

GEOMETRIC (TRANSFORMATIONS)

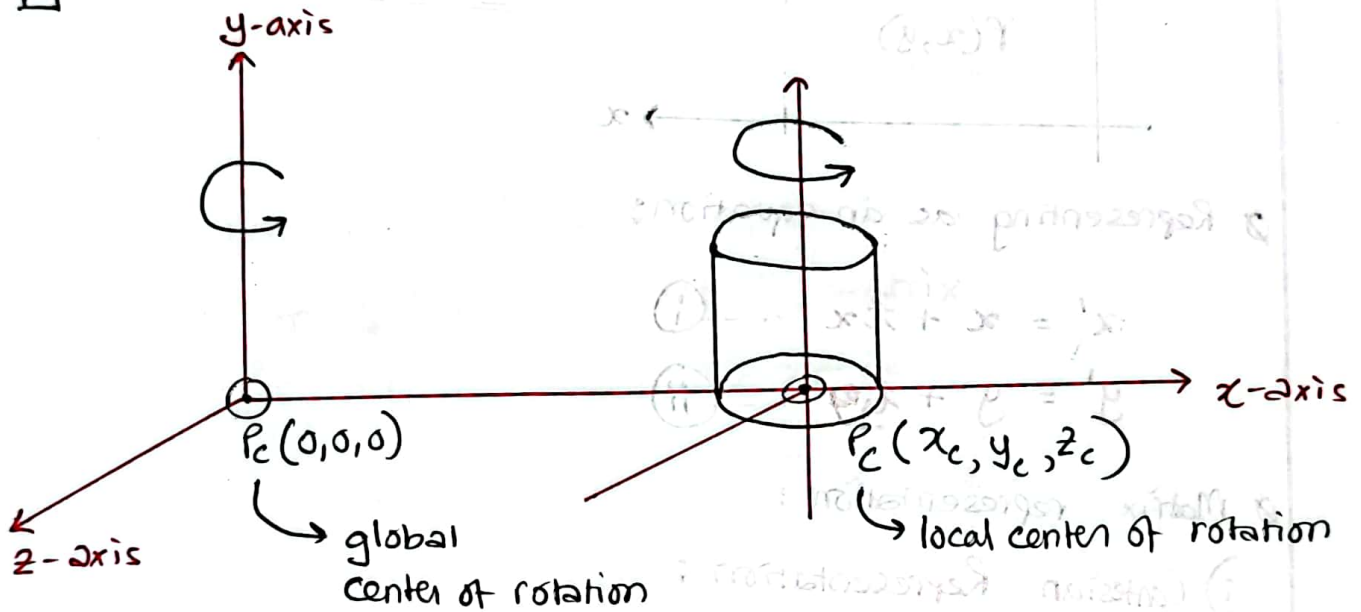
LECTURE 10

→ Repositioning & resizing of an object.

∅ Classification of transformation

- i) Geometric Transformation
(center of rotation is global)
(like revolution)
- ii) Coordinate Transformation
(center of rotation is local)
(like rotation)

Based on Center of Rotation



∅ Categories of Transformation:

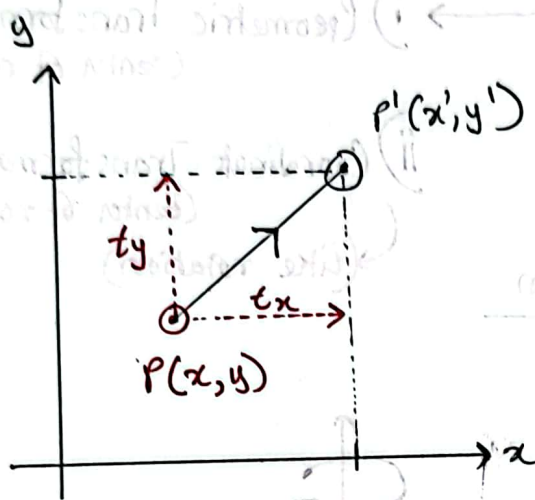
- 1) Translation
- 2) Rotation
- 3) Scaling
- 4) Reflection
- 5) Shearing

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x^2 & 0 & 1 \\ y^2 & 1 & 0 \\ z^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x^2 + x \\ y^2 + y \\ z^2 + z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

1) Translation:

It is the repositioning of an object / pixel in a straight line



t_x, t_y are translation across x axis & y axis

$$\therefore \underline{P' = P + t}$$

Representing as an equation:

$$x' = x + t_x \quad \text{--- (i)}$$

$$y' = y + t_y \quad \text{--- (ii)}$$

Matrix representation:

i) Cartesian Representation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

ii) Homogeneous Representation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Q) We mainly prefer to use Homogeneous coordinate system in Computer Graphics. But why?

→ Using Homogeneous coordinate system adds an extra dimension to the 2D/3D Cartesian system. This allows us to do addition/subtraction, but also allows us to perform multiplication & division. It can be used to generate composite Matrices combining multiple transformations that can be represented into a single matrix to reduce computations.

Ø So in Homogeneous Coordinate system:

→ for 2D we use (3×3) matrix

→ for 3D we use (4×4) matrix

3D Translation matrix representation (x, y, z)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Q Translate the point $(3, 2, 7)$, 3 units in x -axis, 4 units in y -axis & -3 units in the z -axis?

Ans $t_x = 3$, $t_y = 4$, $t_z = -3$

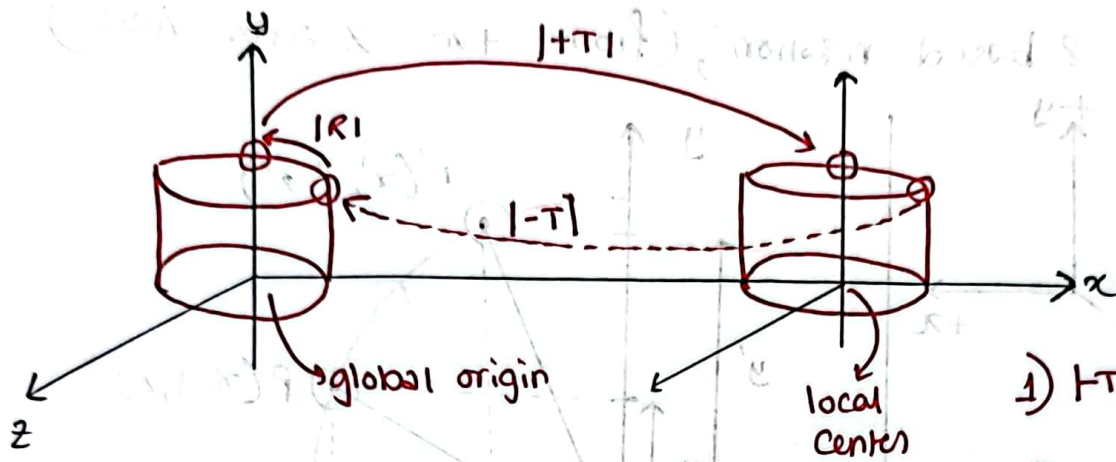
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 \\ 2+4 \\ 7-3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 4 \\ 1 \end{bmatrix} = (6, 6, 4) \quad \text{Ans!}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 & 0 & 0 & 1 \\ y_0 & 0 & 1 & 0 \\ z_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

2) Rotation :

How do we work on Rotation?



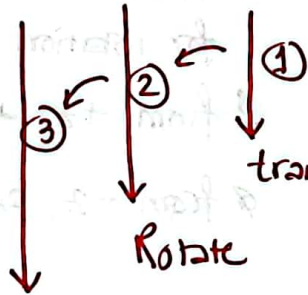
1) $|T|$ translate to origin

2) $|R|$ Rotate

3) $|+T|$ translate to local origin/center.

generates a composite matrix

$$|P'| = |+T| \cdot |R| \cdot |-T| \cdot |P|$$



translate to global origin

Rotate

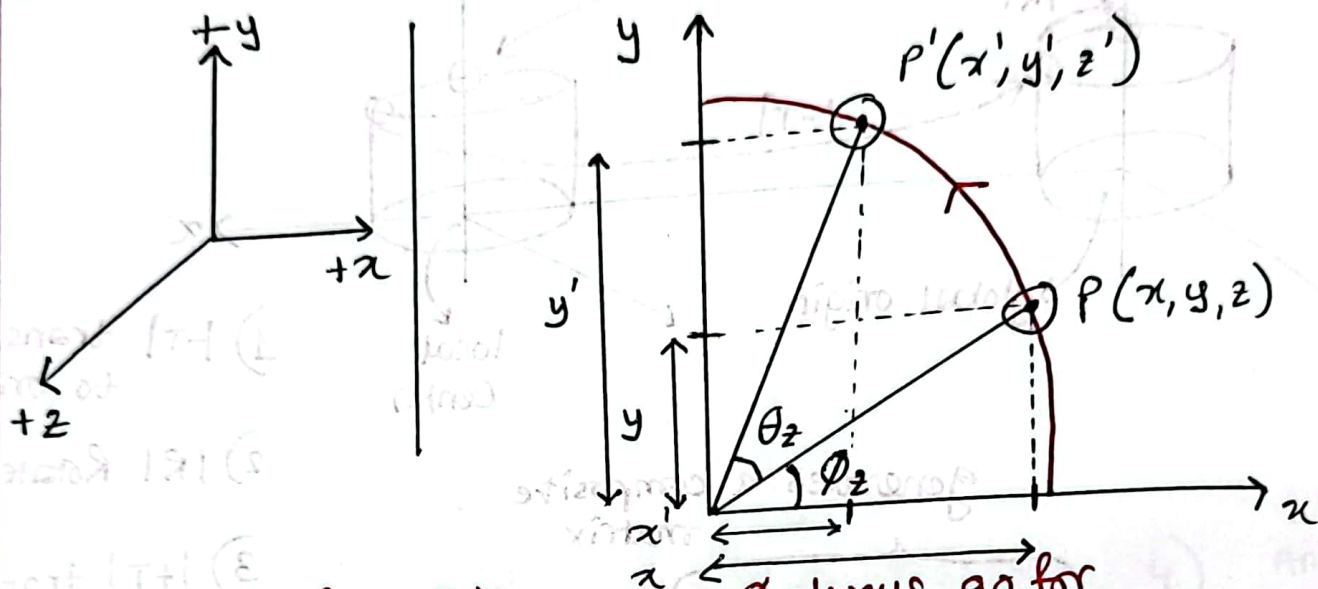
Bring to local origin.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \cos \theta + y' \sin \theta \\ -x' \sin \theta + y' \cos \theta \\ z' \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} (x \cos \theta + y \sin \theta) \\ (-x \sin \theta + y \cos \theta) \\ z \end{bmatrix}$$

(Mathematics of Rotation)

ϕ for z based rotation, (from +ve z axis view)



The value of z -axis is constant, since the rotation is based on z -axis.

ϕ always go for counter clockwise motion for rotation

ϕ from + z axis, counter clockwise

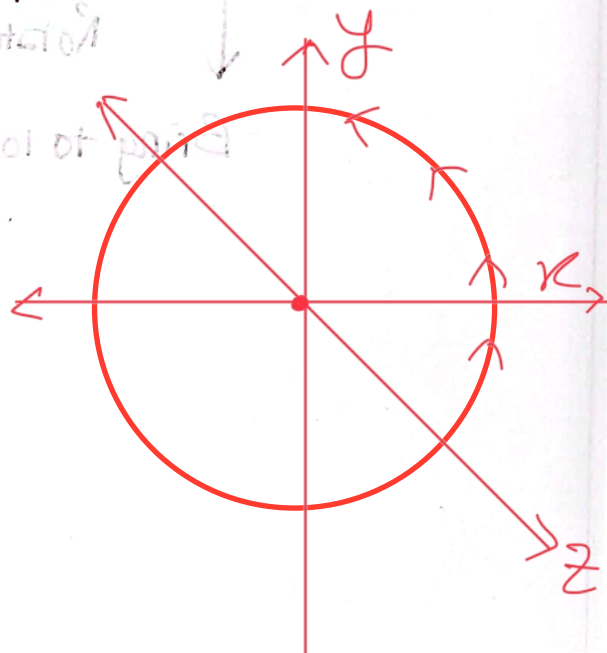
ϕ from - z axis, clockwise

Hence,

$$\left. \begin{aligned} x &= r \cos \phi_z \\ y &= r \sin \phi_z \end{aligned} \right\} \rightarrow P$$

$$\left. \begin{aligned} x' &= r \cos (\theta_z + \phi_z) \\ y' &= r \sin (\theta_z + \phi_z) \end{aligned} \right\} \rightarrow P'$$

$$z' = z \text{ (remains constant)}$$



no change in z -axis

We know,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \cos A \sin B + \sin A \cos B$$

Since,

$$x' = r \cos(\phi + \theta)$$

$$y' = r \sin(\phi + \theta)$$

So,

$$x' = r \{ (\cos \phi \cdot \cos \theta) - (\sin \phi \cdot \sin \theta) \}$$

$$y' = r \{ (\cos \phi \cdot \sin \theta) + (\sin \phi \cdot \cos \theta) \}$$

$$z' = z$$

We know,

$$x = r \cos \phi$$

$$y = r \sin \phi$$

Then,

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

$$z' = z$$

So, the (4×4) representation is: (Homogeneous)

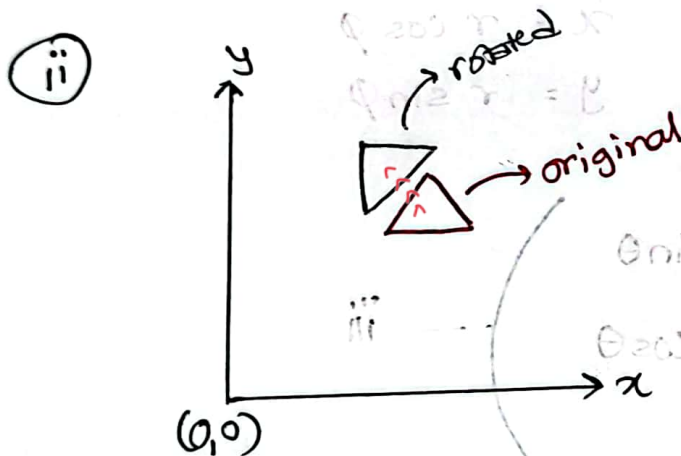
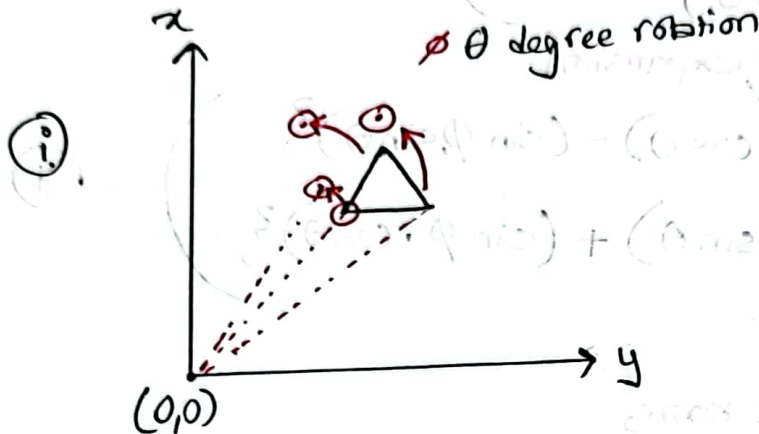
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Remember, this rotation is for Z (rotation with respect to the origin)

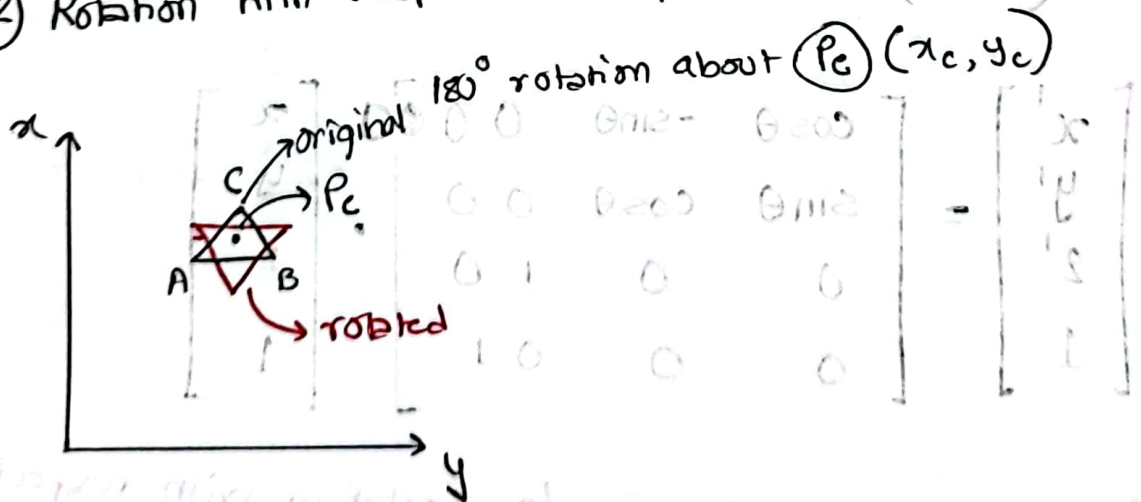
hence we don't need the $|+T|$ & $|-T|$ matrices.

(IDEA behind rotation) \rightarrow 2D

1) rotation with respect to origin:



2) Rotation with respect to a point P_c or (x_c, y_c)



\hookrightarrow In this case for the rotation we need to translate to origin first, rotate, then translate back to local origin.

Q) Rotate the point $(9, 2)$ at an angle of 45° , with respect to origin and find the result?

→ since, we are dealing with a 2D point, we can eliminate the use of z axis for this.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 9\cos 45 - 2\sin 45 \\ 9\sin 45 + 2\cos 45 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 4.95 \\ 7.78 \\ 1 \end{bmatrix} \quad (\text{Ans:})$$

Q) Rotate the point $(6, 2)$ at an angle of 90° with respect to the point $(2, 2)$?

So, $P' = |T| \cdot |R| \cdot |T^{-1}| \cdot |P|$

$\Rightarrow P' = T_{(2,2)} \cdot R_{90} \cdot T_{(-2,-2)} \cdot P$

$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \cos 90 & -\sin 90 & 2 \\ \sin 90 & \cos 90 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \cos 90 & -\sin 90 & -2\cos 90 + 2\sin 90 + 2 \\ \sin 90 & \cos 90 & -2\sin 90 - 2\cos 90 + 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \quad (\text{Ans.})$