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Revisit of the Three-Dimensional Orbital Pursuit-Evasion Game

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Nomenclature

a = semimajor axis
c = effective exhaust velocity
e = eccentricity
g = gravity acceleration
H = Hamiltonian
i = inclination
K = downhill factor
l = phase angle
m = mass
$\boldsymbol{p}$ = vector of problems unknowns
r = position
S_F = switching function
T = thrust
t = time
v = velocity
α, β = thrust angles
Λ = magnitude of velocity adjoint vector
$\boldsymbol{\lambda}_r$ = position adjoint vector
$\boldsymbol{\lambda}_v$ = velocity adjoint vector
λ_m = mass adjoint variable
θ = argument of latitude
ν = true anomaly
Ω = right ascension of ascending node
ω = argument of perigee
$\boldsymbol{\Psi}$ = boundary conditions vector

Subscripts

E = evader
f = final
P = pursuer
u = radial
v = eastward
w = northward
0 = initial

I. Introduction

THE combat of an evasive spacecraft against a pursuing spacecraft is most accurately modeled using two competitive players. The problem becomes a two-sided optimization problem or a zero-sum two-player differential game. This kind of problem was first introduced by Isaacs [1,2]. Useful information can be drawn from this problem, as the solution from the game theory for the optimal strategy of one player can provide the worst possible scenario faced by the other player. Past works have mainly concerned air combat, in which two fighter aircraft in battle maneuver optimally; pursuit-evasion problems between a missile and an aircraft has been solved using an indirect, multiple shooting method by Breitner et al. [3,4]. Only a limited number of studies have concerned orbital pursuit-evasion problems, in particular with realistic dynamics. Pontani and Conway [5] applied the genetic algorithm and a semidirect collocation nonlinear programming method to a three-dimensional orbital pursuit-evasion problem. However, the analytical conditions holding for the control are employed only for a single player in the semi-DCNLP method, which may make one player not maneuver optimally. Also, only the constant acceleration case was considered in their work, neglecting the propellant consumption. The same problem was recently analyzed by Hafer et al. [6] with an approach based on the theory of optimal control solved by means of a homotopic strategy. Although the sensitive method has a computational speed advantage over common direct methods, considerable computation effort is still required, as the method has to run a large number of trajectories. More important, the saddle-point equilibrium for orbital pursuit-evasion may sometimes not be located correctly, when existing methods are used.

In this Note, the three-dimensional orbital pursuit-evasion problem is revisited, through treating the two-sided optimization problem as an extension of an optimal control problem; the optimization of realistic combat, in which both spacecraft maneuver optimally, is solved numerically using an indirect optimization method. The initial guess for the two-sided optimization is obtained easily from a one-sided optimization problem. The hypothesis of constant thrust-to-mass ratio is here removed, as the adopted solution method can deal with variable mass vehicles without difficulties, and seven nonlinear differential equations define the state (position, velocity, and mass) of each spacecraft. As an opposite situation with respect to the pursuit-evasion game, pursuer's interception of a cooperative target is also considered and the results are compared. The extension from pursuer-evader problem to pursuer-cooperative target problem is easy to solve for the indirect procedure proposed here. Results show the effectiveness of the proposed method in finding the correct saddle-point optimal trajectories, which are instead sometimes missed by other commonly employed methods, and detailed analysis is presented to illustrate this point; as a consequence, different and new characteristics of the pursuit-evasion game with respect to existing literature are highlighted.

This Note offers three main contributions: 1) It defines a fast and efficient procedure to apply an indirect optimization method to the three-dimensional orbit pursuit-evasion problem and find local optima, which satisfy analytical necessary conditions for optimality (in this respect, it also solves cases that other methods cannot treat effectively, and new solutions are presented). 2) The proposed method has very light computational burden, as typically below 200 iterations are required to converge; in comparison, the genetic algorithm runs 50,000 trajectories in order to obtain an initial solution, which is then sent to the collocation postprocessor [5],

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whereas the sensitivity method needs to run about 200 trajectories for a single case [6]. 3) Handling of minimum height constraints and mass variation is introduced in order to make the saddle-point solution more practical and accurate.

The Note is organized as follows. Section II presents the orbital pursuit-evasion problem and describes the formulation of dynamic models and constraint conditions. Section III applies the theory of optimal control to the two-sided optimization problem. Several test cases are treated in Sec. IV, where it is shown that the method can be adopted to efficiently locate different saddle-point equilibrium solutions, and it is highlighted that existing methods sometimes fail to find the correct ones. Conclusions are presented in Sec. V.

II. Statement of the Problem

Two-body problem equations are assumed, neglecting orbital perturbations, to allow for comparison with results in existing literature, where the same dynamical model was used. The indirect technique, adopted in the present paper, is, however, capable of dealing with perturbations such as Earth oblateness, lunisolar gravity, and solar radiation pressure [7,8].

For each spacecraft, position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ vectors, in an inertial frame centered at the Earth, and mass m constitute the set of the state variables. The variables are normalized using Earth radius r_{Earth}, the corresponding circular velocity, and the spacecraft masses at departure time as reference values. However, results are presented in dimensional form for a more direct interpretation. For each player, denoted with $i = P$ (pursuer) or E (evader), the nondimensional equations of motion are written as follows:

$$\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i, \qquad \dot{\boldsymbol{v}}_i = \boldsymbol{g}_i + \frac{\boldsymbol{T}_i}{m_i}, \qquad \dot{m}_i = -\frac{T_i}{c_i} \tag{1}$$

A constant thrust-to-mass ratio is equivalent to consider values of c that grow to infinity ($c = \infty$), and so it can be seen as a special case of the general model treated in this Note. The control is performed with thrust magnitude T_i and direction; the latter is described by the two angles, α_i and β_i, that define its components in the Zenith-East-North reference frame:

$$T_{ui} = T_i \cos\alpha_i \cos\beta_i, \quad T_{vi} = T_i \sin\alpha_i \cos\beta_i, \quad T_{wi} = T_i \sin\beta_i \tag{2}$$

The problem is formulated as a zero-sum differential game, in which the pursuer P tries to catch the evader E. The time for interception, t_f, which is assumed as the termination of the game, is to be minimized by P and maximized by E. A maximization problem is here preferred for the pursuer and the objective function is defined as

$$\varphi = -t_f \tag{3}$$

The initial conditions are specified and are related to the orbital elements of the two spacecraft at $t_0 = 0$. The game ends when the pursuer reaches the instantaneous position of the evader, that is, when

$$\boldsymbol{r}_P(t_f) = \boldsymbol{r}_E(t_f) \tag{4}$$

The motion of the two spacecraft is supposed to take place at an altitude greater than 100 km so that atmospheric forces can be neglected. An unconstrained solution is first sought but, when the minimum altitude becomes smaller than the limit value, it is fixed to 100 km by imposing the additional constraints

$$\|\boldsymbol{r}\| = r_{\mathrm{Earth}} + h_{\min} \tag{5}$$

and

$$\boldsymbol{r}^T \boldsymbol{v} = 0 \tag{6}$$

which means that the radial velocity is null at the lowest-altitude point.

III. Two-Sided Optimization via Indirect Method

The theory of optimal control [9] is used to determine the solution that maximizes the performance index, defined by Eq. (3), for P, while it also minimizes the same performance index for E. An adjoint variable is associated with each differential equation to define the Hamiltonian function for each spacecraft as

$$H_i = \boldsymbol{\lambda}_{ri}^T \boldsymbol{v}_i + \boldsymbol{\lambda}_{vi}^T (\boldsymbol{g}_i + \boldsymbol{T}_i / m_i) - \lambda_{mi} T_i / c_i \tag{7}$$

In zero-sum differential games, a pair of optimal strategies corresponds to a saddle-point trajectory in the state space. The sum of the Hamiltonian function of the pursuer and evader, that is,

$$H = H_E + H_P \tag{8}$$

must be maximized by P and minimized by E, in agreement with Pontryagin Maximum Principle (PMP). As the Hamiltonian of the complete system (pursuer plus evader) is separable, that is, the differential equations for each spacecraft do not formally depend on the state and control variables of the other one, it is clear that minimization and maximization commute, that is, $\max_P \min_E J = \min_E \max_P J$. Thus, which side plays first does not change the saddle-point equilibrium. Therefore, one has

$$H = \min_E \max_P H = \min_E H_E + \max_P H_P \tag{9}$$

According to PMP, the optimal controls maximize (pursuer) or minimize (evader) H_i during the whole trajectory; the thrust must therefore be parallel to the velocity adjoint vector $\boldsymbol{\lambda}_v$, or primer vector. The optimal thrust direction is concurrent with the primer vector $\boldsymbol{\lambda}_{vi}$ for the pursuer, whereas it is opposite to the primer vector direction for the evader. Thus, the components of the optimal thrust are

$$T_{uP} = T_P \lambda_{uP} / \Lambda_P, \quad T_{vP} = T_P \lambda_{vP} / \Lambda_P, \quad T_{wP} = T_P \lambda_{wP} / \Lambda_P \tag{10}$$

$$T_{uE} = -T_E \lambda_{uE} / \Lambda_E, \quad T_{vE} = -T_E \lambda_{vE} / \Lambda_E, \quad T_{wE} = -T_E \lambda_{wE} / \Lambda_E \tag{11}$$

where Λ_P and Λ_E are the magnitudes of the velocity adjoint vectors for the pursuer and evader, respectively.

The Hamiltonian can be rewritten as

$$H_i = \boldsymbol{\lambda}_{ri}^T \boldsymbol{v}_i + \boldsymbol{\lambda}_{vi}^T \boldsymbol{g}_i + T_i S_{Fi} \tag{12}$$

where the switching functions $S_{FP} = \lambda_{vP}/m_P - \lambda_{mP}/c_P$ and $S_{FE} = -\lambda_{vE}/m_E - \lambda_{mE}/c_E$ are introduced. It is assumed that there are no singular arcs, as they are rare for practical space trajectories and usually may appear only during atmospheric flight; this means that the switching function is zero at isolated points and a bang-bang control is optimal, as far as the thrust magnitude is concerned. In the light of PMP, the optimal thrust of the pursuer is maximum when $S_{FP} > 0$, whereas the thrust must be set to zero when the switching function is negative. On the other hand, the optimal thrust of the evader is maximum when $S_{FE} < 0$, and zero when the switching function is positive.

The theory of optimal control provides the Euler–Lagrange equations for the adjoint variables

$$\dot{\boldsymbol{\lambda}}_{ri} = -\left[\frac{\partial \boldsymbol{g}_i}{\partial \boldsymbol{r}_i}\right]^T \boldsymbol{\lambda}_{vi} \qquad \dot{\boldsymbol{\lambda}}_{vi} = -\boldsymbol{\lambda}_{ri} \qquad \dot{\lambda}_{mi} = \kappa \frac{T_i \Lambda_i}{m_i^2} \tag{13}$$

where $\kappa = 1$ if $i = P$; $\kappa = -1$ if $i = E$.

With the boundary condition described by Eq. (4) and φ from Eq. (3), and by using PMP, after some algebraic manipulation to eliminate the constant Lagrange multipliers, one obtains the boundary conditions for optimality. At the final point, one deduces

$$\boldsymbol{\lambda}_{rP}(t_f) + \boldsymbol{\lambda}_{rE}(t_f) = 0 \tag{14}$$

In the absence of velocity and mass constraints at the final point one has

$$\lambda_{mi}(t_f) = 0 \tag{15}$$

$$\boldsymbol{\lambda}_{vi}(t_f) = 0 \tag{16}$$

The transversality condition on the final Hamiltonian value becomes

$$H_f = \boldsymbol{\lambda}_{rP_f}^T(\boldsymbol{v}_{Pf} - \boldsymbol{v}_{Ef}) = 1 \tag{17}$$

As discussed in Sec. II, the minimum distance form Earth's surface must be greater than 100 km. The unconstrained solution is sought first. In some of the cases treated in this Note, the pursuer altitude becomes lower than the limit value; in these cases, the trajectory is split into two phases at the point where the minimum altitude is reached, by adding the boundary conditions described by Eqs. (5) and (6) for the pursuer: The conditions for optimality for the constrained solution introduce a discontinuity for the radius adjoint variable, whereas the necessary Hamiltonian continuity implies continuity of the velocity adjoints [10].

The time derivative of the mass adjoint variable is always positive for the pursuer and negative for the evader. Because the final value is null, λ_{mP} is therefore always negative, whereas λ_{mE} is always positive. This fact, in turn, causes the switching function of the pursuer and evader to be always greater than zero or lower than zero, respectively. The thrusters are therefore always on for both spacecraft.

The application of optimal control theory on the pursuit-evasion problem described above transforms the nonlinear optimization problem into a multipoint boundary value problem (BVP), where the initial values of some variables (e.g., the adjoint variables) and problem parameters (e.g., the final time) are unknown and must be determined to satisfy the problem boundary conditions, written in the form $\boldsymbol{\Psi} = 0$. A classic shooting procedure [11] based on Newton's method is used for the solution of the BVP formulated before and therefore solve the nonlinear optimization problem. Provided an initial guess for the problem unknowns $\boldsymbol{p}$ is given, the equations of motion and adjoint differential equations are integrated to determine the errors on the boundary conditions $\boldsymbol{\Psi}$; the tentative solution is then modified to nullify the errors. Each unknown quantity is in turn varied by a small amount δp_i (e.g., 10^{-7}) and the corresponding error change $\delta\boldsymbol{\Psi}_i$ is computed after a new integration: the derivative of the jth error with respect to the ith perturbed unknown is approximated as $(\partial\Psi_{ij}/\partial p_i) = (\delta\Psi_{ij}/\delta p_i)$. Once the error-gradient matrix $[\partial\boldsymbol{\Psi}/\partial\boldsymbol{p}]$ has been computed, the unknowns $\boldsymbol{p}$ are corrected at the rth iteration according to $\boldsymbol{p}^{r+1} = \boldsymbol{p}^r - K[\partial\boldsymbol{\Psi}/\partial\boldsymbol{p}]^{-1}\boldsymbol{\Psi}^r$ assuming a linear behavior; K, ranging from 0.01 to 1, is an introduced downhill factor to obtain easier convergence.

We consider the one-sided optimization problem first, where the evader does not use thrust and its adjoint equations can be neglected; the pursuer-evader game becomes a typical rendezvous problem with a nonmaneuvering target. The one-sided problem is therefore a special case of the two-sided optimization; its solution is easily determined (techniques to define suitable tentative solutions are described in [12]), and it is used as a tentative guess for the actual two-sided missions. Convergence is always obtained rather easily; some difficulties may arise only for very long missions.

The same procedure can deal with cooperative rendezvous maneuvers [7], where the evader becomes a cooperative target and both spacecraft must minimize the transfer time, that is, maximize φ. In this case, the same considerations and equations that hold for the pursuer also hold for the cooperative target.

IV. Numerical Results

Three different cases are considered in this Note. The first two cases are taken from the test cases 4 and 5 of [5], but other values of the acceleration are also used for the two spacecraft to analyze how they influence the solution. The third case refers to GEO operations. Both constant and variable mass solutions are considered and results are compared.

Test case 1 (circular orbits; initial altitude of 400 km for the pursuer, initial altitude of 1500 km for the evader)

The orbit elements are the same as in the test case 4 of [5].

$$P\begin{cases} a_{P0} = 6778.165 \text{ km} \\ e_{P0} = 0 \\ i_{P0} = 150 \text{ deg} \\ \Omega_{P0} = 170 \text{ deg} \\ \theta_{P0} = 0 \text{ deg} \end{cases} \qquad E\begin{cases} a_{E0} = 7878.165 \text{ km} \\ e_{E0} = 0 \\ i_{E0} = 30 \text{ deg} \\ \Omega_{E0} = 30 \text{ deg} \\ \theta_{E0} = 10 \text{ deg} \end{cases} \tag{18}$$

A constant thrust-to-mass ratio is assumed for both spacecraft. One-sided optimization by only maneuvering the pursuer is first studied, to provide tentative solution for the pursuit-evasion game.

A. Pursuer-Inert Target Optimization

Constant acceleration is assumed for the pursuer; different levels are considered. The evader does not use any thrust, $T_E/m_E = 0$ g. The indirect method is employed to solve the one-sided optimization problem in order to minimize the interception time. According to the optimal control theory, maximum thrust is used during the whole mission, as the switching function always remains positive.

The minimum time with $T_P/m_P = 0.1$ g, that is the value used in test case 4 in [5], is 62.78 min, significantly larger than the solution of the pursuit-evasion game of the referenced article (20 min). A similar time, 19.47 min, is obtained only if the acceleration is increased to $T_P/m_P = 0.14$ g. In fact, a $0.1g$ acceleration for 20 min can only raise the pursuer's altitude to about 1000 km (increase close to 600 km), which is still significantly below the evader's circular orbit altitude, that is, 1500 km. P or E should immediately take advantage of any nonoptimal play made by E or P. In particular, the interception time of the pursuit evasion game should be greater than the value that corresponds to a nonmaneuvering target: the obvious reason is that the evader prefers escaping rather than being caught. It is therefore probable that the evader maneuvers against this rule in test case 4 of [5], which appears to be a cooperative rendezvous rather than a pursuit-evasion game.

B. Pursuer-Cooperating Target Optimization

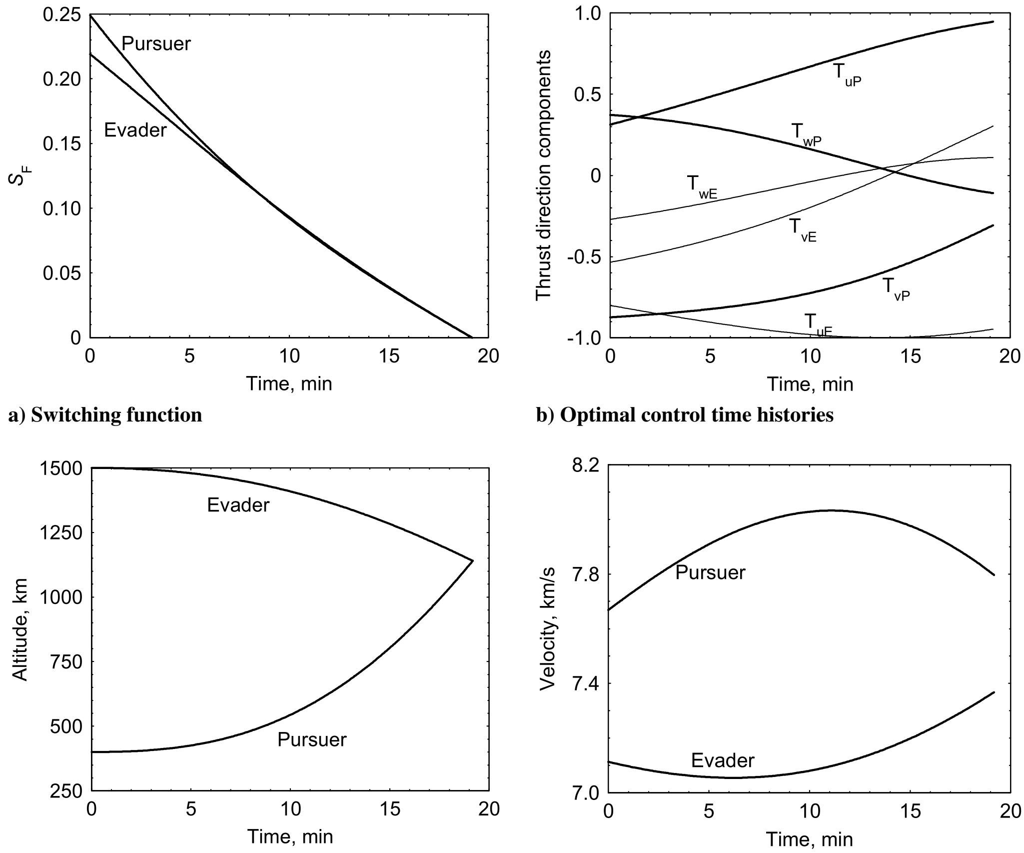
The same constant acceleration values adopted in [5] are used, that is, $T_P/m_P = 0.1$ g, $T_E/m_E = 0.05$ g, and the optimal interception time when the evader benefits the interception is here considered.

The switching functions are positive during the entire mission time for both spacecraft, as shown in Fig. 1a, and the thrusters are again always on. The corresponding optimal control time histories are also shown in Fig. 1b. The altitude and velocity corresponding to the cooperative minimum-time solution are portrayed in Figs. 1c and 1d. Obviously, the evader decreases its altitude, which benefits the pursuer to achieve intercept in only 19.2 min. This solution is quite close to the supposed pursuit-evasion solution of [5], and the altitude time history is similar too.

C. Pursuer-Evader Optimization

Acceleration values are again $T_P/m_P = 0.1$ g and $T_E/m_E = 0.05$ g. The optimal interception time is 94.0 min. Unfortunately, the altitude of the pursuer becomes negative (i.e., -322 km) when the altitude is left free, as shown in Fig. 2; obviously this trajectory is unfeasible, and a minimum height constraint (i.e., 100 km) for the pursuer is introduced for the current case. The constrained solution exhibits a small increase of the final time, to 95.7 min.

Figures 3a and 3b illustrate the optimal control laws for both spacecraft: switching functions and thrust directions are portrayed. The switching function for the pursuer stays positive, and remains negative for the evader, as expected. Note that the thrusting angles are not available at final time, as the final velocity adjoints are null; the same happens for other cases in this Note. Figures 3c and 3d show the saddle-point trajectories leading to interception. The altitude and



a) Switching function

b) Optimal control time histories

c) Altitude

d) Velocity

Fig. 1 Pursuit-cooperating target optimal solution for test case 1: a) switching function; b) optimal control time histories; c) altitude; and d) velocity.

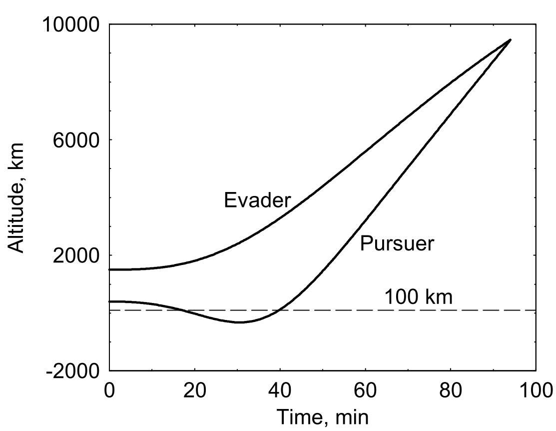


Fig. 2 Pursuit-evasion saddle-point solution for test case 1 without considering altitude constraint.

velocity time histories corresponding to the saddle point solution are shown; the pursuer reaches the minimum height, and the evader tends to raise its altitude during almost the entire game time; increasing the altitude seems the most obvious strategy to avoid being caught by a pursuer coming from a lower orbit. A 3D view of the pursuer-evasion solution of test case 1 is given in Fig. 4a.

Different acceleration levels are also considered for the pursuer. Figure 5 shows two families of saddle-point solutions that occur for progressively increasing values of pursuer's acceleration, with the $0.05g$ acceleration of the evader kept fixed. When the acceleration of the pursuer grows up, the intercept time diminishes. A transition point occurs at about $0.19g$. Beyond this value a new family of short solutions become feasible, with intercept times that drop significantly to values below 20 min; however, this short-time solution family exists only when the pursuer has overwhelming maneuvering capability over that of the evader.

With respect to the satisfaction of the minimum height constraint (MHC), the constraint is not required when the pursuer's acceleration is relatively large, and the pursuer increases its orbit altitude for almost the entire duration of the game; one example is given in Fig. 6a. On the other hand, the MHC must be enforced twice for the cases using modest accelerations for the pursuer, roughly from 0.16 to $0.19g$; the longer locally optimal solution for $0.19g$, is illustrated for the sake of comparison in Fig. 6b.

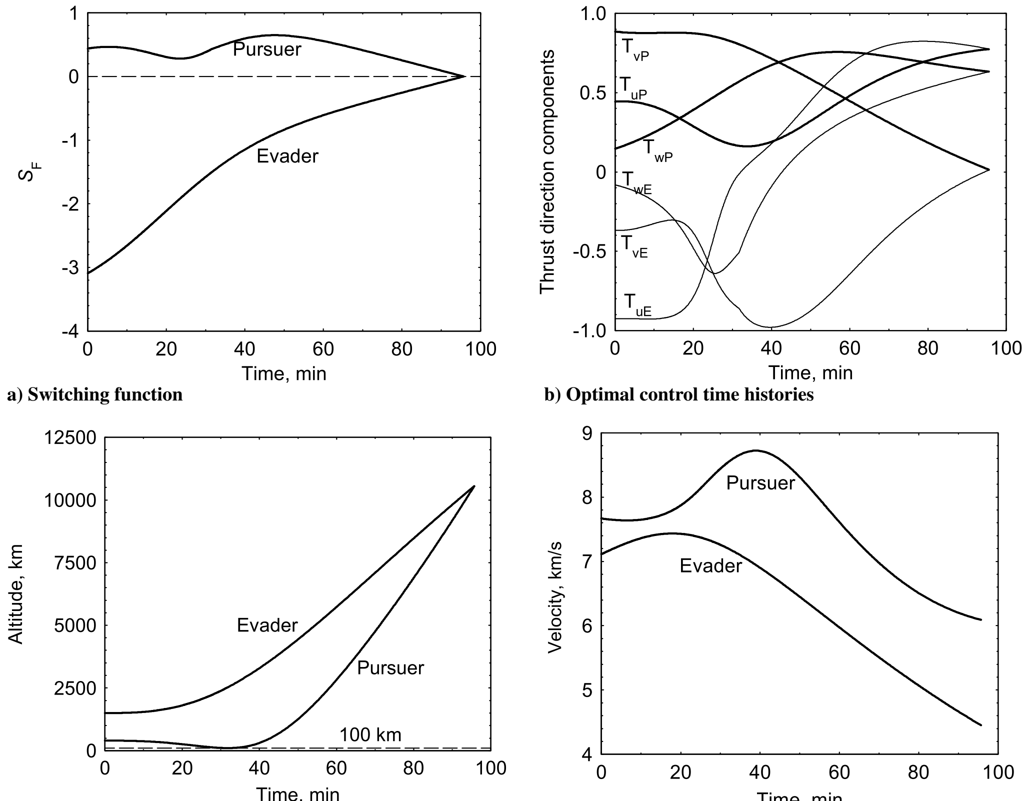
Test case 2 (circular orbits; initial altitude of 400 km for the pursuer, initial altitude of 1500 km for the evader)

The orbit elements are the same as in the test case 5 of [5]. At first, a constant thrust-to-mass ratio is assumed for both spacecraft, that is, $T_P/m_P = 0.12$ g, $T_E/m_E = 0.05$ g.

$$P\begin{cases} a_{P0} = 6778.165 \text{ km} \\ e_{P0} = 0 \\ i_{P0} = 70 \text{ deg} \\ \Omega_{P0} = 170 \text{ deg} \\ \theta_{P0} = 0 \text{ deg} \end{cases} \qquad E\begin{cases} a_{E0} = 7878.165 \text{ km} \\ e_{E0} = 0 \\ i_{E0} = 10 \text{ deg} \\ \Omega_{E0} = -150 \text{ deg} \\ \theta_{E0} = 10 \text{ deg} \end{cases} \tag{19}$$

Figures 7a and 7b show the optimal control laws for both spacecraft. Figures 7c and 7d illustrate the saddle-point trajectories leading to interception, and altitude and velocity of the saddle point solution are portrayed.

For the optimal pursuit-evasion game, 44.0 min are required to interception. A 3D view of the optimal solution is given in Fig. 4b. When mass variations are considered (2000 kg of initial mass and 350 s of specific impulse for both players), a smaller time of 40.1 min is needed. However, the time is still larger than that in [5] (i.e., about 39 min), where a lower acceleration $0.1g$ is used for the pursuer and the same acceleration is used for the evader. Again, this result does not seem to be the correct one for the pursuit-evasion game, as the final time is 39.7 min when the evader does not use any thrust, and the pursuer has a $0.1g$ acceleration. These facts may imply that, in the



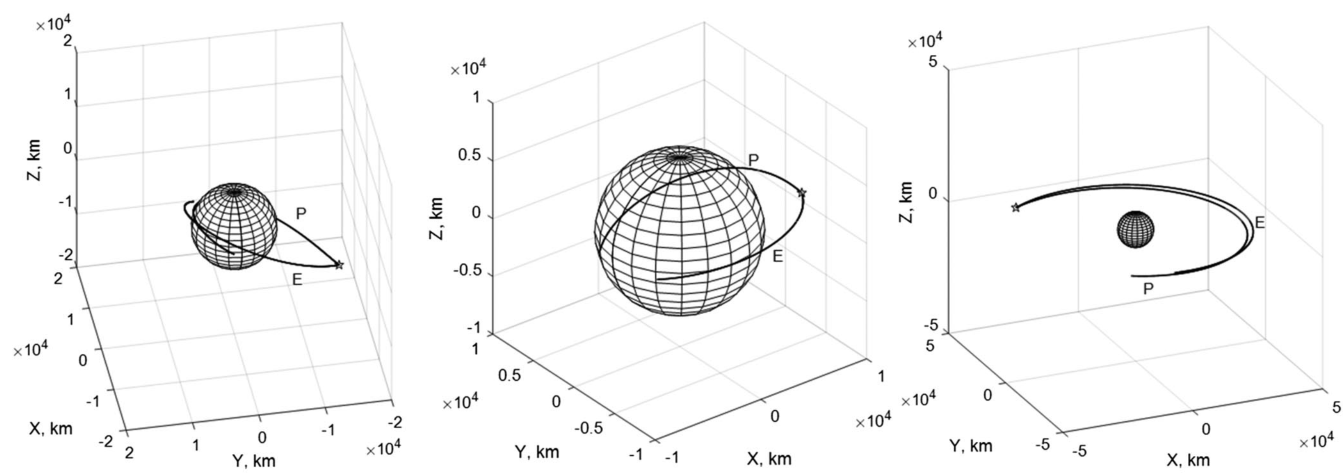
a) Switching function

b) Optimal control time histories

c) Altitude

d) Velocity

Fig. 3 Pursuit-evasion saddle-point solution for test case 1: a) switching function; b) optimal control time histories; c) altitude; and d) velocity.



a) Test case 1

b) Test case 2

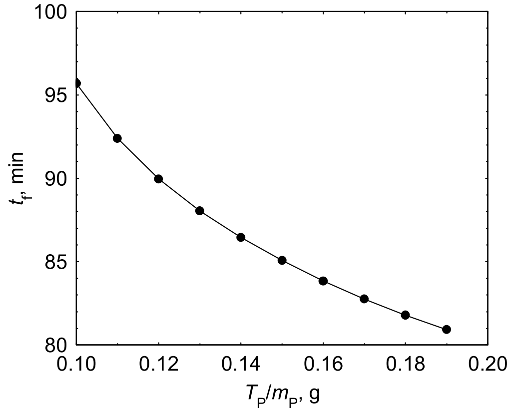
c) Test case 3

Fig. 4 3D view of optimal trajectories: a) test case 1; b) test case 2; and c) test case 3.

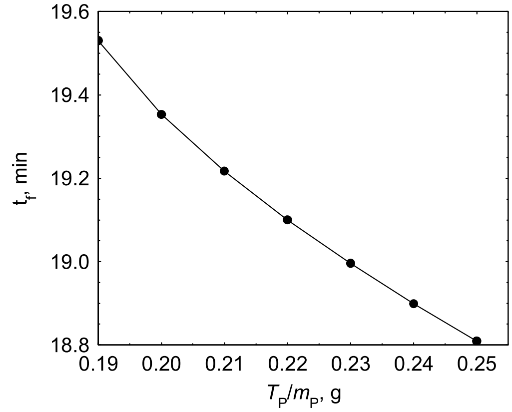
referenced article, the evader does not use the optimal strategy and the pursuer does not face the worst case. The altitude time history may confirm this supposition, as the altitude of the evader almost remains constant in [5], and the thrust direction is perpendicular to the orbital plane for the entire duration of the game; this does not seem the optimal strategy to avoid intercept. In contrast, in-plane maneuvers are used in majority in the solution presented here, as shown in Fig. 7b, where the in-plane thrust components are different from zero for both spacecraft. Figures 8 and 9 show time histories of the relevant osculating elements of the two spacecraft, that is, semimajor axis, eccentricity, inclination, and RAAN. Maneuvers are mainly used to raise the orbit energy and altitude as the evader tries to increase the separation distance; small plane changes still exist in the saddle-point solution, as shown by the RAAN and inclination evolution of the evader.

It is important to note that a key feature of these solutions is that the thrust pointing directions of the two spacecraft tends to be concurrent in pursuit-evasion games, as the pursuer tends to catch up with the evader's strategy. This observation also holds in the other test cases.

The presented solution has a minimum height close to 60 km, which violates the 100-km constraint, as shown in Fig. 7c; however, the short solution does not exist when the minimum height in constrained to be 100 km, as the pursuer has not enough capability to achieve interception in a single revolution. If a slightly larger acceleration for the pursuer is used (e.g., $0.127g$), convergence is easily obtained with the minimum height at 100 km. Therefore, a pursuer's acceleration of $T_P/m_P = 0.12$ g is near to a transition point for this problem; short solutions only exist above this value, in similarity to case 1. Below $0.12g$ the pursuer can obtain intercept only after a longer trajectory using more revolutions. For instance, when considering a constant acceleration of $0.1g$ for the pursuer and the same constant acceleration of $0.05g$ for the evader, the saddle-point solution becomes 205.1 min. If the thrust capability of pursuer is

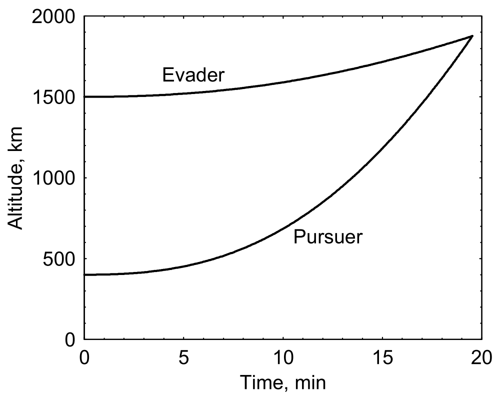


a) $T_p\,/\,m_p \leq 0.19$g long solutions

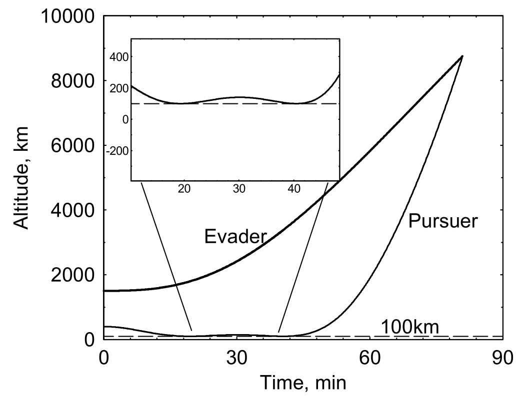


b) $T_p\,/\,m_p \geq 0.19$g short solutions

Fig. 5 Solutions using different accelerations for the pursuer.

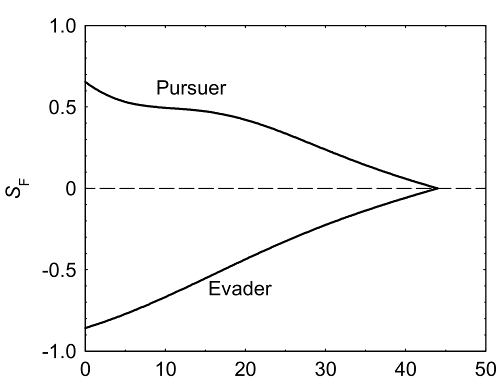


a) Short solution, MHC is inactive

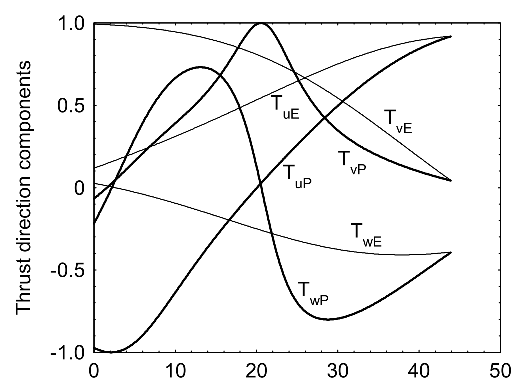


b) Long solution, double MHCs are active

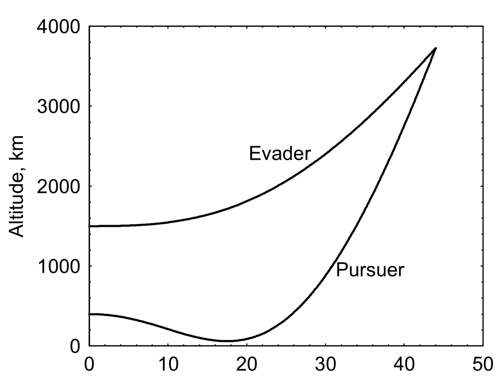
Fig. 6 Altitude time histories of the two local optima using acceleration 0.19g for the pursuer.



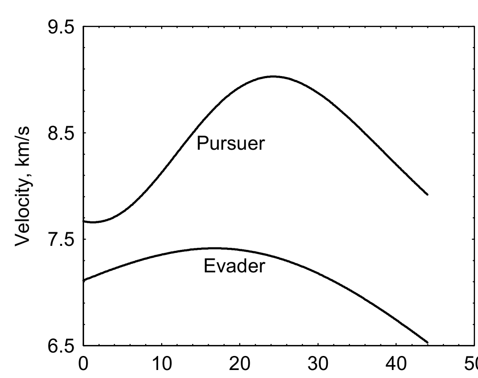
a) Switching function



b) Optimal control time histories



c) Altitude



d) Velocity

Fig. 7 Pursuit-evasion saddle point solution for test case 2: a) switching function; b) optimal control time histories; c) altitude; and d) velocity.

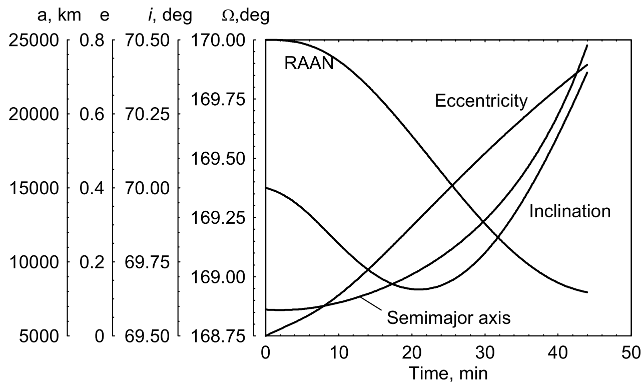


Fig. 8 Orbit elements time histories of the pursuer of case 2.

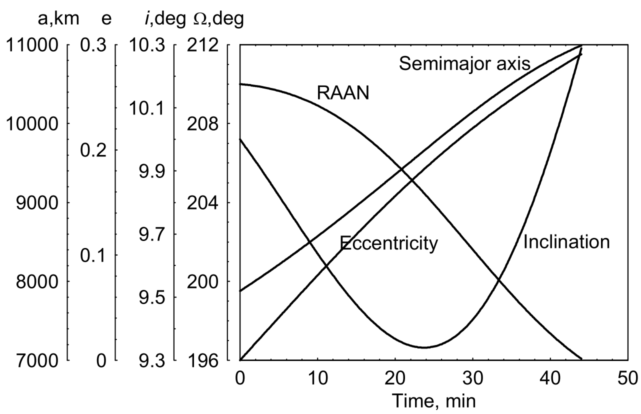


Fig. 9 Orbit elements time histories of the evader of case 2.

further reduced and becomes roughly equal to that of evader, the interception time grows further. For a constant acceleration of $0.06g$ for the pursuer and $0.05g$ for the evader, the interception time becomes 499.9 min; when the acceleration of the pursuer is further decreased to $0.055g$, the saddle solution becomes a quite large value (i.e., 849.4 min). One observes that pursuer's performance is notably degraded when the two players' capability becomes roughly equal. Note that convergence requires much more iterations and therefore is more difficult to obtain for long-time solutions.

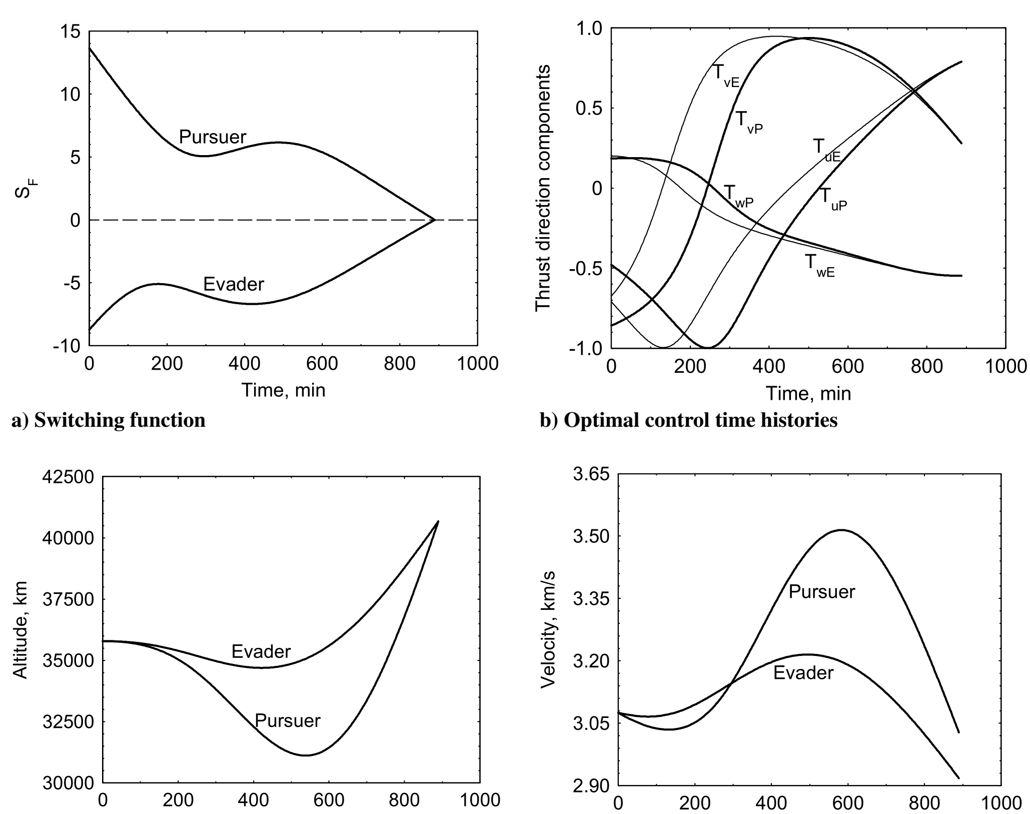
Test case 3 (GEO orbits for the pursuer and the evader)

The orbit elements are given by Eq. (20). The maneuver is mainly a phase change, with minimal adjustments to orbit plane and eccentricity. A practical value for the thrust-to-mass ratio is used for both spacecraft. It is assumed that the maximum thrust values for the two players are $T_P = 20$ N and $T_E = 10$ N, with 2000 kg of initial mass and 350 s of specific impulse for both spacecraft.

$$P\begin{cases} a_{P0} = 42166 \text{ km} \\ e_{P0} = 0.00013 \\ i_{P0} = 1.565 \text{ deg} \\ \Omega_{P0} = 47 \text{ deg} \\ \omega_{P0} = 207.1 \text{ deg} \\ \nu_{P0} = 350 \text{ deg} \end{cases} \quad E\begin{cases} a_{E0} = 42166 \text{ km} \\ e_{E0} = 0.00016 \\ i_{E0} = 0.828 \text{ deg} \\ \Omega_{E0} = 79 \text{ deg} \\ \omega_{E0} = 237 \text{ deg} \\ \nu_{E0} = 309 \text{ deg} \end{cases} \tag{20}$$

Figures 10a and 10b show the optimal control laws. Figures 10c and 10d present the saddle-point trajectories leading to interception, where altitude and velocity corresponding to the saddle point solution are shown. The thrust profiles are similar between the two players, and out-plane maneuvers are again used in minority. A 3D view of the pursuer-evasion solution of test case 3 is given in Fig. 4c.

The optimal interception time is 890 min, with final mass 1688 and 1844 kg for the pursuer and evader, respectively. A larger time (i.e., 913 min) is required when the mass variation is not considered. The



a) Switching function

b) Optimal control time histories

c) Altitude

d) Velocity

Fig. 10 Pursuit-evasion saddle point solution for test case 3: a) switching function; b) optimal control time histories; c) altitude; and d) velocity.

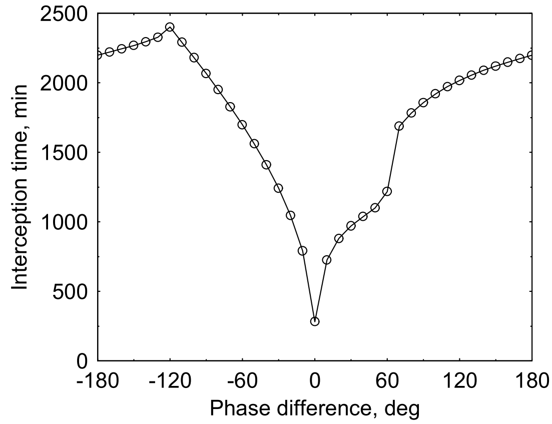


Fig. 11 Phase changes' influences on the optimal interception time.

acceleration grows when propellant consumption is taken into account; this fact benefits the pursuer, which has a larger propellant consumption, more than the evader, and less time to interception is required. This observation also holds in the other test cases. In similarity to the preceding two cases, the pursuer tends to decrease the altitude to obtain a larger velocity, which may be beneficial for a fast interception. This is expected, because a phase adjustment is needed and, in this case, the pursuer must rotate faster to reach the evader. A time delay of evasion is also investigated: assuming that the evader starts maneuvering after 1 h, the intercept time has a decrease of about 23–867 min, whereas for a 2-h delay the decrease is 40–850 min.

The true longitude (phase angle) is defined as $l = \Omega + \omega + \nu$, and the initial values for pursuer and evader are 244.1 and 265 deg, respectively. The initial phase angle difference of the two players is changed from -180 to 180 deg with step 10 deg while the initial phase of the pursuer is kept fixed. The interception time has a minimum (282 min) when the phase difference is 0 and increases sharply when the two players have a small phase difference. For positive differences (evader ahead), the pursuer initially reduces the energy to rotate faster (inner maneuver) and then increases it to match that of the evader. The pursuer's energy is instead initially increased (outer maneuver) for evader behind the pursuer. The maneuvers are, however, nonsymmetrical, as operating at lower altitudes is beneficial for the pursuer and inner maneuvers tend to have a lower final time for the same absolute magnitude difference. The intercept time is maximum (i.e., 2401 min) at phase difference about -120 deg (i.e., 240 deg), as seen in Fig. 11: The cost of recovering -120 deg with an outer maneuver becomes equal to that of recovering 240 deg with an inner one. The abrupt change at 70 deg is related to a switch to a long maneuver, as the thrust difference is too small to obtain the required phase change with the short maneuver that characterizes difference from 0 to 60 deg. As one can expect, the initial conditions have a large influence on the mission characteristics and performance; however, the proposed indirect method has proven capable of dealing with all the considered cases and the general trends (e.g., the effect of the relative acceleration values) remain the same. Given an acceleration margin, the pursuer will always be capable of catching the evader, if sufficient time (and propellant) is available. The specific initial conditions have a large influence on the intercept time, as shown by the previous example, but a detailed analysis of all the cases is beyond the scope of the present Note.

V. Conclusions

A challenging pursuit-evasion game in orbit has been solved with an indirect method. The necessary conditions for optimality have been presented based on Pontryagin Maximum Principle. The method is very efficient and rapid convergence can be obtained in a very short time (seconds). The initial guess to solve the pursuit-evasion game can be provided by the one-sided optimization solution. The proposed method does not suffer from the existence of barrier surfaces and does not exhibit convergence difficulties. The saddle-point solutions satisfy all of the analytical necessary conditions for a local optimum. Different local optima may exist for the same problem, depending on the acceleration levels: with the method presented here one can easily assess their existence and find the corresponding saddle-point solutions. The player competes with opposite objectives (interception time minimization or maximization), and the indirect method provides the means for the optimality verification.

Detailed analysis of cases treated in recent literature shows that other methods may sometimes fail to evaluate the correct strategies for the pursuer or evader; in contrast, the indirect method presented in this Note does not suffer from these shortcomings. The procedure has been extended to consider height constraints and also to deal with the pursuer-cooperating target interception, in addition to the pursuer-inert target problem used to find tentative solutions. Results comparison with the pursuit-evasion problems supports the rightness of the saddle-point solution obtained here.

The results show that the evader tends to raise the altitude when it starts from a higher orbit compared with the pursuer's one; on the contrary, the pursuer tends to increase the velocity by reducing the altitude remarkably during the initial phase of the trajectory, for a more efficient energy increase and faster rotation. In these cases, a minimum-height constraint must be enforced once or twice, according to the acceleration level and number of revolutions. The handling of these constraints has been implemented seamlessly into the method; the pursuer thus uses a skip trajectory around the atmospheric boundary. The existence of short and long families of solutions corresponding to different local optima has also been highlighted. This characteristic has caused convergence issues with existing methods, whereas it is treated flawlessly by the proposed approach. Moreover, as an improvement of the existing works, mass variation is also considered, and the accuracy in assessing the saddle-point solutions is improved.

Acknowledgment

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References

[1] Isaacs, R., *Games of Pursuit*, Rand Corp., Santa Monica, CA, 1951.
[2] Isaacs, R., *Differential Games*, Wiley, New York, 1965, pp. 278–280.
[3] Breitner, M. H., Pesch, H. J., and Grimm, W., "Complex Differential Games of Pursuit–Evasion Type with State Constraints, Part 1: Necessary Conditions for Open-Loop Strategies," *Journal of Optimization Theory and Applications*, Vol. 78, No. 3, 1993, pp. 419–441.
doi:10.1007/BF00939876
[4] Breitner, M. H., Pesch, H. J., and Grimm, W., "Complex Differential Games of Pursuit-Evasion Type with State Constraints, Part 2: Numerical Computation of Open-Loop Strategies," *Journal of Optimization Theory and Applications*, Vol. 78, No. 3, 1993, pp. 443–463.
doi:10.1007/BF00939877
[5] Pontani, M., and Conway, B. A., "Numerical Solution of the Three-Dimensional Orbital Pursuit-Evasion Game," *Journal of Guidance Control and Dynamics*, Vol. 32, No. 2, 2009, pp. 474–487.
doi:10.2514/1.37962
[6] Hafer, W. T., Reed, H. L., Turner, J. D., and Pham, K., "Sensitivity Methods Applied to Orbital Pursuit Evasion," *Journal of Guidance Control and Dynamics*, Vol. 38, No. 6, 2015, pp.1118–1126.
doi:10.2514/1.G000832
[7] Zavoli, A., Simeoni, F., Casalino, L., and Colasurdo, G., "Optimal Cooperative Deployment of a Two-Satellite Formation into a Highly Elliptic Orbit," *2011 AAS/AIAA Astrodynamics Specialist Conference*, AAS Paper 11-641, San Diego, CA, 2011.
[8] Shen, H.-X., and Casalino, L., "High-Accuracy Optimal Finite-Thrust Trajectories for Moon Escape," *Acta Astronautica*, Vol. 131, No. 1, 2017, pp. 102–109.
doi:10.1016/j.actaastro.2016.11.028
[9] Bryson, A. E., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, New York, 1975.
[10] Shen, H.-X., and Casalino, L., "Indirect Optimization of Three-Dimensional Multiple-Impulse Moon-to-Earth Transfers," *Journal of*

Astronautical Sciences, Vol. 61, No. 3, 2014, pp. 255–274. doi:10.1007/s40295-014-0018-9

[11] Casalino, L., Colasurdo, G., and Pastrone, D., "Optimal Low-Thrust Escape Trajectories Using Gravity Assist," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 5, 1999, pp. 637–642. doi:10.2514/2.4451

[12] Colasurdo, G., and Casalino, L., "Tentative Solutions for Indirect Optimization of Spacecraft Trajectories," *Space Engineering Modeling and Optimization with Case Studies*, edited by G. Fasano, and J. D. Pinter, Springer, New York, 2016, pp. 87–102. doi:10.1007/978-3-319-41508-6