ASE 389P.4: Methods of Orbit Determination
Homework 3: The Batch and Sequential Processor

Assigned:	Thursday, March 4, 2021
Due:	Thursday, March 11 @ 12:30pm

This assignment explores the numeric propagation of the state transition matrix (STM) and the batch processor. In the first problem, you will implement the software to generate a STM for a two-dimensional, two-body orbit. You will also verify that your implementation is working by comparing the mapped deviations (computed via the STM) to differences in the nonlinear solution. This is analogous to how one may verify their STM software in the the real world. The second problem illustrates the estimation of a value using the batch processor in a linear system. Finally, you will implement a nonlinear batch processor and the sequential processor for a simple problem.

Turn in your code with your write-up as an appendix.

Problems

1. Assume an orbit plane coordinate system with a gravitational parameter of 1, i.e., $\mu = 1$. The equations of motion are:

$$\begin{aligned} \ddot{x} &= -\frac{x}{r^3} \\ \ddot{y} &= -\frac{y}{r^3} \\ r^2 &= x^2 + y^2 \end{aligned}$$

   a. Generate a "true" solution by numerically integrating the equations of motion for the initial conditions:

$$\underline{X}(t_0) = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}_{t=t_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   Save the values of the state vector $\underline{X}(t_i)$ for $t_i = i \cdot 10$ time units (TU); $i = 0, \ldots, 10$. Provide $\underline{X}(t_i)$ for t_1 and t_{10} in the writeup.

   In your write-up, please indicate which integrator you used, what the tolerance was set to, and any other details necessary. Note, if you use a fixed time-step integrator, set the time-step to be smaller than 10 TU, but only save the data at 10 TU intervals.

   b. Perturb the previous set of initial conditions by an amount

$$\underline{X}^*(t_0) = \underline{X}(t_0) - \delta\underline{X}(t_0)$$

   (notice that the perturbation is subtracted!), where

$$\delta\underline{X}(t_0) = \begin{bmatrix} 1 \times 10^{-6} \\ -1 \times 10^{-6} \\ 1 \times 10^{-6} \\ 1 \times 10^{-6} \end{bmatrix}$$

Numerically integrate this "nominal" trajectory along with the associated state transition matrix to find $\underline{X}^*(t_i)$ and $\underline{\underline{\Phi}}(t_i, t_0)$ at $t_i = i \cdot 10$ TU; $i = 0, \ldots, 10$. Provide $\underline{X}^*(t_i)$ and $\underline{\underline{\Phi}}(t_i, t_0)$ at t_1 and t_{10} in the write-up. Be sure to use the same integrator with the same tolerance as in 1a. Compare to the online solutions before proceeding.

c. For this problem, $\underline{\underline{\Phi}}(t_i, t_0)$ is symplectic. Demonstrate this for $\underline{\underline{\Phi}}(t_{10}, t_0)$ by multiplying it by $\underline{\underline{\Phi}}^{-1}(t_{10}, t_0)$, given by Eq. 4.2.22 in the text. Provide $\underline{\underline{\Phi}}^{-1}(t_{10}, t_0)$ and show that the product with $\underline{\underline{\Phi}}(t_{10}, t_0)$ is the identity matrix.

d. Calculate the perturbation vector, $\delta\underline{X}(t_i)$, by the following methods:

$$
\begin{aligned}
&(1) \quad \delta\underline{X}(t_i) = \underline{X}(t_i) - \underline{X}^*(t_i) \\
&(2) \quad \delta\underline{X}(t_i) = \underline{\underline{\Phi}}(t_i, t_0)\delta\underline{X}(t_0)
\end{aligned}
$$

and compare the results of (1) and (2). Provide the numeric results of (1) and (2) at t_1 and t_{10} in the write-up, along with $\delta\underline{X}(t_i) - \underline{\underline{\Phi}}(t_i, t_0)\delta\underline{X}(t_0)$. How closely do they compare?

State information and differences for $t_0 = 0$ TU (informational purposes):

$$
\begin{aligned}
\underline{X}(t_0) &= \begin{bmatrix} 1.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}^T \\
\underline{X}^*(t_0) &= \begin{bmatrix} 0.999999 & 0.000001 & -0.000001 & 0.999999 \end{bmatrix}^T \\
\underline{\underline{\Phi}}(t_0, t_0) &= \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \\
\delta\underline{X}(t_0) &= \begin{bmatrix} 0.000001 & -0.000001 & 0.000001 & 0.000001 \end{bmatrix}^T \\
\underline{\underline{\Phi}}(t_0, t_0)\delta\underline{X}(t_0) &= \begin{bmatrix} 0.000001 & -0.000001 & 0.000001 & 0.000001 \end{bmatrix}^T \\
\delta\underline{X}(t_0) - \underline{\underline{\Phi}}(t, t_0)\delta\underline{X}(t_0) &= \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}^T
\end{aligned}
$$

State information and differences for $t_1 = 10$ TU:

$$
\begin{aligned}
\underline{X}(t_1) &= \begin{bmatrix} -0.839071529 & -0.544021111 & 0.544021111 & -0.839071529 \end{bmatrix}^T \\
\underline{X}^*(t_1) &= \begin{bmatrix} -0.839031098 & -0.544071486 & 0.544076120 & -0.839041244 \end{bmatrix}^T \\
\underline{\underline{\Phi}}(t_1, t_0) &= \begin{bmatrix} -19.2963174705 & -1.0005919528 & -1.5446240948 & -20.5922746780 \\ 24.5395368984 & 2.5430400375 & 3.3820224390 & 24.9959638293 \\ -26.6284485803 & -1.2470410802 & -2.0860289935 & -27.5413748340 \\ -15.0754226454 & -1.4570972848 & -2.0011442064 & -14.6674122500 \end{bmatrix} \\
\delta\underline{X}(t_1) &= \begin{bmatrix} -0.000040431037 & 0.000050375590 & -0.000055009526 & -0.000030284890 \end{bmatrix}^T \\
\underline{\underline{\Phi}}(t_1, t_0)\delta\underline{X}(t_0) &= \begin{bmatrix} -0.000040432624 & 0.000050374483 & -0.000055008811 & -0.000030286882 \end{bmatrix}^T \\
\delta\underline{X}(t_1) - \underline{\underline{\Phi}}(t_1, t_0)\delta\underline{X}(t_0) &= \begin{bmatrix} 0.000000001587 & 0.000000001107 & -0.000000000715 & 0.000000001992 \end{bmatrix}^T
\end{aligned}
$$

State information and differences for $t_{10} = 100$ TU:

$$\underline{X}(t_{10}) = \begin{bmatrix}0.862318872 & -0.506365641 & 0.506365641 & 0.862318872\end{bmatrix}^T$$
$$\underline{X}^*(t_{10}) = \begin{bmatrix}0.862623360 & -0.505843963 & 0.505845689 & 0.862623303\end{bmatrix}^T$$
$$\underline{\underline{\Phi}}(t_{10}, t_0) = \begin{bmatrix}-151.2840323254 & -0.0696433460 & -0.5751839913 & -152.5394552874\\ -260.2345144322 & 0.8812356066 & 0.0191322895 & -260.6700884451\\ 259.1544475393 & 0.3746434528 & 1.2367484371 & 260.0263802508\\ -152.1279107642 & 0.3667128574 & -0.1388295703 & -151.6392131624\end{bmatrix}$$
$$\delta\underline{X}(t_{10}) = \begin{bmatrix}-0.000304487370 & -0.000521677895 & 0.000519951885 & -0.000304430755\end{bmatrix}$$
$$\underline{\underline{\Phi}}(t_{10}, t_0)\delta\underline{X}(t_0) = \begin{bmatrix}-0.000304329028 & -0.000521766706 & 0.000520042933 & -0.000304272666\end{bmatrix}^T$$
$$\delta\underline{X}(t_{10}) - \underline{\underline{\Phi}}(t_{10},t_0)\delta\underline{X}(t_0)$$
$$= \begin{bmatrix}-0.000000158342 & 0.000000088810 & -0.000000091047 & -0.000000158089\end{bmatrix}^T$$

2. Given the observation state relation $\underline{y} = \underline{\underline{H}}\ x + \underline{\epsilon}$, where x is a scalar and

$$\underline{y} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$
$$\underline{\underline{W}} = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

and

$$\underline{\underline{H}} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$$

with *a priori* information $\overline{x} = 2$ and $\overline{W} = 2$:

a. Using the batch processing algorithm, what is $\hat{x}$? In the write-up, outline the method employed in the code.

b. What is the best estimate of the observation error, $\hat{\epsilon}$?