ASE 389P.4 Methods of Orbit Determination
Homework 3: The Batch and Sequential Processor

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The numeric propagation of the state transition matrix (STM) and the batch processor were explored. A state transition matrix (STM) for a two-dimensional, two-body orbit was first gnerated and verified by comparing the mapped deviations to differences in the nonlinear solution. Finally, a nonlinear batch processor and the sequential processor for a simple problem were implemented.

Problem 1

Assume an orbit plane coordinate system with a gravitational parameter of 1, i.e., $\mu = 1$. The equations of motion are:

$$\begin{aligned} \ddot{x} &= -\frac{x}{r^3} \\ \ddot{y} &= -\frac{y}{r^3} \\ r^2 &= x^2 + y^2 \end{aligned}$$

Problem 1a

Generate a "true" solution by numerically integrating the equations of motion for the initial conditions:

$$\underline{X}(t_0) = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}_{t=t_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Save the values of the state vector $\underline{X}(t_i)$ for $t_i = i \cdot 10$ time units (TU); $i = 0, \ldots, 10$. Provide $\underline{X}(t_i)$ for t_1 and t_{10} in the writeup.

In your write-up, please indicate which integrator you used, what the tolerance was set to, and any other details necessary. Note, if you use a fixed time-step integrator, set the time-step to be smaller than 10 TU, but only save the data at 10 TU intervals.

Solution

The derivation by hand is included in the appendix. The hand-derived and MATLAB results were the same. The final equation for $\partial U/\partial x$ is:

$$\partial U/\partial x = -J_2 \mu R_E^2 \left(\frac{3x}{2}\right) \frac{x^2 + y^2 - 4z^2}{(x^2 + y^2 + z^2)^{7/2}} \tag{1}$$

The MATLAB code used for the entire homework, including calculating the partial derivatives is included in the Appendix, but the snippet used to solve Problem 1a is shown here:

```matlab
syms x y z
global mu RE J2

% constants
% mu = 398600.4;        % G * M1 * M2
% RE = 6378.145;        % Earth radius
% J2 = 0.00108248;      % J2

% for symbolic representation
syms mu RE J2

% radius
r = sqrt(x^2 + y^2 + z^2);

% U point mass
Up = mu/r;

% latitude
phi = asin(z/r);

% U J2
UJ2 = -mu/r * J2 * (RE/r)^2 * ( 3/2 * ( sin(phi) )^2 - 1/2 );

% gradient
d_UJ2 = gradient(UJ2, [x y z]);
d_UJ2 = simplify(d_UJ2);
```

Appendix

$\partial U / \partial x$ derivation

Derivation

$$U = U_{\text{point mass}} + U_{J_2} = \frac{\mu}{r}\left[1 - J_2\left(\frac{R_{\text{Earth}}}{r}\right)^2\left(\frac{3}{2}\sin^2(\phi) - \frac{1}{2}\right)\right] \tag{1}$$

$$U_p = \frac{\mu}{r} \rightarrow \boxed{\nabla U_p = -\frac{\mu}{r^3}\underline{r}}$$ (done in previous homework)

$$\boxed{\frac{\partial U_p}{\partial x} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}}$$

$\phi = \arcsin\left(\frac{z}{r}\right)$

$$U_{J_2} = -\frac{\mu}{r} J_2\left(\frac{R_E}{r}\right)^2\left(\frac{3}{2}\sin^2(\phi) - \frac{1}{2}\right)$$

$$= \left[-J_2\right]\left[\mu r^{-1}\right]\left[R_E^2 r^{-2}\right]\left[\frac{3}{2}\sin^2\left(\arcsin\left(\frac{z}{r}\right)\right) - \frac{1}{2}\right]$$

$$= \left[-J_2 \mu R_E^2\right]\left[r^{-3}\right]\left[\frac{3}{2}\underbrace{\sin^2\left(\arcsin\left(\frac{z}{r}\right)\right)}_{\left(\cancel{\sin}\left(\cancel{\arcsin}\left(\frac{z}{r}\right)\right)\right)^2 = \frac{z^2}{r^2}} - \frac{1}{2}\right]$$

$$= \left[-J_2 \mu R_E^2\right]\left[r^{-3}\right]\left[\frac{3}{2}z^2 r^{-2} - \frac{1}{2}\right]$$

$$= \left[-J_2 \mu R_E^2\right]\underbrace{\left[\frac{3}{2}z^2 r^{-5} - \frac{1}{2}r^{-3}\right]}_{A}$$

$$\nabla U_{J_2} = \frac{\partial U_{J_2}}{\partial x}\hat{i} + \frac{\partial U_{J_2}}{\partial y}\hat{j} + \frac{\partial U_{J_2}}{\partial z}\hat{k}$$

$$A = \frac{3}{2}z^2 r^{-5} - \frac{1}{2}r^{-3} = \frac{3}{2}z^2\left(x^2+y^2+z^2\right)^{-5/2} - \frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial A}{\partial x} = \left(\frac{3}{\cancel{2}}z^2\right)\left(-\frac{5}{2}\right)\left(x^2+y^2+z^2\right)^{-7/2}(\cancel{2}x) + \left(\frac{1}{\cancel{2}}\right)\left(+\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(\cancel{2}x)$$

$$= -\frac{15xz^2}{2}\left(x^2+y^2+z^2\right)^{-7/2} + \frac{3x}{2}\left(x^2+y^2+z^2\right)^{-5/2}$$

$$\frac{\partial A}{\partial x} = \left(\frac{3x}{2}\right)\frac{\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}} \quad \hat{i} \quad \rightarrow \quad \frac{3x}{2}\;\frac{r^2-5z^2}{r^7}$$

$$\frac{\partial A}{\partial y} = \left(\frac{3y}{2}\right)\frac{\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}} \quad \hat{j} \quad \rightarrow \quad \frac{3y}{2}\;\frac{r^2-5z^2}{r^7}$$

$$\frac{\partial A}{\partial z} = \frac{3}{\cancel{2}}z^2\left(-\frac{5}{2}\right)\left(x^2+y^2+z^2\right)^{-7/2}(\cancel{2}z) + 3z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$+\left(\frac{1}{\cancel{2}}\right)\left(\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(\cancel{2}z)$$

$$= -\frac{15z^3}{2}\left(x^2+y^2+z^2\right)^{-7/2} + \underset{\frac{6}{2}}{\cancel{3}}z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$+\frac{3z}{2}\left(x^2+y^2+z^2\right)^{-5/2}$$

$$\frac{\partial A}{\partial z} = \frac{3z}{2}\left[\frac{3\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}}\right] \quad \hat{k} \quad \rightarrow \quad \frac{3z}{2}\;\frac{3r^2-5z^2}{r^7}$$

$$\nabla A = \frac{\partial A}{\partial x}\hat{i} + \frac{\partial A}{\partial y}\hat{j} + \frac{\partial A}{\partial z}\hat{k}$$

$$\boxed{\begin{aligned}
\frac{\partial U_{J_2}}{\partial x} &= \left[-J_2\mu R_E^2\right]\left(\frac{3x}{2}\;\frac{r^2-5z^2}{r^7}\right)\\
\frac{\partial U_{J_2}}{\partial x} &= \left[-J_2\mu R_E^2\right]\left(\frac{3y}{2}\;\frac{r^2-5z^2}{r^7}\right)\\
\frac{\partial U_{J_2}}{\partial x} &= \left[-J_2\mu R_E^2\right]\left(\frac{3z}{2}\;\frac{3r^2-5z^2}{r^7}\right)
\end{aligned}}$$

$$\frac{\partial U_p}{\partial x} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial U_{J_2}}{\partial x} = \left[-J_2 \mu R_E^{\,2}\right]\left(\frac{3x}{2}\,\frac{r^2-5z^2}{r^7}\right)$$

$$= \left[-J_2 \mu R_E^{\,2}\right]\left[\frac{3x}{2}\,\frac{x^2+y^2-4z^2}{(x^2+y^2+z^2)^{7/2}}\right] \quad \checkmark$$

matches Matlab output

$$\frac{\partial U}{\partial x} = \frac{\partial U_p}{\partial x} + \frac{\partial U_{J_2}}{\partial x}$$

HW2 MATLAB code

```
% ASE 389 Orbit Determination
% HW 3
% Junette Hsin
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References

[1] Bob Schutz, G. H. B., Byron Tapley, *Statistical Orbit Determination*, Academic Press, 2004.

[2] Jah, M. K., "ASE 389P.4 Methods of Orbit Determination Module 3," , January 2021.