

# ASE 389P.4 Methods of Orbit Determination

## Homework 2: Orbit Propagation with Perturbations

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**New force models have been added to the orbit propagator created in Homework 1. The effects that these new forces have on the Keplerian orbit elements and the total specific energy were analyzed.**

### I. Introduction

In this assignment, a tool was created to numerically propagate a circular orbit about the Earth and to convert between Cartesian and Keplerian orbital elements. 2 solutions for the derivation of the gradient potential function are given. 2-body equations of motion and constants of motion are also explored. The software implemented in this assignment will be used later in the course provides additional background and/or a review of basic orbital mechanics.

### II. Problem 1

#### A. Statement

Given the Earth orbiting spacecraft position and velocity vectors in Cartesian coordinates

$$\underline{R} = -2436.45\hat{i} - 2436.45\hat{j} + 6891.037\hat{k} \text{ km} \quad (1)$$

$$\underline{V} = \dot{\underline{R}} = 5.088611\hat{i} - 5.088611\hat{j} + 0.0\hat{k} \text{ km/s} \quad (2)$$

solve for the Keplerian elements  $(a, e, i, \Omega, \omega, \nu)$ . Provide your values in the write-up. See the lecture notes

Assume  $\mu = 398600.5 \text{ km}^3/\text{s}^2$ .

#### B. Solution

The algorithms to convert orbital elements to Cartesian and back were taken from References [1] and [2]. First, the specific angular momentum vector,  $h$ , and perpendicular node vector (to the plane of the orbit),  $n$ , were calculated. Inclination, eccentricity, and the rest of the orbital elements followed while checking for equatorial orbits, NaNs, and quadrants of angles. Given the spacecraft position and velocity vectors from the problem statement, the Keplerian elements are:

$$\begin{aligned} a &= 7.712184983762814e + 03 \\ e &= 0.447229247404423 \\ i &= 1.570796326794897 \\ \omega &= 3.139593866862924 \\ \Omega &= 3.926990816987241 \\ \nu &= 2.032461649676350 \end{aligned} \quad (1)$$

where  $a$  is the semi-major axis,  $e$  is the eccentricity,  $i$  is the orbit inclination,  $\omega$  is the argument of perigee,  $\Omega$  is the right ascension of the ascending node, and  $\nu$  is the true anomaly.

### III. Problem 2

#### A. Statement

Convert the Keplerian elements from Problem 1 back to position and velocity and provide the values in the write-up. See the lecture notes:

#### B. Solution

The expression for position in the perifocal frame is the following:

$$\underline{R} = r \cos(\nu) \hat{P} + r \sin(\nu) \hat{Q} \quad (2)$$

where the scalar magnitude  $r$  can be determined from the polar equation of a conic:

$$r = \frac{p}{1 + e \cos(\nu)} \quad (3)$$

where

$$p = a(1 - e^2) = \frac{h^2}{\mu} \quad (4)$$

$\underline{R}$  then needs to be transformed from the perifocal frame to the ECI frame, which can be done through a series of transformation matrices as outlined in References [1] and [3]. The resulting position and velocity vectors returned are:

$$\begin{aligned} r &= [-2.416951611809028e + 03, 2.416951611809029e + 03, -6.904756183437363e + 03] \\ v &= [-5.088570340542163, 5.088570340542164, -0.028767991782603] \end{aligned} \quad (5)$$

### IV. Problem 3

#### A. Statement

Given the gravity potential function  $U = \mu/R$ , solve for the two-body acceleration due to gravity, i.e.,

$$\nabla U = \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k} \quad (3)$$

where  $R = \underline{R} \bullet \underline{R}$ . Include your derivation in your solution to the assignment.

#### B. Solution 1

To solve for the two-body acceleration due to gravity, first calculate  $R$ , which is given in the problem statement as  $R = \underline{R} \bullet \underline{R}$ . Let us first define  $\underline{R}$  as the following:

$$\underline{R} = x\hat{i} + y\hat{j} + z\hat{k} \quad (6)$$

The dot product of the vector  $\underline{R}$  with itself is equal to the square of its magnitude:

$$R = \underline{R} \bullet \underline{R} = x^2 + y^2 + z^2 \quad (7)$$

Given the gravity potential function  $U = \mu/R$ , the gradient of the gravity potential function  $\nabla U$  is:

$$\nabla U = \frac{\delta U}{\delta x} \hat{i} + \frac{\delta U}{\delta y} \hat{j} + \frac{\delta U}{\delta z} \hat{k} \quad (8)$$

First derive  $\frac{\delta U}{\delta x} \hat{i}$ :

$$\frac{\delta U}{\delta x} \hat{i} = \frac{\delta\left(\frac{\mu}{R}\right)}{\delta x} \hat{i} = \frac{\delta\left(\frac{\mu}{x^2 + y^2 + z^2}\right)}{\delta x} \hat{i} \quad (9)$$

Take the partial derivative:

$$\frac{\delta\left(\frac{\mu}{x^2 + y^2 + z^2}\right)}{\delta x} \hat{i} = \frac{\delta\left(\mu(x^2 + y^2 + z^2)^{-1}\right)}{\delta x} \hat{i} = -\mu(x^2 + y^2 + z^2)^{-2}(2x)\hat{i} \quad (10)$$

Now simplify:

$$-\mu(x^2 + y^2 + z^2)^{-2}(2x)\hat{i} = -\frac{2\mu x}{(x^2 + y^2 + z^2)^2} \hat{i} \quad (11)$$

Thus:

$$\frac{\delta U}{\delta x} \hat{i} = -\frac{2\mu x}{(x^2 + y^2 + z^2)^2} \hat{i} \quad (12)$$

$\frac{\delta U}{\delta y} \hat{j}$  and  $\frac{\delta U}{\delta z} \hat{k}$  can be derived through the same process, which result in the following:

$$\frac{\delta U}{\delta y} \hat{j} = -\frac{2\mu y}{(x^2 + y^2 + z^2)^2} \hat{j} \quad (13)$$

$$\frac{\delta U}{\delta z} \hat{k} = -\frac{2\mu z}{(x^2 + y^2 + z^2)^2} \hat{k} \quad (14)$$

The gradient of the gravity potential function is thus:

$$\nabla U = -\frac{2\mu x}{(x^2 + y^2 + z^2)^2} \hat{i} - \frac{2\mu y}{(x^2 + y^2 + z^2)^2} \hat{j} - \frac{2\mu z}{(x^2 + y^2 + z^2)^2} \hat{k} \quad (15)$$

Plug in  $x = -2436.45$ ,  $y = -2436.45$ ,  $z = 6891.037$ , and  $\mu = 398600.5$  from Problem 1 into Equation 15 to solve for the two-body acceleration due to gravity:

$$\nabla U = 5.512551407304731e - 07 \hat{i} - 5.512551407304731e - 07 \hat{j} - 1.559120676071291e - 06 \hat{k} \quad (16)$$

### C. Solution 2

There may have been a typo with the gravitational potential function in the problem statement. The gravitational potential function is commonly simplified as  $U = \mu/r$ , where  $r$  is the distance between two bodies [1]:

$$r = \sqrt{R} = \sqrt{\underline{R} \cdot \underline{R}} = (\underline{R} \cdot \underline{R})^{1/2} = (x^2 + y^2 + z^2)^{1/2} \quad (17)$$

If we use Equation 17 to calculate the gradient for the potential function instead of Equation 7, then  $\frac{\delta U}{\delta x} \hat{i}$  becomes:

$$\frac{\delta U}{\delta x} \hat{i} = \frac{\delta\left(\frac{\mu}{r}\right)}{\delta x} \hat{i} = \frac{\delta\left(\frac{\mu}{(x^2 + y^2 + z^2)^{1/2}}\right)}{\delta x} \hat{i} = \frac{\delta\left(\mu(x^2 + y^2 + z^2)^{-1/2}\right)}{\delta x} \hat{i} \quad (18)$$

Take the derivative and simplify:

$$\frac{\delta\left(\mu(x^2 + y^2 + z^2)^{-1/2}\right)}{\delta x} = \mu\left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}(2x)\hat{i} = -\frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \hat{i} \quad (19)$$

Thus:

$$\frac{\delta U}{\delta x} \hat{i} = -\frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \hat{i} = -\frac{\mu x}{r^3} \hat{i} \quad (20)$$

And:

$$\frac{\delta U}{\delta y} \hat{j} = -\frac{\mu y}{r^3} \hat{j} \quad (21)$$

$$\frac{\delta U}{\delta z} \hat{k} = -\frac{\mu z}{r^3} \hat{k} \quad (22)$$

Equation 15 then becomes:

$$\nabla U = -\frac{\mu x}{r^3} \hat{i} - \frac{\mu y}{r^3} \hat{j} - \frac{\mu z}{r^3} \hat{k} \quad (23)$$

Plug in  $x = -2436.45$ ,  $y = -2436.45$ ,  $z = 6891.037$ , and  $\mu = 398600.5$  from Problem 1 into Equation 23 to solve for the two-body acceleration due to gravity:

$$\nabla U = 0.002123566317530 \hat{i} + 0.002123566317530 \hat{j} - 0.006006104810708 \hat{k} \quad (24)$$

## V. Problem 4

### A. Statement

Develop the necessary code to numerically integrate the equations of motion using the position and velocity from Problem 1 as the initial conditions. Compute the future position and velocity at 20-second intervals for two full orbits. Plot the magnitude of the position, velocity, and acceleration as a function of time for two full orbits and provide the figure. Compute the specific orbital angular momentum vector for these two full orbits and plot that as well, as a function of time, as a 3D scatter plot ( $\underline{h} = \underline{R} \times \underline{V}$ ). Assume that the motion is only due to the accelerations derived from Eq(3)

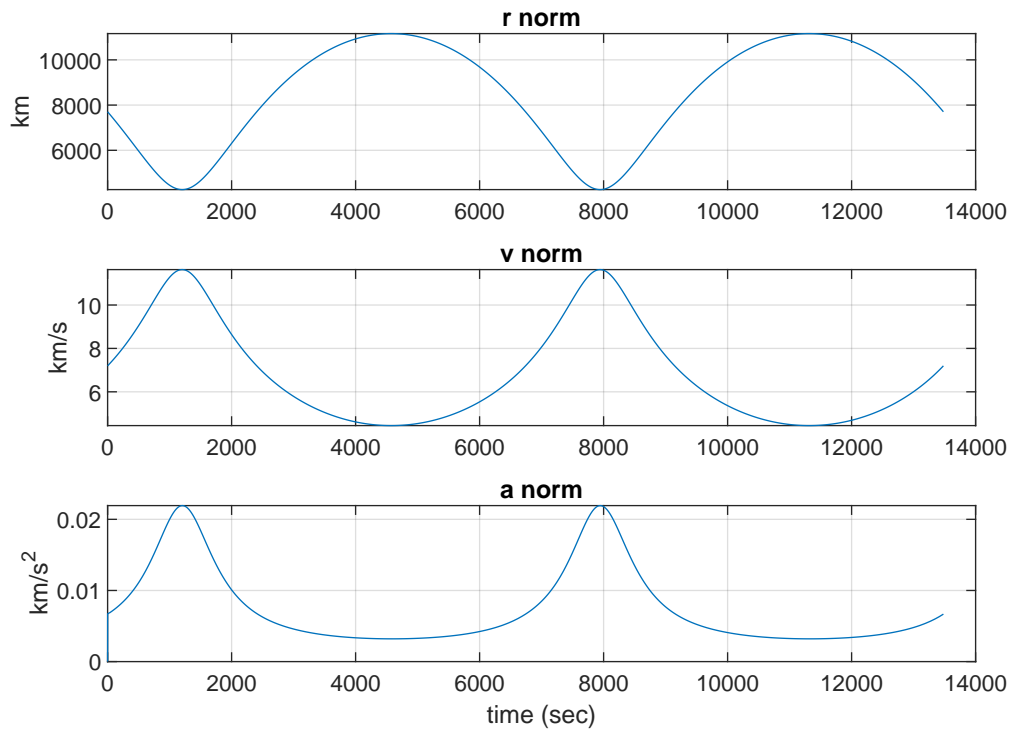
### B. Solution

The period for the orbit was found by using the semi-major axis in the following equation:

$$T = \left| 2\pi \sqrt{a^3/\mu} \right| \quad (25)$$

The orbit was propagated by numerically integrating the equations of motion 23 for 2 periods using `ode45`. The relative and absolute tolerances were set to  $1e-8$ . The acceleration was calculated by differentiating velocity with respect to time, and then the magnitudes of the position, velocity, and acceleration were plotted in Figure 1.

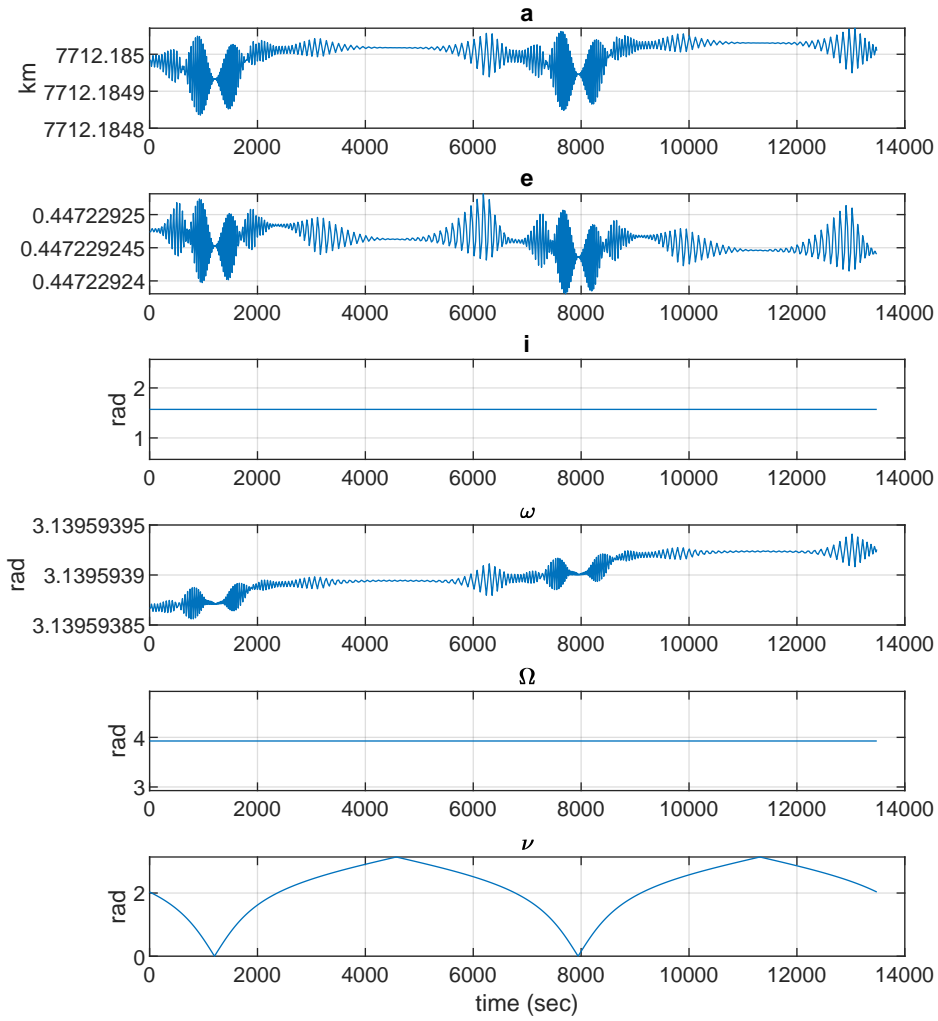
## Problem 4: 2-Body EOM



**Fig. 1**

The magnitudes of the velocity and acceleration vary throughout the orbit, which indicates that the orbit is not circular. The position and velocity vectors were converted into orbital elements which are shown in 2. All of the orbital elements with the exception of true anomaly remain essentially constant, revealing the predictable and Keplerian nature of the orbit. The eccentricity shows that the orbit is elliptical; an eccentricity of 0 forms a perfectly circular orbit, a 1 forms a parabolic escape orbit, and greater than 1 forms a hyperbolic orbit.

## Problem 4: 2-Body Orbital Elements



**Fig. 2**

Figure 3 illustrates that the specific angular momentum essentially remains constant throughout the entire orbit. The difference between the maximum and minimum of **h norm** is  $7.204597714007832e-04$ , which is incredibly small especially when considering that the order of magnitude for all values of **h norm** is 4.

## Problem 4: 2-Body Specific Angular Momentum

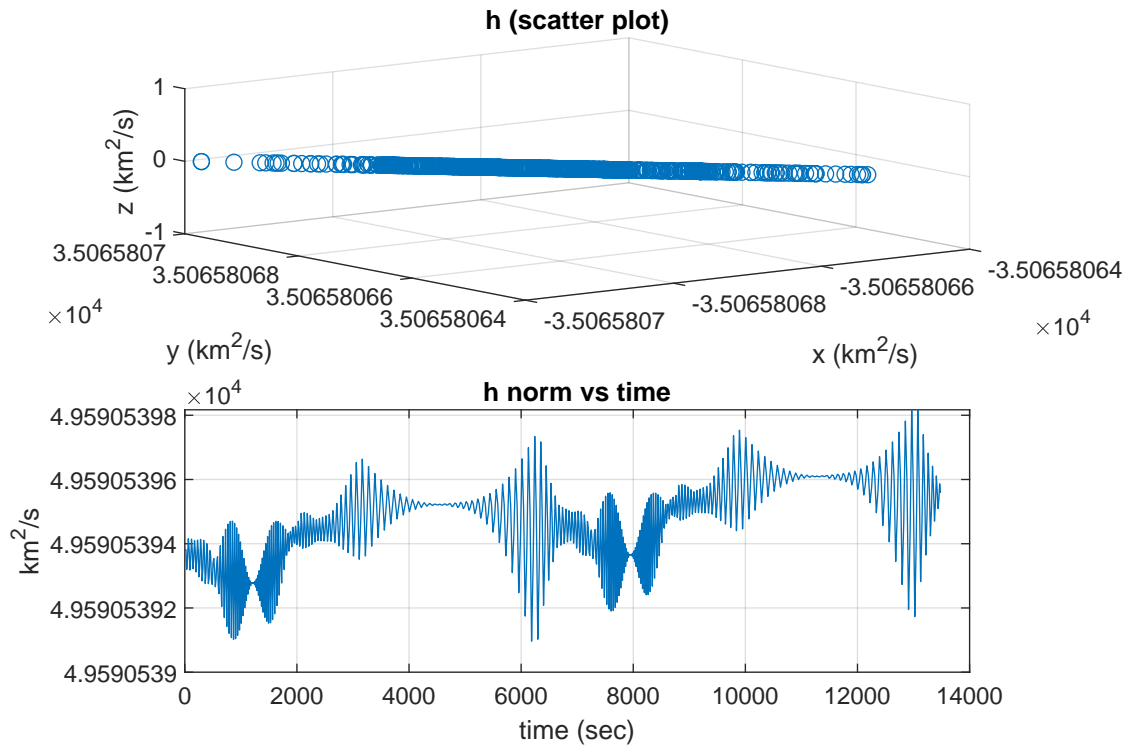


Fig. 3

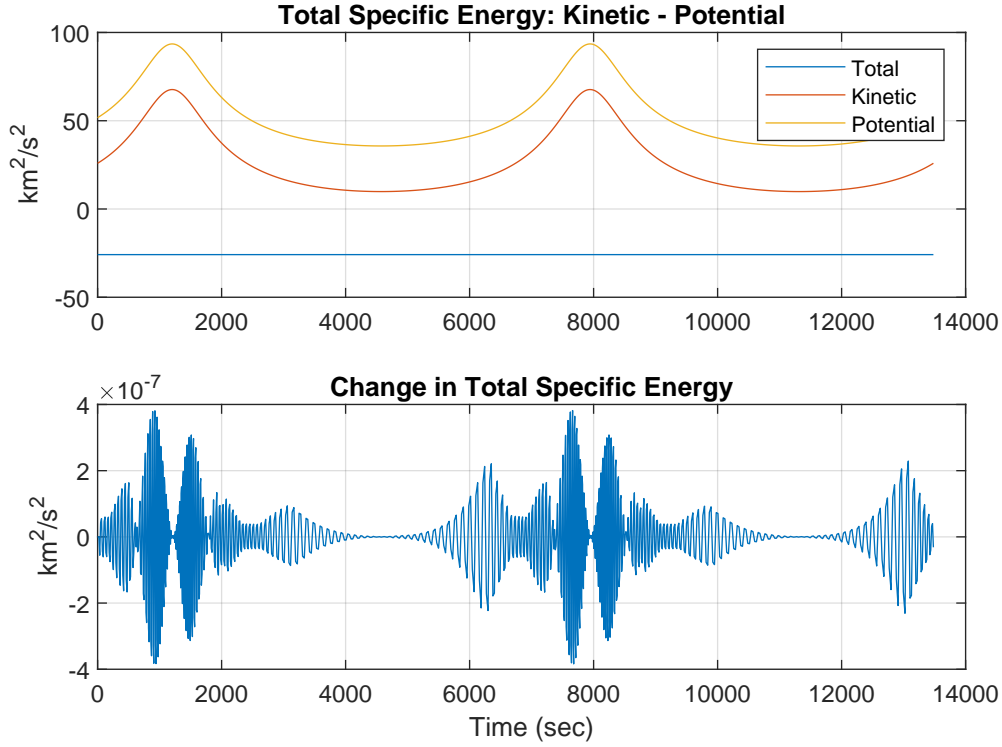
## VI. Problem 5

### A. Statement

Compute the specific kinetic energy and specific potential energy as a function of time and plot the change in total specific energy to show that it remains constant over the two orbits. (i.e. plot  $dE = E(t) - E(t_0)$ ). Include the image in your write-up. Why is the change in total specific energy not constant?

## B. Solution

### Problem 4: 2-Body Specific Energy



**Fig. 4**

The energy constant of motion is given in Reference [1]:

$$E = \frac{v^2}{2} - \frac{\mu}{r} \quad (26)$$

in which the first term is the "kinetic energy per unit mass," or specific kinetic energy, and the last term is the specific potential energy. The potential energy will always be negative due to setting the zero reference of potential energy at the center of a massive body and the work done when moving from one point in space to another against the force of gravity. The specific mechanical energy,  $E$ , is the sum of the specific kinetic and potential energy and remains constant along its orbit.

## VII. Conclusion

Calculating orbital elements to and from Cartesian position and velocity vectors can be tricky because of the quadrant ambiguities when performing trigonometric computations. Problem 3 might have included a typo in its problem statement for the gravity potential function; the potential is commonly given as  $U = \mu/r$  in which  $r$  is the distance between 2 bodies, not  $U = \mu/(\underline{R} \cdot \underline{R})$ . Solutions were given for both potential equations. Though position and velocity may vary along an orbit, there are constants of motion such as angular momentum and specific mechanical energy which govern the motion of celestial bodies.



## VIII. Appendix

### A. HW1 MATLAB code

```
1  % ASE 389 Orbit Determination
2  % HW 1
3  % Junette Hsin
4
5  %% Problem 1
6
7  global mu
8
9  mu = 398600.5 ;
10 r = [ -2436.45; -2436.45; 6891.037 ];
11 v = [ 5.088611; 5.088611; 0 ];
12
13 rv = [ r; v];
14 oe = rv2oe(rv);
15
16 %% Problem 2
17
18 rv = oe2rv(oe);
19
20 %% Problem 3
21
22 x = -2436.45;
23 y = -2436.45;
24 z = 6891.037;
25 mu = 398600.5;
26
27 dux = -2*mu*x / ( x^2 + y^2 + z^2 )^2;
28 duy = -2*mu*y / ( x^2 + y^2 + z^2 )^2;
29 duz = -2*mu*z / ( x^2 + y^2 + z^2 )^2;
30
31 rnorm = sqrt( x^2 + y^2 + z^2 );
32
33 dux = -mu*x / ( rnorm )^3;
34 duy = -mu*y / ( rnorm )^3;
35 duz = -mu*z / ( rnorm )^3;
36
37 %% Problem 4
38
39 a = oe(1);
40 T = abs(2 * pi * sqrt(a^3 / mu)); % period
41
42 toler = 1e-8; % 1e-14 accurate; 1e-6 coarse
43 options = odeset('reltol', toler, 'abstol', toler );
44 [t,x] = ode45(@TwoBod_6states, [0 2*T], [r; v], options);
45
46 for i = 1:length(t)
47 rnorm(i) = norm(x(i, 1:3));
48 vnorm(i) = norm(x(i, 4:6));
49 H(i, :) = cross(x(i, 1:3), x(i, 4:6));
50 hnorm(i) = norm(H(i, :));
51 end
52
53 anorm = 0;
54 for i = 2:length(t)
55 a = (x(i, 4:6) - x(i-1, 4:6)) / ( t(i) - t(i-1) );
56 anorm(i) = norm(a);
57 end
58
59 % -----
60
61 name = 'Problem 4: 2-Body EOM';
62 h = figure('name', name);
63
64 % position
```

```

65 subplot(3,1,1)
66 plot(t, rnorm); grid on
67 title('r norm')
68 ylabel('km')
69
70 % velocity
71 subplot(3,1,2)
72 plot(t, vnorm); grid on
73 title('v norm')
74 ylabel('km/s')
75
76 % acceleration
77 subplot(3,1,3)
78 plot(t, anorm); grid on
79 title('a norm');
80 ylabel('km/s^2')
81 xlabel('time (sec)')
82
83 sgtitle(name)
84
85 save_pdf(h, 'prob4_2bodeom');
86
87 % -----
88
89 name = 'Problem 4: 2-Body EOM Orbit';
90 h = figure('name', name);
91 plot3(x(:,1), x(:,2), x(:,3)); hold on; grid on;
92 plot3(x(1,1), x(1,2), x(1,3), 'o')
93 plot3(x(end,1), x(end,2), x(end,3), 'x')
94 xlabel('x (km)')
95 ylabel('y (km)')
96 zlabel('z (km)')
97 legend('orbit', 'start', 'end')
98
99 sgtitle(name)
100
101 save_pdf(h, 'prob4_2bodeom_orbit');
102
103 % -----
104
105 clear oe
106 for i = 1:length(t)
107     oe(i,:) = rv2oe(x(i,:));
108 end
109
110 labels = {'a', 'e', 'i', '\omega', '\Omega', '\nu'};
111 units = {'km', '', 'rad', 'rad', 'rad', 'rad'};
112 name = 'Problem 4: 2-Body Orbital Elements';
113 h = figure('name', name, 'position', [100 100 500 600]);
114 for i = 1:6
115     subplot(6,1,i)
116     plot(t, oe(:, i)); grid on
117     title(labels{i});
118     ylabel(units{i});
119 end
120 xlabel('time (sec)')
121 sgtitle(name)
122
123 save_pdf(h, 'prob4_2bdoes');
124
125 % -----
126
127 name = 'Problem 4: 2-Body Specific Angular Momentum';
128 h = figure('name', name);
129 subplot(2,1,1)
130 scatter3(H(:,1), H(:,2), H(:,3)); grid on
131 xlabel('x (km^2/s)')
132 ylabel('y (km^2/s)')

```

```

133 zlabel('z (km^2/s)')
134 title('h (scatter plot)')
135
136 subplot(2,1,2)
137 plot(t, hnorm); grid on
138 xlabel('time (sec)')
139 ylabel('km^2/s')
140 title('h norm vs time')
141
142 sgtitle(name)
143
144 save_pdf(h, 'prob4_angmom')
145
146 %% Problem 5
147
148 % specific kinetic energy
149 for i = 1:length(t)
150 T(i) = 0.5 * vnorm(i)^2;
151 U(i) = mu / rnorm(i);
152 end
153 E = T - U;
154
155 name = 'Problem 4: 2-Body Specific Energy';
156 h = figure('name', name);
157 subplot(2,1,1)
158 plot(t, E); grid on; hold on;
159 plot(t, T);
160 plot(t, U);
161 ylabel('km^2/s^2')
162 legend('Total', 'Kinetic', 'Potential')
163 title('Total Specific Energy: Kinetic - Potential')
164 subplot(2,1,2)
165 plot(t, [0 diff(E)]); grid on
166 title('Change in Total Specific Energy')
167 xlabel('Time (sec)')
168 ylabel('km^2/s^2')
169 sgtitle(name)
170
171 save_pdf(h, 'prob5_energy')
172
173 %% subfunctions
174
175 function save_pdf(h, name)
176
177 % save as cropped pdf
178 set(h,'Units','Inches');
179 pos = get(h,'Position');
180 set(h,'PaperPositionMode','Auto','PaperUnits','Inches','PaperSize',[pos(3), pos(4)])
181 print(h,name,'-dpdf','-r0')
182
183 end

```

## B. rv2oe function

```

1 function oe = rv2oe(rv)
2 % -----
3 % Inputs
4 %   rv = [6x1] position and velocity states vector
5 %
6 % Outputs
7 %   oe = [6x1] orbital elements: a, e, i, w, Omega, nu
8 %       a      = semimajor axis
9 %       e      = eccentricity
10 %       i      = inclination
11 %       w      = argument of perigee
12 %       Omega  = right ascension of ascending node
13 %       nu     = true anomaly

```

```

14  % -----
15
16  global mu
17
18  r = rv(1:3);
19  v = rv(4:6);
20
21  % angular momentum
22  h = cross(r,v);
23
24  % node vector
25  nhat = cross([0 0 1], h);
26
27  % eccentricity
28  evec = ( (norm(v)^2 - mu/norm(r))*r - dot(r,v)*v ) / mu;
29  e = norm(evec);
30
31  % specific mechanical energy
32  energy = norm(v)^2/2 - mu/norm(r);
33
34  % semi-major axis and p
35  if abs(e-1.0)>eps
36  a = -mu/(2*energy);
37  p = a*(1-e^2);
38  else
39  p = norm(h)^2/mu;
40  a = inf;
41  end
42
43  % inclination
44  i = acos(h(3)/norm(h));
45
46  % right ascension of ascending node (check for equatorial orbit)
47  if i > 0.000001
48  Omega = acos( nhat(1)/norm(nhat) );
49  else
50  Omega = 0;
51  end
52  if isnan(Omega)
53  Omega = 0;
54  end
55  if nhat(2)<0
56  Omega = 2*pi - Omega;
57  end
58
59  % argument of perigee
60  if e > 0.000001
61  w = acos( dot(nhat,evec)/(norm(nhat)*e) );
62  else
63  w = 0;
64  end
65  if isnan(w)
66  w = 0;
67  end
68  % if e(3)<0
69  %   argp = 360-argp
70  % end
71
72  % true anomaly
73  nu = acos( dot(evec,r) / (e*norm(r)) );
74  % if dot(r,v)<0
75  %   nu = 360 - nu
76  % end
77
78  oe = [a; e; i; w; Omega; nu];
79
80  end

```

### C. oe2rv function

```

1 function [rv] = oe2rv(oe)
2 % -----
3 % Purpose: Convert orbital elements and time past epoch to the classic
4 % Cartesian position and velocity
5 %
6 % Inputs:
7 %   oe      = [6x1] or [1x6] orbital elements
8 %   delta_t = t - t0 time interval
9 %   mu      = Gravity * Mass (of Earth) constant
10 %
11 % Outputs:
12 %   rv      = position and velocity state vector
13 % -----
14
15 % global delta_t
16 global mu
17
18 a      = oe(1);
19 e      = oe(2);
20 i      = oe(3);
21 w      = oe(4);
22 LAN    = oe(5);
23 % M0    = oe(6);
24 nu     = oe(6);
25
26 % nu is TRUE ANOMALY --> use Kepler's to calculate MEAN ANOMALY
27 % E = 2*atan( sqrt( (1-e)/(1+e) ) * tan(nu/2) );
28 % M = M0 + sqrt( mu/a^3 ) * (delta_t);
29 % E = keplerEq(M, e, eps);
30 % E = kepler(M, e);
31 % nu = 2*atan( sqrt( (1+e)/(1-e) ) * tan(E/2) );
32
33 p = a * ( 1 - e^2 );           % intermediate variable
34 r = p / ( 1 + e*cos(nu) );     % r_magnitude, polar coordinates
35
36 % Perifocal position and velocity
37
38 r_pf = [ r * cos(nu); r * sin(nu); 0 ];
39 v_pf = [ -sqrt(mu/p) * sin(nu); sqrt(mu/p) * (e + cos(nu)); 0 ];
40
41 % Perifocal to ECI transformation, 3-1-3 rotation
42 R11 = cos(LAN)*cos(w) - sin(LAN)*sin(w)*cos(i);
43 R12 = -cos(LAN)*sin(w) - sin(LAN)*cos(w)*cos(i);
44 R13 = sin(LAN)*sin(i);
45
46 R21 = sin(LAN)*cos(w) + cos(LAN)*sin(w)*cos(i);
47 R22 = -sin(LAN)*sin(w) + cos(LAN)*cos(w)*cos(i);
48 R23 = -cos(LAN)*sin(i);
49
50 R31 = sin(w)*sin(i);
51 R32 = cos(w)*sin(i);
52 R33 = cos(i);
53
54 R = [R11 R12 R13; R21 R22 R23; R31 R32 R33];
55
56 % Transform perifocal to ECI frame
57 r_vec = R * r_pf;
58 v_vec = R * v_pf;
59
60 % Position and state vector
61 rv = [r_vec; v_vec];
62
63 end
64
65 %% Kepler equation solvers
66

```

```

67 function E = keplerEq(M,e,eps)
68 % Function solves Kepler's equation  $M = E - e \sin(E)$ 
69 % Input - Mean anomaly M [rad] , Eccentricity e and Epsilon
70 % Output eccentric anomaly E [rad].
71 En = M;
72 Ens = En - (En - e*sin(En) - M)/(1 - e*cos(En));
73 while ( abs(Ens-En) > eps )
74 En = Ens;
75 Ens = En - (En - e*sin(En) - M)/(1 - e*cos(En));
76 end
77 E = Ens;
78 end
79
80 function E = kepler(M, e)
81 f = @(E) E - e * sin(E) - M;
82 E = fzero(f, M); % <-- I would use M as the initial guess instead of 0
83 end

```

s

## References

- [1] Donald D. Mueller, J. W., and Bate, R. R., *Fundamentals of Astrodynamics*, Dover Publications, Inc., 1971.
- [2] Jah, M. K., “ASE 389P.4 Methods of Orbit Determination Module 3,” , January 2021.
- [3] Jah, M. K., “ASE 389P.4 Methods of Orbit Determination Module 2,” , January 2021.