Time and Reference Systems

The physical and numerical models presented so far have tacitly assumed the availability of a unique time and reference system for the equation of motion. In practice, however, one faces a multitude of historically grown concepts and definitions, which are employed along with each other. Whereas the definition of both time and the fundamental reference systems has traditionally been based on the rotational and translational motion of the Earth, one has now advanced to ideally uniform atomic time scales and an ideally non-rotating quasar-tied celestial reference frame. Nevertheless, a thorough understanding of the Earth's motion and rotation remains essential for a rigorous description of satellite orbits and even more the accurate modeling of ground based measurements.

5.1 Time

Despite the apparent familiarity and its use in everyday life, time has remained an issue that requires careful attention in the description of astronomical, physical, and geodetic phemomena. In accordance with the advance of physical theories, observational methods, and measuring devices, the underlying concepts and definitions have undergone continued revisions and refinements up to the present date.

Time is traditionally measured in days of 86 400 seconds duration, where the length of the solar day is determined from subsequent meridian transits of the Sun. Because of the orbital motion of the Earth around the Sun, the Sun's right ascension changes by approximately one degree per day and the solar day is thus about 4 minutes longer than the period of the Earth's rotation. The latter time interval, which is also known as a sidereal day, amounts to $23^{h}56^{m}4\overset{s}{.}1$ (solar time) and is equal to time between successive meridian passages of the vernal equinox.

In view of the eccentricity of the Earth's orbit and the resulting seasonal variations of the Sun's apparent motion, the real Sun is not, however, well suited for time reckoning purposes. Instead it had to be replaced by the concept of a mean Sun, that moves uniformly in right ascension at a rate determined from observations and analytical ephemerides. Based on a conventional expression for the right ascension of the mean Sun that was derived from Newcomb's Tables of the Motion of the Earth, the *Greenwich Mean Time GMT* or *Universal Time UT* was established in 1925 as an international time scale for astronomical and civil purposes.

When imperfections in the UT time scale became apparent that are due to irregularities and secular variations in the Earth's rotation, it was decided to establish a new time scale in 1960 that was based exclusively on the orbital motion of solar system bodies. This time scale, known as *Ephemeris Time ET*, defined time as the independent argument of planetary and lunar ephemerides. Based on this definition ET could be determined by comparing observed positions of the Sun, the planets, or the Earth's Moon with tabulated data predicted from analytical or numerical theories of motion. Ephemeris Time is thus the prototype of a dynamical time scale, which considers time as a continuously and uniformly passing physical quantity in the dynamical theories of motion.

With the advent of atomic clocks *Atomic Time* was introduced as a new timing system that was more easily accessible by laboratory standards and free from deficiencies of dynamical models. More recently a set of time scales has been defined that accounts for the effects of general relativity in the framework of a four-dimensional space-time.

Today the following time scales are of prime relevance in the precision modeling of Earth orbiting satellites:

- *Terrestrial Time (TT)*, a conceptually uniform time scale that would be measured by an ideal clock on the surface of the geoid. TT is measured in days of 86 400 SI[1] seconds and is used as the independent argument of geocentric ephemerides.
- *International Atomic Time (TAI)*, which provides the practical realization of a uniform time scale based on atomic clocks and agrees with TT except for a constant offset of 32.184 s and the imperfections of existing clocks.
- *GPS Time*, which like TAI is an atomic time scale but differs in the chosen offset and the choice of atomic clocks used in its realization.
- *Greenwich Mean Sidereal Time (GMST)*, the Greenwich hour angle of the vernal equinox.
- *Universal Time (UT1)*, today's realization of a mean solar time, which is derived from GMST by a conventional relation.
- *Coordinated Universal Time (UTC)*, which is tied to the International Atomic Time TAI by an offset of integer seconds that is regularly updated to keep UTC in close agreement with UT1.

For a description of planetary and lunar motion as well as solar system events within a general relativistic context, the above time scales are further supplemented by Geocentric and Barycentric Coordinate Time (TCG and TCB) as well as Dynamical Barycentric Time (TDB).

The mutual relation of the above time scales and their historical evolution is outlined in Fig. 5.1. Here distinction is made between dynamical time scales that serve as independent argument in the equations of motion, atomic time scales that provide the practical realization of a uniform clock, and the non-uniform solar time scales that are tied to the motion of the Sun and the rotation of the Earth.

[1] Système International (cf. Goodman & Bell 1986)

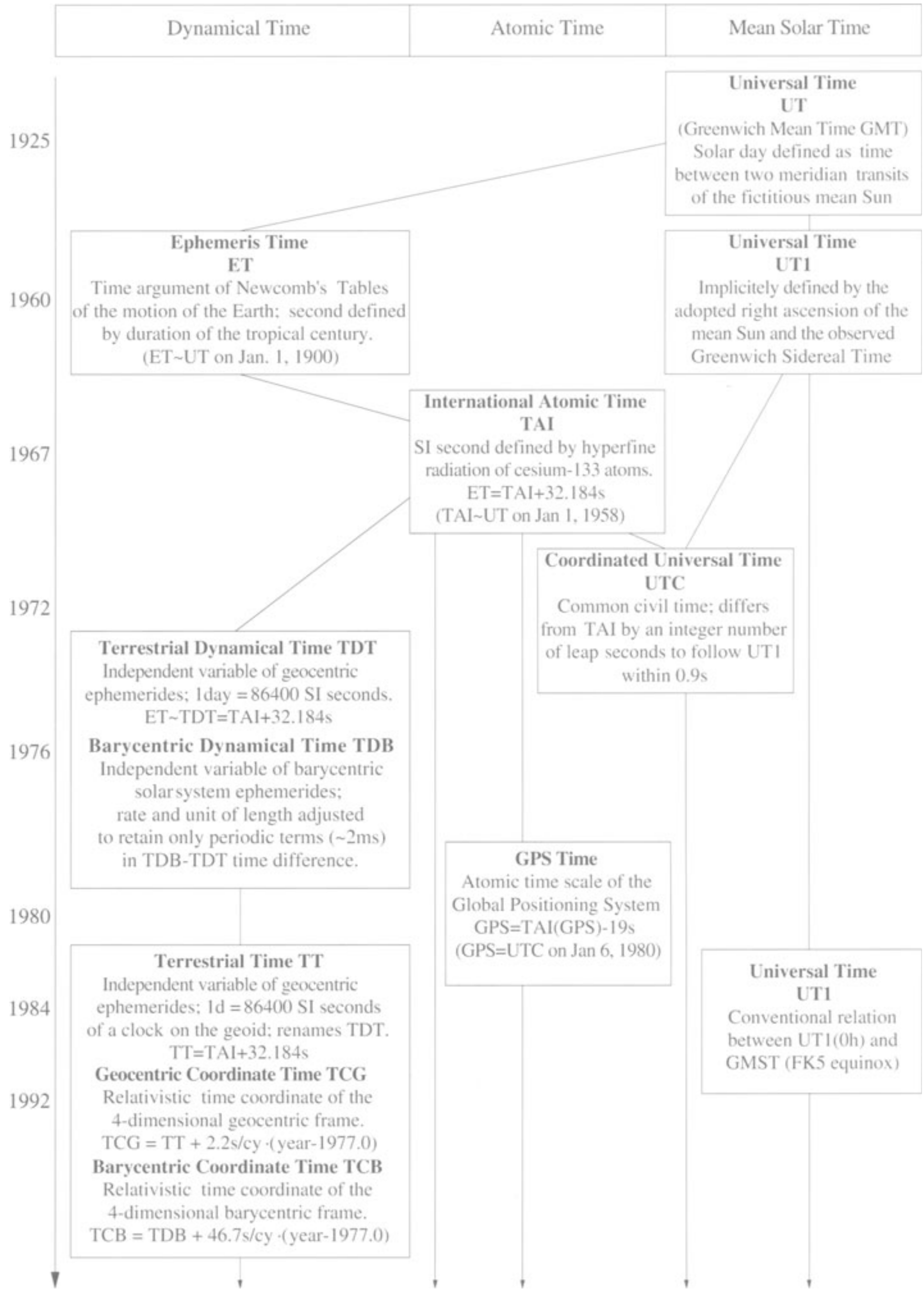

Fig. 5.1. Evolution of conventional time scales

5.1.1 Ephemeris Time

Ephemeris Time was adopted in 1960[2] to cope with irregularities in the Earth's rotation that had been found to affect the flow of mean solar time. Even though its definition is based on Newtonian physics and has meanwhile been replaced by TT, TCG, and TCB within a relativistic framework, Ephemeris Time still represents the prototype of a dynamical time scale and provides a useful link to historical planetary observations.

The definition of Ephemeris Time is based on Newcomb's analytical theory of the Earth's motion around the Sun (Newcomb 1898). In his analytical solution of the equations of motion, Newcomb expressed the relative motion of the Earth-Moon barycenter and the Sun by a set of secularly perturbed Keplerian elements and superimposed periodic perturbations. Based on his theory and an adjustment to obervations, he derived the expression

$$L_\odot = 279°41'48\overset{''}{.}04 + 129\,602\,768\overset{''}{.}13 \cdot T + 1\overset{''}{.}089 \cdot T^2 \tag{5.1}$$

for the geometric mean longitude of the Sun with respect to the Earth-Moon barycenter. Here $L_\odot$ refers to the mean equinox of date while T measures time from noon 1900 January 0 (JD 2415020.0) GMT in Julian centuries of 36525 days.

While a day was originally meant to represent a mean solar day in Newcomb's computations, the above relation was later adopted as a conventional expression in the definition of Ephemeris Time. To this end, the instant at which the geometric mean longitude of the Sun had a value of $279°41'48\overset{''}{.}04$ near the beginning of the calendar year AD 1900 was defined as 1900 January 0, 12^{h} Ephemeris Time (ET). The rate of change $dL_\odot/dT$ at this epoch is given by the linear term in (5.1), which corresponds to an orbital period of

$$P = \frac{360 \cdot 3600''}{129\,602\,768\overset{''}{.}13} \cdot 36525 \cdot 86400\,\mathrm{s} = 31\,556\,925.9747\,\mathrm{s} \quad . \tag{5.2}$$

Accordingly, the ephemeris second was defined as the fraction $1/31556925.9747$ of the tropical year at 1900 January 0.5 ET, where a tropical year specifies the time during which the Sun's mean longitude, as referred to the mean equinox of date, increases by 360°.

Even though Ephemeris Time provides a conceptually smooth and uniform time scale it is more difficult to measure than mean solar time that is closely related to the Earth's rotation. In practice Ephemeris Time has to be determined by comparing observations of the Sun, Moon or planets with precomputed ephemerides. Among these bodies the Moon exhibits the fastest orbital motion and has therefore formed the basis for the actual implementation of Ephemeris Time. Soon, however, Ephemeris Time became superseded by the use of atomic time scales, which provided a much better short-term availability together with an excellent long-term stability.

[2]A preliminary definition of Ephemeris Time was actually devised about ten years earlier (see Seidelmann 1992), but was refined and revised in subsequent resolutions.

5.1.2 Atomic Time

Atomic (or molecular) clocks are based on the periodic oscillation of a microwave signal that is in resonance with a low-energy state transition of a specific atom or molecule. While the first clock built at the National Bureau of Standards in 1948 used an ammonia (NH_3) absorption line to control the frequency generation (Forman 1985), today's atomic clocks are generally based on cesium (^{133}Cs), hydrogen (^{1}H), or rubidium (^{87}Rb) (McCoubrey 1996). Among these types, cesium clocks provide the best long-term stability and are therefore used as primary standards in the practical realization of atomic time scales.

The principle of a cesium-beam atomic clock is illustrated in Fig. 5.2. A beam of cesium-133 atoms leaves an oven through a thin hole and enters the inhomogenous field of a Stern–Gerlach magnet. It then passes through a microwave resonator and a second magnet before it is finally collected by a detector (cf. Vessot 1974).

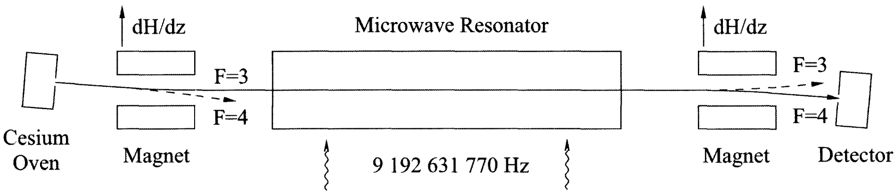

Fig. 5.2. Schematic view of a cesium-beam atomic clock

Depending on the nucleus and outer electron spins, the cesium atoms exhibit a total angular momentum of either $F = 3$ or $F = 4$. Both states are separated by a small energy of about 0.04 meV and are almost equally populated in thermal equilibrium. Atoms in the $F = 3$ state experience an acceleration along the gradient of an external magnetic field, while atoms in the $F = 4$ state are deflected into the opposite direction. The first Stern–Gerlach magnet thus acts as a state selector, which allows only atoms in the $F = 3$ state to enter the subsequent microwave resonator. Here the beam passes through an electromagnetic field with a nominal frequency of 9 192 631 770 Hz, which corresponds to the energy difference between the two states. Accordingly, atoms in the $F = 3$ state may absorb a microwave photon and change the electron spin from anti-parallel to parallel orientation with respect to the spin of the nucleus. Upon leaving the resonator, the atoms pass a second Stern–Gerlach magnet (analyzer) that separates the $F = 3$ and $F = 4$ states and allows only the latter (i.e. those with modifed electron spin) to enter the detector. A maximum signal is thus obtained, if the microwave radiation frequency is properly centered to the hyperfine transition. The detector signal can, therefore, be used to adjust the resonator frequency to a fraction of the natural linewidth and obtain a highly stable frequency reference. Upon continued subdivision, lower frequencies of equal stability are obtained that ultimately provide the desired clock signal.

Due to the sharpness of the absorption line, the resonance frequency can be matched with great precision and is thus ideally suited as an accurate time reference.

Typical accuracies achieved with present cesium clocks range from 10^{-12} to 10^{-14} (Guinot 1989) with prospects for stabilities down to 10^{-16} (Wolf & Petit 1995). This may be compared to a stability of the Earth's rotation of about 10^{-8} (0.3 s/year) and an accuracy of ephemeris time determination in the range of 10^{-10} (0.05 s in 10 years).

In comparison with Ephemeris Time as derived from lunar observations, the cesium resonance frequency was determined as $9\,192\,631\,770 \pm 20$ Hz by Markowitz (1958). The numerical value was finally adopted in 1967 to independently define one second in the Système International (SI) as the duration of exactly 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.

At the French *Bureau International de l'Heure BIH* atomic clocks were used as early as 1955 in addition to traditional astronomical time keeping procedures. In 1972, the BIH atomic time scale was adopted as a world-wide standard time under the name *International Atomic Time TAI.* The unit of time of TAI is defined as the SI second and the origin has arbitrarily been chosen such that TAI closely matches Universal Time on 1958 January 1.0, yielding the relation

$$\mathrm{ET} = \mathrm{TAI} + 32.184\,\mathrm{s} \quad . \tag{5.3}$$

Today TAI is established at the French *Bureau International des Poids et Mesures BIPM* using an elaborate stability algorithm and clock readings from a large number of atomic clocks (Guinot 1989).

In addition to TAI, the atomic time scale established by the Global Positioning Satellite (GPS) system has become of great significance in the past decade due to the common availability of GPS receivers. Besides serving the direct needs of geodetic and navigational measurements, GPS provides high-precision timing signals with a near-instantaneous and worldwide availability. GPS time is realized by an independent set of atomic clocks and is maintained to follow the United States Naval Observatory (USNO) atomic clock time with an accuracy of 1 μs, which itself differs from TAI by less than 5 μs. The origin of GPS time was arbitrarily chosen to coincide with UTC on 1980 January 6.0 UTC, i.e. GPS time differs from TAI by a constant offset of

$$\mathrm{GPS} = \mathrm{TAI} - 19\,\mathrm{s} \tag{5.4}$$

aside from the aforementioned clock offsets on the micro-second level.

5.1.3 Relativistic Time Scales

While time is an absolute quantity in the context of Newtonian physics, which does not depend on the location and the motion of a clock, the same is no longer true in a general relativistic framework. Instead, different proper times apply for each clock, that are related to each other by a four-dimensional space-time transformation. This transformation requires knowledge of the space-time metric, which itself depends on the location and motion of the gravitating masses. Within the solar system, a first-order post-Newtonian approximation is generally adequate for a treatment of relativistic effects in view of moderate velocities and gravitational potentials (Soffel & Brumberg 1991).

In the vicinity of the Earth, it is possible to choose a rotation-free system of four-dimensional space-time coordinates ($x^0 = ct, \boldsymbol{x} = (x^1, x^2, x^3)$) in such a way that the invariant space-time distance between two events is given by

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2U}{c^2}\right)(dx^0)^2 + \left(1 + \frac{2U}{c^2}\right)(d\boldsymbol{x})^2 \tag{5.5}$$

to lowest order. Here c denotes the speed of light, τ is the proper time (as opposed to coordinate time t) and U is the sum of the Earth's gravitational potential and the tidal potential generated by external bodies. Eqn. (5.5) implies that the rate of a clock at rest on the surface of the Earth differs from the rate of coordinate time by a factor of

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2U}{c^2} - \frac{v^2}{c^2}} \approx 1 - \frac{GM_\oplus}{R_\oplus c^2} - \frac{v^2}{2c^2} \approx 1 - 7 \cdot 10^{-10} \quad , \tag{5.6}$$

where $v \approx \omega_\oplus R_\oplus \cos\varphi$ is the clock's speed in the non-rotating frame for a given latitude φ. Likewise, clocks at different altitudes will have different proper times and experience a rate difference in long-term comparisons.

The conceptual difference between proper time and coordinate time has led the International Astronomical Union (IAU) to adopt two different time scales for use since 1992, which are named as *Terrestrial Time TT* (formerly Terrestrial Dynamical Time TDT)[3] and *Geocentric Coordinate Time TCG*. Terrestrial Time has as its unit the SI second as measured on the geoid and provides a smooth continuation of Ephemeris Time, i.e.

$$\mathrm{TT} = \mathrm{TDT} = \mathrm{ET} = \mathrm{TAI} + 32.184\,\mathrm{s} \quad . \tag{5.7}$$

Geocentric Coordinate Time TCG in contrast represents the time coordinate of a four-dimensional reference system and differs from TT by a constant scale factor $1 - L_\mathrm{G}$ with

$$L_\mathrm{G} = 6.9692903 \cdot 10^{-10} \tag{5.8}$$

(Wolf & Petit 1995). By convention TCG agress with TT on 1977 January 1.0, yielding the relation

$$\mathrm{TCG} = \mathrm{TT} + L_\mathrm{G} \cdot (\mathrm{JD} - 2443144.5) \cdot 86400\,\mathrm{s} \quad . \tag{5.9}$$

Around the epoch J2000, the difference TCG–TT amount to roughly 0.5 s.

[3] The word *dynamical* was originally used to emphasize its nature as the argument of dynamical theories of motion in contrast to atomic time scales governed by the laws of quantum mechanics. It was eventually dropped in 1992, since for practical purposes Terrestrial Time is actually derived from the atomic TAI time scale.

Supplementary to TCG, the *Barycentric Coordinate Time TCB* has been introduced to describe the motion of solar-system objects in a non-rotating relativistic frame centered at the solar-system barycenter. Both time scales are defined to match each other on 1977 January 1.0 TAI but exhibit a rate difference

$$\frac{d(\mathrm{TCB}-\mathrm{TCG})}{d\mathrm{TCG}} \approx \frac{GM_\odot}{ac^2}+\frac{v_\oplus^2}{2c^2} \approx \frac{3}{2}\frac{GM_\odot}{ac^2} \approx 1.5\cdot 10^{-8} \quad , \tag{5.10}$$

that depends on the gravitational potential of the Sun at the mean Earth-Sun distance $a = 1$ AU and the Earth's orbital velocity $v_\oplus$. Due to the eccentricity of the Earth's orbit and the associated variations of the heliocentric distance and velocity, the rigorous transformation involves additional periodic terms and is given by

$$\mathrm{TCB} = \mathrm{TCG} + L_\mathrm{C}\cdot(\mathrm{JD}-2443144.5)\cdot 86400\,\mathrm{s} + P \tag{5.11}$$

with

$$L_\mathrm{C} = 1.4808268457\cdot 10^{-8} \tag{5.12}$$

(McCarthy 1996) and

$$\begin{aligned} P &\approx +0\overset{s}{.}0016568\cdot\sin(35999\overset{\circ}{.}37T + 357\overset{\circ}{.}5) \\ &\quad +0\overset{s}{.}0000224\cdot\sin(32964\overset{\circ}{.}5T + 246^\circ) \\ &\quad +0\overset{s}{.}0000138\cdot\sin(71998\overset{\circ}{.}7T + 355^\circ) \\ &\quad +0\overset{s}{.}0000048\cdot\sin(3034\overset{\circ}{.}9T + 25^\circ) \\ &\quad +0\overset{s}{.}0000047\cdot\sin(34777\overset{\circ}{.}3T + 230^\circ) \\ T &= (\mathrm{JD}-2451545.0)/36525 \end{aligned} \tag{5.13}$$

(Seidelmann & Fukushima 1992). The leading periodic term is of 1.7 ms amplitude and varies with the sine of the Earth's mean anomaly. All other terms are about two orders of magnitude smaller. In view of the significant rate difference between TCB and TCG/TT the accumulated TCB–TT time difference amounts to roughly 11 s around the epoch J2000 (cf. Fig. 5.3).

TCB supersedes a time scale known as *Barycentric Dynamical Time TDB*, which was introduced by the IAU in 1976 and defined to differ from TDT (now TT) by periodic terms, only. Accordingly TDB and TCB are related by

$$\mathrm{TCB} = \mathrm{TDB} + L_\mathrm{B}\cdot(\mathrm{JD}-2443144.5)\cdot 86400\,\mathrm{s} \quad , \tag{5.14}$$

where the scale difference

$$L_\mathrm{B} = L_\mathrm{C} + L_\mathrm{G} = 1.5505197487\cdot 10^{-8} \tag{5.15}$$

(McCarthy 1996) synchronizes the average rate of Barycentric Dynamical Time with that of Terrestrial (Dynamical) Time. While the definition of TDB appeared to be useful at first sight in view of the small amplitude of the TDB-TT time difference it has a subtle implication for models of solar system dynamics. While the post-Newtonian equations hold irrespective of the use of TCB or TDB time, the TDB

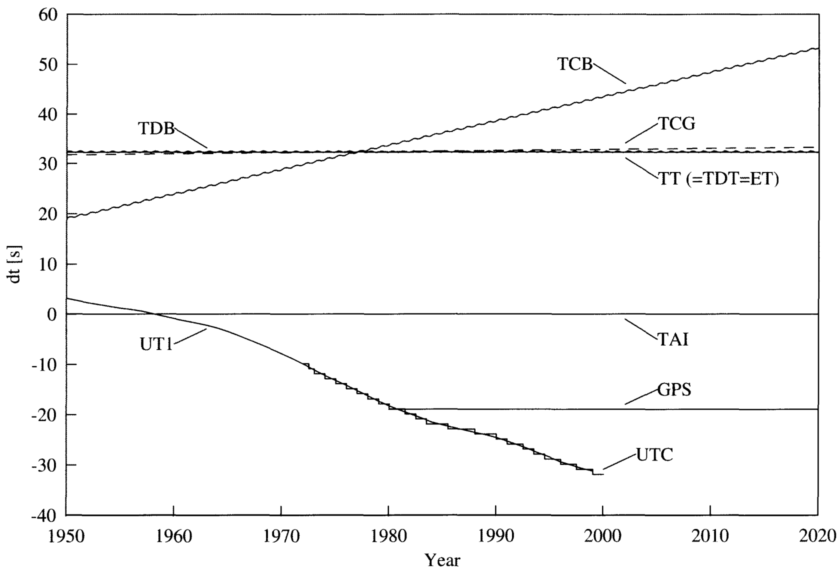

Fig. 5.3. Difference of atomic, dynamical, and solar time scales between 1950 and 2020. Periodic terms in TCB and TDB have been exaggerated by a factor of 100 to make them discernible. (Adapted from Seidelmann & Fukushima 1992)

second is longer than the TCB second by a factor L_B. Furthermore, in order to maintain the adopted numerical value

$$c = 299\,792\,458\,\mathrm{m/s} \tag{5.16}$$

of the speed of light, the length of a meter is likewise different in the TCB and TDB system. In a similar manner derived quantities like the masses of the Sun, Earth, and planets are affected by the scaling difference (Hellings 1986). Considering, however, that all precise solar system ephemerides are so far based on a TDB time scale, the continued use of TDB is still accepted by the current IAU resolutions.

5.1.4 Sidereal Time and Universal Time

Greenwich Mean Sidereal Time GMST, also known as Greenwich Hour Angle, denotes the angle between the mean vernal equinox of date and the Greenwich meridian. It is a direct measure of the Earth's rotation and may jointly be expressed in angular units or units of time with 360° (2π) corresponding to 24^{h}. In terms of SI seconds, the length of a sidereal day (i.e the Earth's spin period) amounts to $23^{\mathrm{h}}56^{\mathrm{m}}4\overset{\mathrm{s}}{.}091 \pm 0\overset{\mathrm{s}}{.}005$, making it about four minutes shorter than a 24^{h} solar day. Due to length-of-day variations with an amplitude of several milliseconds, sidereal time cannot be computed from other time scales with sufficient precision but must be derived from astronomical and geodetic observations.

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                                                          BULLETIN B 135
                                                          03 May 1999

   Contents described in the Explanatory Supplement, mailed with Bulletin B133

 1 - EARTH ORIENTATION PARAMETERS (IERS evaluation).
     The values in this section are samplings of section 2 given at five-day
     intervals.

   Date     MJD       x        y     UT1R-UTC     UT1R-TAI     dPsi dEpsilon
   1999               "        "        s            s       0.001"   0.001"
  (0h UTC)

 Final Bulletin B values.

  MAR   2  51239   .06888  .24160   .651265  -31.348735   -45.4    -5.7
  MAR   7  51244   .06436  .24260   .646766  -31.353234   -46.6    -5.6
  MAR  12  51249   .05871  .24135   .641753  -31.358247   -45.8    -5.9
  MAR  17  51254   .05110  .23914   .636039  -31.363961   -44.7    -6.2
  MAR  22  51259   .04643  .24049   .629993  -31.370007   -45.3    -6.4

  MAR  27  51264   .03623  .24148   .623165  -31.376835   -44.2    -7.0
  APR   1  51269   .02603  .24102   .616594  -31.383407   -44.1    -7.1

 Preliminary extension, to be updated weekly in Bulletin A and monthly
  in Bulletin B.

  APR   6  51274   .01733  .23957   .609575  -31.390425   -45.1    -7.2
  APR  11  51279   .01051  .24108   .602247  -31.397753   -43.6    -7.0
  APR  16  51284   .00731  .24620   .595119  -31.404881   -44.6    -7.4
...

  JUN  10  51339  -.04182  .28130   .519993  -31.480007   -45.1    -7.3
  JUN  15  51344  -.04460  .28498   .514725  -31.485275   -46.4    -8.1
  JUN  20  51349  -.04711  .28882   .509953  -31.490048   -45.4    -7.4
  JUN  25  51354  -.04935  .29281   .505676  -31.494324   -47.9    -7.4
  JUN  30  51359  -.05133  .29693   .501881  -31.498119   -47.4    -7.7

 Note. In UT1R, the effects of zonal tides with  periods shorter than 35 days
 are removed; UT1-UT1R ( smaller than  0.0025s  in absolute value ) should be
 added after quadratic interpolation of UT1R.Section 2 of this Bulletin gives
 the daily interpolation of x, y, UT1, duration of day, dPsi, and dEpsilon.


 2 - SMOOTHED VALUES OF X, Y, UT1, D, DPSI, DEPSILON (IERS EVALUATION)
 at one-day intervals. For smoothing characteristics, see Table2 in the
 explanatory supplement. The reference system is described in the 1997
 IERS Annual Report.
   1999      MJD       x         y       UT1-UTC  UT1-UT1R   D     dPsi dEpsilon
   (0 h UTC)           "         "          s         ms     ms   0.001"  0.001"
  MAR   1    51238   .06996    .24187    .652011   -.154    .960  -45.4   -5.7
  MAR   2    51239   .06888    .24160    .651078   -.187    .944  -45.4   -5.7
  MAR   3    51240   .06806    .24156    .650161   -.277    .930  -45.7   -5.7
  MAR   4    51241   .06740    .24173    .649232   -.391    .990  -46.0   -5.7
  MAR   5    51242   .06657    .24214    .648193   -.490   1.053  -46.3   -5.6
...

  APR  26    51294  -.00689    .25211    .580721   -.475   1.379  -42.7   -6.9
  APR  27    51295  -.00808    .25277    .579283   -.552   1.425  -42.0   -6.8
  APR  28    51296  -.00831    .25398    .577792   -.641   1.475  -42.4   -6.8
  APR  29    51297  -.00846    .25501    .576290   -.705   1.469  -42.8   -6.8
  APR  30    51298  -.00871    .25641    .574839   -.713   1.405  -42.9   -6.9
```

Fig. 5.4. Sample set of Earth Orientation Parameters as provided by the Bulletin B of the IERS International Earth Rotation Service

Universal Time UT1 is the presently adopted realization of a mean solar time scale with the purpose of achieving a constant average length of the solar day of 24 hours. As a result, the length of one second of Universal Time is not constant, because the actual mean length of a day depends on the rotation of the Earth and the apparent motion of the Sun (i.e. the length of the year). Similar to sidereal time, it is not possible to determine Universal Time by a direct conversion from e.g. atomic time, because the rotation of the Earth cannot be predicted accurately. Every change in the Earth's rotation alters the length of the day, and must therefore be taken into account in UT1. Universal Time is therefore defined as a function of sidereal time, which directly reflects the rotation of the Earth. For any particular day, 0^{h} UT1 is defined as the instant at which Greenwich Mean Sidereal Time has the value

$$\begin{aligned} \mathrm{GMST}(0^{\mathrm{h}}\mathrm{UT1}) &= 24110\overset{\mathrm{s}}{.}54841 + 8640184\overset{\mathrm{s}}{.}812866 \cdot T_0 \\ &\quad + 0\overset{\mathrm{s}}{.}093104 \cdot T_0^2 - 0\overset{\mathrm{s}}{.}0000062 \cdot T_0^3 \end{aligned} \tag{5.17}$$

(Aoki et al. 1982). In this expression the time argument

$$T_0 = \frac{\mathrm{JD}(0^{\mathrm{h}}\mathrm{UT1}) - 2451545}{36525} \tag{5.18}$$

denotes the number of Julian centuries of Universal Time that have elapsed since 2000 Jan. 1.5 UT1 at the beginning of the day. For an arbitrary time of the day, the expression may be generalized to obtain the relation

$$\begin{aligned} \mathrm{GMST} &= 24110\overset{\mathrm{s}}{.}54841 \\ &\quad + 8640184\overset{\mathrm{s}}{.}812866\, T_0 + 1.002737909350795\,\mathrm{UT1} \\ &\quad + 0\overset{\mathrm{s}}{.}093104\, T^2 - 0\overset{\mathrm{s}}{.}0000062\, T^3 \quad , \end{aligned} \tag{5.19}$$

where the time argument

$$T = \frac{\mathrm{JD}(\mathrm{UT1}) - 2451545}{36525} \tag{5.20}$$

specifies the time in Julian centuries of Universal Time elapsed since 2000 Jan. 1.5 UT1.

The difference between Universal Time and Terrestrial Time or International Atomic Time can only be determined retrospectively. At the end of the 20th century $\Delta T = \mathrm{TT} - \mathrm{UT1}$ amounts to roughly 65 s and increases by about 0.5 to 1.0 seconds per year (cf. Fig. 5.3). In addition to the secular variation, which is caused by tidal friction in the Earth-Moon system, UT1 is subject to periodic variations on the 1 ms level that are caused by tidal perturbations of the polar moment of inertia (see McCarthy (1996) and references therein). By convention, zonal tide terms with periods between 5 and 35 days are removed from UT1 to obtain the regularized Universal Time UT1R. Values of the UT1R–TAI time difference are published on a monthly basis in Bulletin B of the International Earth Rotation Service (IERS) (cf. Fig. 5.4), while the adopted expression for UT1-UT1R is given in McCarthy (1996). Aside from reconstructed, post-facto values of the Earth orientation parameters, the bulletin provides approximate forecasts over a two month time frame at 5-day and 1-day intervals. Using quadratic interpolation of the tabulated data, UT1 may be obtained for arbitrary instants from given TAI (or TT), which then allows GMST to be computed as a function of TAI using the conventional relation (5.19).

Clock time, which is used for everyday purposes, is derived from *Coordinated Universal Time* (UTC). Since 1972, UTC is obtained from atomic clocks running at the same rate as International Atomic Time and Terrestrial Time. By the use of leap seconds, which may be inserted at the end of June and/or the end of December, care is taken to ensure that UTC never deviates by more than 0.9 seconds from Universal Time UT (cf. Fig. 5.3). Between 1972 and 1999, a total of 23 leap seconds have been introduced as summarized in Table 5.1. New leap seconds are announced in Bulletin C of the IERS (cf. Fig. 5.5) about half a year in advance of their implementation.

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                 INTERNATIONAL EARTH ROTATION SERVICE (IERS)
               SERVICE INTERNATIONAL DE LA ROTATION TERRESTRE

BUREAU CENTRAL DE L'IERS
OBSERVATOIRE DE PARIS
61, Av. de l'Observatoire 75014 PARIS (France)
Tel.       : 33 (0) 1 40 51 22 26
FAX        : 33 (0) 1 40 51 22 91
Internet   : iers@obspm.fr

                                             Paris, 17 July 1998

                                             Bulletin C 16

                                             To authorities responsible for
                                             the measurement and distribution
                                             of time

                                  UTC TIME STEP
                            on the 1st of January 1999

 A positive leap second will be introduced at the end of December 1998.
 The sequence of dates of the UTC second markers will be:

                           1998 December 31,     23h 59m 59s
                           1998 December 31,     23h 59m 60s
                           1999 January   1,      0h  0m  0s

 The difference between UTC and the International Atomic Time TAI is:

  from 1997 July 1,     0h UTC, to 1999 January 1, 0h UTC  : UTC-TAI = - 31s
  from 1999 January 1, 0h UTC, until further notice        : UTC-TAI = - 32s

 Leap seconds can be introduced in UTC at the end of the months of
 December or June, depending on the evolution of UT1-TAI. Bulletin C mailed
 every six months, either to announce a time step in UTC, or to confirm that
 there will be no time step at the next possible date.
```

Fig. 5.5. Announcement of new UTC leap seconds in Bulletin C of the IERS International Earth Rotation Service

Table 5.1. Leap seconds introduced in Coordinated Universal Time (UTC) since 1972.

From	UTC–TAI	From	UTC–TAI	From	UTC–TAI
1972 Jan. 1	−10 s	1981 July 1	−20 s	1996 Jan. 1	−30 s
1972 July 1	−11 s	1982 July 1	−21 s	1997 July 1	−31 s
1973 Jan. 1	−12 s	1983 July 1	−22 s	1999 Jan. 1	−32 s
1974 Jan. 1	−13 s	1985 July 1	−23 s		
1975 Jan. 1	−14 s	1988 Jan. 1	−24 s		
1976 Jan. 1	−15 s	1990 Jan. 1	−25 s		
1977 Jan. 1	−16 s	1991 Jan. 1	−26 s		
1978 Jan. 1	−17 s	1992 July 1	−27 s		
1979 Jan. 1	−18 s	1993 July 1	−28 s		
1980 Jan. 1	−19 s	1994 July 1	−29 s		

5.2 Celestial and Terrestrial Reference Systems

The equation of motion as derived in Chap. 3 describes the orbit of a satellite with respect to a quasi-inertial or Newtonian reference system, i.e. with respect to a coordinate system that moves with the center of the Earth but is free of rotation. Satellite observations on the other hand are commonly obtained from an observing site on the surface of the Earth, which is not at rest with respect to this reference system. In order to compare ground-based measurements with the computed satellite position, a concise definition of celestial and terrestrial reference systems is required and their mutual relation has to be established.

Traditionally, celestial reference frames have been tied to the Earth's rotation and its annual revolution around the Sun. In view of the apparent constancy of both the orbital plane and the rotation axis of the Earth, two global coordinate systems can be defined in a straightforward manner. The first one gives the position of a point in space with respect to the ecliptic (the Earth's orbital plane), while the other one refers to the Earth's equatorial plane (the plane perpendicular to the rotation

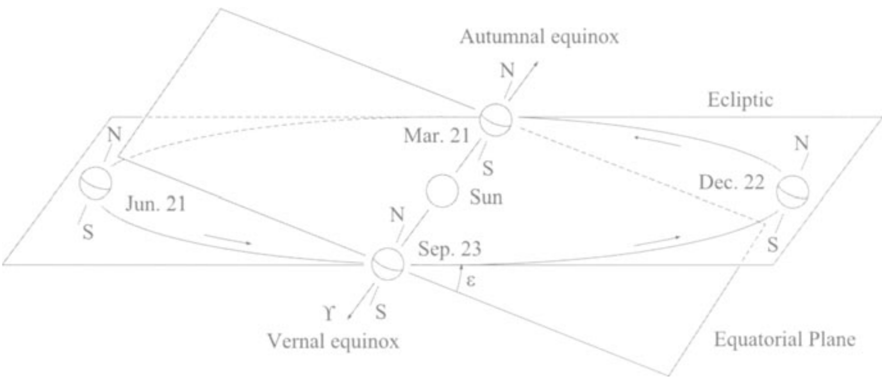

Fig. 5.6. Ecliptic and equator

axis). These planes are inclined at an angle $\varepsilon \approx 23.5°$ and the line of intersection is a common axis of both coordinate systems (cf. Fig. 5.6). The x/x'-axis is defined as being the direction of the vernal equinox or First Point of Aries, designated by Υ. It is perpendicular to both the North Celestial Pole (the z-axis) and the north pole of the ecliptic (the z'-axis). According to their definition the equatorial coordinates $\boldsymbol{r}$ and the ecliptic coordinates $\boldsymbol{r}'$ of a given point are related by a rotation

$$\boldsymbol{r}' = \boldsymbol{R}_x(\varepsilon)\boldsymbol{r} \quad , \tag{5.21}$$

where the precise value of the obliquity ε is given in (5.42). The choice between ecliptic and equatorial coordinates is mainly a question of vividness and convenience. Planetary orbits, for example, are inclined at small angles to the Earth's orbital plane and are therefore commonly described in ecliptic coordinates. Equatorial coordinates, on the other hand, are closely related to geographical coordinates and provide a natural link to an Earth-fixed reference system.

While the orbital plane of a body around a central mass is fixed in space as long as the attractive force is parallel to the radius vector, this condition does not hold for the Earth due to the presence of other solar system planets. This results in a small secular variation of the orbital plane which is known as planetary precession. At the same time the Earth's axis of rotation is perturbed by the torque exerted on the equatorial bulge by the Sun and Moon. This torque tries to align the equator with the ecliptic and results in a gyroscopic motion of the Earth's rotation axis around the pole of the ecliptic with a period of about 26 000 years. As a result of this lunisolar precession the vernal equinox recedes slowly on the ecliptic, whereas the obliquity of the ecliptic remains essentially constant. In addition to precession some minor periodic perturbations of the Earth's rotation axis may be observed that are known as nutation and reflect variations of the solar and lunar torques on time scales larger than a month. In view of the time-dependent orientation of equator and ecliptic a standard reference frame is usually based on the mean equator, ecliptic, and equinox of some fixed epoch, which is currently selected as the beginning of the year 2000. Access to the *Earth Mean Equator and Equinox of J2000 (EME2000)* is provided by the FK5 star catalog (Fricke et al. 1988), which provides precise positions and proper motions of some 1 500 stars for the epoch J2000 as referred to the given reference frame.

In view of conceptual difficulties related to the dynamical definition of the ecliptic and equinox (see e.g. Kinoshita & Aoki 1983), it was decided by the IAU in 1991 to establish a new *International Celestial Reference System (ICRS)*[4] and adopt it for use from 1998 onwards (Feissel & Mignard 1998). The origin of the ICRS is defined as the solar-system barycenter within a relativistic framework and its axes are fixed with respect to distant extragalactic radio objects. These are supposed to have no proper motion, thus ensuring that the ICRS exhibits no net rotation. For a smooth transition to the new system, the ICRS axes are chosen in

[4]Here, the term *Reference System* means the set of basic concepts and models used to define at any instant the orientation of the reference axes. A *Reference Frame*, in contrast, means a specific realization in accordance with the concepts.

such a way as to be consistent with the previous FK5 system to within the accuracy of the latter. The fundamental plane of the ICRS is closely aligned with the mean Earth equator at J2000 and the origin of right ascension is defined by an adopted right ascension of the quasar 3C273.

The practical realization of the ICRS is designated the *International Celestial Reference Frame (ICRF)* and is jointly maintained by the IERS and the IAU Working Group on Reference Frames (cf. Arias et al. 1995). It is mainly based on high-precision observations of extragalactic radio sources using Very Long Baseline Interferometry (VLBI) and may be accessed through a catalog providing source coordinates of 608 objects (cf. McCarthy 1996). Links to existing optical catalogs are provided by radio stars (Seidelmann 1998), while the ICRS and planetary frame tie is provided by VLBI observations of planetary spacecraft as well as lunar laser ranging (LLR) (Folkner et al. 1994, Standish 1998).

Complementary to the ICRS, the *International Terrestrial Reference System (ITRS)* provides the conceptual definition of an Earth-fixed reference system (McCarthy 1996). Its origin is located at the Earth's center of mass (including oceans and atmosphere) and its unit of length is the SI meter (consistent with the TCG time coordinate). The orientation of the *IERS Reference Pole (IRP)* and *Meridian (IRM)* are consistent with the previously adopted BIH system at epoch 1984.0 and the former Conventional International Origin (CIO) (cf. Sect. 5.4.3). The time evolution of the ITRS is such that it exhibits no net rotation with respect to the Earth's crust. Realizations of the ITRS are given by the *International Terrestrial Reference Frame (ITRF)* that provides estimated coordinates and velocities of selected observing stations under authority of the IERS. Observational techniques used in their determination include satellite laser ranging (SLR), lunar laser ranging (LLR), Global Positioning System (GPS), and VLBI measurements. New versions of the ITRF are published annually and exhibit global differences at the centimeter level.

The transformation between the International Celestial Reference System and the International Terrestrial Reference System is accomplished by conventional models for

- precession (Lieske et al. 1977), describing the secular change in the orientation of the Earth's rotation axis and the equinox,
- nutation (Seidelmann 1982), describing the periodic and short-term variation of the equator and the vernal equinox, and
- Sidereal Time in relation to UT1 (Aoki et al. 1982), describing the Earth's rotation about its axis.

These models are supplemented by the IERS Earth Observation Parameters (EOP), comprising

- observations of the UT1-TAI difference and
- measured coordinates of the rotation axis relative to the IERS Reference Pole

(IERS 1998). The resulting transformation may be expressed as

$$\boldsymbol{r}_{\mathrm{ITRS}} = \boldsymbol{\Pi}(t)\,\boldsymbol{\Theta}(t)\,\boldsymbol{N}(t)\,\boldsymbol{P}(t)\,\boldsymbol{r}_{\mathrm{ICRS}} \tag{5.22}$$

where the rotation matrices $\boldsymbol{P}$, $\boldsymbol{N}$, $\boldsymbol{\Theta}$, and $\boldsymbol{\Pi}$ describe the coordinate changes due to precession, nutation, Earth rotation, and polar motion, respectively. A detailed account of the underlying concepts of these transformations and the adopted numerical expressions is presented in the subsequent sections.

5.3 Precession and Nutation

5.3.1 Lunisolar Torques and the Motion of the Earth's Rotation Axis

In order to describe the precession of the Earth's rotation axis, the Earth is considered as a rotationally symmetric gyroscope with an angular momentum $\boldsymbol{l}$ that changes with time under the influence of an external torque $\boldsymbol{D}$ according to

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{D} \quad . \tag{5.23}$$

Even though the direction of the angular momentum may, in general, differ from the symmetry axis of a gyroscope and the instantaneous axis of rotation, one may neglect these differences in the discussion of precession and nutation and assume that $\boldsymbol{l}$ is parallel to the unit vector $\boldsymbol{e}_z$ that defines the Earth's axis (cf. Fig. 5.7). Then

$$\boldsymbol{l} = C\omega_\oplus \boldsymbol{e}_z \tag{5.24}$$

where

$$\omega_\oplus \approx 7.29 \cdot 10^{-5}\,\text{rad/s} \tag{5.25}$$

is the angular velocity of the Earth's rotation and C is the moment of inertia. For a spherical body of homogeneous density with mass $M_\oplus$ and radius $R_\oplus$ the moment of inertia is given by

$$I = \frac{2}{5} M_\oplus R_\oplus^2 \tag{5.26}$$

for an arbitrary axis of rotation. Due to the Earth's flattening and its internal structure the actual moments of inertia are given by slightly differing values

$$A = 0.329 M_\oplus R_\oplus^2 \quad \text{and} \quad C = 0.330 M_\oplus R_\oplus^2 \tag{5.27}$$

for a rotation around an axis in the equatorial plane and a rotation around the polar axis, respectively. It may be noted that these quantities are related to the C_{20} geopotential coefficient by

$$C - A = -C_{20} M_\oplus R_\oplus^2 \quad . \tag{5.28}$$

The torque $\boldsymbol{D}$ due to a point mass m (i.e. the Sun or Moon) at a geocentric position $\boldsymbol{r}$ is given by

$$\boldsymbol{D} = -m\,(\boldsymbol{r} \times \ddot{\boldsymbol{r}}) \quad , \tag{5.29}$$

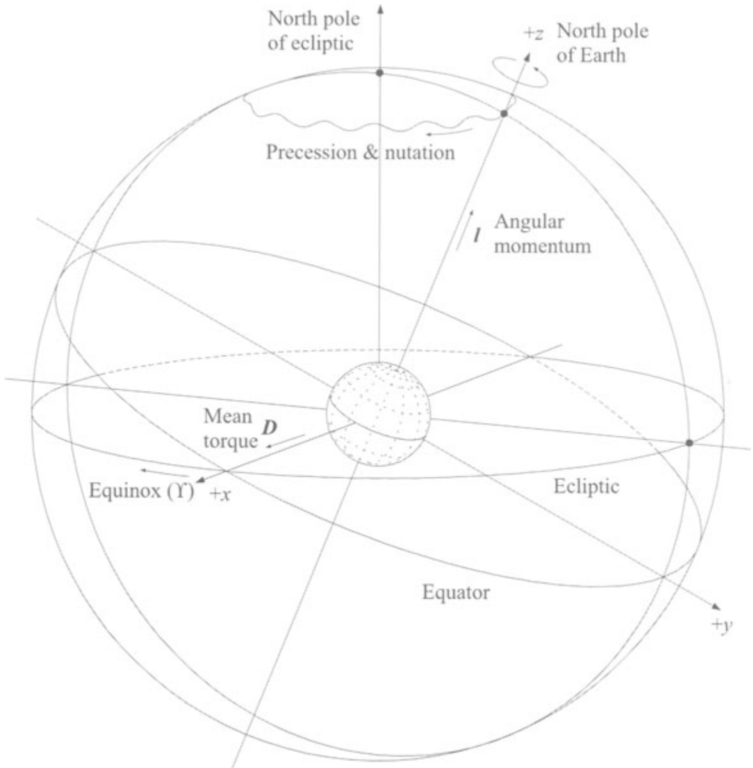

Fig. 5.7. Motion of the Earth's axis under the influence of solar and lunar torques

if $\ddot{\boldsymbol{r}}$ designates the acceleration of m by the gravitational force of the Earth. Neglecting higher-order zonal terms in the expansion of the geopotential, $\ddot{\boldsymbol{r}}$ is obtained as

$$\ddot{\boldsymbol{r}} = -\frac{GM_\oplus}{r^3}\boldsymbol{r} - \frac{3}{2}\frac{GM_\oplus R_\oplus^2 C_{20}}{r^7}\left[\,(5z^2 - r^2)\boldsymbol{r} - 2(zr^2)\boldsymbol{e}_z\,\right] \tag{5.30}$$

for a rotationally symmetric Earth (cf. Sect. 3.2), where $z = \boldsymbol{r}\boldsymbol{e}_z$ is the distance of m from the equatorial plane. All terms of the acceleration that are parallel to the radius vector affect the Earth's center of mass, only, and the resulting torque is given by the simple expression

$$\boldsymbol{D} = Gm(C - A)\frac{3z(\boldsymbol{r} \times \boldsymbol{e}_z)}{r^5} \quad . \tag{5.31}$$

The Sun moves around the Earth in a near-circular orbit that is inclined at an angle ε with respect to the equator and the resulting torque vanishes whenever the Sun crosses the equator ($z = 0$). Introducing the unit vector $\boldsymbol{e}_x$ in the direction of the vernal equinox (cf. Fig. 5.7), the torque of the Sun at right angles to the line of nodes is found to be

$$\boldsymbol{D}_\odot = GM_\odot(C - A)\frac{3\sin\varepsilon\cos\varepsilon}{r_\odot^3}\boldsymbol{e}_x \quad , \tag{5.32}$$

irrespective of whether the Sun is above or below the equatorial plane. This results in a mean solar torque

$$\overline{\boldsymbol{D}}_\odot = GM_\odot(C - A)\frac{3\sin\varepsilon\cos\varepsilon}{2r_\odot^3}\boldsymbol{e}_x \tag{5.33}$$

in the direction of the vernal equinox during the course of a year, whereas the mean component in the direction perpendicular to $\boldsymbol{e}_x$ vanishes. Making use of Kepler's third law, the last expression may further be written as

$$\overline{\boldsymbol{D}}_\odot = \frac{3}{2}(C - A)\sin\varepsilon\cos\varepsilon\, n_\odot^2\,\boldsymbol{e}_x \quad , \tag{5.34}$$

where $n_\odot$ is the mean motion of the Sun in its orbit around the Earth.

Similar considerations hold for the Moon, with the exception that the inclination of the lunar orbit with respect to Earth's equator is not fixed, but varies between 18° and 28° during a period of about 18 years. Since this period is small compared to the time scale of precession, one may, however, assume that the Moon moves in the ecliptic just like the Sun. This yields a total mean torque of

$$\overline{\boldsymbol{D}} = \frac{3}{2}(C - A)\sin\varepsilon\cos\varepsilon\left(n_\odot^2 + \frac{M_\mathrm{M}}{M_\oplus}n_\mathrm{M}^2\right)\boldsymbol{e}_x \quad , \tag{5.35}$$

which changes neither the Earth's total angular momentum nor the obliquity ε but forces $\boldsymbol{l}$ to move around the pole of the ecliptic at an angular velocity

$$\Omega_\mathrm{prec} = \frac{|\overline{\boldsymbol{D}}|}{\sin(\varepsilon)|\boldsymbol{l}|} = \frac{3}{2}\frac{C - A}{C}\cos(\varepsilon)\frac{n_\odot^2 + n_\mathrm{M}^2 M_\mathrm{M}/M_\oplus}{\omega_\oplus} \tag{5.36}$$

of one revolution in 26 000 years.

5.3.2 Coordinate Changes due to Precession

The combined effects of precession on the orientation of the ecliptic and the equator are illustrated in Fig. 5.8, where the motion of both planes is described with respect to the mean equator and ecliptic of the reference epoch J2000 (2000 January 1.5).

Due to lunisolar precession the intersection of the mean equator of epoch t and the mean ecliptic of J2000 lags behind the vernal equinox Υ_{2000} of J2000 by an angle

$$\psi = 5038''\!.8 \cdot T - 1''\!.1 \cdot T^2 \tag{5.37}$$

that increases almost linearly with time, while the inclination of the mean equator with respect to the ecliptic of J2000 is nearly constant:

$$\omega = 23°26'21'' + 0''\!.05 \cdot T^2 \quad . \tag{5.38}$$

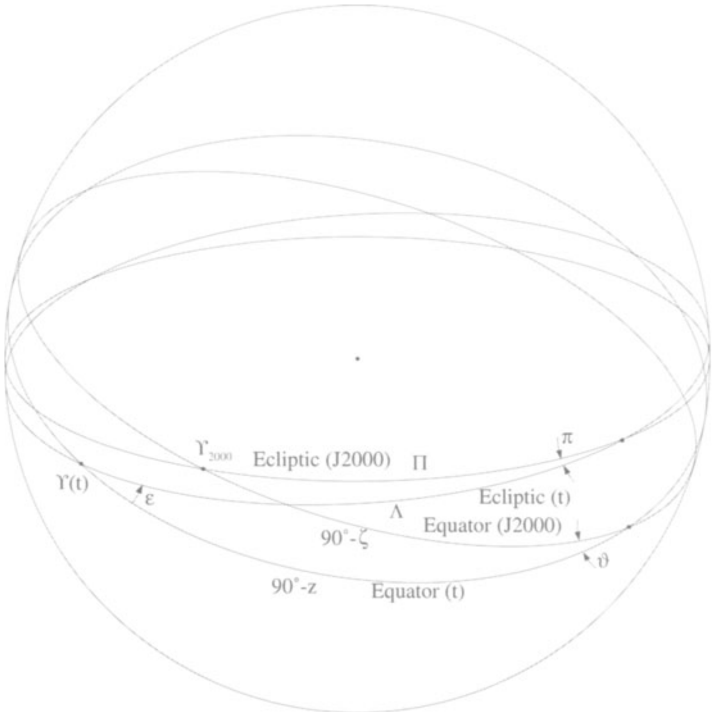

Fig. 5.8. The effects of precession on the ecliptic, equator, and vernal equinox

Here

$$T = (\text{JD} - 2451545.0)/36525.0 \tag{5.39}$$

is measured in Julian centuries Terrestrial Time[5] since J2000 TT.

While the gravitational pull of the Sun and Moon changes the direction of the Earth's axis and the equatorial plane, it does not affect the orientation of the ecliptic. Long-term changes of the mean orbit of the Earth around the Sun do, however, arise from the influence of the planets, which results in a corresponding motion of the ecliptic. With respect to the ecliptic of J2000 the ecliptic at another epoch is inclined at an angle of

$$\pi = 47\overset{''}{.}0029 \cdot T - 0\overset{''}{.}03302 \cdot T^2 + 0\overset{''}{.}000060 \cdot T^3 \quad , \tag{5.40}$$

where the line of intersection is described by the angle

$$\Pi = 174\overset{\circ}{.}876383889 - 869\overset{''}{.}8089 \cdot T + 0\overset{''}{.}03536 \cdot T^2 \quad . \tag{5.41}$$

These values follow from a theory of the secular changes of the Earth's orbital elements and have been derived by Lieske et al. (1977) following earlier calculations by Newcomb.

[5]Following a recommendation of the IERS (McCarthy 1996), the expressions for precession and nutation are to be evaluated in terms of Terrestrial Time, instead of Barycentric Dynamical Time.

As a result of planetary precession the obliquity of the ecliptic is slightly decreasing and amounts to

$$\varepsilon = 23.^\circ 43929111 - 46.''8150\,T - 0.''00059\,T^2 + 0.''001813\,T^3 \quad . \tag{5.42}$$

The combined precession in longitude

$$p = \Lambda - \Pi = 5029.''0966 \cdot T + 1.''11113 \cdot T^2 - 0.''000006 T^3 \tag{5.43}$$

is somewhat smaller, therefore, than the lunisolar precession ψ alone.

The orientation of the mean equator and equinox of epoch T with respect to the equator and equinox of J2000 is defined by the three angles

$$\begin{aligned} \zeta &= 2306.''2181\,T + 0.''30188\,T^2 + 0.''017998\,T^3 \\ \vartheta &= 2004.''3109\,T - 0.''42665\,T^2 - 0.''041833\,T^3 \\ z &= \zeta + 0.''79280\,T^2 + 0.''000205\,T^3 \end{aligned} \tag{5.44}$$

that follow from the fundamental quantities π, Π, p, and ε.

According to Fig. 5.8 the transformation from coordinates $\boldsymbol{r}_{\mathrm{ICRF}}$ (referred to the mean equator and equinox of J2000) to coordinates referred to the mean equator and equinox of some other epoch ("mean-of-date") may now be written as

$$\boldsymbol{r}_{\mathrm{mod}} = \boldsymbol{P}\,\boldsymbol{r}_{\mathrm{ICRF}} \tag{5.45}$$

where the matrix $\boldsymbol{P}$ is the product of three consecutive rotations:

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{R}_z(-90^\circ - z)\,\boldsymbol{R}_x(\vartheta)\,\boldsymbol{R}_z(90^\circ - \zeta) \\ &= \boldsymbol{R}_z(-z)\,\boldsymbol{R}_y(\vartheta)\,\boldsymbol{R}_z(-\zeta) \quad . \end{aligned} \tag{5.46}$$

Evaluating the matrix product, one obtains the following expression for $\boldsymbol{P} = (p_{ij})$:

$$\begin{aligned} p_{11} &= -\sin z \sin\zeta + \cos z \cos\vartheta \cos\zeta \\ p_{21} &= +\cos z \sin\zeta + \sin z \cos\vartheta \cos\zeta \\ p_{31} &= +\sin\vartheta \cos\zeta \\[1ex] p_{12} &= -\sin z \cos\zeta - \cos z \cos\vartheta \sin\zeta \\ p_{22} &= +\cos z \cos\zeta - \sin z \cos\vartheta \sin\zeta \\ p_{32} &= -\sin\vartheta \sin\zeta \\[1ex] p_{13} &= -\cos z \sin\vartheta \\ p_{23} &= -\sin z \sin\vartheta \\ p_{33} &= +\cos\vartheta \quad . \end{aligned} \tag{5.47}$$

Since $\boldsymbol{P}$ is a rotation matrix, its inverse $\boldsymbol{P}^{-1}$ is equal to the transpose $\boldsymbol{P}^T$:

$$\boldsymbol{P}^{-1} = \boldsymbol{P}^T = \boldsymbol{R}_z(+\zeta)\boldsymbol{R}_y(-\vartheta)\boldsymbol{R}_z(+z) \quad . \tag{5.48}$$

The precession transformation between arbitrary epochs T_1 and T_2 is thus obtained from

$$\boldsymbol{r}_2 = \boldsymbol{P}(T_2)\,\boldsymbol{P}^T(T_1)\,\boldsymbol{r}_1 \quad . \tag{5.49}$$

Here $\boldsymbol{P}(T)$ denotes the rotation from the mean equator and equinox of J2000 to the mean equator and equinox of epoch T.

Alternatively, the generalized expressions

$$\begin{aligned}
\zeta(T,t) &= (+2306''\!.2181 + 1''\!.39656\,T - 0''\!.000139\,T^2)\,t \\
&\qquad +(+0''\!.30188 - 0''\!.000344\,T)\,t^2 + 0''\!.017998\,t^3 \\
z(T,t) &= (+2306''\!.2181 + 1''\!.39656\,T - 0''\!.000139\,T^2)\,t \\
&\qquad +(+1''\!.09468 + 0''\!.000066\,T)\,t^2 + 0''\!.018203\,t^3 \\
\vartheta(T,t) &= (+2004''\!.3109 - 0''\!.85330\,T - 0''\!.000217\,T^2)\,t \\
&\qquad +(-0''\!.42665 - 0''\!.000217\,T)\,t^2 - 0''\!.041833\,t^3
\end{aligned} \tag{5.50}$$

(Lieske et al.1977, Lieske 1979) with

$$\begin{aligned}
T &= T_1 &&= (\mathrm{JD}_1(\mathrm{TT}) - 2451545.0)/36525.0 \\
t &= T_2 - T_1 &&= (\mathrm{JD}_2(\mathrm{TT}) - \mathrm{JD}_1(\mathrm{TT}))/36525.0
\end{aligned} \tag{5.51}$$

can be used to compute the transformation matrix

$$\boldsymbol{P}(T_2, T_1) = \boldsymbol{R}_z(-z(T,t))\boldsymbol{R}_y(\vartheta(T,t))\boldsymbol{R}_z(-\zeta(T,t)) \tag{5.52}$$

directly from the mean equator and equinox of epoch T_1 to the mean equator and equinox of epoch T_2.

The 3rd-order polynomials[6] for the precession angles given in (5.50) obey the identities

$$\begin{aligned}
z(T+t,-t) &= -\zeta(T,t) \\
\zeta(T+t,-t) &= -z(T,t) \\
\vartheta(T+t,-t) &= -\vartheta(T,t) \quad .
\end{aligned} \tag{5.53}$$

Accordingly,

$$\begin{aligned}
\boldsymbol{P}(T_1, T_2) &= \boldsymbol{R}_z(-z(T+t,\,-t))\boldsymbol{R}_y(+\vartheta(T+t,\,-t))\boldsymbol{R}_z(-\zeta(T+t,\,-t)) \\
&= \boldsymbol{R}_z(\zeta(T,t))\boldsymbol{R}_y(-\vartheta(T,t))\boldsymbol{R}_z(z(T,t)) \\
&= \boldsymbol{P}^T(T_2, T_1)
\end{aligned} \tag{5.54}$$

yields the rigorous inverse of $\boldsymbol{P}(T_2, T_1)$. On the other hand, the transitivity relation

$$\boldsymbol{P}(T_3, T_1) = \boldsymbol{P}(T_3, T_2)\boldsymbol{P}(T_2, T_1) \tag{5.55}$$

is not maintained exactly by the generalized precession angles. It is therefore better in practical applications of expression (5.50) to avoid the sequential use of precession matrices. Otherwise, errors typically of the order of 10^{-11} rad, or $10^{-6}{}''$, will arise for epochs lying within one century from the reference epoch J2000.

[6]The coefficient $\zeta_2' = \frac{1}{2}\partial^3\zeta/\partial^2 t\partial T$ is originally given as -0.000345 in Lieske (1977) and has been replaced by the proper value -0.000344 in Lieske (1979).

5.3.3 Nutation

Aside from the secular precessional motion the orientation of the Earth's rotation axis is affected by small periodic perturbations that are known as nutation. They are due to the monthly and annual variations of the lunar and solar torque that have been averaged in the treatment of precession. The main contribution to nutation arises from the varying orientation of the lunar orbit with respect to the Earth's equator as expressed by the longitude of the Moon's ascending node Ω. It induces a periodic shift

$$\Delta\psi \approx -17\overset{''}{.}200 \cdot \sin(\Omega) \tag{5.56}$$

of the vernal equinox and a change

$$\Delta\varepsilon \approx +9\overset{''}{.}203 \cdot \cos(\Omega) \tag{5.57}$$

of the obliquity of the ecliptic during the 18.6-year nodal period of the Moon. As a result the true celestial pole performs an elliptic motion around the mean position as affected by the lunisolar precession.

The currently adopted IAU 1980 nutation series is based on theories of Kinoshita (1977) and Wahr (1981). It expresses the nutation angles

$$\begin{aligned} \Delta\Psi &= \sum_{i=1}^{106} (\Delta\Psi)_i \cdot \sin(\phi_i) \\ \Delta\varepsilon &= \sum_{i=1}^{106} (\Delta\varepsilon)_i \cdot \cos(\phi_i) \end{aligned} \tag{5.58}$$

by a total of 106 terms, which are summarized in Table 5.2 (Seidelmann 1982). Each term describes a periodic function of the mean elements of the lunar and solar orbit with argument

$$\phi_i = p_{l,i} l + p_{l',i} l' + p_{F,i} F + p_{D,i} D + p_{\Omega,i} \Omega \tag{5.59}$$

and integer coefficients $p_{l,i}$, $p_{l',i}$, $p_{F,i}$, $p_{D,i}$, and $p_{\Omega,i}$. The other parameters are the Moon's mean anomaly (l), the Sun's mean anomaly (l'), the mean distance of the Moon from the ascending node (F), the difference between the mean longitudes of the Sun and the Moon (D), and the mean longitude of the ascending node of the lunar orbit (Ω). Numerical values for use with the IAU 1980 theory of nutation are originally given as

$$\begin{aligned} l &= 134°57'46\overset{''}{.}733 + 477198°52'02\overset{''}{.}633\,T + 31\overset{''}{.}310\,T^2 + 0\overset{''}{.}064\,T^3 \\ l' &= 357°31'39\overset{''}{.}804 + 35999°03'01\overset{''}{.}224\,T - 0\overset{''}{.}577\,T^2 - 0\overset{''}{.}012\,T^3 \\ F &= 93°16'18\overset{''}{.}877 + 483202°01'03\overset{''}{.}137\,T - 13\overset{''}{.}257\,T^2 + 0\overset{''}{.}011\,T^3 \\ D &= 297°51'01\overset{''}{.}307 + 445267°06'41\overset{''}{.}328\,T - 6\overset{''}{.}891\,T^2 + 0\overset{''}{.}019\,T^3 \\ \Omega &= 125°02'40\overset{''}{.}280 - 1934°08'10\overset{''}{.}539\,T + 7\overset{''}{.}455\,T^2 + 0\overset{''}{.}008\,T^3 \end{aligned} \tag{5.60}$$

Table 5.2. The IAU 1980 nutation theory

p_l	$p_{l'}$	p_F	p_D	p_Ω	$\Delta\Psi$ [0.0001″]	$\Delta\varepsilon$ [0.0001″]	i	p_l	$p_{l'}$	p_F	p_D	p_Ω	$\Delta\Psi$	$\Delta\varepsilon$	i
0	0	0	0	1	$-171996-174.2T$	$+92025+8.9T$	1	1	0	2	2	2	−8	3	54
0	0	0	0	2	$2062+0.2T$	$-895+0.5T$	2	1	0	0	0	0	6	0	55
−2	0	2	0	1	46	−24	3	2	0	2	−2	2	6	−3	56
2	0	−2	0	0	11	0	4	0	0	0	2	1	−6	3	57
−2	0	2	0	2	−3	1	5	0	0	2	2	1	−7	3	58
1	−1	0	−1	0	−3	0	6	1	0	2	−2	1	6	−3	59
0	−2	2	−2	1	−2	1	7	0	0	0	−2	1	−5	3	60
2	0	−2	0	1	1	0	8	1	−1	0	0	0	5	0	61
0	0	2	−2	2	$-13187-1.6T$	$5736-3.1T$	9	2	0	2	0	1	−5	3	62
0	1	0	0	0	$1426-3.4T$	$54-0.1T$	10	0	1	0	−2	0	−4	0	63
0	1	2	−2	2	$-517+1.2T$	$224-0.6T$	11	1	0	−2	0	0	4	0	64
0	−1	2	−2	2	$217-0.5T$	$-95+0.3T$	12	0	0	0	1	0	−4	0	65
0	0	2	−2	1	$129+0.1T$	−70	13	1	1	0	0	0	−3	0	66
2	0	0	−2	0	48	1	14	1	0	2	0	0	3	0	67
0	0	2	−2	0	−22	0	15	1	−1	2	0	2	−3	1	68
0	2	0	0	0	$17-0.1T$	0	16	−1	−1	2	2	2	−3	1	69
0	1	0	0	1	−15	9	17	−2	0	0	0	1	−2	1	70
0	2	2	−2	2	$-16+0.1T$	7	18	3	0	2	0	2	−3	1	71
0	−1	0	0	1	−12	6	19	0	−1	2	2	2	−3	1	72
−2	0	0	2	1	−6	3	20	1	1	2	0	2	2	−1	73
0	−1	2	−2	1	−5	3	21	−1	0	2	−2	1	−2	1	74
2	0	0	−2	1	4	−2	22	2	0	0	0	1	2	−1	75
0	1	2	−2	1	4	−2	23	1	0	0	0	2	−2	1	76
1	0	0	−1	0	−4	0	24	3	0	0	0	0	2	0	77
2	1	0	−2	0	1	0	25	0	0	2	1	2	2	−1	78
0	0	−2	2	1	1	0	26	−1	0	0	0	2	1	−1	79
0	1	−2	2	0	−1	0	27	1	0	0	−4	0	−1	0	80
0	1	0	0	2	1	0	28	−2	0	2	2	2	1	−1	81
−1	0	0	1	1	1	0	29	−1	0	2	4	2	−2	1	82
0	1	2	−2	0	−1	0	30	2	0	0	−4	0	−1	0	83
0	0	2	0	2	$-2274-0.2T$	$977-0.5T$	31	1	1	2	−2	2	1	−1	84
1	0	0	0	0	$712+0.1T$	−7	32	1	0	2	2	1	−1	1	85
0	0	2	0	1	$-386-0.4T$	200	33	−2	0	2	4	2	−1	1	86
1	0	2	0	2	−301	$129-0.1T$	34	−1	0	4	0	2	1	0	87
1	0	0	−2	0	−158	−1	35	1	−1	0	−2	0	1	0	88
−1	0	2	0	2	123	−53	36	2	0	2	−2	1	1	−1	89
0	0	0	2	0	63	−2	37	2	0	2	2	2	−1	0	90
1	0	0	0	1	$63+0.1T$	−33	38	1	0	0	2	1	−1	0	91
−1	0	0	0	1	$-58-0.1T$	32	39	0	0	4	−2	2	1	0	92
−1	0	2	2	2	−59	26	40	3	0	2	−2	2	1	0	93
1	0	2	0	1	−51	27	41	1	0	2	−2	0	−1	0	94
0	0	2	2	2	−38	16	42	0	1	2	0	1	1	0	95
2	0	0	0	0	29	−1	43	−1	−1	0	2	1	1	0	96
1	0	2	−2	2	29	−12	44	0	0	−2	0	1	−1	0	97
2	0	2	0	2	−31	13	45	0	0	2	−1	2	−1	0	98
0	0	2	0	0	26	−1	46	0	1	0	2	0	−1	0	99
−1	0	2	0	1	21	−10	47	1	0	−2	−2	0	−1	0	100
−1	0	0	2	1	16	−8	48	0	−1	2	0	1	−1	0	101
1	0	0	−2	1	−13	7	49	1	1	0	−2	1	−1	0	102
−1	0	2	2	1	−10	5	50	1	0	−2	2	0	−1	0	103
1	1	0	−2	0	−7	0	51	2	0	0	2	0	1	0	104
0	1	2	0	2	7	−3	52	0	0	2	4	2	−1	0	105
0	−1	2	0	2	−7	3	53	0	1	0	1	0	1	0	106

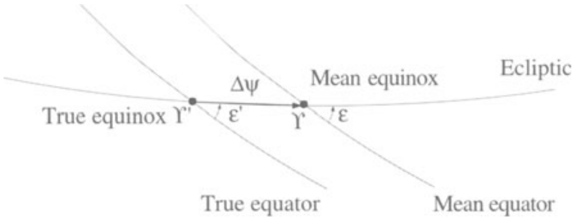

Fig. 5.9. The shift in the positions of the equator, the ecliptic and the vernal equinox, caused by nutation

in Seidelmann (1982), while slightly modified expressions recommended by the IERS are given in McCarthy (1996).

Following Fig. 5.9, the transformation from mean-of-date coordinates (referred to the mean equator and equinox) to true-of-date coordinates (referred to the true equator and equinox) may be written as

$$\boldsymbol{r}_{\text{tod}} = \boldsymbol{N}(T)\,\boldsymbol{r}_{\text{mod}}, \tag{5.61}$$

with

$$\boldsymbol{N}(T) = \boldsymbol{R}_x(-\varepsilon-\Delta\varepsilon)\,\boldsymbol{R}_z(-\Delta\psi)\,\boldsymbol{R}_x(\varepsilon) \quad . \tag{5.62}$$

The elements of the transformation matrix $\boldsymbol{N} = (n_{ij})$ in equatorial coordinates are given by

$$\begin{array}{lcl}
n_{11} & = & +\cos(\Delta\Psi) \\
n_{21} & = & +\cos(\varepsilon')\cdot\sin(\Delta\Psi) \\
n_{31} & = & +\sin(\varepsilon')\cdot\sin(\Delta\Psi) \\[1ex]
n_{12} & = & -\cos(\varepsilon)\cdot\sin(\Delta\Psi) \\
n_{22} & = & +\cos(\varepsilon)\cdot\cos(\varepsilon')\cdot\cos(\Delta\Psi)+\sin(\varepsilon)\cdot\sin(\varepsilon') \\
n_{32} & = & +\cos(\varepsilon)\cdot\sin(\varepsilon')\cdot\cos(\Delta\Psi)-\sin(\varepsilon)\cdot\cos(\varepsilon') \\[1ex]
n_{13} & = & -\sin(\varepsilon)\cdot\sin(\Delta\Psi) \\
n_{23} & = & +\sin(\varepsilon)\cdot\cos(\varepsilon')\cdot\cos(\Delta\Psi)-\cos(\varepsilon)\cdot\sin(\varepsilon') \\
n_{33} & = & +\sin(\varepsilon)\cdot\sin(\varepsilon')\cdot\cos(\Delta\Psi)+\cos(\varepsilon)\cdot\cos(\varepsilon') \quad ,
\end{array} \tag{5.63}$$

where ε and $\varepsilon' = \varepsilon + \Delta\epsilon$ are the mean and true obliquity of the ecliptic at time $T = (\text{JD(TT)} - 2451545.0)/36525$.

From VLBI and LLR observations, the IAU 1980 theory of nutation is known to be in error on the level of several milli-arcseconds and an improved nutation theory (IERS 1996) due to T. Herring has been made available in McCarthy (1996). Nevertheless, the IAU 1980 series is retained as official standard in the IERS conventions and the existing deficiencies are compensated for by observed values of the celestial pole offsets $\delta\Delta\psi$ and $\delta\Delta\varepsilon$. Improved nutation angles are obtained by adding these corrections to the IAU 1980 values:

$$\begin{array}{lcl}
\Delta\psi & = & \Delta\psi_{\text{IAU1980}} \;+\; \delta\Delta\psi \\
\Delta\varepsilon & = & \Delta\varepsilon_{\text{IAU1980}} \;+\; \delta\Delta\varepsilon \quad .
\end{array} \tag{5.64}$$

The corresponding correction to the nutation matrix is obtained from

$$N = \begin{pmatrix} 1 & -\delta\Delta\psi\cos\varepsilon & -\delta\Delta\psi\sin\varepsilon \\ +\delta\Delta\psi\cos\varepsilon & 1 & -\delta\Delta\varepsilon \\ +\delta\Delta\psi\sin\varepsilon & +\delta\Delta\varepsilon & 1 \end{pmatrix} N_{\mathrm{IAU1980}} \tag{5.65}$$

(McCarthy 1996). Post-facto determinations and short-term predictions of the celestial pole offsets are published on a monthly basis in Bulletin B of the IERS (cf. Fig. 5.4).

5.4 Earth Rotation and Polar Motion

5.4.1 Rotation About the Celestial Ephemeris Pole

The IAU precession and nutation theories yield the instantaneous orientation of the Earth's rotation axis, or, more precisely, the orientation of the *Celestial Ephemeris Pole* (CEP)[7] with respect to the International Celestial Reference System. The rotation about the CEP axis itself is described by the Greenwich Mean Sidereal Time (GMST) that measures the angle between the mean vernal equinox and the Greenwich Meridian (cf. Sect. 5.1.4). Given the UT1–UTC or UT1–TAI time difference as monitored and published by the IERS, the Greenwich Mean Sidereal Time at any instant can be computed from the conventional relation (5.19).

Similar to GMST, the Greenwich Apparent Sidereal Time (GAST) measures the hour angle of the *true* equinox. Both values differ by the nutation in right ascension

$$\mathrm{GAST} - \mathrm{GMST} = \Delta\psi\cos\epsilon \quad , \tag{5.66}$$

which is also known as the *equation of the equinoxes*[8]. Given the apparent sidereal time, the matrix

$$\boldsymbol{\Theta}(t) = \boldsymbol{R}_z(\mathrm{GAST}) \tag{5.67}$$

yields the transformation between the true-of-date coordinate system (as defined by the adopted precession–nutation theory) and a system aligned with the Earth equator and Greenwich meridian.

[7] The Celestial Ephemeris Pole differs slightly from the instantaneous rotation axis which was used in the earlier nutation theory of Woolard (1953). The adoption of the CEP is related to the fact that the rotation axis performs a predictable daily motion around the CEP under the action of Sun and Moon and is not, therefore, a proper reference pole for theoretical and observational reasons. On the Earth's surface the difference between both poles amounts to approximately 0.6 m. For a detailed discussion the reader is referred to Seidelmann (1982) Groten (1984), and Capitaine et al. (1985).

[8] If milliarcsecond accuracy is required in the equation of the equinoxes, two additional terms $+0\overset{''}{.}002649\sin\Omega - 0\overset{''}{.}000013\cos\Omega$ with Ω denoting the longitude of the Moon's ascending node should be added to the right-hand side of (5.66). These terms represent a second-order correction resulting from a coupling between precession in longitude and nutation in obliquity in a kinematical definition of apparent sidereal time (cf. Capitaine & Gontier 1993, McCarthy 1996).

The common z-axis of both systems points to the Celestial Ephemeris Pole, which is not, however, fixed with respect to the surface of the Earth, but performs a periodic motion around its mean position from which it differs by at most 10 m. This motion is known as *polar motion* and can be understood by considering a rotationally symmetric gyroscope, in which the rotation axis moves around the axis of figure in the absence of external torques.

5.4.2 Free Eulerian Precession

In a body-fixed coordinate system ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$) that is aligned with the principal axes of inertia the angular momentum $\boldsymbol{l}'$ of a symmetric gyroscope is given by

$$\boldsymbol{l}' = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \boldsymbol{\omega} \quad , \tag{5.68}$$

where $\boldsymbol{\omega}$ is the instantaneous rotation axis and where A and C are the moments of inertia for a rotation around the $\boldsymbol{e}_1$- or $\boldsymbol{e}_2$-axis and the $\boldsymbol{e}_3$-axis, respectively. Without external torques the angular momentum $\boldsymbol{l}$ is constant in an inertial reference system, but since $\boldsymbol{l}'$ refers to a rotating system it obeys the relation

$$\frac{d\boldsymbol{l}}{dt} = \frac{d\boldsymbol{l}'}{dt} + \boldsymbol{\omega} \times \boldsymbol{l}' = \boldsymbol{0} \quad . \tag{5.69}$$

Upon insertion this yields Euler's equations

$$\begin{aligned} A\frac{d\omega_1}{dt} + (C-A)\,\omega_2\omega_3 &= 0 \\ A\frac{d\omega_2}{dt} - (C-A)\,\omega_1\omega_3 &= 0 \\ C\frac{d\omega_3}{dt} &= 0 \end{aligned} \tag{5.70}$$

for the motion of $\boldsymbol{\omega}$ with respect to the body-fixed coordinate system in the specialized case of a symmetric gyroscope. While the third equation implies a constant component of $\boldsymbol{\omega}$ around the symmetry axis $\boldsymbol{e}_3$, the solution of the first two equations is given by

$$\begin{aligned} \omega_1 &= a\cos\left(\frac{C-A}{A}\omega_3 t + b\right) \\ \omega_2 &= a\sin\left(\frac{C-A}{A}\omega_3 t + b\right) \quad . \end{aligned} \tag{5.71}$$

The instantaneous rotation vector therefore describes a circle around the $\boldsymbol{e}_3$-axis, where the radius a and phase b are fixed by the initial conditions. The period

$$P = \frac{2\pi}{\omega_3}\frac{A}{C-A} \tag{5.72}$$

depends on the angular velocity and the flattening of the gyroscope as expressed by the fraction of the moments of inertia. For the Earth, the *dynamical flattening* $C/(C - A)$ as derived from the observed precession rate (cf. Sect. 5.3.1) amounts to 0.00326, which yields a period of 305 days.

5.4.3 Observation and Extrapolation of Polar Motion

Observations show that the Earth's polar motion is actually a superposition of two components. One is the free precession with a period of about 435 days (the Chandler period) that is not, however, in accord with the expected 305 day period and can only be explained by a non-rigid Earth model. The second part is an annual motion that is induced by seasonal changes of the Earth's mass distribution due to air and water flows.

In contrast to precession and nutation the motion of the rotation axis with respect to the surface of the Earth cannot, therefore, be predicted from theory but has to be monitored by continuous observations. For this purpose the mean position of the pole of rotation during the years 1900 to 1905 is usually chosen as the origin for polar motion measurements. Historically two slightly differing reference points have been employed by various institutions. The CIO (Conventional International Origin) is defined by the location of five stations of the *International Latitude Service* (ILS) that has been involved in polar motion measurements from the beginning of the century, whereas the BIH pole was later adopted by the *Bureau International de l'Heure*. The difference between the two definitions is estimated to be less than 1 m (Groten 1984). Following the introduction of the International Terrestrial Reference System (ITRS) all polar motion data have consistently been referred to the IERS Reference Pole (IRP), which was initially aligned with the BIH pole in 1984.0.

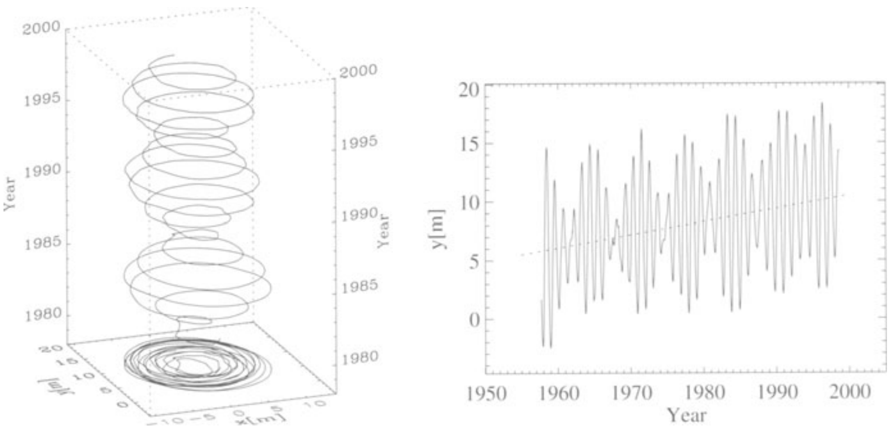

Fig. 5.10. Due to polar motion the Celestial Ephemeris Pole (CEP) performs a periodic oscillation around the IERS Reference Pole (IRP). The superposition of the annual oscillation and the Chandlerian free precession results in a pronounced beat frequency of roughly 5–6 six years. In addition, the CEP exhibits a secular motion in the y-direction

Examples of polar motion observations are given in Fig. 5.10 which shows the coordinates x_p and y_p of the Celestial Ephemeris Pole with respect to the IERS Reference Pole as a function of time. The x and y-axes are aligned with the IERS Reference Meridian (Greenwich meridian) and the 90°-West meridian. A displacement of $0\rlap{.}''1$ corresponds to 3 m on the surface of the Earth. Since 1900 the mean position of the pole has shifted by about 10 m due to small changes in the Earth's mass distribution (cf. Fig. 5.10). As a result of this motion, which is known as *polar wander*, the observed oscillation of the rotation axis is no longer symmetric with respect to the adopted CIO/BIH-pole. The superposition of the annual oscillation and the free precession is evident from the frequency spectrum of polar motion shown in Fig. 5.11. Both contributions are of near-equal magnitude and almost cancel each other with a beat period of 5 to 6 years (cf. Fig. 5.10). Current values of the pole coordinates are published on a monthly basis in Bulletin B (cf. Fig. 5.4) of the International Earth Rotation Service with a resolution of one and five days, respectively. From these data intermediate values for any time may be obtained by quadratic interpolation with sufficient accuracy.

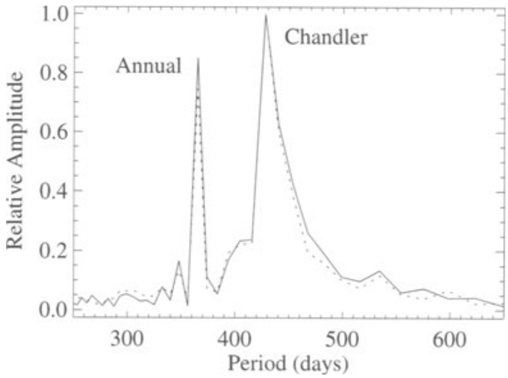

Fig. 5.11. The polar motion frequency spectrum for the x_p coordinate (continuous line) and the y_p coordinate (dashed line) clearly shows the annual and the Chandlerian period

Even though polar motion cannot rigorously be predicted, an extrapolation over a certain interval is nevertheless possible from previous data. For this purpose the motion of the pole may be modeled as a superposition of a linear motion (polar wander), an oscillation with a period of 365.25 days (annual term), and an oscillation with a period of 435 days (Chandler term). Appropriate coefficients that provide an extrapolation of tabulated polar motion data with an accuracy of about $0\rlap{.}''01$ over one month are published twice per week in IERS Bulletin A issued jointly by the IERS and US National Earth Orientation Service (NEOS). Similar predictions are also provided by the US National Imagery and Mapping Agency (NIMA) as part of the GPS precise ephemeris generation process (NIMA 1999).

For a prediction over longer time scales a more flexible model has been proposed by Chao (1985). The two components of polar motion are represented by time-dependent functions

$$\begin{aligned} x_\mathrm{p} &= a_x + b_x t + c_{\mathrm{a}x} \cos(2\pi t/P_{\mathrm{a}x} + \phi_{\mathrm{a}x}) + c_{\mathrm{c}x} \cos(2\pi t/P_{\mathrm{c}x} + \phi_{\mathrm{c}x}) \\ y_\mathrm{p} &= a_y + b_y t + c_{\mathrm{a}y} \sin(2\pi t/P_{\mathrm{a}y} + \phi_{\mathrm{a}y}) + c_{\mathrm{c}y} \sin(2\pi t/P_{\mathrm{c}y} + \phi_{\mathrm{c}y}) \end{aligned} \tag{5.73}$$

with a total of 16 free parameters $a_x, \ldots, \phi_{\mathrm{c}y}$ that are obtained from a least-squares fit to six years of past polar motion data. By allowing for different annual and Chandlerian periods P_a and P_c as well as different phases ϕ_a and ϕ_c in the x- and y-component of polar motion some additional degrees of freedom are introduced in this model that improve the prediction in times of notable period changes (e.g. starting in 1977). Within a one-year prediction interval an accuracy of $0\overset{\prime\prime}{.}025$ can thus be achieved.

5.4.4 Transformation to the International Reference Pole

Based on the previous discussion, the transformation from true-of-date coordinates (as defined by the theory of precession and nutation) to the International Terrestrial Reference System may be expressed as

$$\boldsymbol{r}_\mathrm{ITRF} = \boldsymbol{\Pi}(t)\,\boldsymbol{\Theta}(t)\,\boldsymbol{r}_\mathrm{tod} \quad . \tag{5.74}$$

Here $\boldsymbol{\Theta}$ (cf. (5.67)) describes the Earth's rotation about the CEP axis, while

$$\boldsymbol{\Pi} = \boldsymbol{R}_y(-x_\mathrm{p})\,\boldsymbol{R}_x(-y_\mathrm{p}) \approx \begin{pmatrix} 1 & 0 & +x_\mathrm{p} \\ 0 & 1 & -y_\mathrm{p} \\ -x_\mathrm{p} & +y_\mathrm{p} & 1 \end{pmatrix} \tag{5.75}$$

accounts for polar motion and describes the subsequent transition to the International Reference Pole and Meridian. In view of the small angles involved ($0\overset{\prime\prime}{.}3 \approx 1.5\,\mu\mathrm{rad}$), second order terms can safely be neglected in the expansion of the trigonometric functions and the linearized form of $\boldsymbol{\Pi}$ is fully adequate for all applications.

5.5 Geodetic Datums

Besides the International Terrestrial Reference System and its annually updated realizations ITRF*yy*, a variety of other global geodetic datums are in widespread use. Common to all systems is the goal of establishing a global coordinate system that originates at the Earth's center of mass and is closely aligned with the Greenwich meridian and the adopted pole.

The *World Geodetic System* 1972 (WGS72) and 1984 (WGS84) have been established by the United States Department of Defense (DoD) and the Defence Mapping Agency[9] (DMA) for use with the TRANSIT and GPS satellite navigation systems. WGS84, in its initial realization, was itself based on reference station

[9]Now: National Geospatial-Intelligence Agency (NGA); previously National Imagery and Mapping Agency (NIMA; 1996-2004)

coordinates obtained by TRANSIT Doppler measurements and achieved a global accuracy of 1–2 meters. To improve its precision, two new realizations named WGS84 (G730) and WGS84 (G873) were established (Malys & Slater 1994, Malys et al. 1997) based on accurate GPS positioning techniques. The revised systems are considered to agree with the ITRF on the decimeter and centimeter level (cf. NIMA 1997). Similar to the use of WGS84 in GPS applications, the Russian GLONASS system employs a specific datum known as PZ-90 reference frame (ICD-GLONASS 1998).

Table 5.3. Helmert transformation parameters for global geodetic datums. References: (a) McCarthy 1992, (b) McCarthy 1996, (c) Cunningham & Curtis 1996, (d) Mitrikas et al. 1998

From	To	T_1 [cm]	T_2 [cm]	T_3 [cm]	D 10^{-9}	R_1 $0''.001$	R_2 $0''.001$	R_3 $0''.001$	Ref.
ITRF90	WGS72	+6.0	−51.7	−472.3	−231	+18.3	−0.3	+547.0	(a)
ITRF90	WGS84	+6.0	−51.7	−22.3	−11	+18.3	−0.3	−7.0	(a)
ITRF90	ITRF88	+0.0	−1.2	−6.2	+6	+0.1	0.0	0.0	(a)
ITRF94	ITRF88	+1.8	0.0	−9.2	+7.4	+0.1	0.0	0.0	(b)
ITRF94	ITRF90	+1.8	1.2	−3.0	+0.9	0.0	0.0	0.0	(b)
ITRF94	ITRF92	+0.8	0.2	−0.8	−0.8	0.0	0.0	0.0	(b)
ITRF94	WGS84 (G730)	−2	+2	−1	+0.2	+2.5	+1.9	−2.5	(c)
ITRF94	WGS84 (G873)	+1	−1	−2	+0.3	+0.6	+1.2	+0.7	(c)
WGS84	PZ-90	+47	+51	+156	−22	+15.7	+3.5	−356	(d)

Except for statistical errors in the associated station coordinates, the relation between different datums may be expressed by an infinitesimal seven-parameter transformation. This is known as Helmert transformation and accounts for an offset in the adopted origin ($T_{1,2,3}$), a scale difference (D) and a misalignment of the coordinate axes ($R_{1,2,3}$). Given the coordinates $\boldsymbol{r}$ in the original system, the coordinates in another system may be expressed as

$$\boldsymbol{r}' = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + \begin{pmatrix} +D & -R_3 & +R_2 \\ +R_3 & +D & -R_1 \\ -R_2 & +R_1 & +D \end{pmatrix} \boldsymbol{r} \ . \tag{5.76}$$

Sample parameters for common transformations are provided in Table 5.3. In view of different conventions for the names and signs of the transformation parameters, care should be taken when applying the above equation with other parameter sets.

Supplementary to the Cartesian coordinates in the terrestrial reference system the location of points on or near the surface of the Earth is commonly expressed in terms of geodetic coordinates relative to a chosen reference ellipsoid. The geodetic longitude λ is identical to the geocentric longitude and measures the angle between the Greenwich meridian (or the International Reference Meridian) and the meridian through the point. By convention λ is counted positive towards the east of Greenwich. Unlike the geocentric latitude φ' that specifies the inclination of the position vector with respect to the equatorial plane, the geodetic latitude φ gives the angle between the Earth's equator and the normal to the reference ellipsoid. It thus equals the elevation of the north celestial pole above the local tangent plane.

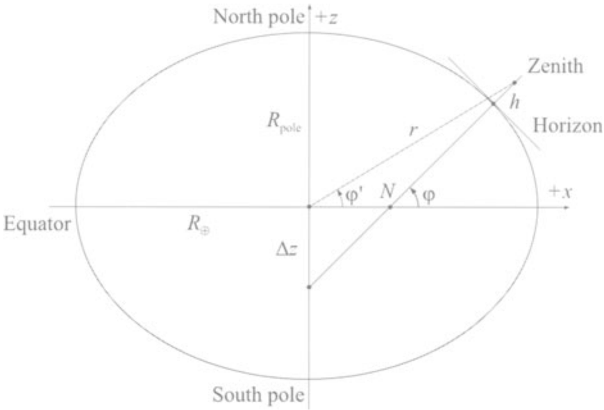

Fig. 5.12. Geocentric and geodetic latitude

The reference ellipsoid is rotationally symmetric and any plane through the symmetry axis intersects the ellipsoid in an ellipse of flattening f which is defined by the relative difference of the equatorial radius and the polar radius:

$$f = \frac{R_\oplus - R_{\text{pole}}}{R_\oplus} \quad . \tag{5.77}$$

All points on the Greenwich meridian therefore obey the relation

$$x^2 + \frac{z^2}{(1-f)^2} = R_\oplus^2 \quad , \tag{5.78}$$

which may also be written in the differenced form as

$$\frac{dz}{dx} = -(1-f)^2 \frac{x}{z} \quad . \tag{5.79}$$

On the other hand

$$\frac{dz}{dx} = -\frac{1}{\tan\varphi} \tag{5.80}$$

according to the definition of φ, and by equating both expressions one obtains

$$z = x\,(1-f)^2 \tan\varphi \quad . \tag{5.81}$$

Inserting this relation into the equation of the ellipse and solving for x finally leads to

$$\begin{aligned} x &= R_\oplus \frac{1}{\sqrt{1+(1-f)^2\tan^2\varphi}} = R_\oplus \frac{\cos\varphi}{\sqrt{1-f(2-f)\sin^2\varphi}} \\ z &= R_\oplus \frac{(1-f)^2\tan\varphi}{\sqrt{1+(1-f)^2\tan^2\varphi}} = R_\oplus \frac{(1-f)^2\sin\varphi}{\sqrt{1-f(2-f)\sin^2\varphi}} \end{aligned} \quad . \tag{5.82}$$

This relation between Cartesian and geodetic coordinates may easily be generalized for arbitrary points yielding

$$\boldsymbol{r} = \begin{pmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ ((1-f)^2N+h)\sin\varphi \end{pmatrix} \quad , \tag{5.83}$$

where

$$N = \frac{R_\oplus}{\sqrt{1-f(2-f)\sin^2\varphi}} \tag{5.84}$$

is an auxiliary quantity that is illustrated in Fig. 5.12 and where h is the height above the reference ellipsoid.

While the computation of Cartesian coordinates from given geodetic coordinates is fairly simple, the inverse transformation is slightly more involved. Besides direct methods that involve the solution of a quartic equation (Borkowski 1989, Bowring 1985) there are several iterative methods, which usually converge rapidly. The method described here utilizes the quantity

$$\Delta z = (N+h)\ \sin\varphi - z = N\,e^2\ \sin\varphi \quad , \tag{5.85}$$

where

$$e = \sqrt{1-(1-f)^2} \tag{5.86}$$

stands for the eccentricity of the reference ellipsoid. Initially Δz is set to $e^2 z$, which is a good approximation for all points that are reasonably close to the surface of the Earth. Improved values are then calculated from

$$\begin{aligned} \sin\varphi &= \frac{z+\Delta z}{\sqrt{x^2+y^2+(z+\Delta z)^2}} \\ N &= \frac{R_\oplus}{\sqrt{1-e^2\sin^2\varphi}} \\ \Delta z &= N\,e^2\ \sin\varphi \quad , \end{aligned} \tag{5.87}$$

until the iteration converges. The geodetic longitude and latitude and the height above the reference ellipsoid may then be calculated from

$$\begin{aligned} \lambda &= \arctan\left(\frac{y}{x}\right) \\ \varphi &= \arctan\left(\frac{z+\Delta z}{\sqrt{x^2+y^2}}\right) \\ h &= \sqrt{x^2+y^2+(z+\Delta z)^2} - N \quad , \end{aligned} \tag{5.88}$$

Table 5.4. Common reference ellipsoids

Datum	$R_\oplus$	$1/f$	Reference
GEM-10B	6378 138 m	298.257	McCarthy 1992
GEM-T3	6378 137 m	298.257	McCarthy 1992
WGS72	6378 135 m	298.26	McCarthy 1992
WGS84	6378 137 m	298.257223563	NIMA 1997
ITRF (GRS-80)	6378 137 m	298.257222101	McCarthy 1996, Moritz 1980
PZ-90	6378 136 m	298.257839303	ICD-GLONASS 1998

which follows immediately from Fig. 5.12. It is noted that the above relations are singular for points on the z-axis, which is likewise the case for many direct methods (see e.g. Seidelmann 1992).

Since the difference between the Earth's equatorial and polar radii is less than 22 km, the flattening $f \approx 1/298.257$ is a very small quantity and the difference between geodetic and geocentric latitudes amounts to twelve arcminutes at most. To a first approximation

$$\varphi = \varphi' + f \sin(2\varphi') \quad , \tag{5.89}$$

which shows that the difference between φ and φ' reaches its maximum for intermediate latitudes but vanishes at the poles and the equator. Numerical values of the inverse flattening for various datums and reference ellipsoid are presented in Table 5.4.

Exercises

Exercise 5.1 (ICRS to ITRS Transformation) Compute the transformation from the International Celestial Reference System (or the mean equator and equinox of J2000) to the International Terrestrial Reference System (or the reference pole and Greenwich meridian) for the epoch 1999 March 4, 0^{h} UTC.

Hint: Obtain Terrestrial Time (TT) and Universal Time 1 (UT1) as well as pole coordinates at the time of interest from the respective IERS bulletins (cf. Figs. 5.4 and 5.5). Employ the IAU 1976 precession theory and the IAU 1980 nutation theory to compute the instantaneous orientation of the Celestial Ephemeris Pole (neglecting any corrections to the nutation angles). In computing the Earth rotation transformation account for the conventional relation between UT1 and GMST as well as the first-order term of the equation of the equinoxes.

Solution: The IERS Bulletins B (No. 135) and C (No. 16) provide the following Earth orientation parameters and derived quantities:

$$\begin{array}{lcl} \mathrm{UTC}-\mathrm{TAI} & = & -32\overset{\mathrm{s}}{.}0 \\ \mathrm{TT}-\mathrm{UTC} & = & +64\overset{\mathrm{s}}{.}184 \\ \mathrm{UT1}-\mathrm{UTC} & = & +0\overset{\mathrm{s}}{.}649232 \\ x_{\mathrm{p}} & = & +0\overset{\prime\prime}{.}06740 \\ y_{\mathrm{p}} & = & +0\overset{\prime\prime}{.}24173 \end{array}$$

Using the above assumptions, the following step-by-step transformation matrices for precession ($\boldsymbol{P}$), nutation ($\boldsymbol{N}$), Earth rotation ($\boldsymbol{\Theta}$), and polar motion ($\boldsymbol{\Pi}$) are obtained:

$$\boldsymbol{P} = \begin{pmatrix} +0.99999998 & +0.00018581 & +0.00008074 \\ -0.00018581 & +0.99999998 & -0.00000001 \\ -0.00008074 & -0.00000001 & +1.00000000 \end{pmatrix}$$

$$\boldsymbol{N} = \begin{pmatrix} +1.00000000 & +0.00004484 & +0.00001944 \\ -0.00004484 & +1.00000000 & +0.00003207 \\ -0.00001944 & -0.00003207 & +1.00000000 \end{pmatrix}$$

$$\boldsymbol{\Theta} = \begin{pmatrix} -0.94730417 & +0.32033547 & +0.00000000 \\ -0.32033547 & -0.94730417 & +0.00000000 \\ +0.00000000 & +0.00000000 & +1.00000000 \end{pmatrix}$$

$$\boldsymbol{\Pi} = \begin{pmatrix} +1.00000000 & +0.00000000 & +0.00000033 \\ +0.00000000 & +1.00000000 & -0.00000117 \\ -0.00000033 & +0.00000117 & +1.00000000 \end{pmatrix}$$

Multiplication then yields the matrix

$$\boldsymbol{U}^{\mathrm{ICRS}}_{\mathrm{ITRS}} = \boldsymbol{\Pi}\,\boldsymbol{\Theta}\,\boldsymbol{N}\,\boldsymbol{P} = \begin{pmatrix} -0.94737803 & +0.32011696 & -0.00008431 \\ -0.32011696 & -0.94737803 & -0.00006363 \\ -0.00010024 & -0.00003330 & +0.99999999 \end{pmatrix}$$

that describes the full ICRS to ITRS transformation.

Exercise 5.2 (Velocity in the Earth-fixed Frame) The GPS precise ephemerides of the National Imagery and Mapping Agency (NIMA) provide the state vectors of the GPS satellites in an Earth-fixed reference system (presently WGS84 (G873)). This frame is considered as *rotating*, which implies that the rotation of the axes must be considered in the transformation of the velocity vector, i.e.

$$\begin{aligned} \boldsymbol{r}_{\mathrm{WGS}} &= \boldsymbol{U}_{\mathrm{WGS}}^{\mathrm{ICRS}}(t)\,\boldsymbol{r}_{\mathrm{ICRS}} \\ \boldsymbol{v}_{\mathrm{WGS}} &= \boldsymbol{U}_{\mathrm{WGS}}^{\mathrm{ICRS}}(t)\,\boldsymbol{v}_{\mathrm{ICRS}} + \frac{d\boldsymbol{U}_{\mathrm{WGS}}^{\mathrm{ICRS}}(t)}{dt}\,\boldsymbol{r}_{\mathrm{ICRS}} \quad . \end{aligned} \tag{5.90}$$

Given the state vector

$$\begin{aligned} \boldsymbol{r}_{\mathrm{WGS}} &= (\ \ 19440.953805,\ \ 16881.609273,\ -6777.115092)\ \mathrm{km} \\ \boldsymbol{v}_{\mathrm{WGS}} &= (-0.8111827456, -0.2573799137, -3.0689508125)\ \mathrm{km/s} \end{aligned}$$

of satellite PRN 15 at epoch 1999 March 4, 0^{h} GPS time, compute the position and velocity vector in the International Celestial Reference System (mean equator and equinox of J2000). Check your result by showing that the corresponding orbital elements describe a near-circular orbit with a twelve-hour period ($a \approx 26560\,\mathrm{km}$) and an inclination of about 56°.

Hint: The WGS84 (G873) frame is identical to the International Terrestrial Reference Frame within an accuracy of a few centimeters. In computing the derivative of the ICRS to ITRS transformation, the precession, nutation and polar motion matrix may be considered as constant, i.e.

$$\frac{d\boldsymbol{U}_{\mathrm{ITRS}}^{\mathrm{ICRS}}(t)}{dt} \approx \boldsymbol{\Pi}\,\frac{d\boldsymbol{\Theta}}{dt}\,\boldsymbol{N}\,\boldsymbol{P} \quad . \tag{5.91}$$

Furthermore, the time derivative of the Earth rotation matrix is given by

$$\frac{d\boldsymbol{\Theta}(t)}{dt} = \omega_{\oplus} \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\Theta}(t) \tag{5.92}$$

where

$$\omega_{\oplus} = \frac{d(\mathrm{GAST})}{dt} \approx 1.002737909350795\frac{2\pi}{86400\,\mathrm{s}} = 7.2921158553{\cdot}10^{-5}\,\mathrm{s}^{-1}$$

(cf. (5.19)) is the Earth's angular velocity. IERS Earth orientation parameters for the date of interest are provided in the previous exercise.

Solution: The GPS–UTC time difference amounts to 13 s, which results in the ICRS-ITRS transformation matrix

$$\boldsymbol{U} = \begin{pmatrix} -0.94707414 & +0.32101491 & -0.00008425 \\ -0.32101491 & -0.94707414 & -0.00006371 \\ -0.00010024 & -0.00003330 & +0.99999999 \end{pmatrix}$$

and in its derivative

$$\dot{U} = \begin{pmatrix} -0.23408779 & -0.69061744 & -0.00004561 \\ +0.69061743 & -0.23408779 & +0.00006167 \\ +0.00000089 & -0.00000005 & +0.00000000 \end{pmatrix} \cdot 10^{-4}/\mathrm{s} \quad .$$

Then

$$\begin{aligned} \boldsymbol{r}_{\mathrm{ICRS}} &= \boldsymbol{U}^T \boldsymbol{r}_{\mathrm{WGS}} \\ &= (-23830.593,\ -9747.074,\ -6779.829)\ \mathrm{km} \\ \boldsymbol{v}_{\mathrm{ICRS}} &= \boldsymbol{U}^T \boldsymbol{v}_{\mathrm{WGS}} + \dot{\boldsymbol{U}}^T \boldsymbol{r}_{\mathrm{WGS}} \\ & \quad (+1.561964,\ -1.754346,\ -3.068851)\ \mathrm{km/s} \end{aligned}$$

is the state vector in the inertial celestial reference system. The associated osculating orbital elements of the GPS satellite are obtained as

Semimajor axis	a	26561.013 km
Eccentricity	e	0.0070606
Inclination	i	56.338°
RA ascend. node	Ω	12.146°
Arg. of perigee	ω	87.617°
Mean anomaly	M	109.435° ,

which matches the specified orbital characteristics of the GPS space segment.

Exercise 5.3 (Geodetic coordinates) The Cartesian coordinates of the NIMA GPS receiver at Diego Garcia are given by

$$\boldsymbol{r}_{\mathrm{WGS84(G873)}} = (+1917032.190,\ +6029782.349,\ -801376.113)\ \mathrm{m}$$

at epoch 1997.0 (Cunningham & Curtis 1996). Compute the corresponding geodetic coordinates using the WGS84 reference ellipsoid.

Solution:

$$\begin{array}{lcll} \text{East longitude} & \lambda & = & +72.36312094^\circ \\ \text{Latitude} & \varphi & = & -7.26654999^\circ \\ \text{Height} & h & = & -63.667\ \mathrm{m} \quad . \end{array}$$