ASE 389P.4 Methods of Orbit Determination
Homework 5: Setting Up the Term Project

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The theory and algorithms are derived and computer program to establish the trajectory of an Earth-orbiting satellite is developed. The assumptions for the study are:

- **Three tracking stations taking apparent range and range-rate data are available for tracking the satellite. Apparent quantities imply that the one-way light time between signal transmission and reception were modeled into the measurement (i.e. the effect is dealt with).**
- **The force model used to generate the truth is the EGM96 gravity field of degree and order 20, attitude-dependent solar radiation pressure, and atmospheric drag.**
- **The satellite is a box-wing shaped with one Sun-pointed solar panel with known component sizes, material properties, and orientation. The spacecraft -Z axis (in the spacecraft body reference frame) is always Nadir-pointed and has the antenna.**

Problem

For the state vector containing

$$\boldsymbol{X} = \begin{bmatrix} \underline{r} \\ \underline{\dot{r}} \\ C_D \\ C_{solar} \\ \cdots \\ \underline{b}_{Range_i} \end{bmatrix} \tag{1}$$

where $\underline{b}_{Range_i}$ are the Range biases for each tracking station, $i = 1, ..., n$.

Problem 1

Derive the $\underline{\underline{A}}_{n\times n}$ and $\underline{\underline{\tilde{H}}}_{m\times n}$ matrices for the linearized system and implement the partials in your computer language of choice. We recommend using a symbolic solver, e.g., MATLAB's symbolic toolbox, due to the large number of partial derivatives required. Compare to the numeric solutions online (found on canvas) at t_0. Compute the relative difference for each non-zero element, for example (in MATLAB)

```
>> relDiff = abs((yourHtilde-solutionHtilde)./solutionHtilde)
```

For the $\underline{\underline{\tilde{H}}}$ solution, provide the numeric values for the relative difference in your write-up. For the $\underline{\underline{A}}$ matrix, include a histogram of the exponents, e.g., (in MATLAB)

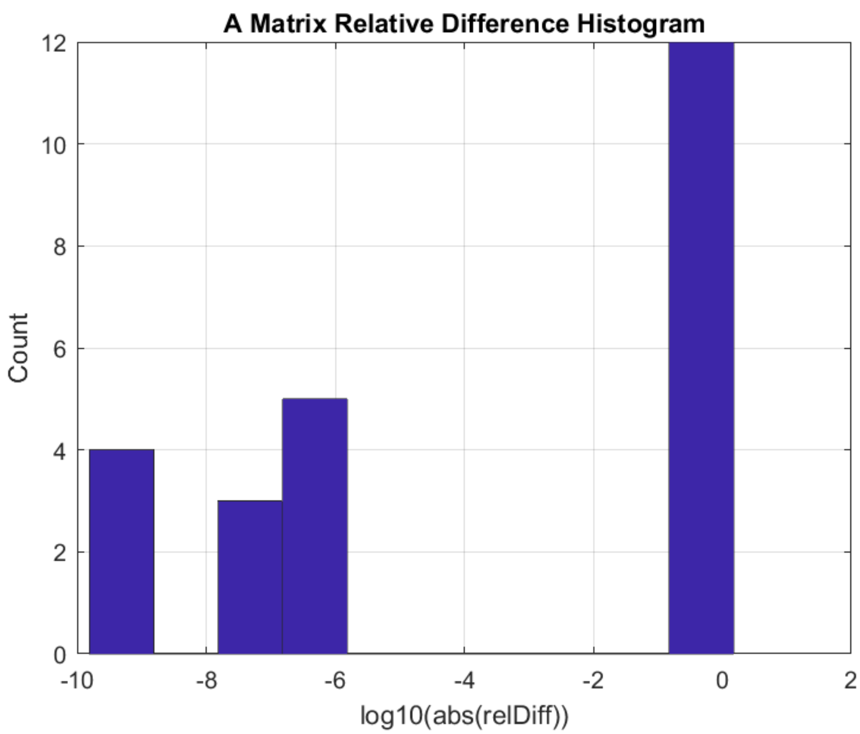
```
>> hist(reshape(log10(abs(relDiff)),n*m,1))
```

Solution

Relative difference for A matrix:

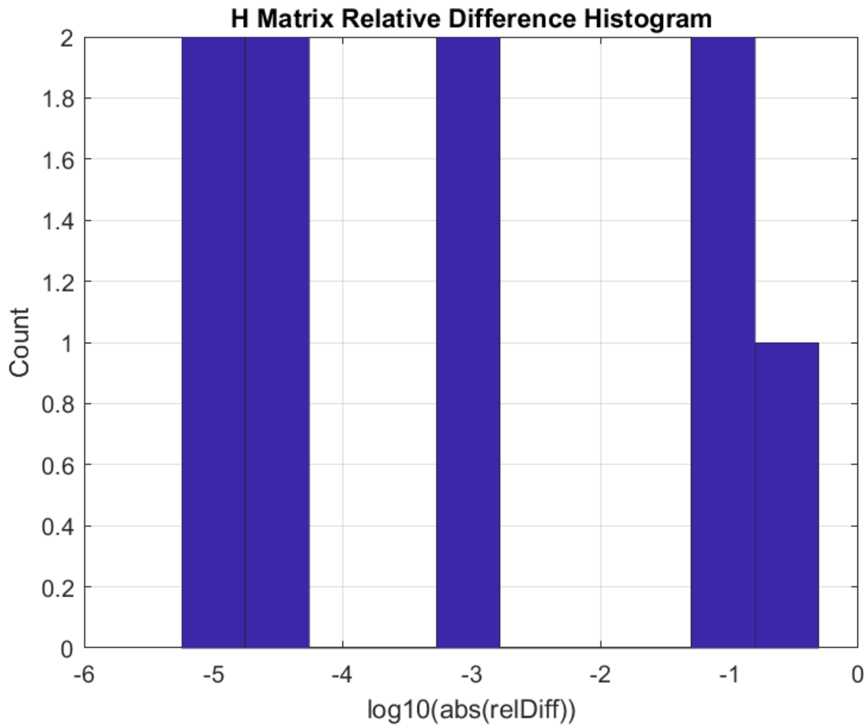
```
>> relDiff = abs((Ajune - Ajah)./Ajah)
```

```
relDiff =

NaN      NaN      NaN      0          NaN        NaN        NaN
NaN      NaN      NaN      NaN        0          NaN        NaN
NaN      NaN      NaN      NaN        NaN        0          NaN
3.0388e-08   2.1018e-08   1.8969e-07   1.35   1.35   1.35   1.5
3.5538e-07   6.6569e-08   6.4491e-07   1.35   1.35   1.35   1.5
7.7416e-07   1.2355e-06   1.5541e-10   1.35   1.35   1.35   1.5
NaN      NaN      NaN      NaN        NaN        NaN        NaN
```



Relative difference for H matrix:

```
>> relDiff = abs((Htjune - Htjah)./Htjah)

relDiff =

3.6195e-05   0.0011651   5.703e-06    NaN          NaN         NaN         NaN
0.053035     0.12866     0.49245      3.6195e-05   0.0011651   5.703e-06   NaN
```



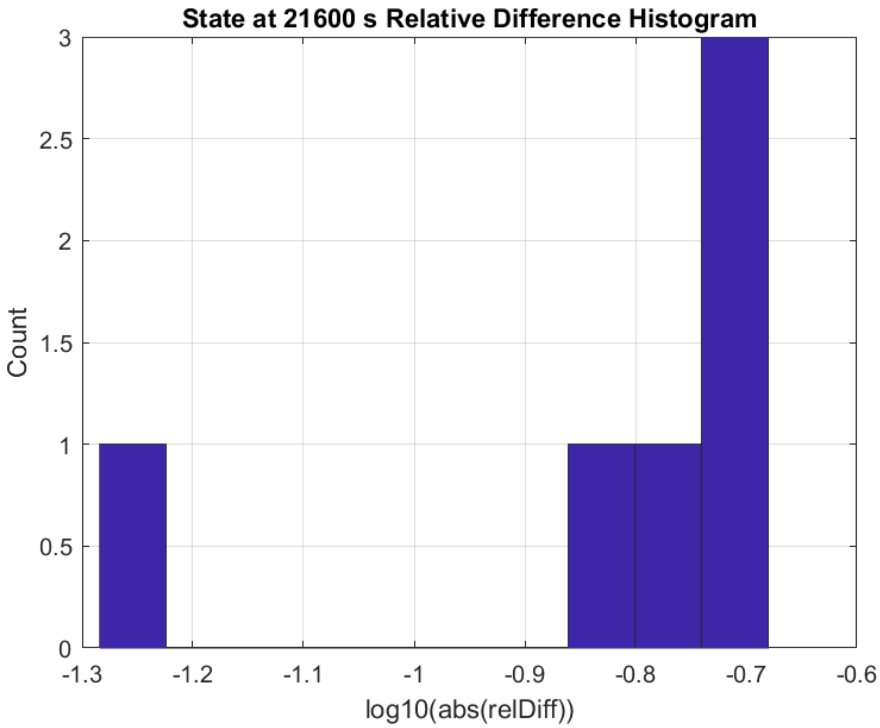
Problem 2

Integrate position, velocity, and $\underline{\underline{\Phi}}_{(t_i,t_0)}$ from $t = 0, \ldots, 21600$ seconds. Store the results in 60 second intervals. Compare your results with those on the web (found on ther Canvas site) and compute the relative difference of the top left relevant portion of $\underline{\underline{\Phi}}_{\Phi(21600,0)}$. Like the previous question, provide a histogram of the exponents for the STM comparison.

Solution

Relative difference for the end state at 21600 seconds:

```
>> relDiff = abs((X(end,:)' - X_GMAT)./X_GMAT)

relDiff =

0.16065
0.18772
0.20901
0.19632
0.1421
0.05198
```



Problem 3

Calculate the predicted range and range-rate for the appropriate tracking station at each observation time (use the data provided on the Canvas site). Compare and plot the range and range-rate residuals (post-fit). Calculate the range residual RMS and range-rate residual RMS using:

$$RMS = \sqrt{\frac{\sum_{i=0}^{n} \left(Y_{\text{Obs}} - Y_{\text{Comp}}\right)^2}{n}}$$

Provide the values in your write-up.

Solution

The range and rate-rate residuals were calculated as the following:

Atoll range RMS:

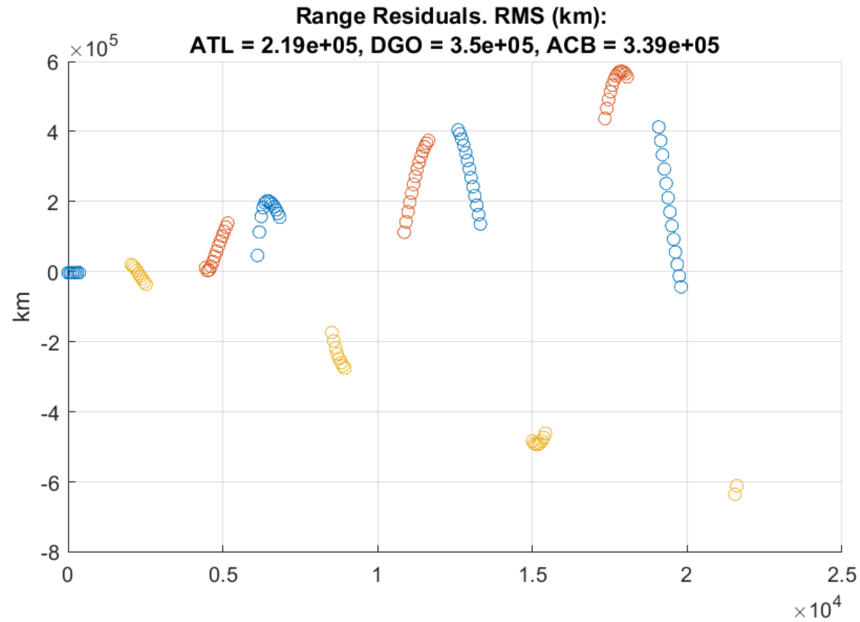
```
>> d_rms_ATL

d_rms_ATL =

2.1942e+05
```

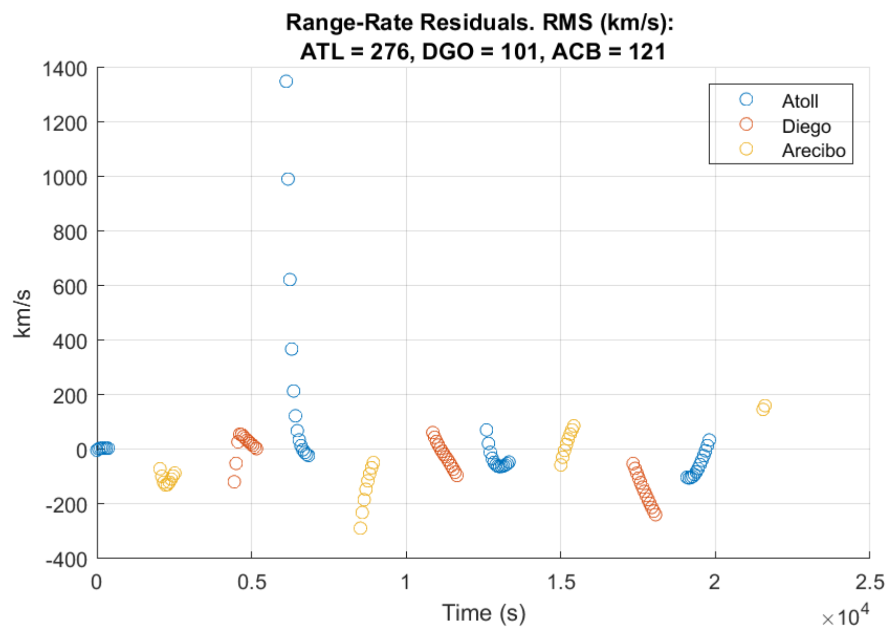
Atoll range-rate RMS:

```
>> v_rms_ATL

v_rms_ATL =

275.66
```

Diego Garcia range RMS:

```
>> d_rms_DGO
```


Range Residuals. RMS (km):
ATL = 2.19e+05, DGO = 3.5e+05, ACB = 3.39e+05
×10^5
6
4
2
0
-2
-4
-6
-8
km
0
0.5
1
1.5
2
2.5
×10^4


Range-Rate Residuals. RMS (km/s):
ATL = 276, DGO = 101, ACB = 121
1400
1200
1000
800
600
400
200
0
-200
-400
km/s
Atoll
Diego
Arecibo
0
0.5
1
1.5
2
2.5
Time (s)
×10^4

```

d_rms_DGO =

3.5046e+05
```

Diego Garcia range-rate RMS:

```
>> v_rms_DGO

v_rms_DGO =

100.68
```

Arecibo range RMS:

```
>> d_rms_ACB

d_rms_ACB =

3.3891e+05
```

Arecibo range-rate RMS:

```
>> v_rms_ACB

v_rms_ACB =

120.98
```

Appendix

HW5 MATLAB code

Note: Functions to transform from ECI to ECEF and back were submitted as part of Homework 4, and so are left out of this appendix.

```
% HW 5
% Junette Hsin

clear; clc
addpath(genpath('mice'));
addpath(genpath('spice_data'));

% Load SPICE kernel file
cspice_furnsh( 'spice_data/naif0011.tls' )
cspice_furnsh( 'spice_data/de421.bsp' )
cspice_furnsh( 'spice_data/pck00010.tpc ')


%% Parameters

% initial state initial guess (M --> KM)
CD  = 1.88;
X0 = [ 6990.077798814194  ;
1617.465311978378  ;
22.679810569245355 ;
-1.67513972506056   ;
7.27372441330686   ;
0.252688512916741  ;
CD ];

% initialize STM
STM0 = eye(7);
STM0 = reshape(STM0, [49 1]);
XSTM0 = [X0; STM0];

global muE RE muS AU muM eE wE J2 J3 J4 Cd Cs eop_data
global A m p p0 r0_drag H

% epoch = 1 Feb 2018, 05:00:00 UTC;
JD = 2458150.70833;

% Constants
muE = 398600.4415;          % Earth Gravitational Parameter (km^3/s^2)
RE  = 6378.1363;            % Earth Radius (km)
muS = 132712440018;         % Sun's Gravitational Parameter (km^3/s^2)
AU  = 149597870.7;          % 1 Astronomical Unit (km)
muM = 4902.800066;          % Moon's Gravitational Parameter (km^3/s^2)
eE  = 0.081819221456;       % Earth's eccentricity
wE  = 7.292115146706979e-5; % Earth's rotational velocity (rad/s)
m   = 2000;                 % satellite mass (kg)
Cd  = 0.04;                 % diffuse reflection
Cs  = 0.04;                 % specular reflection

J2 = 1.08262617385222e-3;
J3 = -2.53241051856772e-6;
J4 = -1.61989759991697e-6;

global LEO_DATA_Apparent
% Load observation data
load('LEO_DATA_Apparent.mat')

eop_data = load('finals_iau1980.txt');

% Atmospheric drag
r   = norm(X0(1:3));            % km
H   = 88667.0 / 1000;           % m --> km
```

```
r0_drag   = (700 + RE);            % m --> km
p0   = 3.614e-13 * 1e9;            % kg/m3 --> kg/km^3
p    = p0*exp( -(r-r0_drag)/H );
A    = 15 / 1e6;                   % km^2


%% Convert t0 to ET, i.e. seconds past J2000, the base time variable for SPICE. function calls.

%  Epoch for initial conditions
t0       = 'Feb 1, 2018, 05:00:00 UTC';
abcorr   = 'NONE';

%  Convert the epoch to ephemeris time.
et_t0    = cspice_str2et( t0 );

% extract observation epochs
epochs = LEO_DATA_Apparent(:,2);
epochs = et_t0 + epochs;


%% Derive A matrix

X  = sym('X', [7 1]);
dX = fn.EOM(et_t0, X);

% compute partials
Amat     = jacobian( dX, X );
Amat_fn = matlabFunction(Amat);

% test Amat_fn at t0
Ajune = Amat_fn( X0(1), X0(2), X0(3), X0(4), X0(5), X0(6), X0(7))    ;
Ajah  = load('A_t0.mat'); Ajah = Ajah.A;

disp('Ajah ./ Ajune: ')
disp(Ajah ./ Ajune);


%% integrate EOM

% set ode45 params
rel_tol = 1e-10;          % 1e-14 accurate; 1e-6 coarse
abs_tol = 1e-10;
options = odeset('reltol', rel_tol, 'abstol', abs_tol );

% Set run state
run_state = 2;
disp('Running sim ...')

if run_state == 1
% fuck the STM for now. integrate
[t, X] = ode45(@fn.EOM, [epochs(1) : 60 : epochs(end)], X0, options);
elseif run_state == 2
[t, XSTM] = ode45(@(t, XSTM) fn.EOM_STM(t, XSTM, Amat_fn), [epochs(1) : 60 : epochs(end)], XSTM0,
     options);
X = XSTM(:, 1:6);
end
disp('Pos and Vel end: ')
disp(X(end, 1:6)');

%% Plot what u see

% create figure
ftitle = 'Orbit';
figure('name', ftitle);

r_x = X(:,1); r_y = X(:,2); r_z = X(:,3);
plot3(r_x, r_y, r_z); hold on; grid on;
scatter3(r_x(1), r_y(1), r_z(1));
```

```matlab
scatter3(r_x(end), r_y(end), r_z(end), 'kx');
xlabel('x (LU)'); ylabel('y (LU)'); zlabel('z (LU)');
title(ftitle)

ftitle = 'A Matrix Relative Difference Histogram';
figure('name', ftitle);
relDiff = abs((Ajune - Ajah)./Ajah);
hist(reshape(log10(abs(relDiff)),7*7,1));
title('A Matrix Relative Difference Histogram')
ylabel('Count')
xlabel('log10(abs(relDiff))')

% Compare with solution
X_GMAT = [
-5153.790483
-4954.421472
-144.8250293
5.178059443
-5.38748629
-0.211928207];

ftitle = 'State at 21600 s Relative Difference Histogram';
figure('name', ftitle);
relDiff = abs((X(end,:)' - X_GMAT)./X_GMAT);
hist(log10(abs(relDiff)));
title(ftitle)
ylabel('Count')
xlabel('log10(abs(relDiff))')


%% Convert coordinates ECEF <--> ECI

global eopdata const

SAT_Const

% read Earth orientation parameters
fid = fopen('eop19620101.txt','r');
% ----------------------------------------------------------------------------------------------------
% |  Date    MJD      x         y       UT1-UTC      LOD       dPsi    dEpsilon     dX        dY        DAT
% |(0h UTC)           "         "          s           s          "        "          "         "         s
% ----------------------------------------------------------------------------------------------------
eopdata = fscanf(fid,'%i %d %d %i %f %f %f %f %f %f %f %f %i',[13 inf]);
fclose(fid);

% Station coords. Convert M --> KM
r_ATL_ECEF = [-6143584   1364250   1033743]' / 1000;  % Atoll
r_DGO_ECEF = [ 1907295   6030810  -817119 ]' / 1000;  % Diego
r_ACB_ECEF = [ 2390310 -5564341   1994578]' / 1000;  % Arecibo
v_ATL_ECEF = [0; 0; 0];
v_DGO_ECEF = [0; 0; 0];
v_ACB_ECEF = [0; 0; 0];

JD_UTC     = cspice_et2utc(et_t0, 'J', 10);
JD_UTC     = str2num(extractAfter(JD_UTC, 'JD '));

% Convert to ECI frame
r_ATL_ECI  = fn.ECEFtoECI(JD, r_ATL_ECEF);
v_ATL_ECI  = v_ATL_ECEF + cross([ 0 0 wE ]', r_ATL_ECEF);
v_ATL_ECI  = fn.ECEFtoECI(JD, v_ATL_ECI); % Technically wrong. Look in Vallado

v_ATL_ECI  = fn.ECEFtoECI(JD, v_ATL_ECEF) + cross([ 0 0 wE]', r_ATL_ECEF);
```

```
r_DGO_ECI   = fn.ECEFtoECI(JD, r_DGO_ECEF);
v_DGO_ECI   = v_ATL_ECEF + cross([ 0 0 wE ]', r_DGO_ECEF);
v_DGO_ECI   = fn.ECEFtoECI(JD, v_DGO_ECI); % Technically wrong. Look in Vallado

r_ACB_ECI   = fn.ECEFtoECI(JD, r_ACB_ECEF);
v_ACB_ECI   = v_ATL_ECEF + cross([ 0 0 wE ]', r_ACB_ECEF);
v_ACB_ECI   = fn.ECEFtoECI(JD, v_ACB_ECI); % Technically wrong. Look in Vallado

% KM
r0_ECEF = fn.ECItoECEF(JD, [X0(1) X0(2) X0(3)]');
v0_ECEF = X0(4:6) + cross([ 0 0 -wE ]', [X0(1) X0(2) X0(3)]');
X0_ECEF = [r0_ECEF; v0_ECEF];


% %% Mahooti check
%
% MJD_UTC      = JD_UTC - 2400000.5;
%
% % Atoll
% Y            = ECEF2ECI(MJD_UTC, [r_ATL_ECEF; v_ATL_ECEF]');
% r_ATL_ECI   = Y(1:3);
% v_ATL_ECI   = Y(4:6);
%
% % Diego
% Y            = ECEF2ECI(MJD_UTC, [r_DGO_ECEF; v_DGO_ECEF]');
% r_DGO_ECI   = Y(1:3);
% v_DGO_ECI   = Y(4:6);
%
% % Arecibo
% Y            = ECEF2ECI(MJD_UTC, [r_ACB_ECEF; v_ACB_ECEF]');
% r_ACB_ECI   = Y(1:3);
% v_ACB_ECI   = Y(4:6);


%% DO EVERYTHING IN ECI FRAME

X = sym('X', [7; 1]);

% Atoll
XS = [ r_ATL_ECI; v_ATL_ECI ];
Ht_ATL_fn = fn.Ht_fn(XS);

% Diego
XS = [ r_DGO_ECI; v_DGO_ECI ];
Ht_DGO_fn = fn.Ht_fn(XS);

% Arecibo
XS = [ r_ACB_ECI; v_ACB_ECI ];
Ht_ACB_fn = fn.Ht_fn(XS);

X0_H    = X0;
Htjune  = Ht_ATL_fn( X0_H(1), X0_H(2), X0_H(3), X0_H(4), X0_H(5), X0_H(6))    ;
Htjah   = load('H_Tilde_t0.mat'); Htjah = Htjah.H_TILDA;

disp('Htjune: ')
disp(Htjune)
disp('Htjah: ')
disp(Htjah)
disp('Htjune - Htjah: ')
disp(Htjune - Htjah)

ftitle = 'H Matrix Relative Difference Histogram';
figure('name', ftitle);
relDiff = abs((Htjune - Htjah)./Htjah);
hist(reshape(log10(abs(relDiff)),2*7,1));
title('H Matrix Relative Difference Histogram')
ylabel('Count')
```

```
xlabel('log10(abs(relDiff))')


%% Calculate residuals

% Reset t to start incrementing at 0
t_XSTM = t - t(1);

% Atoll
[t_ATL, d_err_ATL, d_rms_ATL, v_err_ATL, v_rms_ATL] = ...
fn.Y_residuals(1, t_XSTM, XSTM, Ht_ATL_fn);

% Diego
[t_DGO, d_err_DGO, d_rms_DGO, v_err_DGO, v_rms_DGO] = ...
fn.Y_residuals(2, t_XSTM, XSTM, Ht_DGO_fn);

% Arecibo
[t_ACB, d_err_ACB, d_rms_ACB, v_err_ACB, v_rms_ACB] = ...
fn.Y_residuals(3, t_XSTM, XSTM, Ht_ACB_fn);

ftitle = 'Residuals';
figure('name', ftitle);
subplot(2,1,1)
scatter(t_ATL, d_err_ATL); hold on; grid on;
scatter(t_DGO, d_err_DGO);
scatter(t_ACB, d_err_ACB);
title({'Range Residuals. RMS (km): '; ...
        sprintf('ATL = %.3g, DGO = %.3g, ACB = %.3g', d_rms_ATL, d_rms_DGO, d_rms_ACB) });
        ylabel('km')
        subplot(2,1,2)
        scatter(t_ATL, v_err_ATL); hold on; grid on;
        scatter(t_DGO, v_err_DGO);
        scatter(t_ACB, v_err_ACB);
        title({'Range-Rate Residuals. RMS (km/s): '; ...
                sprintf('ATL = %.3g, DGO = %.3g, ACB = %.3g', v_rms_ATL, v_rms_DGO, v_rms_ACB) })
                    ;
                xlabel('Time (s)')
                ylabel('km/s')
                legend('Atoll', 'Diego', 'Arecibo', 'color', 'none');
```

1. EOM

```
function dX = EOM(et, X)
% ----------------------------------------------------------------------
% Purpose: Generate EOM for satellite orbiting earth due to geopotential,
% lunisolar, SRP, and drag perturbations
%
% Inputs
%   t   = [1x1] time (ET epoch) vector
%   X   = [7x1] state vector in ECI frame (inertial)
%
% Outputs
%   dX  = [7x1] derivative of state vector
% ----------------------------------------------------------------------

global wE muE
global A m p0 r0_drag H

% force column vector. Check if X is numeric or symbolic
if isnumeric(X)
dX = zeros(7, 1);
else
dX = sym(zeros(7,1));
end

% Set velocity and CD
dX(1:3) = X(4:6);
```

```
CD = X(7);

% accel due to point mass (not needed when geopotential gravity is present)
r       = norm(X(1:3));
% dX(4:6) = ( - muE / r^3 ) * X(1:3);

% accel due to gravity
% if isnumeric(X); g = fn.a_spherical(et, X); else g = fn.g_J2J3J4(X); end
% g = fn.a_spherical(et, X);
g = fn.g_J2J3J4(X);
dX(4:6) = dX(4:6) + g;

% accel due to lunisolar perturbation
[a_sun, a_moon] = fn.lunisolar(et, X);
dX(4:6) = dX(4:6) + a_sun + a_moon;

% accel due to SRP
a_srp    = fn.a_SRP(et, X);
dX(4:6) = dX(4:6) + a_srp;

% CHECK UNITS. USE KM !!!
% accel due to drag
pA       = p0 * exp( -(r - r0_drag)/H );
VA       = X(4:6) - cross( [0; 0; wE], X(1:3) );
VAnorm   = norm(VA);
a_drag   = - 1/2 * CD * A/m * pA * VAnorm * VA;
% a_drag  = a_drag / 26.5;  % Correct to match Jah's Amat???
dX(4:6) = dX(4:6) + a_drag;

end
```

2. EOM_STM

```
function dX = EOM_STM(et, X, Amat_fn)
% ------------------------------------------------------------------------
% Purpose: Generate EOM for satellite orbiting earth due to gravity (EGM 96),
% lunisolar perturbations, SRP, and drag
%
% Inputs
%   t    = [7x1] time (ET epoch) vector
%   rv   = [49x1] state vector in ECI frame (inertial)
%
% Outputs
%   drv = [49x1] derivative of state vector
% ------------------------------------------------------------------------

% force column vector
dX = zeros(7+49, 1);

% EOM
dX(1:7) = fn.EOM(et, X);

% STM stuff
Amat = Amat_fn(X(1), X(2), X(3), X(4), X(5), X(6), X(7));
STM  = X(8:7+49);
STM  = reshape(STM, [7 7]);
dSTM = Amat*STM;
dSTM = reshape(dSTM, [49,1]);

dX(8:7+49) = dSTM;

end
```

3. Ht_fn

```matlab
function Ht_fn_out = Ht_fn(XS)

X = sym('X', [7; 1]);

r_site = [X(1)-XS(1); X(2)-XS(2); X(3)-XS(3)];
v_site = [X(4)-XS(4); X(5)-XS(5); X(6)-XS(6)];
d      = norm(r_site);
v      = dot(v_site, r_site/norm(r_site));

Htmat        = sym(zeros(2,7));
Htmat(1,:)   = simplify(gradient(d, X));
Htmat(2,:)   = simplify(gradient(v, X));
Ht_fn_out    = matlabFunction(Htmat);

end
```

4. Y_residuals

```matlab
function [t_STA, d_err_STA, d_rms_STA, v_err_STA, v_rms_STA] = ...
 Y_residuals(ID_STA, t_XSTM, XSTM, Ht_STA_fn)

global LEO_DATA_Apparent

% Station data
i_STA     = find(LEO_DATA_Apparent(:, 1) == ID_STA);
Yobs_STA  = LEO_DATA_Apparent(i_STA, :);
t_STA     = Yobs_STA(:, 2);

% Calculate Y = H * x
Ycalc_STA = zeros(size(Yobs_STA));
Ycalc_STA(:, 1:2) = Yobs_STA(:, 1:2);
for i = 1:length(i_STA)

% find t index
ti     = Yobs_STA(i,2);
i_XSTM = find(t_XSTM == ti);

% Extract states
Xi   = XSTM( i_XSTM, 1:7)';
STMi = XSTM( i_XSTM, 8:7+49 );
STMi = reshape(STMi, [7 7]);

% compute H
Hti = Ht_STA_fn(Xi(1), Xi(2), Xi(3), Xi(4), Xi(5), Xi(6)) * STMi;

% Y = H * x
Ycalc_STA(i,3:4) = Hti * Xi;

end

% Calculate residuals
d_err_STA = Yobs_STA(:,3) - Ycalc_STA(:,3);
d_rms_STA = rms(d_err_STA);
v_err_STA = Yobs_STA(:,4) - Ycalc_STA(:,4);
v_rms_STA = rms(v_err_STA);

end
```

References

[1] Bob Schutz, G. H. B., Byron Tapley, *Statistical Orbit Determination*, Academic Press, 2004.

[2] Vallado, D. A., and McClain, W. D., *Fundamentals of Astrodynamics and Applications*, 4th ed., Microcosm Press, 2013.

[3] Vallado, D. A., "Fundamentals of Astrodynamics and Applications 4th Ed Consolidated Errata," `https://celestrak.com/software/vallado/ErrataVer4.pdf`, 2019.

[4] Montenbruck, O., and Gill, E., *Satellite Orbits: Models, Methods and Applications*, Springer, Berlin, Heidelberg, 2000.

[5] "Standard Rapid EOP Data since 02. January 1973 (IAU1980)," `https://datacenter.iers.org/data/7/finals.all`, 2021.