ASE 389P.4 Methods of Orbit Determination
Homework 2: Orbit Propagation with Perturbations

Assigned: Thursday, February 11, 2021
Due: Thursday, February 18 2021 @ 12:30pm

With this homework, you will add new force models to the orbit propagator created in Homework 1. You will then analyze the effects that these new forces have on the Keplerian orbit elements and the total specific energy.

Please include all software as an appendix to the turned in write-up. Turn in your solution as a PDF uploaded to Canvas.

Problems

1. Integrate the equations of motion for one day using the same initial conditions as in Homework #1. However, now include the Earth's oblateness, i.e., the J_2 term. The equations of motion are still $\ddot{\underline{r}} = \nabla U$, but now U includes J_2:

$$U = U_{\text{point mass}} + U_{J_2} = \frac{\mu}{r}\left[1 - J_2\left(\frac{R_{\text{Earth}}}{r}\right)^2\left(\frac{3}{2}\sin^2(\phi) - \frac{1}{2}\right)\right] \tag{1}$$

Use the following (note that some of these values are different from Homework #1!):

$$\begin{aligned} J_2 &= 0.00108248 \\ \mu &= 398600.4 \text{ km}^3/\text{s}^2 \\ R_{\text{Earth}} &= 6378.145 \text{ km} \\ \phi &= \text{Latitude} \rightarrow \sin(\phi) = \frac{z}{r} \\ r &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

Use the initial conditions from Homework #1, which are:

$$\underline{r} = -2436.45\hat{\imath} - 2436.45\hat{\jmath} + 6891.037\hat{k}\ km \tag{2}$$

$$\underline{v} = \dot{\underline{r}} = 5.088611\hat{\imath} - 5.088611\hat{\jmath} + 0.0\hat{k}\ km/s \tag{3}$$

The state at the final time is posted below to verify that your integrator is working properly.

Integrator Check

Before proceeding, let's make sure you have your integrator working properly. Below is a table of states at one day past t_0 for problems 1 and 2. It was generated with `ode45()` in MATLAB with a relative tolerance of 3×10^{-14} and an absolute tolerance of 1×10^{-16}. You should match these results to 8 digits or better.

	Two Body Only	Two Body + J_2	Two Body + Drag	Two Body + J_2 + Drag
X (km)	-5971.19544867343	-5751.49900721589	-5971.19402510054	-5751.50585441435
Y (km)	3945.58315019255	4721.14371040552	3945.54021911748	4721.10759954747
Z (km)	2864.53021742433	2046.03583664311	2864.58889466593	2046.09502951715
$\dot{X}$ (km/s)	0.049002818030	-0.797658631074	0.049055321740	-0.797610370476
$\dot{Y}$ (km/s)	-4.185030861883	-3.656513108387	-4.185065899722	-3.656553079577
$\dot{Z}$ (km/s)	5.848985672439	6.139612016678	5.848960931651	6.139595227675

Problems

1a. Use the MATLAB symbolic toolbox to compute the Cartesian partial derivatives of U. Compute $\partial U/\partial x$ by hand (include the derivation in the write-up) and compare your results with MATLAB. The full derivation may be scanned and turned in as an appendix, but include the final equation in the typed write-up.

1b. Plot the orbital elements a, e, i, Ω, ω, and T_p for one day at 20 second intervals, where T_p is the time of perigee passage and include the figure in your write-up. Be sure to label your axes and ensure that everything in each figure is readable. Using your insight from the two-body model, what conclusions can you draw about the J_2 effect on Keplerian orbital elements? Compute the period of the orbit ($P = 2\pi\sqrt{a^3/\mu}$) based on the initial state (Eqs (2)-(3)). How does the trend in the plots compare to the period?

1c. Compute the specific energy (energy/mass) and show that it is conserved around the orbit by plotting $dE = E(t) - E(t_0)$. Use the following equation with U from above:

$$E = \frac{v^2}{2} - U \tag{4}$$

1d. Compute h_k, the k-component of the angular momentum vector

$$\underline{h} = \underline{r} \times \underline{v} \tag{5}$$

and plot $dh_k = h_k(t) - h_k(t_0)$ to show that it remains constant.

2. Integrate the equations of motion with the conditions given in Problem 1 that include the Earth point-mass, J_2, and now also drag. Use the following relationship for the acceleration due to drag:

$$\ddot{\underline{r}} = -\frac{1}{2} C_D \frac{A}{m} \rho_A V_A \underline{V}_A \tag{6}$$

where

$$\begin{aligned}
C_D &= 2.0 \\
A &= 3.6 \text{ m}^2 \\
m &= 1350 \text{ kg} \\
\rho_0 &= 4 \times 10^{-13} \text{ kg/m}^3 \\
r_0 &= 7298.145 \text{ km} \\
H &= 200.0 \text{ km} \\
\dot{\theta} &= 7.29211585530066 \times 10^{-5} \text{ rad/s} \\
\rho_A &= \rho_0 e^{-(r-r_0)/H} \\
\underline{V}_A &= \begin{bmatrix} \dot{x} + \dot{\theta} y \\ \dot{y} - \dot{\theta} x \\ \dot{z} \end{bmatrix} \\
V_A &= \sqrt{\left(\dot{x} + \dot{\theta} y\right)^2 + \left(\dot{y} - \dot{\theta} x\right)^2 + \dot{z}^2}
\end{aligned}$$

Make sure you have agreement in all of your units when computing the drag acceleration! Compare your result to the online values to ensure that your integrator is working properly.

2a. Compute the specific energy at 20 second intervals and plot $dE = E(t) - E(t_0)$ (using Eq. (4)). Include the plot in your write-up. What can you infer from the plot? Is the total energy conserved? Why or why not?

2b. Compute the same Keplerian orbital elements generated in Problem 1b at 20 second intervals (a, e, i, Ω, ω, and T_p) and plot the differences in these elements from those computed in Problem 1. That is, generate time histories of each orbital element with and without drag and plot the differences, e.g., $a_{2B+J2+Drag} - a_{2B+J2}$. Include the results in your write-up. What can you observe in these plots? Which orbital elements are impacted by drag and how are they affected?