is $o_P(1)$ similar to Equation (A.5)-(A.8), given that $\hat{e} \in (\xi, 1-\xi)$, $\mathbb{E}[(Y_i - \hat{\mu})^2|X_i] < K$ and $\|e_0 - \hat{e}\|\|\mu_0 - \hat{\mu}\| = o_P(n^{-1/2})$. For the fourth term (A.12), we have

$$
\begin{aligned}
&\mathbb{E}[((\psi - \hat{\nu})^2 - (\psi - \nu_0)^2)^2] \\
= &\mathbb{E}[\mathbb{E}[(2(\psi - \nu_0)(\tau_0 - \hat{\nu}) + (\nu_0 - \hat{\nu})^2)^2|X_i]] \\
= &\mathbb{E}[4Var(\psi|X_i)(\nu_0 - \hat{\nu})^2 + (\nu_0 - \hat{\nu})^4] \\
< &4K\mathbb{E}[(\nu_0 - \hat{\nu})^2] + \mathbb{E}[(\nu_0 - \hat{\nu})^4]
\end{aligned}
$$

which is $o_P(1)$ given that $\|\nu_0 - \hat{\nu}\| = o_P(n^{-1/4})$. Similarly, the third term (A.11) is also $o_P(1)$. The above conditions imply that Equation (A.3) holds, and thus $E_n = o_P(n^{-1/2})$.

For POVT:

$$
\begin{aligned}
E_n &= (P_n - P_0)(IF_\lambda(O_i, \hat{P}) - IF_\lambda(O_i, P_0)) \\
&= (P_n - P_0)\left(\hat{\phi}^1 - \phi_0^1 - 2P_0(\hat{\psi}^1)\hat{\psi}^1 + 2P_0(\psi_0^1)\psi_0^1\right) && \text{(A.13)} \\
&\quad - P_0(\hat{\phi}^1 - \phi_0^1) + 2(P_0(\hat{\psi}^1) - P_0(\psi_0^1))(P_0(\hat{\psi}^1) + P_0(\psi_0^1)) && \text{(A.14)} \\
&\quad - (P_n - P_0)\left(\hat{\phi}^0 - \phi_0^0 - 2P_0(\hat{\psi}^0)\hat{\psi}^0 + 2P_0(\psi_0^0)\psi_0^0\right) && \text{(A.15)} \\
&\quad + P_0(\hat{\phi}^0 - \phi_0^0) - 2(P_0(\hat{\psi}^0) - P_0(\psi_0^0))(P_0(\hat{\psi}^0) + P_0(\psi_0^0)). && \text{(A.16)}
\end{aligned}
$$

First, note that the quantity $\|\hat{\phi}^1 - \phi_0^1\|_P$ is similar to (A.4) and (A.5) in the E_n of CPT, with ψ^1 replaced by ϕ^1. Following the same derivation as (A.4) and (A.5), it is straightforward to show that $\|\hat{\phi}^1 - \phi_0^1\|_P = o_P(1)$. For the quantity $\|P_0(\hat{\psi}^1)\hat{\psi}^1 - P_0(\psi_0^1)\psi_0^1\|_P$, we can write it as

$$
\begin{aligned}
\|P_0(\hat{\psi}^1)\hat{\psi}^1 - P_0(\psi_0^1)\psi_0^1\|_P &= \|P_0(\hat{\psi}^1 - \psi_0^1)\hat{\psi}^1 + P_0(\psi_0^1)(\hat{\psi}^1 - \psi_0^1)\|_P \\
&\le \|P_0(\hat{\psi}^1 - \psi_0^1)\hat{\psi}^1\|_P + \|P_0(\psi_0^1)(\hat{\psi}^1 - \psi_0^1)\|_P \\
&\le \|\hat{\psi}^1\|_P|P_0(\hat{\psi}^1 - \psi_0^1)| + |P_0(\psi_0^1)|\|\hat{\psi}^1 - \psi_0^1\|_P \\
&\le \|\hat{\psi}^1\|_P\|\hat{\psi}^1 - \psi_0^1\|_P + |P_0(\psi_0^1)|\|\hat{\psi}^1 - \psi_0^1\|_P \\
&= o_P(1)
\end{aligned}
$$

where the first inequality follows from triangle inequality, the last inequality follows from Jensen's and Cauchy-Schwarz inequalities. The last equality holds as long as $\|\hat{\psi}^1\|_P$ is $O_P(1)$ and $\|\hat{\psi}^1 - \psi_0^1\|_P$ is $o_P(1)$, and the latter condition is already proved by (A.4) and (A.5).

The above conditions imply that the term (A.13) is $o_P(n^{-1/2})$. Next, we prove that the term (A.14) is $o_P(n^{-1/2})$.

The first term in (A.14) is $o_P(n^{-1/2})$ given that $\|\hat{\psi}^1 - \psi_0^1\|_P = o_P(n^{-1/2})$. For the second term, we have

$$
\begin{aligned}
P_0(\hat{\phi}^1 - \phi_0^1) &= P_0\left(\frac{D_i}{\hat{e}(X_i)}(Y_i^2 - \hat{\mu}^2(1, X_i)) + \hat{\mu}^2(1, X_i) - \mu_0^2(1, X_i)\right) \\
&= P_0\left(\frac{D_i}{\hat{e}(X_i)}(Y_i^2 - \mu_0^2(1, X_i) + \mu_0^2(1, X_i) - \hat{\mu}^2(1, X_i)) + \hat{\mu}^2(1, X_i) - \mu_0^2(1, X_i)\right) \\
&= P_0\left(\left(\frac{D_i}{\hat{e}(X_i)} - 1\right)(\mu_0^2(1, X_i) - \hat{\mu}^2(1, X_i))\right) \\
&= P_0\left(\left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1\right)(\mu_0^2(1, X_i) - \hat{\mu}^2(1, X_i))\right) \\
&\leq \frac{1}{\xi} P_0((e_0(X_i) - \hat{e}(X_i))(\mu_0^2(1, X_i) - \hat{\mu}^2(1, X_i))) \\
&\leq \frac{1}{\xi}\|e_0 - \hat{e}\|_P \|\mu_0^2 - \hat{\mu}^2\|_P \\
&= o_P(n^{-1/2})
\end{aligned} \tag{A.17}
$$

where the second equality follows from the fact that $D_i(Y_i^2 - \mu_0^2(1, X_i)) = 0$, third equality follows from the law of iterated expectations, the last inequality follows from the Cauchy-Schwarz inequality.

For the second term in (A.14), $P_0(\hat{\psi}^1 - \psi_0^1) = o_P(n^{-1/2})$ is similar to $P_0(\hat{\phi}^1 - \phi_0^1)$ with Y_i^2 replaced by Y_i, and μ_0^2 replaced by μ_0. Then the second term is $o_P(n^{-1/2})$ as long as $P_0(\hat{\psi}^1)$ is $O_P(1)$.

Similar bounds holds for $d = 0$, so (A.15) and (A.16) are also $o_P(n^{-1/2})$. All terms in E_n are $o_P(n^{-1/2})$, and thus $E_n = o_P(n^{-1/2})$.

The remainder term R_n:

For CPT:

$$
\begin{aligned}
R_n &= \beta(\hat{P}) - \beta(P_0) + P_0(IF_\beta(O_i, \hat{P})) \\
&= P_0(IF_\beta(O_i, \hat{P}) - \beta(P_0)) \\
&= \mathbb{E}[b(X_i)' b(X_i)]^{-1} P_0 \Bigg(b(X_i)' \left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1 \right) (\mu_0(1, X_i) - \hat{\mu}(1, X_i)) \\
&\quad - b(X_i)' \left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1 \right) (\mu_0(0, X_i) - \hat{\mu}(0, X_i)) \Bigg).
\end{aligned}
$$

The derivation is similar to Equation (A.17). Consider the quantity $\mathbb{E}[b(X_i)' b(X_i)] R_n$. If this quantity is $o_P(n^{-1/2})$, then R_n is $o_P(n^{-1/2})$ since $\mathbb{E}[b(X_i)' b(X_i)]$ is positive definite.

$$
\begin{aligned}
\|\mathbb{E}[b(X_i)' b(X_i)] R_n\| &\leq P_0 \left(\left\| b(X_i)' \left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1 \right) (\mu_0(1, X_i) - \hat{\mu}(1, X_i)) \right\| \right) \\
&\quad + P_0 \left(\left\| b(X_i)' \left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1 \right) (\mu_0(0, X_i) - \hat{\mu}(0, X_i))' \right\| \right) \\
&= P_0 \left(\|b(X_i)'\| \left| \left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1 \right) (\mu_0(1, X_i) - \hat{\mu}(1, X_i)) \right| \right) \\
&\quad + P_0 \left(\|b(X_i)'\| \left| \left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1 \right) (\mu_0(0, X_i) - \hat{\mu}(0, X_i))' \right| \right) \\
&\leq K P_0 \left(\left| \left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1 \right) (\mu_0(1, X_i) - \hat{\mu}(1, X_i)) \right| \right) \\
&\quad + K P_0 \left(\left| \left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1 \right) (\mu_0(0, X_i) - \hat{\mu}(0, X_i)) \right| \right) \\
&\leq \frac{K}{\xi} \|e_0 - \hat{e}\|_P \|\mu_0 - \hat{\mu}\|_P + \frac{K}{\xi} \|\hat{e} - e_0\|_P \|\mu_0 - \hat{\mu}\|_P \\
&= o_P(n^{-1/2})
\end{aligned}
$$

where the first inequality follows from triangle and Jensen's inequality. Then we have the desired result $R_n = o_P(n^{-1/2})$.

For CVT:

$$
\begin{aligned}
R_n &= \theta(\hat{P}) - \theta(P_0) + P_0(IF_\theta(O_i, \hat{P})) \\
&= P_0(\theta(\hat{P}) - \theta(P_0) + (\psi(O_i, \hat{P}) - \tau(\hat{P}))^2 - (\psi(O_i, \hat{P}) - \nu(X_i, \hat{P}))^2 - \theta(\hat{P})) \\
&= P_0((\psi(O_i, \hat{P}) - \tau(\hat{P}))^2 - (\psi(O_i, \hat{P}) - \nu(X_i, \hat{P}))^2 - (\nu(X_i, P_0) - \tau(P_0))) \\
&= P_0((\nu(X_i, \hat{P}) - \tau(\hat{P}))^2 - (\nu(X_i, P_0) - \tau(P_0))^2 + 2(\psi(O_i, \hat{P}) - \nu(X_i, \hat{P}))(\nu(X_i, \hat{P}) - \tau(\hat{P}))) \\
&= P_0((\tau(P_0) - \tau(\hat{P}))^2 - (\nu(X_i, P_0) - \nu(X_i, \hat{P}))^2 + 2(\nu(X_i, \hat{P}) - \nu(X_i, P_0))(\nu(X_i, \hat{P}) - \tau(\hat{P})) \\
&\quad + 2(\psi(O_i, \hat{P}) - \nu(X_i, \hat{P}))(\nu(X_i, \hat{P}) - \tau(\hat{P}))) \\
&= P_0((\tau(P_0) - \tau(\hat{P}))^2 - (\nu(X_i, P_0) - \nu(X_i, \hat{P}))^2 + 2(\psi(O_i, \hat{P}) - \nu(X_i, P_0))(\nu(X_i, \hat{P}) - \tau(\hat{P}))).
\end{aligned}
$$

The first term $P_0((\tau(P_0) - \tau(\hat{P}))^2 - (\nu(X_i, P_0) - \nu(X_i, \hat{P}))^2)$ is $o_P(n^{-1/2})$ given that both $\|\tau_0 - \tau(\hat{P})\|_P$ and $\|\nu_0 - \hat{\nu}\|_P$ are $o_P(n^{-1/4})$. For the second term, consider the quantity

$$
\begin{aligned}
&P_0((\psi(O_i, \hat{P}) - \nu(X_i, P_0))^2(\nu(X_i, \hat{P}) - \tau(\hat{P}))^2) \\
\leq\ &P_0((\psi(O_i, \hat{P}) - \nu(X_i, P_0))^2)P_0((\nu(X_i, \hat{P}) - \tau(\hat{P}))^2) \\
\leq\ &KP_0((\psi(O_i, \hat{P}) - \nu(X_i, P_0))^2)
\end{aligned}
$$

where the first inequality follows from Cauchy-Schwarz inequality, and the second inequality follows from the condition that $(\hat{\nu}(X_i) - \tau(\hat{P}))^2 < K$. The remaining term is similar to the R_n of CPT. Specifically,

$$
\begin{aligned}
(\psi(O_i, \hat{P}) - \nu(X_i, P_0))^2 &= \Bigg(\left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1\right)(\mu_0(1, X_i) - \hat{\mu}(1, X_i)) \\
&\quad - \left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1\right)(\mu_0(0, X_i) - \hat{\mu}(0, X_i)) \Bigg)^2 \\
&\leq 2\left(\frac{e_0(X_i)}{\hat{e}(X_i)} - 1\right)^2 (\mu_0(1, X_i) - \hat{\mu}(1, X_i))^2 \\
&\quad + 2\left(\frac{1 - e_0(X_i)}{1 - \hat{e}(X_i)} - 1\right)^2 (\mu_0(0, X_i) - \hat{\mu}(0, X_i))^2 \\
&\leq \frac{2}{\xi^2}(e_0(X_i) - \hat{e}(X_i))^2((\mu_0(1, X_i) - \hat{\mu}(1, X_i))^2) \\
&\quad + \frac{2}{\xi^2}(e_0(X_i) - \hat{e}(X_i))^2((\mu_0(0, X_i) - \hat{\mu}(0, X_i))^2).
\end{aligned}
$$

Therefore, $P_0((\psi(O_i, \hat{P}) - \nu(X_i, P_0))^2) = o_P(n^{-1})$ by Cauchy-Schwarz inequality, which implies $R_n = o_P(n^{-1/2})$.

For POVT:

$$R_n = \lambda(\hat{P}) - \lambda(P_0) + P_0(IF_\lambda(O_i, \hat{P}))$$

For $d = 1$, the remainder term is

$$R_n^1 = P_0(\hat{\phi}^1) - P_0(\phi^1) - (P_0(\hat{\psi}^1))^2 + (P_0(\psi^1))^2 \tag{A.18}$$
$$+ P_0(\hat{\phi}^1 - P_0(\hat{\phi}^1) - 2P_0(\hat{\psi}^1)(\hat{\psi}^1 - P_0(\hat{\psi}^1))) \tag{A.19}$$

Note that (A.19) is zero. The term

$$(\text{A.18}) = P_0(\hat{\phi}^1 - \phi^1) - (P_0(\hat{\psi}^1) + P_0(\psi^1))P_0(\hat{\psi}^1 - \psi^1) = o_P(n^{-1/2})$$

with the bounds of all terms already proved in previous derivations. Similar bounds holds for $d = 0$, and thus $R_n = o_P(n^{-1/2})$.

All proofs are complete.

□

B Appendix: Tables and Figures