Derivation

$$U = U_{\text{point mass}} + U_{J_2} = \frac{\mu}{r}\left[1 - J_2\left(\frac{R_{\text{Earth}}}{r}\right)^2\left(\frac{3}{2}\sin^2(\phi) - \frac{1}{2}\right)\right] \tag{1}$$

$U_p = \frac{\mu}{r} \rightarrow \boxed{\nabla U_p = -\frac{\mu}{r^3}\underline{r}}$ (done in previous homework)

$$\boxed{\frac{\partial U_p}{\partial x} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}}$$

$\phi = \arcsin\left(\frac{z}{r}\right)$

$$U_{J_2} = -\frac{\mu}{r} J_2 \left(\frac{R_E}{r}\right)^2 \left(\frac{3}{2}\sin^2(\phi) - \frac{1}{2}\right)$$

$$= \left[-J_2\right]\left[\mu r^{-1}\right]\left[R_E^2 r^{-2}\right]\left[\frac{3}{2}\sin^2\left(\arcsin\left(\frac{z}{r}\right)\right) - \frac{1}{2}\right]$$

$$= \left[-J_2 \mu R_E^2\right]\left[r^{-3}\right]\left[\frac{3}{2}\sin^2\left(\arcsin\left(\frac{z}{r}\right)\right) - \frac{1}{2}\right]$$

$\left(\sin(\arcsin(\frac{z}{r}))\right)^2 = \frac{z^2}{r^2}$

$$= \left[-J_2 \mu R_E^2\right]\left[r^{-3}\right]\left[\frac{3}{2}z^2 r^{-2} - \frac{1}{2}\right]$$

$$= \left[-J_2 \mu R_E^2\right]\left[\frac{3}{2}z^2 r^{-5} - \frac{1}{2}r^{-3}\right]$$

(A)

$$\nabla U_{J_2} = \frac{\partial U_{J_2}}{\partial x}\hat{i} + \frac{\partial U_{J_2}}{\partial y}\hat{j} + \frac{\partial U_{J_2}}{\partial z}\hat{k}$$

$$(A) = \frac{3}{2}z^2 r^{-5} - \frac{1}{2}r^{-3} = \frac{3}{2}z^2\left(x^2+y^2+z^2\right)^{-5/2} - \frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial A}{\partial x} = \left(\frac{3}{2}z^2\right)\left(-\frac{5}{2}\right)\left(x^2+y^2+z^2\right)^{-7/2}(2x) + \left(\frac{1}{2}\right)\left(+\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2x)$$

$$= -\frac{15xz^2}{2}\left(x^2+y^2+z^2\right)^{-7/2} + \frac{3x}{2}\left(x^2+y^2+z^2\right)^{-5/2}$$

$$\frac{\partial A}{\partial x} = \left(\frac{3x}{2}\right)\frac{\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}}\ \hat{i} \rightarrow \frac{3x}{2}\,\frac{r^2-5z^2}{r^7}$$

$$\frac{\partial A}{\partial y} = \left(\frac{3y}{2}\right)\frac{\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}}\ \hat{j} \rightarrow \frac{3y}{2}\,\frac{r^2-5z^2}{r^7}$$

$$\frac{\partial A}{\partial z} = \frac{3}{2}z^2\left(-\frac{5}{2}\right)\left(x^2+y^2+z^2\right)^{-7/2}(2z) + 3z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2z)$$

$$= -\frac{15z^3}{2}\left(x^2+y^2+z^2\right)^{-7/2} + \frac{6}{2}z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$+\frac{3z}{2}\left(x^2+y^2+z^2\right)^{-5/2}$$

$$\frac{\partial A}{\partial z} = \frac{3z}{2}\left[\frac{3\left(x^2+y^2+z^2\right)-5z^2}{\left(x^2+y^2+z^2\right)^{7/2}}\right]\ \hat{k} \rightarrow \frac{3z}{2}\,\frac{3r^2-5z^2}{r^7}$$

$$\nabla A = \frac{\partial A}{\partial x}\hat{i} + \frac{\partial A}{\partial y}\hat{j} + \frac{\partial A}{\partial z}\hat{k}$$

$$\frac{\partial U_{J_2}}{\partial x} = \left[-J_2 \mu R_E^2\right]\left(\frac{3x}{2}\,\frac{r^2-5z^2}{r^7}\right)$$

$$\frac{\partial U_{J_2}}{\partial x} = \left[-J_2 \mu R_E^2\right]\left(\frac{3y}{2}\,\frac{r^2-5z^2}{r^7}\right)$$

$$\frac{\partial U_{J_2}}{\partial x} = \left[-J_2 \mu R_E^2\right]\left(\frac{3z}{2}\,\frac{3r^2-5z^2}{r^7}\right)$$

$$\frac{\partial U_p}{\partial x} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial U_{J_2}}{\partial x} = \left[-J_2 \mu R_E^2\right]\left(\frac{3x}{2}\,\frac{r^2 - 5z^2}{r^7}\right)$$

$$= \left[-J_2 \mu R_E^2\right]\left[\frac{3x}{2}\,\frac{x^2+y^2-4z^2}{(x^2+y^2+z^2)^{7/2}}\right]$$ ✓ matches matlab output

$$\frac{\partial U}{\partial x} = \frac{\partial U_p}{\partial x} + \frac{\partial U_{J_2}}{\partial x}$$