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A Closed-Form Solution to the Minimum $\Delta V_{\rm tot}^2$ Lambert's Problem

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Abstract A closed form solution to the minimum $\Delta V_{\rm tot}^2$ Lambert problem between two assigned positions in two distinct orbits is presented. Motivation comes from the need of computing optimal orbit transfer matrices to solve re-configuration problems of satellite constellations and the complexity associated in facing these problems with the minimization of $\Delta V_{\rm tot}$. Extensive numerical tests show that the difference in fuel consumption between the solutions obtained by minimizing $\Delta V_{\rm tot}^2$ and $\Delta V_{\rm tot}$ does not exceed 17%. The $\Delta V_{\rm tot}^2$ solution can be adopted as starting point to find the minimum $\Delta V_{\rm tot}$. The solving equation for minimum $\Delta V_{\rm tot}^2$ Lambert problem is a quartic polynomial in term of the angular momentum modulus of the optimal transfer orbit. The root selection is discussed and the singular case, occurring when the initial and final radii are parallel, is analytically solved. A numerical example for the general case (orbit transfer "pork-chop" between two non-coplanar elliptical orbits) and two examples for the singular case (Hohmann and GTO transfers) are provided.

1 Introduction

Impulsive orbit transfer [1–3] is an important research area in celestial mechanics with an existing rich literature. The variety of studies performed in

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this area is wide, covering different problems, such as the existence of an overall optimal solution for the n-impulse maneuver problem [4], a different proof of the Hohmann transfer [5], as well as a technique based on pre-computed tables for the coplanar orbit transfer case [6].

Lambert's problem is a particular instance of the 2-impulse orbit transfer problem, a well-known problem in celestial mechanics. The problem consists of finding the orbital elements of a transfer orbit connecting a departure position (radius $\mathbf{R}_1$) with an arrival position (radius $\mathbf{R}_2$) under the assumption that the orbit transfer must be performed on an assigned time interval ΔT (time of flight). In a more compact form, the problem can be stated as it follows: "Given initial and final radius vectors, find the conic section with prescribed time-of-flight between initial and final positions." This problem is extensively solved to find the optimal departure and time-of-flight for a generic two-impulse orbit transfer, where the optimality usually implies the minimization of the fuel consumption. A typical example is the building of the so called "pork-chop" plots [14] allowing the preliminary analysis to identify departure and arrival times of rendezvous problems. Because of its importance, since its initial formulation, several mathematical approaches have been developed to solve this problem in the most efficient and robust way. Among all the important contributions given to this problem, we can list [7], providing a universal solution using second order Halley's iteration, [15], presenting a uniform method (independent from the orbit), and [16, 17] showing an iterative technique to compute the solution with 6 digits of accuracy.

Alternatively, references [8–12] have faced the optimal two-impulse orbit transfer problem in a slightly different way, known as the minimum $\Delta V_{\rm tot}$ Lambert problem, where the constraint of performing the orbit transfer in a prescribed time interval is substituted by finding the orbit transfer associated with minimum fuel consumption. The existing proposed techniques to solve this problem can be classified in two distinct categories. The first one proposes iterative techniques characterized by different convergence speed and computational load, while the approaches belonging to the second category all arrive at a complicate equation whose solution requires numerical technique (e.g., sixth degree polynomial root solver). So far no closed form solution has been found to the minimum $\Delta V_{\rm tot}$ Lambert problem, and even in the most recent attempts to solve it (e.g., Ref. [13]), the solution is obtained through numerical techniques whose convergence (or divergence) is a function of the initial point selection. On the other hand, the convergence speed is also a function of the particular problem to be solved. The geometry of this "minimum $\Delta V_{\rm tot}$ Lambert problem," is provided in Fig. 1.

The optimal orbit transfer problem is a problem that must be extensively solved in satellite re-configuration problems, where the transfer orbit cost matrices must be evaluated to solve the optimal combinatory problem that is finding the destination orbit to each initial orbit. Motivated by this specific reason, this paper proposes to solve the "$\Delta V_{\rm tot}^2$ Lambert problem," a problem we prove that can be solved in a closed form. The purpose is to minimize the following cost function:

$$\Delta V_{\rm tot}^2 = |\Delta \mathbf{V}_1|^2 + |\Delta \mathbf{V}_2|^2, \tag{1}$$

where $\Delta\mathbf{V}_1$ and $\Delta\mathbf{V}_2$ represent the initial and final orbit transfer impulses, respectively.

Minimization of ΔV_{tot} means fuel minimization while ΔV_{tot}^2 minimization does not. However, the question of how much more fuel the ΔV_{tot}^2 minimization requires is a question that is here addressed. Assume that the minimum ΔV_{tot} and ΔV_{tot}^2 require two different two-impulse maneuvers, p and q, respectively. It is clear that $\Delta V_{\text{tot}}(p) \leq \Delta V_{\text{tot}}(q)$ and that $\Delta V_{\text{tot}}^2(q) \leq \Delta V_{\text{tot}}^2(p)$. Using the elementary inequalities, $|a| + |b| \leq \sqrt{2(a^2 + b^2)}$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$, we can derive the following estimation

$$\Delta V_{\text{tot}}(p) \leq \Delta V_{\text{tot}}(q) \leq \sqrt{2\Delta V_{\text{tot}}^2(q)} \leq \sqrt{2\Delta V_{\text{tot}}^2(p)} \leq \sqrt{2}\Delta V_{\text{tot}}(p). \tag{2}$$

From a strictly mathematical point of view, Eq. (2) means that the manoeuver minimizing ΔV_{tot}^2 requires up to 41.5% more fuel that the manoeuver minimizing ΔV_{tot}. This upper bound estimation is, indeed, substantially far from what experienced in practical cases. Extensive numerical tests have shown that Eq. (2) overestimates the real difference by obtaining an upper bound value of less than 17%. In addition, this upper bound value appears to be associated with those cases where the orbital plane change is dramatic. Most of the numerical tests show minimal differences and in many cases, as for the Hohmann transfer case, the minimizations of ΔV_{tot} and ΔV_{tot}^2 give the same solution.

2 Problem definition

Let us consider the minimum-fuel two-impulse orbit transfer problem between orbit "1" and orbit "2", whose geometry in provided in Fig. 1. Initially, the spacecraft is on orbit "1" with radius $\mathbf{R}_1 = R_1\,\mathbf{r}_1$ and velocity $\mathbf{V}_1$, where it applies the first impulse $\Delta\mathbf{V}_1 = \mathbf{W}_1 - \mathbf{V}_1$ and moves into the transfer orbit with velocity $\mathbf{W}_1$. Finally, it reaches the target orbit "2" at radius $\mathbf{R}_2 = R_2\,\mathbf{r}_2$ with velocity $\mathbf{W}_2$, where it applies the second impulse, $\Delta\mathbf{V}_2 = \mathbf{V}_2 - \mathbf{W}_2$.

The transfer orbit plane is assigned as the plane where the departure and arrival vectors lie. In this plane we have two transfer trajectories allowing the spacecraft to go from $\mathbf{R}_1$ to $\mathbf{R}_2$. In terms of true anomaly variations, these two transfer trajectories complement each other since the variation can be greater than π or less than π, depending on the selected direction of the angular momentum. In other words, the computation of the true anomaly variation must be consistent with the definition of the angular momentum direction.

We have two distinct directions for the angular momentum

$$\mathbf{h}^{\pm} = \pm\frac{\mathbf{R}_1 \times \mathbf{R}_2}{|\,\mathbf{R}_1 \times \mathbf{R}_2\,|} \tag{3}$$

The direction of the angular momentum also establishes the transfer direction and, consequently, the departure and arrival true anomalies, φ_1 and φ_2,

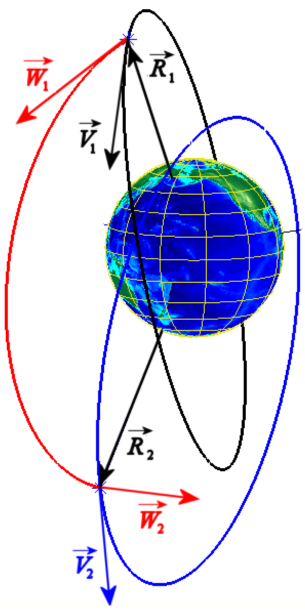

Fig. 1 Radii and velocities definitions

respectively. In order to be consistent with the angular momentum direction, the difference between the two true anomalies, $\Delta\varphi = \varphi_2 - \varphi_1$, must satisfy

$$\sin \Delta\varphi^{\pm} = \mathbf{r}_2 \cdot (\mathbf{h}^{\pm} \times \mathbf{r}_1) \qquad \text{and} \qquad \cos \Delta\varphi = \mathbf{r}_1 \cdot \mathbf{r}_2 \tag{4}$$

where $\Delta\varphi^{+} + \Delta\varphi^{-} = 2\pi$.

Initial and final velocities, $\mathbf{V}_1$ and $\mathbf{V}_2$, can be split in radial, tangential, and normal components, as follows

$$\begin{cases} \mathbf{V}_1 = (\mathbf{V}_1 \cdot \mathbf{r}_1)\mathbf{r}_1 + (\mathbf{V}_1 \cdot \mathbf{s}_1)\mathbf{s}_1 + (\mathbf{V}_1 \cdot \mathbf{h})\mathbf{h} = V_{1r}\,\mathbf{r}_1 + V_{1s}\,\mathbf{s}_1 + V_{1n}\,\mathbf{h} \\ \mathbf{V}_2 = (\mathbf{V}_2 \cdot \mathbf{r}_2)\mathbf{r}_2 + (\mathbf{V}_2 \cdot \mathbf{s}_2)\mathbf{s}_2 + (\mathbf{V}_2 \cdot \mathbf{h})\mathbf{h} = V_{2r}\,\mathbf{r}_2 + V_{2s}\,\mathbf{s}_2 + V_{2n}\,\mathbf{h} \end{cases} \tag{5}$$

where $\mathbf{s}_1 = \mathbf{h} \times \mathbf{r}_1$ and $\mathbf{s}_2 = \mathbf{h} \times \mathbf{r}_2$.

This minimum ΔV^2_{tot} Lambert problem implies the minimization of the following quantity

$$\begin{aligned} \Delta V^2_{\text{tot}} &= (\mathbf{W}_1 - \mathbf{V}_1) \cdot (\mathbf{W}_1 - \mathbf{V}_1) + (\mathbf{V}_2 - \mathbf{W}_2) \cdot (\mathbf{V}_2 - \mathbf{W}_2) = \\ &= V_1^2 + V_2^2 - 2\,(\mathbf{V}_1 \cdot \mathbf{W}_1 + \mathbf{V}_2 \cdot \mathbf{W}_2) + W_1^2 + W_2^2 \end{aligned} \tag{6}$$

From the energy equation applied to the transfer orbit we obtain

$$W_1^2 = \frac{2\mu}{R_1} - \frac{\mu}{a} \qquad \text{and} \qquad W_2^2 = \frac{2\mu}{R_2} - \frac{\mu}{a} \tag{7}$$

where a is the unknown semi-major axis of the transfer orbit. Substituting Eq. (7) in Eq. (6), the transfer cost function to minimize becomes

$$\Delta V^2_{\text{tot}} = C - 2\,(\mathbf{V}_1 \cdot \mathbf{W}_1 + \mathbf{V}_2 \cdot \mathbf{W}_2) - \frac{2\mu}{a} \tag{8}$$

where $C = V_1^2 + V_2^2 + \dfrac{2\mu}{R_1} + \dfrac{2\mu}{R_2}$ includes all known constants. This means that our problem becomes to maximize the cost function

$$G = \mathbf{V}_1 \cdot \mathbf{W}_1 + \mathbf{V}_2 \cdot \mathbf{W}_2 + \frac{\mu\,(1-e^2)}{p} \tag{9}$$

where e and p are the eccentricity and the semi-parameter of the transfer orbit. The polar representation of the orbital radius allows us to write

$$R_1 = \frac{p}{1 + e\,\cos\varphi_1} \qquad \rightarrow \qquad e\,\cos\varphi_1 = \frac{p}{R_1} - 1 \tag{10}$$

and

$$R_2 = \frac{p}{1 + e\,\cos(\varphi_1 + \Delta\varphi)} \qquad \rightarrow \qquad e\,\cos(\varphi_1 + \Delta\varphi) = \frac{p}{R_2} - 1 \tag{11}$$

By expanding $\cos(\varphi_1 + \Delta\varphi) = \cos\varphi_1 \cos\Delta\varphi - \sin\varphi_1 \sin\Delta\varphi$, we can combine Eq. (10) and Eq. (11) into

$$e\,\sin\varphi_1 = \frac{1}{\sin\Delta\varphi}\left[\left(\frac{p}{R_1} - 1\right)\cos\Delta\varphi - \left(\frac{p}{R_2} - 1\right)\right] \tag{12}$$

Equation (10) can be rewritten as $e\,\cos\varphi_1 = e\,\cos[(\varphi_1 + \Delta\varphi) - \Delta\varphi]$; therefore we can write

$$e\,[\cos(\varphi_1 + \Delta\varphi)\,\cos\Delta\varphi + \sin(\varphi_1 + \Delta\varphi)\,\sin\Delta\varphi] = \frac{p}{R_1} - 1$$

from which we obtain

$$e\,\sin(\varphi_1 + \Delta\varphi) = \frac{1}{\sin\Delta\varphi}\left[\left(\frac{p}{R_1} - 1\right) - \left(\frac{p}{R_2} - 1\right)\cos\Delta\varphi\right] \tag{13}$$

The two scalar products appearing in Eq. (9) can be evaluated in any reference frame. For instance, they can be evaluated in the orbital reference frame of the transfer orbit. This yields

$$\begin{aligned} \mathbf{V}_1 \cdot \mathbf{W}_1 &= \begin{Bmatrix} V_{1r}\,\cos\varphi_1 - V_{1s}\,\sin\varphi_1 \\ V_{1r}\,\sin\varphi_1 + V_{1s}\,\cos\varphi_1 \\ V_{1n} \end{Bmatrix}^{\mathrm{T}} \sqrt{\frac{\mu}{p}} \begin{Bmatrix} -\sin\varphi_1 \\ e + \cos\varphi_1 \\ 0 \end{Bmatrix} = \\ &= \sqrt{\frac{\mu}{p}}\,[\,V_{1r}\,e\,\sin\varphi_1 + V_{1s}\,(1 + e\,\cos\varphi_1)\,] \end{aligned} \tag{14}$$

Substituting Eq. (10) and Eq. (12) in Eq. (14) we obtain

$$\mathbf{V}_1 \cdot \mathbf{W}_1 = K_1\,\sqrt{p} + \frac{L_1}{\sqrt{p}} \tag{15}$$

where

$$\begin{cases} K_1 = \dfrac{\sqrt{\mu}}{\sin\Delta\varphi}\left(\dfrac{\cos\Delta\varphi}{R_1} - \dfrac{1}{R_2}\right) V_{1r} + \dfrac{\sqrt{\mu}}{R_1}\,V_{1s} \\ L_1 = \dfrac{(1 - \cos\Delta\varphi)\sqrt{\mu}}{\sin\Delta\varphi}\,V_{1r} \end{cases} \tag{16}$$

Analogously, for the $\mathbf{V}_2 \cdot \mathbf{W}_2$ we obtain

$$\begin{aligned} \mathbf{V}_2 \cdot \mathbf{W}_2 &= \begin{Bmatrix} V_{2r} \cos\varphi_2 - V_{2s} \sin\varphi_2 \\ V_{2r} \sin\varphi_2 + V_{2s} \cos\varphi_2 \\ V_{2n} \end{Bmatrix}^{\mathrm{T}} \sqrt{\frac{\mu}{p}} \begin{Bmatrix} -\sin\varphi_2 \\ e + \cos\varphi_2 \\ 0 \end{Bmatrix} = \\ &= \sqrt{\frac{\mu}{p}} \, \{V_{2r}\, e \sin(\varphi_1 + \Delta\varphi) + [1 + e \cos(\varphi_1 + \Delta\varphi)]\, V_{2s}\} \end{aligned} \tag{17}$$

Substituting Eq. (11) and Eq. (13) in Eq. (17) we obtain

$$\mathbf{V}_2 \cdot \mathbf{W}_2 = K_2 \sqrt{p} + \frac{L_2}{\sqrt{p}} \tag{18}$$

where

$$\begin{cases} K_2 = \dfrac{\sqrt{\mu}}{\sin\Delta\varphi} \left(\dfrac{1}{R_1} - \dfrac{\cos\Delta\varphi}{R_2} \right) V_{2r} + \dfrac{\sqrt{\mu}}{R_2} V_{2s} \\ L_2 = \dfrac{\sqrt{\mu}\,(\cos\Delta\varphi - 1)}{\sin\Delta\varphi} V_{2r} \end{cases} \tag{19}$$

The eccentricity can also be expressed in terms of the semi-parameter p of the transfer orbit. Using Eq. (10) and Eq. (12) we can evaluate the quantity

$$1 - e^2 = 1 - (e \sin\varphi_1)^2 - (e \cos\varphi_1)^2 = \alpha\, p^2 + \beta\, p + \gamma \tag{20}$$

where

$$\begin{cases} \alpha = -\dfrac{1}{R_1^2} - \dfrac{1}{\sin^2\Delta\varphi} \left[\left(\dfrac{\cos\Delta\varphi}{R_1} - \dfrac{1}{R_2} \right) \right]^2 \\ \beta = \dfrac{2}{R_1} - \dfrac{2(1 - \cos\Delta\varphi)}{\sin^2\Delta\varphi} \left(\dfrac{\cos\Delta\varphi}{R_1} - \dfrac{1}{R_2} \right) \\ \gamma = -\dfrac{(1 - \cos\Delta\varphi)^2}{\sin^2\Delta\varphi} \end{cases} \tag{21}$$

Using Eq. (15), Eq. (18), and Eq. (20), we can now re-write Eq. (9) in terms of only the semi-parameter p,

$$G = (K_1 + K_2) \sqrt{p} + \frac{L_1 + L_2}{\sqrt{p}} + \mu \left(\alpha\, p + \beta + \frac{\gamma}{p} \right). \tag{22}$$

The semi-parameter is related to the modulus of the angular momentum as $h = \sqrt{\mu\, p}$. Therefore, we can re-write Eq. (22) as follows

$$G = \frac{K_1 + K_2}{\sqrt{\mu}}\, h + \frac{\sqrt{\mu}\,(L_1 + L_2)}{h} + \alpha\, h^2 + \mu\, \beta + \frac{\mu^2 \gamma}{h^2} \tag{23}$$

Stationary conditions imply $G'(h) = 0$. Multiplying $G'(h)$ by $h^3/(2\alpha)$ we get a quartic with missing quadratic term

$$F(h) = h^4 + c_3\, h^3 + c_1\, h + c_0 = 0 \tag{24}$$

where

$$c_3 = \frac{K_1 + K_2}{2\,\alpha \sqrt{\mu}} \quad c_1 = -\frac{\sqrt{\mu}\,(L_1 + L_2)}{2\,\alpha}, \quad \text{and} \quad c_0 = -\frac{\mu^2 \gamma}{\alpha}. \tag{25}$$

This quartic equation is solved using Ferrari's method as described in the appendix. A simple inspection of Eq. (21) shows that both α and γ are negative. Therefore the independent coefficient c_0 of the quartic is also negative, by Eq. (25). This implies that Eq. (24) has at least one positive and one negative root, because $F(0) = c_0 < 0$ and $\lim_{h\to\infty} F(h) = +\infty$. For the same reason, one of the positive solutions satisfies $F'(h) > 0$, which means that $G''(h) = 2\alpha F'(h)/h^3 < 0$; i.e., it is a local maximum of G as we needed. We can discard the negative and complex roots, since they carry no physical meaning (h represents the magnitude of the angular momentum vector).

As we initially said, we have to consider two possible orientations $\mathbf{h}^{\pm}$ for the angular momentum. Consequently, we have to follow this procedure twice and eventually solve two quadratic equations. However, a simple analysis of the whole algorithm reveals that these two quartic equations are identical except the coefficients c_1 and c_3 that appear with a different sign. More precisely, if $F(h) = 0$ is the quartic associated with $\mathbf{h}^+$, then $F(-h) = 0$ is the quartic associated with $\mathbf{h}^-$. From this observation we conclude that we only need to solve one quartic equation, and keep all the real solutions. The positive roots correspond to the magnitude of the angular momentum (which is pointing in the direction $\mathbf{h}^+$) and the negative roots correspond (if we change them into positive) to the magnitude of the angular momentum (pointing as $\mathbf{h}^-$).

When the optimal solution $h_{\rm opt}$ has been found, the semi-parameter can be recovered as $p = h_{\rm opt}^2/\mu$, and the eccentricity can be computed by

$$e = \sqrt{1 - \alpha\, p^2 - \beta\, p - \gamma} \tag{26}$$

using the values of α, β, and γ provided by Eq. (21). The value for φ_1 is computed using Eq. (10) and Eq. (12). Finally $\varphi_2 = \varphi_1 + \Delta\varphi$. Special care is needed here when $h_{\rm opt}$ is negative: do not forget to use $-\Delta\varphi$ instead of $\Delta\varphi$.

2.1 Pork-chop example

Using the theory so far explained (and the solution provided for the singular case that is presented in the next section) it is possible to produce "pork-chops" to identify the best departure and arrival orbital positions for two-impulse orbit transfer problem between two generic elliptic orbits. To make an example with real satellites we have selected the orbits associated with the two satellites "ALSAT 1" and "ARIANE 44L," respectively, whose Two-Line Elements[1] (TLE) are given as it follows

```
ALSAT 1
1 27559U 02054A   08259.52685948 -.00000002  00000-0  84653-5 0  6025
2 27559 097.9807 137.4784 0009664 216.5494 143.5047 14.62977897309534
ARIANE 44L
1 28576U 91075N   08351.94568414  .00000179  00000-0  64019-2 0  6927
2 28576 006.5534 128.0629 6595687 237.3611 042.0029 02.83587463 72170
```

[1] NORAD Two-Line Element Sets are available for most of the existing satellites in the non-archival publications `http://www.celestrak.com/` and explained in Section 2.4.2 of Ref. [2].

$\lvert\Delta V_1\rvert = 2.1256$ km/s	Time (UTC)	15-Sep-2008 at 12:38:40
$R_x = 3160.1254$	$R_y = -3850.6707$	$R_z = -5011.9852$
$V_x = -4.458$	$V_y = 3.1012$	$V_z = -5.1916$
$\Delta V_x = -1.3612$	$\Delta V_y = 0.14785$	$\Delta V_z = -1.6258$
$\Delta V_r = 0.46381$	$\Delta V_t = 1.9923$	$\Delta V_h = -0.57762$
$\lvert\Delta V_2\rvert = 4.534$ km/s	Time (UTC)	15-Sep-2008 at 14:05:00
$R_x = -16875.8926$	$R_y = 14279.1834$	$R_z = 516.0392$
$V_x = -1.2765$	$V_y = 1.7995$	$V_z = 3.0439$
$\Delta V_x = -2.7982$	$\Delta V_y = -2.4082$	$\Delta V_z = -2.6321$
$\Delta V_r = 0.519$	$\Delta V_t = -3.2595$	$\Delta V_h = -3.1086$

Table 1 Impulses Δv (km/s) 6.6595 km/s

Figure 2 shows the three dimensional (left) and contour (right) plots for the minimum ΔV_{tot}^2 transfer cost for the two selected orbits. The ΔV_{tot} (km/s) is provided as a function of the departure and arrival mean anomalies, respectively.

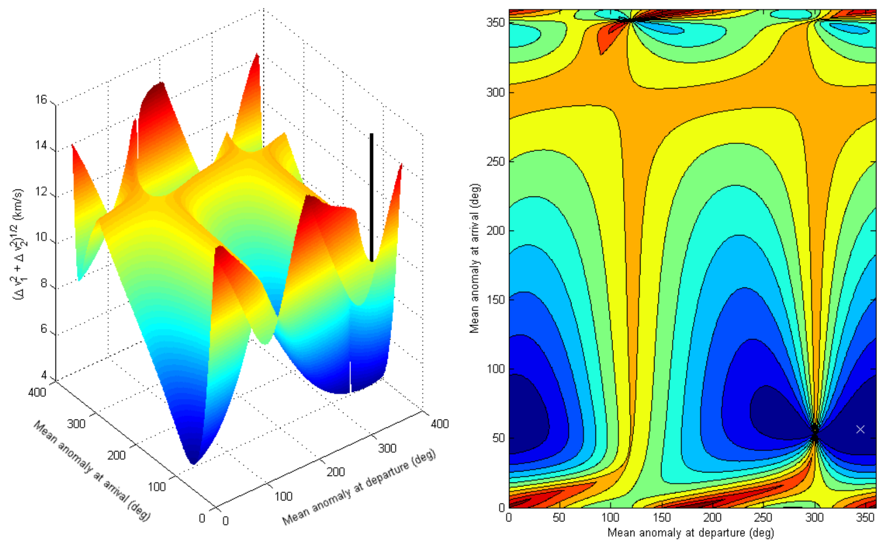

Fig. 2 3-D and contour map "porkchop" of $\sqrt{\Delta v_1^2 + \Delta v_2^2}$

The optimal orbit transfer is identified in Fig. 2 by a black vertical bar in the 3-D plot and by a white "X" in the contour plot. The details of the optimal orbit transfer are given in Table 1. These are: 1) the impulse magnitude and time (UTC), 2) the Cartesian position $[R_x, R_y, R_z]$, 3) the Cartesian velocity $[V_x, V_y, V_z]$, 4) the Cartesian components of the impulse $[\Delta V_x, \Delta V_y, \Delta V_z]$, and 5) the radial, tangential, and normal component of the impulse $[\Delta V_r, \Delta V_t, \Delta V_h]$.

Finally, Fig. 3 shows the three-dimensional geometry of the selected orbits and the optimal orbit transfer trajectory.

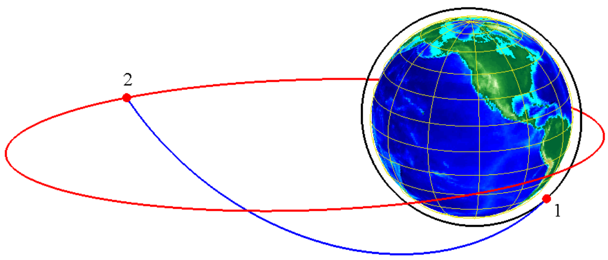

Fig. 3 3-D view of the optimal orbit transfer for the "porkchop" example

3 Singularity

The method previously described becomes singular when departure and arrival radii are parallel. In this case the vector product in Eq. (3) vanishes and the direction of the angular momentum becomes undefined. This happens, for instance, in the Hohmann transfer case. It is clear that if $\Delta\varphi = 0$ and $R_1 \neq R_2$ then it is impossible to go from $\mathbf{R_1}$ to $\mathbf{R_2}$ by means of a simple two-impulse manoeuver. The remaining singular case, when $\Delta\varphi = \pi$, is solved in this section.

If $\Delta\varphi = \pi$ then $\varphi_2 = \varphi_1 + \pi$, and the semi-parameter p of any orbit passing through $\mathbf{R_1}$ and $\mathbf{R_2}$ is determined

$$R_1 = \frac{p}{1 + e\,\cos\varphi_1} \qquad \text{and} \qquad R_2 = \frac{p}{1 - e\,\cos\varphi_1} \tag{27}$$

from which we obtain

$$p = \frac{2\,R_1\,R_2}{R_1 + R_2} \qquad \text{and} \qquad e\,\cos\varphi_1 = \frac{R_2 - R_1}{R_2 + R_1}. \tag{28}$$

The scalar product between radius and velocity vector satisfies

$$\mathbf{R}_1 \cdot \mathbf{W}_1 = \frac{h\,e\,\sin\varphi_1}{1 + e\,\cos\varphi_1} = \frac{(R_2 + R_1)\,h\,e\,\sin\varphi_1}{2\,R_2} \tag{29}$$

and

$$\mathbf{R}_2 \cdot \mathbf{W}_2 = -\frac{(R_2 + R_1)\,h\,e\,\sin\varphi_1}{2\,R_1} \tag{30}$$

from which we derive

$$\mathbf{r}_1 \cdot \mathbf{W}_1 + \mathbf{r}_2 \cdot \mathbf{W}_2 = 0. \tag{31}$$

Equation (31) is useful to solve the singularity case because it tells us that the components of the transfer velocities along the direction $\mathbf{r}_1$ (or $\mathbf{r}_2$) are identical. Equation (31) implies the geometrical situation depicted in Fig. 4,

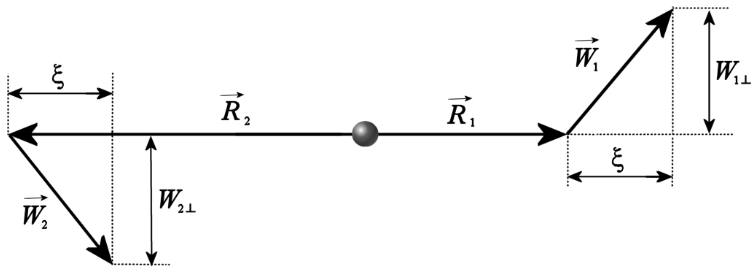

Fig. 4 Singularity case: geometric property provided by Eq. (31)

showing the transfer orbit velocities components in the (unknown) transfer orbit plane.

In other words we can write

$$\xi = |\,\mathbf{W}_{1r}\,| = |\,\mathbf{W}_{2r}\,| \tag{32}$$

for the radial components while the normal components can be derived from the angular momentum

$$h = R_1\, W_{1\perp} = R_2\, W_{2\perp} \tag{33}$$

Let ϑ be the angle between the departure orbit plane and the orbit transfer plane. We can introduce the $[\,\mathbf{r}_1,\, \mathbf{s}_1,\, \mathbf{h}_1\,]$ reference frame, where $\mathbf{h}_1 = \dfrac{\mathbf{R}_1 \times \mathbf{V}_1}{|\,\mathbf{R}_1 \times \mathbf{V}_1\,|}$ and $\mathbf{s}_1 = \mathbf{h}_1 \times \mathbf{r}_1$. The transformation matrix moving from this reference frame to the inertial reference frame is

$$C = [\,\mathbf{r}_1 \,\vdots\, \mathbf{s}_1 \,\vdots\, \mathbf{h}_1\,] \tag{34}$$

In the $[\,\mathbf{r}_1,\, \mathbf{s}_1,\, \mathbf{h}_1\,]$ reference frame the initial and final velocities, $\mathbf{V}_1$ and $\mathbf{V}_2$, have components

$$\mathbf{V}_1^{(1)} = \begin{Bmatrix} V_{1r} \\ V_{1s} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_1 \cdot \mathbf{r}_1 \\ \mathbf{V}_1 \cdot \mathbf{s}_1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{V}_2^{(1)} = \begin{Bmatrix} V_{2r} \\ V_{2s} \\ V_{2h} \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_2 \cdot \mathbf{r}_1 \\ \mathbf{V}_2 \cdot \mathbf{s}_1 \\ \mathbf{V}_2 \cdot \mathbf{h}_1 \end{Bmatrix} \tag{35}$$

while the transfer velocity vectors can be expressed as

$$\mathbf{W}_1^{(1)} = R_1(\vartheta) \begin{Bmatrix} \xi \\ W_{1\perp} \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{W}_2^{(1)} = -R_1(\vartheta) \begin{Bmatrix} \xi \\ W_{2\perp} \\ 0 \end{Bmatrix} \tag{36}$$

where $R_1(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}$ is the matrix performing rigid rotation about the first axis (direction $\mathbf{r}_1$) by the angle ϑ.

The cost function to minimize is provided by Eq. (6). Using Eq. (33) and Eq. (36), the term $(W_1^2 + W_2^2)$ can be written as

$$W_1^2 + W_2^2 = 2\,\xi^2 + \frac{h^2}{R_1^2} + \frac{h^2}{R_2^2} \tag{37}$$

while the $\mathbf{V}_1 \cdot \mathbf{W}_1$ and $\mathbf{V}_2 \cdot \mathbf{W}_2$ scalar products can be evaluated in the $[\,\mathbf{r}_1,\,\mathbf{s}_1,\,\mathbf{h}_1\,]$ reference frame as

$$\begin{cases} \mathbf{V}_1^{(1)} \cdot \mathbf{W}_1^{(1)} = +V_{1r}\,\xi + V_{1s}\,\dfrac{h}{R_1}\,\cos\vartheta \\ \mathbf{V}_2^{(1)} \cdot \mathbf{W}_2^{(1)} = -V_{2r}\,\xi - V_{2s}\,\dfrac{h}{R_2}\,\cos\vartheta - V_{2h}\,\dfrac{h}{R_2}\,\sin\vartheta \end{cases} \tag{38}$$

Substituting Eq. (37) and Eq. (38) in Eq. (6) we obtain a cost function in term of the two unknowns, ξ and ϑ, only. The derivatives with respect to the variables are

$$\frac{\partial \Delta V_{\text{tot}}^2}{\partial \vartheta} = 0 \quad \rightarrow \quad \left(\frac{V_{1s}}{R_1} - \frac{V_{2s}}{R_2}\right)\sin\vartheta = -\frac{V_{2h}}{R_2}\,\cos\vartheta \tag{39}$$

$$\frac{\partial \Delta V_{\text{tot}}^2}{\partial \xi} = 0 \quad \rightarrow \quad \xi = \frac{V_{1r} + V_{2r}}{2} \tag{40}$$

The knowledge of the angle ϑ and the velocity ξ, allow us to obtain the solution for the singular case

$$\mathbf{W}_1 = C\,\mathbf{W}_1^{(1)} \qquad \text{and} \qquad \mathbf{W}_2 = C\,\mathbf{W}_2^{(1)} \tag{41}$$

where $\mathbf{W}_1^{(1)}$ and $\mathbf{W}_2^{(1)}$ are given by Eq. (36) and C by Eq. (34).

3.1 Hohmann transfer example

An important example that can be solved using the theory presented for the singular case is the Hohmann transfer. For this case we have

$$p = \frac{2\,R_1\,R_2}{R_1 + R_2} \qquad \text{and} \qquad h = \sqrt{\mu\,p} = \sqrt{\frac{2\mu\,R_1\,R_2}{R_1 + R_2}}.$$

The velocities satisfy

$$V_{1r} = V_{2r} = V_{2h} = 0, \quad V_{1s} = \sqrt{\mu/R_1}, \quad \text{and} \quad V_{2s} = -\sqrt{\mu/R_2}$$

Using these values, Eq. (40) gives $\xi = 0$ while Eq. (39) provides $\vartheta = 0$. Finally, Eq. (33) and Eq. (36) provide

$$W_{1\perp} = \frac{h}{R_1} = \sqrt{\frac{2\mu\,R_2}{R_1\,(R_1 + R_2)}} \qquad \text{and} \qquad \mathbf{W}_1 = \begin{Bmatrix} 0 \\ W_{1\perp} \\ 0 \end{Bmatrix} \tag{42}$$

which is the correct solution of the Hohmann transfer.

3.2 GTO example

The problem of orbit transfer from an inclined circular parking orbit (inclination i, Radius R_1) to the Geo-Synchronous equatorial circular orbit (Radius R_2), known as the GTO transfer problem, is here considered as another example of application of the theory developed for the singular case.

For this example, the departure and arrival velocities vectors as expressed in the $[\,\mathbf{r}_1,\, \mathbf{s}_1,\, \mathbf{h}_1\,]$ reference frame are

$$\mathbf{V}_1^{(1)} = \sqrt{\frac{\mu}{R_1}} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \text{and} \qquad \mathbf{V}_2^{(1)} = \sqrt{\frac{\mu}{R_2}} \begin{Bmatrix} 0 \\ -\cos i \\ \sin i \end{Bmatrix} \tag{43}$$

which implies

$$\xi = 0 \qquad \text{and} \qquad \tan\vartheta = \frac{-\sin i}{(R_2/R_1)^{3/2} + \cos i} \tag{44}$$

The angle ϑ represents the orbit plane change contribution during the first impulse (the plane change at second impulse will be $i+\vartheta$). For a 28° inclined (Cape Kennedy) 500 km parking orbit, Eq. (44) provides the correct value of $\vartheta = -1.6624°$.

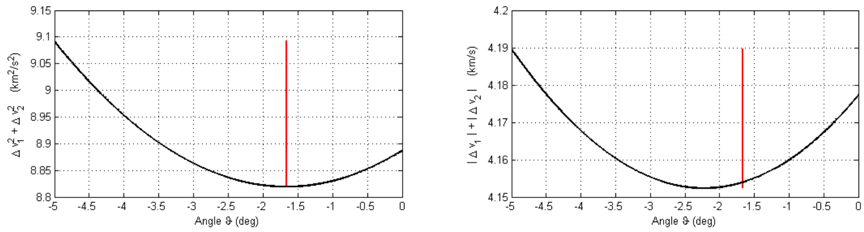

Fig. 5 GTO transfer. Difference between $\min(\Delta v_1^2 + \Delta v_2^2)$ and $\min(|\Delta v_1| + |\Delta v_2|)$. The red line indicates the $\min(\Delta v_1^2 + \Delta v_2^2)$ solution.

Figure 5 shows, for the GTO transfer problem, the optimal solution obtained by minimizing $|\Delta v_1| + |\Delta v_2|$ (right plot) and the one obtained by minimizing $\Delta v_1^2 + \Delta v_2^2$ (left plot). The red vertical line indicates the solution obtained by minimizing $\Delta v_1^2 + \Delta v_2^2$.

4 Conclusions

This paper presents a closed-form solution for the minimum $\Delta V_{\rm tot}^2$ Lambert problem. Motivation comes from the need of building transfer cost matrices to solve the combinatory problems of reconfiguring satellite constellations. The resulting procedure presented constitutes an easy tool to implement that can be useful when extensive minimum $\Delta V_{\rm tot}$ problems must be solved, as in the re-configuration problems of satellite constellations.

The solution provided by minimizing ΔV_{tot}^2 does not always minimize the fuel consumption, but it represents a good approximation, and it can be used as the starting point for finding (e.g., using gradient descent approach) the true optimal two-impulse manoeuver with minimum ΔV_{tot}. The difference between ΔV_{tot}^2 and ΔV_{tot} minimizations problems is numerically found to be bounded with a maximum value lower than 17%.

Since the method does not require any iterations, the number of steps for the whole procedure (complexity) is constant. This is a crucial property when the algorithm has to be extensively used within other optimization programs. An example is the building of fast "pork-chop" plots for two-impulse transfer between assigned orbits. Two examples for the singular case (occurring when initial and final radii are parallel) are provided for Hohmann and GTO transfers.

Appendix: Solution of Algebraic Quartic Equation

Consider again Eq. (24), here repeated

$$h^4 + c_3\, h^3 + c_1\, h + c_0 = 0$$

This equation can be solved using Ferrari's method, which is summarized in this appendix for the specific case of a quartic with missing quadratic coefficient, $c_2 = 0$. By setting

$$a = -\frac{3\, c_3^2}{8}, \qquad b = \frac{c_3^2}{8} + c_1, \quad \text{and} \quad c = -\frac{3\, c_3^2}{256} + \frac{c_1\, c_3}{4} + c_0 \tag{45}$$

then, if $b = 0$, the four solutions are provided by

$$h = -\frac{c_3}{4} \pm \sqrt{\frac{-a \pm \sqrt{a^2 - 4\, c}}{2}} \qquad (\text{if } b = 0). \tag{46}$$

Otherwise, after evaluating the quantities

$$P = -\frac{a^2}{12} - c, \quad Q = -\frac{a^3}{108} + \frac{a\, c}{3} - \frac{b^2}{8}, \quad \text{and} \quad R = \frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}, \tag{47}$$

we can compute

$$U = \sqrt[3]{R} \qquad \text{and} \qquad y = \begin{cases} = 5\, a/6 - \sqrt[3]{Q} & (\text{if } U = 0) \\ = 5\, a/6 + U - P/(3\, U) & (\text{if } U \neq 0) \end{cases} \tag{48}$$

Finally, the four solutions are

$$\begin{cases} h_{1,2} = -\dfrac{c_3}{4} + \dfrac{\sqrt{a + 2\, y} \pm \sqrt{-(3\, a + 2\, y + 2\, b/\sqrt{a + 2\, y})}}{2} \\ h_{3,4} = -\dfrac{c_3}{4} - \dfrac{\sqrt{a + 2\, y} \pm \sqrt{-(3\, a + 2\, y - 2\, b/\sqrt{a + 2\, y})}}{2} \end{cases} \tag{49}$$

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