ASE 389P.4 Methods of Orbit Determination
Homework 1: Basic Orbit Propagation

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This assignment is designed to provide a basic introduction on how to propagate an orbit. MATLAB was used to complete the assignment.

I. Introduction

In this assignment, a tool was created to numerically propagate a circular orbit about the Earth and to convert between Cartesian and Keplerian orbital elements. The software implemented in this assignment will be used later in the course provides additional background and/or a review of basic orbital mechanics.

II. Problem 1

A. Statement

Given the Earth orbiting spacecraft position and velocity vectors in Cartesian coordinates

$$\underline{R} = -2436.45\hat{\imath} - 2436.45\hat{\jmath} + 6891.037\hat{k}\ km \tag{1}$$

$$\underline{V} = \dot{\underline{R}} = 5.088611\hat{\imath} - 5.088611\hat{\jmath} + 0.0\hat{k}\ km/s \tag{2}$$

solve for the Keplerian elements (a, e, i, Ω, ω, ν). Provide your values in the write-up. See the lecture notes

Assume $\mu = 398600.5$ km^3/s^2.

B. Solution

The algorithms to convert orbital elements to Cartesian and back were taken from Reference [1]. Given the spacecraft position and velocity vectors from the problem statement, the Keplerian elements are:

$$\begin{aligned} a &= 7.712184983762814e+03 \\ e &= 0.447229247404423 \\ i &= 1.570796326794897 \\ \omega &= 3.139593866862924 \\ \Omega &= 3.926990816987241 \\ \nu &= 2.032461649676350 \end{aligned} \tag{1}$$

where a is the semi-major axis, e is the eccentricity, i is the orbit inclination, ω is the argument of perigee, Ω is the right ascension of the ascending node, and ν is the true anomaly.

III. Problem 2

A. Statement

Convert the Keplerian elements from Problem 1 back to position and velocity and provide the values in the write-up. See the lecture notes:

B. Solution

IV. Problem 3

A. Statement

Given the gravity potential function $U = \mu/R$, solve for the two-body acceleration due to gravity, i.e.,

$$\nabla U = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k} \tag{3}$$

where $R = \underline{R} \bullet \underline{R}$. Include your derivation in your solution to the assignment.

B. Solution 1

To solve for the two-body acceleration due to gravity, first calculate R, which is given in the problem statement as $R = \underline{R} \cdot \underline{R}$. Let us first define $\underline{\text{R}}$ as the following:

$$\underline{R} = x\hat{\imath} = y\hat{\jmath} + z\hat{k} \tag{2}$$

The dot product of the vector $\underline{\text{R}}$ with itself is equal to the square of its magnitude:

$$R = \underline{R} \cdot \underline{R} = x^2 + y^2 + z^2 \tag{3}$$

Given the gravity potential function $U = \mu/R$, the gradient of the gravity potential function ∇U is:

$$\nabla U = \frac{\delta U}{\delta x}\hat{\imath} + \frac{\delta U}{\delta y}\hat{\jmath} + \frac{\delta U}{\delta z}\hat{k} \tag{4}$$

First derive $\frac{\delta U}{\delta x}\hat{\imath}$:

$$\frac{\delta U}{\delta x}\hat{\imath} = \frac{\delta\left(\frac{\mu}{R}\right)}{\delta x}\hat{\imath} = \frac{\delta\left(\frac{\mu}{x^2 + y^2 + z^2}\right)}{\delta x}\hat{\imath} \tag{5}$$

Take the partial derivative:

$$\frac{\delta\left(\frac{\mu}{x^2 + y^2 + z^2}\right)}{\delta x}\hat{\imath} = \frac{\delta\left(\mu(x^2 + y^2 + z^2)^{-1}\right)}{\delta x}\hat{\imath} = -\mu(x^2 + y^2 + z^2)^{-2}(2x)\hat{\imath} \tag{6}$$

Now simplify:

$$-\mu(x^2 + y^2 + z^2)^{-2}(2x)\hat{\imath} = -\frac{2\mu x}{(x^2 + y^2 + z^2)^2}\hat{\imath} \tag{7}$$

Thus:

$$\frac{\delta U}{\delta x}\hat{\imath} = -\frac{2\mu x}{(x^2 + y^2 + z^2)^2}\hat{\imath} \tag{8}$$

$\frac{\delta U}{\delta y}\hat{\jmath}$ and $\frac{\delta U}{\delta z}\hat{k}$ can be derived through the same process, which result in the following:

$$\frac{\delta U}{\delta y}\hat{\jmath} = -\frac{2\mu y}{(x^2 + y^2 + z^2)^2}\hat{\jmath} \tag{9}$$

$$\frac{\delta U}{\delta z}\hat{k} = -\frac{2\mu z}{(x^2 + y^2 + z^2)^2}\hat{k} \tag{10}$$

The gradient of the gravity potential function is thus:

$$\nabla U = -\frac{2\mu x}{(x^2+y^2+z^2)^2}\hat{i} - \frac{2\mu y}{(x^2+y^2+z^2)^2}\hat{j} - \frac{2\mu z}{(x^2+y^2+z^2)^2}\hat{k} \tag{11}$$

Plug in $x = -2436.45$, $y = -2436.45$, $z = 6891.037$, and $\mu = 398600.5$ from Problem 1 into Equation 11 to solve for the two-body acceleration due to gravity:

$$\nabla U = 5.512551407304731e-07\hat{i} - 5.512551407304731e-07\hat{j} - -1.559120676071291e-06\hat{k} \tag{12}$$

C. Solution 2

The gravitational potential function is commonly simplified as $\nabla U = \mu/r$, where r is the distance between two bodies [1]. Therefore:

$$r = \sqrt{R} = \sqrt{\underline{R}\cdot\underline{R}} = (\underline{R}\cdot\underline{R})^{1/2} = (x^2+y^2+z^2)^{1/2} \tag{13}$$

If we use Equation 13 to calculate the gradient for the potential function, then $\frac{\delta U}{\delta x}\hat{i}$ becomes:

$$\frac{\delta U}{\delta x}\hat{i} = \frac{\delta\left(\frac{\mu}{r}\right)}{\delta x}\hat{i} = \frac{\delta\left(\frac{\mu}{(x^2+y^2+z^2)^{1/2}}\right)}{\delta x}\hat{i} = \frac{\delta\left(\mu(x^2+y^2+z^2)^{-1/2}\right)}{\delta x} \tag{14}$$

Take the derivative and simplify:

$$\frac{\delta\left(\mu(x^2+y^2+z^2)^{-1/2}\right)}{\delta x} = \mu\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x)\hat{i} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}\hat{i} \tag{15}$$

Thus:

$$\frac{\delta U}{\delta x}\hat{i} = -\frac{\mu x}{(x^2+y^2+z^2)^{3/2}}\hat{i} = -\frac{\mu x}{r^3}\hat{i} \tag{16}$$

And:

$$\frac{\delta U}{\delta y}\hat{j} = -\frac{\mu y}{r^3}\hat{j} \tag{17}$$

$$\frac{\delta U}{\delta z}\hat{k} = -\frac{\mu z}{r^3}\hat{k} \tag{18}$$

Equation 11 then becomes:

$$\nabla U = -\frac{\mu x}{r^3}\hat{i} - \frac{\mu y}{r^3}\hat{j} - \frac{\mu z}{r^3}\hat{k} \tag{19}$$

Plug in $x = -2436.45$, $y = -2436.45$, $z = 6891.037$, and $\mu = 398600.5$ from Problem 1 into Equation 19 to solve for the two-body acceleration due to gravity:

$$\nabla U = 0.002123566317530\hat{i} + 0.002123566317530\hat{j} - 0.006006104810708\hat{k} \tag{20}$$

V. Problem 4

A. Statement

Develop the necessary code to numerically integrate the equations of motion using the position and velocity from Problem 1 as the initial conditions. Compute the future position and velocity at 20-second intervals for two full orbits. Plot the magnitude of the position, velocity, and acceleration as a function of time for two full orbits and provide the figure. Compute the specific orbital angular momentum vector for these two full orbits and plot that as well, as a function of time, as a 3D scatter plot ($\underline{h} = \underline{R}\ X\ \underline{V}$). Assume that the motion is only due to the accelerations derived from Eq(3)

B. Solution

The period for the orbit was found by using the semi-major axis in the following equation:

$$T = \left| 2\pi \sqrt{a^3/\mu} \right| \tag{21}$$

The orbit was propagated by numerically integrating the equations of motion 19 for 2 periods using `ode45`. The relative and absolute tolerances were set to 1e-8. The acceleration was calculated by differentiating velocity with respect to time, and then the magnitudes of the position, velocity, and acceleration were plotted in Figure 1.

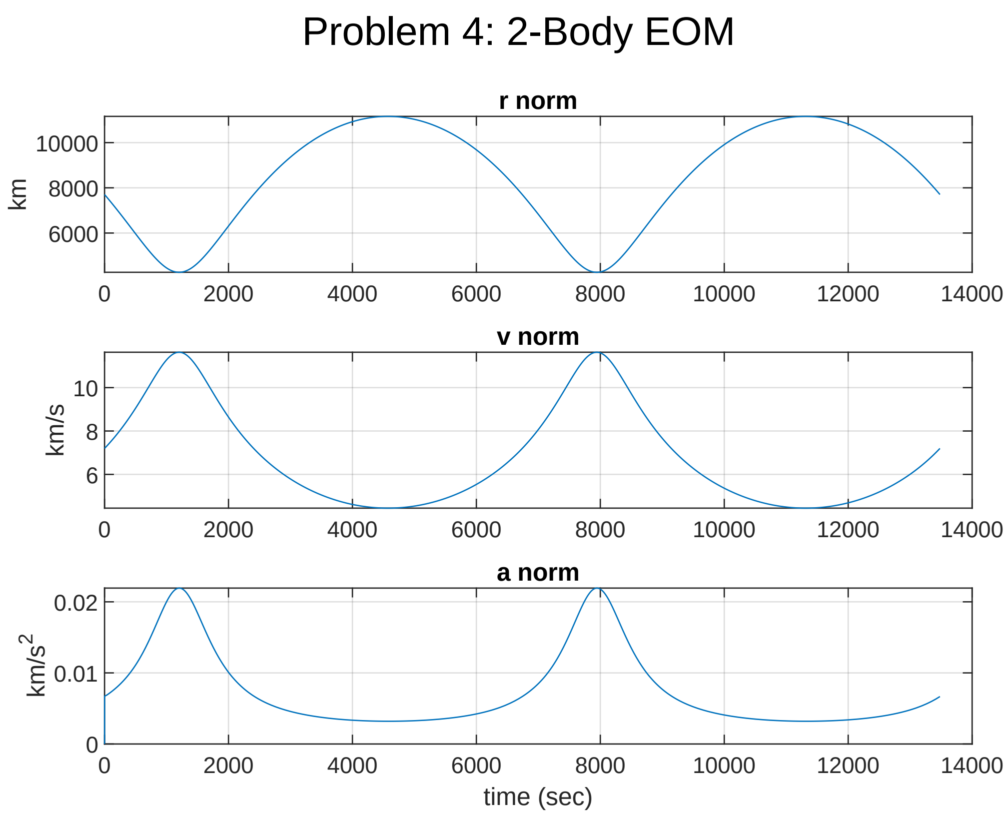

Fig. 1

The magnitudes of the velocity and acceleration vary throughout the orbit, which indicates that the orbit is not circular. The position and velocity vectors were converted into orbital elements which are shown in 2. All of the orbital elements with the exception of true anomaly remain essentially constant, revealing the predictable and Keplerian nature of the orbit. The eccentricity shows that the orbit is elliptical; an eccentricity of 0 forms a perfectly circular orbit and a 1 forms a parabolic escape orbit.

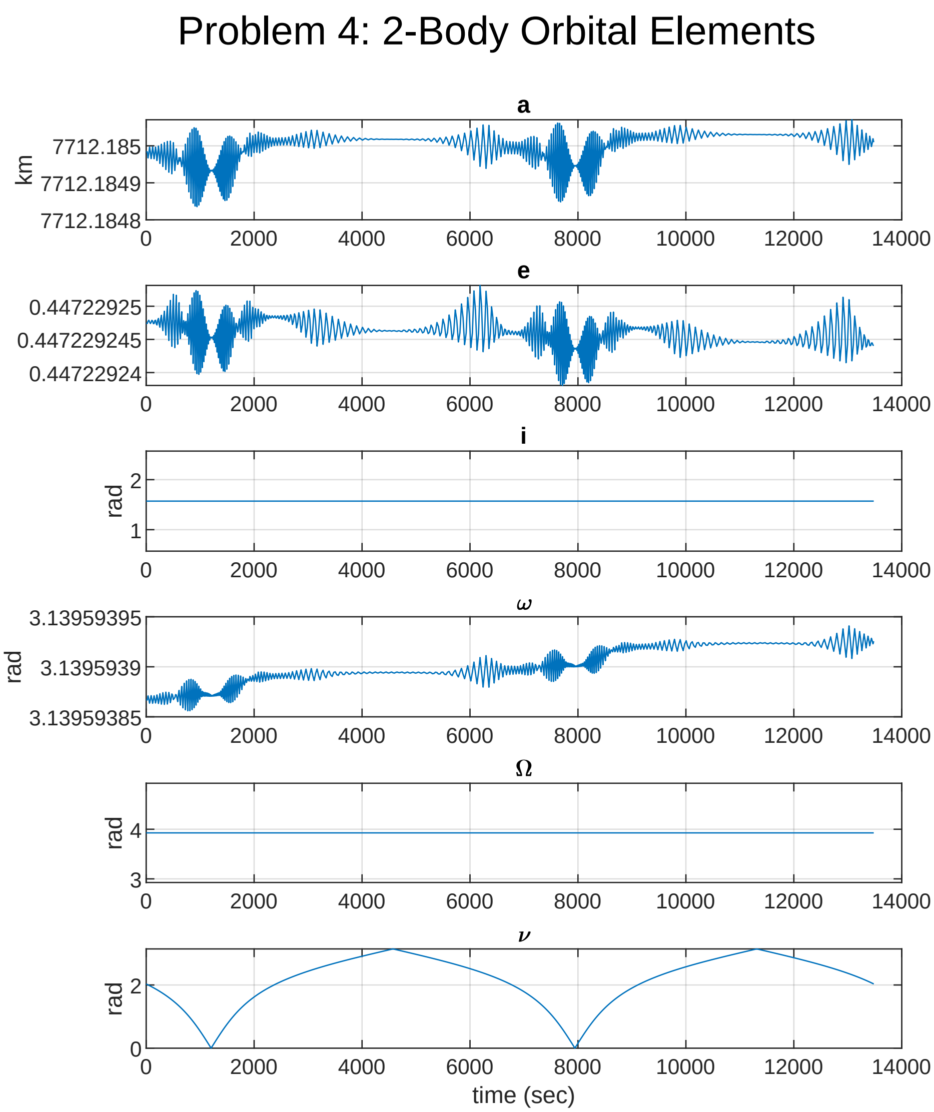

Fig. 2

Figure 3 illustrates that the specific angular momentum essentially remains constant throughout the entire orbit. The difference between the maximum and minimum of **h norm** is 7.204597714007832e-04, which is incredibly small especially when considering that the order of magnitude for all values of **h norm** is 4.

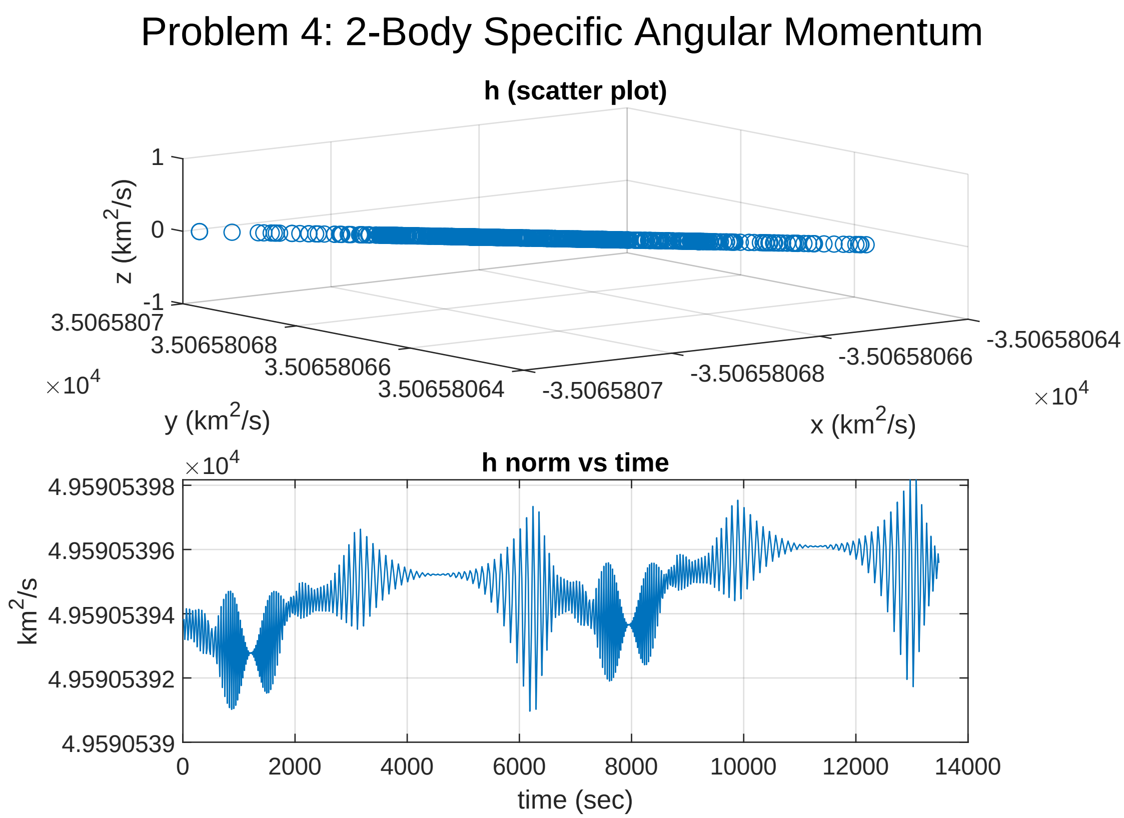

Fig. 3

VI. Problem 5

A. Statement

Compute the specific kinetic energy and specific potential energy as a function of time and plot the change in total specific energy to show that it remains constant over the two orbits. (i.e. plot $dE = E(t) - E(t_0)$). Include the image in your write-up. Why is the change in total specific energy not constant?

B. Solution

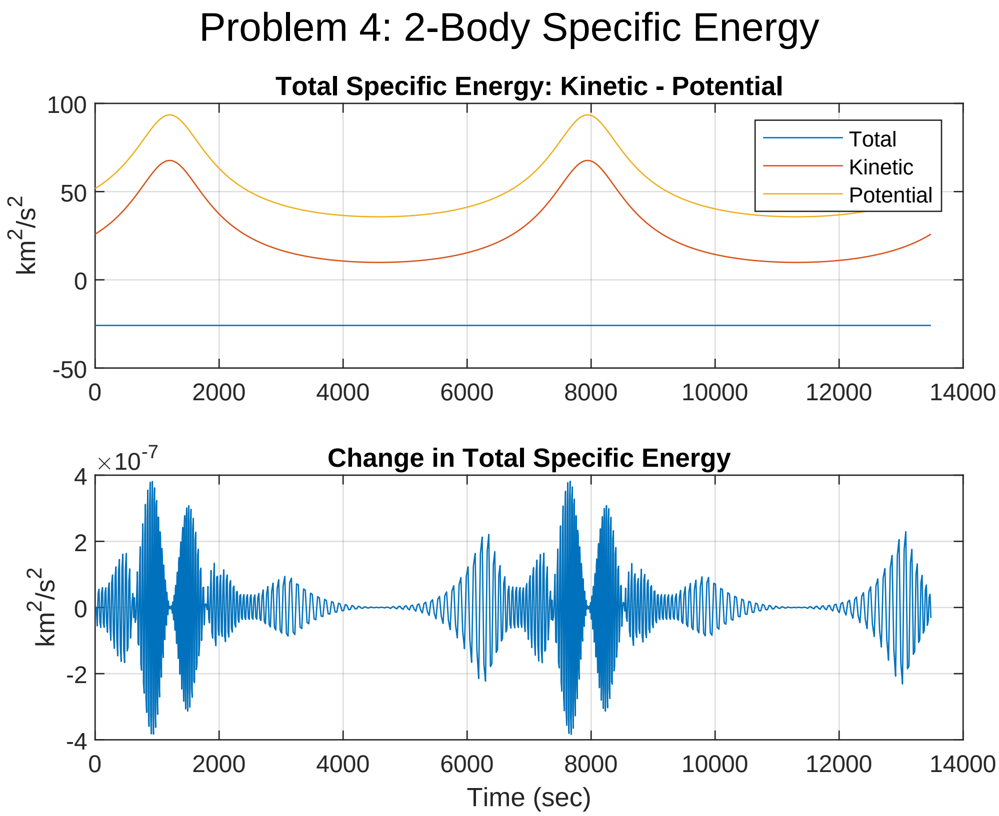

Fig. 4

The energy constant of motion is given in Reference [1]:

$$E = \frac{v^2}{2} - \frac{\mu}{r} \tag{22}$$

in which the first term is the "kinetic energy per unit mass," or specific kinetic energy, and the last term is the specific potential energy. The specific mechanical energy, E, is the sum of the specific kinetic and potential energy and remains constant along its orbit.

VII. Conclusion

VIII. Appendix

HW1 MATLAB code:

```matlab
% ASE 389 Orbit Determination
% HW 0
% Junette Hsin


%% Problem 1

t    = linspace( 0, 20, 101 );
t    = 0 : 0.2 : 400;
A    = 1.34;
phi  = pi/3;
km   = 1;

x    = A * cos( sqrt( km )*t + phi );

name = 'Problem 1: harmonic oscillator x(t)';
figure('name', name);
plot(t, x);
ylabel('x(t)');
xlabel('t');
title(name);
set(gca, 'fontsize', 12);


%% Problem 2

y0 = zeros(2,1);
y0(1) = A * cos(phi);
y0(2) = - A * sqrt( km ) * sin(phi);

reltol = 1e-12;
abstol = 1e-20;
myoptions   = odeset( 'RelTol', reltol, 'AbsTol', abstol);
[t, y]      = ode45( @harmoscillator, t, y0, myoptions, km);

% numerical - analytical
error = y(:,1)' - x;

name = 'Problem 2: x(t) error';
figure('name', name);
plot(t, error);
xlabel('t')
ylabel('numerical - analytical')
title({name ;
sprintf('RelTol = %.2e, AbsTol = %.2e ', reltol, abstol) });
set(gca, 'fontsize', 12);


%% subfunctions

function dx = harmoscillator( t, x, km )

dx     = zeros(2, 1);
dx(1) = x(2);
dx(2) = -km * x(1);

end
```

References

[1] Donald D. Mueller, J. W., and Bate, R. R., *Fundamentals of Astrodynamics*, Dover Publications, Inc., 1971.