Theory of Neural Dynamics and Application to ML based on RC

Exercise sheet 1 (25.04.2024)

Part 1: The Hodgkin-Huxley neuron model and its simulations

We consider the Hodgkin-Huxley neuron model described by the coupled differential equations:

$$\begin{cases} \dfrac{dV}{dt} = \dfrac{1}{C_m}\left[\overline{g}_K n^4(V_K - V) + \overline{g}_{Na} m^3 h(V_{Na} - V) + \overline{g}_L(V_L - V) + I^{ext}\right], \\ \dfrac{dn}{dt} = \alpha_n(1-n) - \beta_n n, \\ \dfrac{dm}{dt} = \alpha_m(1-m) - \beta_m m, \\ \dfrac{dh}{dt} = \alpha_h(1-h) - \beta_h h. \end{cases} \tag{1}$$

The variable $V = V(t)$, represents the membrane potential of the neuron (measured in mV), and t is the time measured in msec and $\{n = n(t), m = m(t), h = h(t)\}$ represent auxiliary dimensionless $[0,1]$-valued dynamical variables representing the sodium activation, sodium inactivation, and potassium activation, respectively. The initial condition of the differential equations given by $V(0) = V_0$, $n(0) = n_0$, $m(0) = m_0$, and $h(0) = h_0$. need to be specified in the simulations.

The capacitance of the membrane of the neuron is represented by $C_m = 1.0\ \mu\text{F/cm}^3$.

The conductances $\overline{g}_K = 36\ \text{mS/cm}^2$, $\overline{g}_{Na} = 120\ \text{mS/cm}^2$, and $\overline{g}_L = 0.3\ \text{mS/cm}^2$, respectively, denote the maximal sodium, potassium, and leakage conductance when all ion channels are open.

The potentials $V_K = -12$ mV, $V_{Na} = 115$ mV, and $V_L = 10$ mV are the reversal potentials for sodium, potassium and leak channels, respectively.

I^{ext} (measured in $\mu\text{A/cm}^2$) represents a constant external current exciting into the neuron.

The dynamics of the gating variables $x = \{m, h, n\}$, depending on the voltage-dependent opening and closing rate functions $\alpha_x(V)$ and $\beta_x(V)$, are given by:

$$\begin{cases} \alpha_m(V) &=& \dfrac{25-V}{10\left[e^{(25-V)/10} - 1\right]} \\ \beta_m(V) &=& 4e^{-V/18} \\ \alpha_h(V) &=& \dfrac{7}{100}e^{-V/20} \\ \beta_h(V) &=& \dfrac{1}{1+e^{(30-V)/10}} \\ \alpha_n(V) &=& \dfrac{10-V}{100\left[e^{(10-V)/10} - 1\right]} \\ \beta_n(V) &=& \dfrac{1}{8}e^{-V/80} \end{cases} \tag{2}$$

Problem 1:

By fixing the external stimulus at $I^{ext} = 5.2$ and the initial conditions at $V(0) = 0$, $n(0) = 0.35$, $m(0) = 0.06$, and $h(0) = 0.6$, use an ODE integrator package in Python of your choice (e.g., *solve_ivp*, *odeint*, *ode*, all in module *scipy.integrate*) to simulate the Hodgkin-Huxley neuron model in eq:1 with the parameter values given above. Plot (using, e.g., *Matplotlib*) the time series of the membrane potential ($V(t)$) and each gating variable ($m(t), h(t), n(t)$) for a duration of $T = 200$ milliseconds. Label the axes and provide a legend for each of your plots. Your submission should include the Python code used to simulate the model and generate the plots. Briefly (not more than 10 sentences) discuss the results and any observations you made.

Problem 2:

Repeat Problem 1 with new initial conditions given by $V(0) = 70$, $n(0) = 0.25$, $m(0) = 0.07$, and $h(0) = 0.65$. Note the appropriateness of initial conditions for the gating variables (recall from the lecture that the gating variable probability measures.)

Problem 3:

Repeat Problem 1 with a new external stimulus given by $I^{ext} = 6.8$ and new initial conditions given by $V(0) = 0$, $n(0) = 0.35$, $m(0) = 0.06$, and $h(0) = 0.6$.

Problem 4:

Repeat Problem 3 with new initial conditions given by $V(0) = 70$, $n(0) = 0.15$, $m(0) = 0.02$, and $h(0) = 0.4$.

Part 2: Timescale separation and dimension reduction

Problem 1

Consider the following set of coupled ODEs, each characterized by the timescales ε_1 and ε_2,

$$\begin{cases} \dfrac{dx}{dt} & = & \varepsilon_1^{-1}\big[-x(t)+y(t)+I(t)\big] \\ \dfrac{dy}{dt} & = & \big[-y(t)+x(t)^2+A\big]/\varepsilon_2 \end{cases} \tag{3}$$

(a) If $\varepsilon_1 \ll \varepsilon_2$, then the system can be reduced to:

$$\varepsilon_1 \frac{dx}{dt} = -x(t)+x(t)^2+I(t)+A \tag{4}$$

(b) If $\varepsilon_2 \ll \varepsilon_1$, then the system can be reduced to:

$$\frac{dy}{dt} = \frac{1}{\varepsilon_2}\big[-y(t)+\big[y(t)+I(t)\big]^2+A\big] \tag{5}$$

(c) If $\varepsilon_1 = \varepsilon_2$, then none of the ODEs in (a) and (b) above can be correct.

Problem 2

A channel with gating variable $R(t)$, given by $\dfrac{dR}{dt} = \varepsilon_1^{-1}\big[-R(t)+R_0(V)\big]$ influences the voltage $V(t)$ given by $\varepsilon_2 \dfrac{dV}{dt} = -\big[V(t)-V_0\big]+R(t)^2 A$.

A reduction of dimension

(a) is possible, and the result is $\varepsilon_1 \dfrac{dR}{dt} = -R(t)+R_0\big[V_0+R(t)^2 A\big]$ if $\varepsilon_2 \ll \varepsilon_1$.

(b) is impossible if $\varepsilon_1 = \varepsilon_2$.

(c) is possible, and the result is $\varepsilon_2 \dfrac{dV}{dt} = -V(t)+V_0+R_0(V)^2 A$ if $\varepsilon_2 \gg \varepsilon_1$.