1 Asymptotic analysis

- Notations
  - $O(f(x))$: upper bound.
  - $\Omega(f(x))$: lower bound.
  - $\Theta(f(x))$: both upper and lower bound.
  - Not every $g(x)$ has $\Theta(f(x))$.
    - $g(x) = (1 + \sin x)x^3 + x^2$
- Master theorem
  - $T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$
  - Case 1: $f(n) = O(n^c)$, $c < \log_b a$.
    Then $T(n) = \Theta(n^{\log_b a})$.
    - Example: $T(n) = 8T(n/2) + 10n^2$.
      $a = 8, b = 2, c = 2, \log_b a = 3 > c$.
      Thus $T(n) = \Theta(n^3)$
  - Case 2: $f(n) = \Theta(n^c \log^k n)$, $c = \log_b a$.
    Then $T(n) = \Theta(n^c log^{k+1} n)$.
    - Example: $T(n) = 2T(n/2) + 10n$.
      $a = 2, b = 2, c = 1, k = 0, \log_b a = 1 = c$.
      Thus $T(n) = \Theta(n \log n)$
  - Case 3: $f(n) = \Omega(n^c)$, $c > \log_b a$.
    Then $T(n) = \Theta(f(n))$.
    - Example: $T(n) = 2T(n/2) + n^2$.
      $a = 2, b = 2, c = 2, \log_b a = 1 < c$.
      Thus $T(n) = \Theta(n^2)$

2 Divide and conquer

- Example: peak finding
  - 1-d [$O(\lg n)$]:
    - a[m-1] > a[m]: search left.
    - a[m+1] > a[m]: search right.
  - 2-d [$O(n \lg n)$]:
    - Given mth col, find max in ith row.
    - a[i, m-1] > a[i, m]: search left.
    - a[i, m+1] > a[i, m]: search right.

3 Heap

- Priority queue
  - insert(S, x)
  - max(S)
  - extract_max(S)
  - increase_key(S, x, k)

3.1 Max heap

- Parent is greater than children (and descendants).
- build_max_heap[$O(n)$]: bottom up max_heapify.
  Denote h as height. $\frac{n}{4}$th node has height 1.

$$T(n) = \sum_{h=1}^{H} \frac{n}{2^{h+1}} h = \frac{n}{2} \sum_{h=1}^{H} \frac{h}{2^h} = O(n)$$

$\sum_{l=0}^{\infty} hx^h = \frac{x}{(1-x)^2}$ when $x < 1$. Here $x = 1/2$.

$$s = \sum_{h=0}^{\infty} hx^h$$

$$t = \int \frac{s}{x} dx = x \sum_{h} x^h = \frac{x}{1-x}$$

$$\frac{s}{x} = \frac{dt}{dx} = \frac{1}{(1-x)^2}$$

- max_heapify[$O(\log h)$]: sink down a node, provided this node is not max heap, but both children are max heap. This process makes the node a max heap.
- insert[$O(\log H)$]: insert to the end and bubble up.
- pop[$O(\log H)$]: replace root with last and heapify.

3.2 Median heap

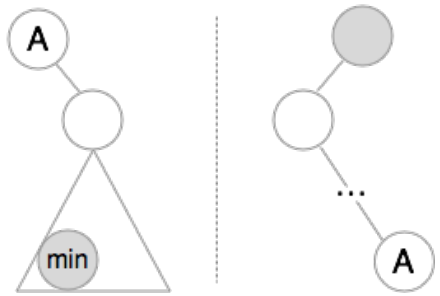
- Maintain MaxHeap on the left and MinHeap on the right.
- Insert:
  - Insert into MaxHeap, if key < root of MaxHeap.
  - Insert into MinHeap, otherwise.
  - Balance if $size_{max} - size_{min} > 1$ by pop-and-insert.

4 Tree

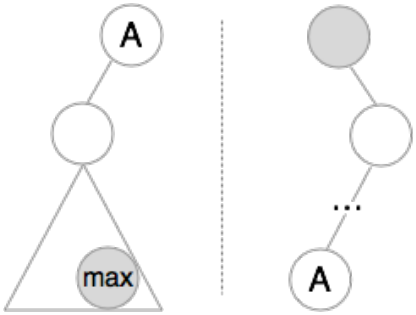
- Height of tree: length of longest path from root to leaf.
- Height of node: length of longest path from node to leaf.
- Depth of node: length from root to node.
- Augmented tree: augment nodes with some property (size, height, etc.).
- Balanced tree: $h = O(\lg n)$.
  - AVL tree
  - Red-black (2-3-4) tree
  - Skip list
  - Treap
  - 2-3 tree
  - B-tree

4.1 Binary search tree

- Example: lane reservation.
- Def: left is smaller, right is larger.
- search, min, max $[O(h)]$
- succ $[O(h)]$
  - Case 1: has right child.
  - Case 2: doesn't have right child.

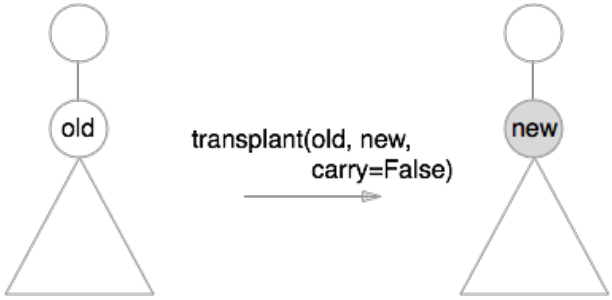


- pred $[O(h)]$
  - Case 1: has left child.
  - Case 2: doesn't have left child.

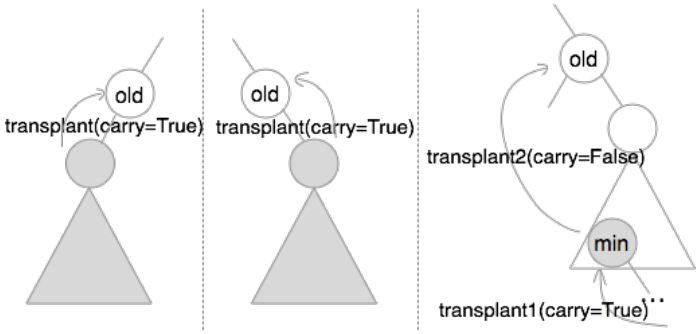


- insert
  - Probe parent.
  - Insert node as leaf.
- delete
  - Transplant
    - Carry

    - Non-carry



  - Case 1: node with single left (or nill).
  - Case 2: node with single right (or nill).
  - Case 3: node with both children.
  - Candidate strategy:
    - Case 1/2: transplant its only child.
    - Case 3, strategy 1: transplant min of right subtree. Min of right subtree falls into case 1 or 2, and needs one more transplant.
    - Case 3, strategy 2: transplant max of left subtree. Max of left subtree falls into case 1 or 2, and needs one more transplant.



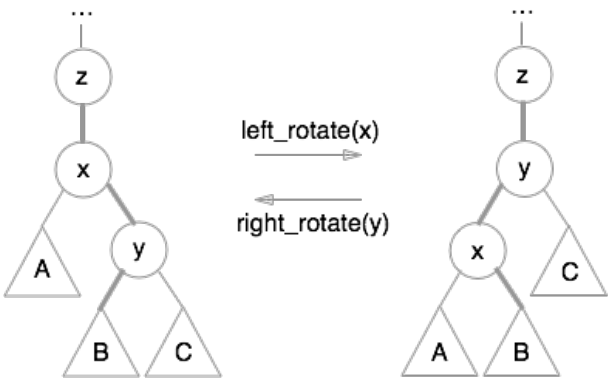
4.2 AVL tree

- Def: $|h_{left} - h_{right}| \leq 1, \quad (h_{null} = -1)$.
- AVL tree is balanced tree. Proof: Denote N_h as min # of nodes to form an AVL tree, then

$$\begin{aligned} N_h &= 1 + N_{h-1} + N_{h-2} \\ &> 2N_{n-2} = \Theta(2^{h/2}) \\ h &= \Theta(\lg N_h) = O(\lg n) \end{aligned}$$

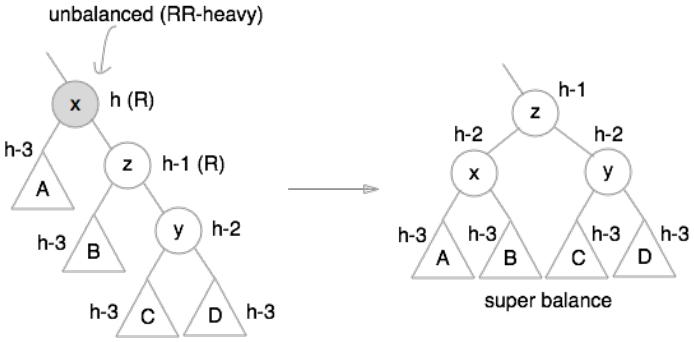
- insert
  - Intuitions:
    - left_rotate on the right-heavy node.
    - right_rotate on the left-heavy node.
    - Only ancestors of the inserted node can become unbalanced after insertion.
    - Ensure only ancestors need rotation, and keep subtrees untainted.
  - left_rotate: rotate node from parent to left child.

– right_rotate: rotate node from parent to right child.



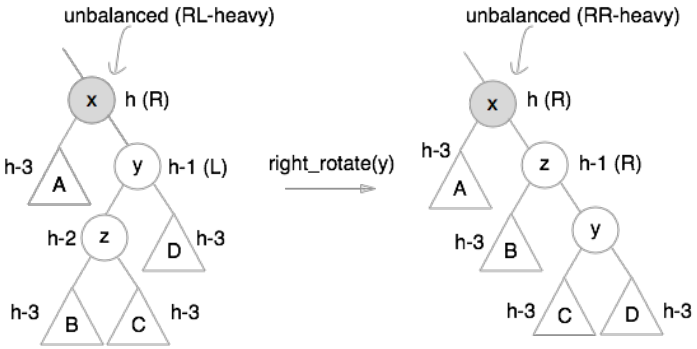
– Case 1: RR-heavy.

– Strategy 1: left rotate the second right-heavy node.



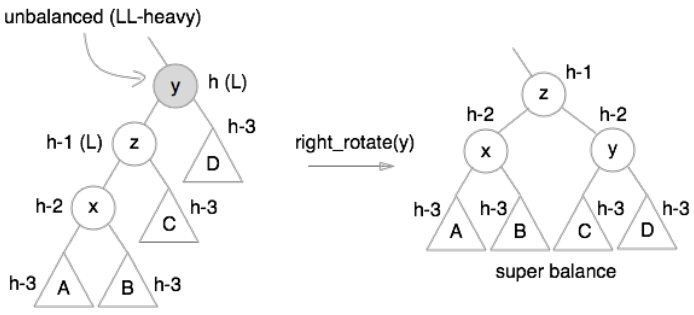
– Case 2: RL-heavy.

– Strategy 2: right rotate the left-heavy node and reduce to case 1. Brief: right_rotate then left_rotate.



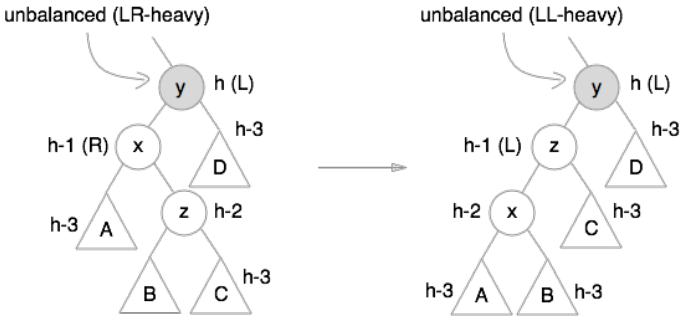
– Case 3: LL-heavy.

– Strategy 3: right rotate the second left-heavy node.

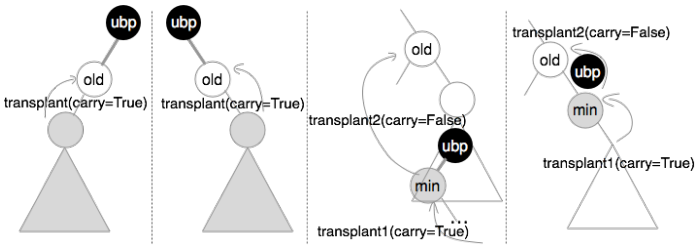


– Case 4: LR-heavy.

– Strategy 4: left rotate the right-heavy node and reduce to case 3. Brief: left_rotate then right_rotate.



- delete
  - Delete as binary search tree.
  - Unbalance point
    - Case 1: the deleted has single left (or nil).
    - Case 2: the deleted has single right (or nil).
    - Case 3: the deleted has both children, and the candidate (successor or predecessor depending on the strategy) is not son of it.
    - Case 4: the deleted has both children, and the candidate is son of it.



4.3 Red-black tree

- Properties
  - All nodes are either red or black.
  - End nodes are extended with nil leaves.
  - Root and leaves (nil's) are black.
  - Red node has black parent.
  - All simple paths from node x to a leaf have same # of black nodes.
    - black_height: # of black's to leaf, excluding start, including leaf.
- Claim: $h \leq 2\log(n+1)$. Proof:
  - Merge red's into black's (2-3-4 tree with height h').
  - # leaves = n+1
  - $2^{h'} \leq$ # leaves $\leq 4^{h'}$
  - $h \leq 2h'$
- insert
  - Insert as red node.

- Resolve the nodes down to the root.
- TODO: cases to switch color or rotate.

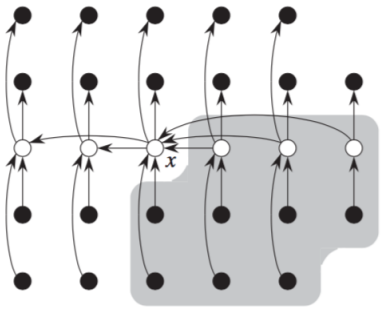
- Versus AVL tree
  - More cases to consider.
  - Asymptotically the same complexity.
  - Faster insert/delete (fewer rotations).
  - Slower lookup (greater height).

5 Sort

- Goal: output n sorted items.
- Comparison model
  - Decision tree: every comparison model can be represented as a decision tree.
  - # leaves $\geq$ # possible outcomes $= n!$
  - height $\geq \log n! = n \log n - O(n)$ (Sterling).
  - Lower bound: $\Omega(n \log n)$
- $O(n \log n)$ sorting
  - Quick sort: Pivot. Divide-and-conquer.
  - Merge sort: Divide-and-conquer.
  - Heap/BST.
- Linear-time (integer) sorting
  - Counting sort
    - L[i]: counter or list of items for integer i.
    - $O(n + k)$, where n is number of items, k is the largest integer.
  - Radix sort
    - Base b repr: $(\overline{d_{D-1}...d_1 d_0})_b$, where $D = \log_b k$.
    - Sort each digit by counting sort.
    - $O((n + b) \log_b k)$
    - $O(c\,n)$ if $k = n^c$, $b = \Theta(n)$.

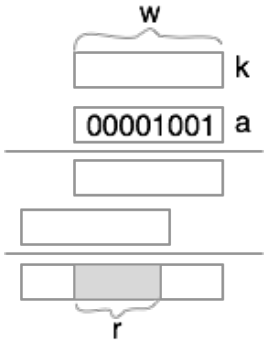
6 Order statistics

- Find kth smallest element.
- $pivot(p, q, k)$: find kth element within $[p, q)$ recursively.
  - Worst case: $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$
  - Expectation: $E[T(n)] = \Theta(n)$
    - X_i: indicates if split into $(i, n-i-1)$
    - $T_i = T(\max\{i, n-i-1\}) + \Theta(n)$
    - $T(n) = \sum X_i T_i$
    - $E[T(n)] = \sum_{i=0}^{n-1} T_i/n \leq \Theta(n) + \frac{2}{n} \sum_{k=n/2}^{n} T(k)$
    - $E[T(n)] \leq c\,n$ by mathematical induction: claim for n, prove for $n+1$.
- Median pivot
  - Recursively find median of median.
  - Choose median of median as pivot.
  - Partition like above.
  - $T(n) \leq T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$
  - $T(n) \leq \Theta(n)$ by induction.



7 Hash

- Prehash: map keys to non-negative integers.
- Hash: map the large key space to smaller space.
- Chaining: chain up the collided items.
- Simple uniform hashing
  - Assumption: a key is equally likely to be mapped into any slot. The hash value is $r.v.$. This assumption generally does NOT hold true, especially when hash function is determinant.
  - Load factor $\alpha = n/m$: expected length of chain, where n is # keys, m is # slots.
  - Complexity: $O(1 + \alpha)$.
- Hash functions:
  - Mod: $h(k) = k\%m$, where m is prime.
  - Mul: $h(k) = [(a \cdot k)\,\%\,2^w] >> (w - r)$, where w is # bits in k (key), and r is # bits in m (# slots).



  - Universal: $h_{a,b}(k) = [(a\,k + b)\,\%\,p]\,\%\,m$, where $k \in \mathcal{K}$, $p > |\mathcal{K}|$ is picked as prime, and a, b are randomly picked within $\{0, ..., p-1\}$.
    - $Pr\{h_1(k_1) = h_2(k_2)\} = 1/m$, given k_1, k_2.

- Applications
  - Dict: random hash + dynamic table
  - String match: Rabin-Karp
    - $h(s) = (\sum ord(s_i) \cdot base^i) \,\%\, m$
    - $h(s_{(i+1):(i+l)}) = \{[h(s_{i:(i+l-1)}) - ord(s_i)] \cdot base + ord(s_{i+l})\} \,\%\, m$

7.1 Universal hashing

- Adversary model: given hash function is not random, adversary can always find a set of collided keys.
- Example: $h_{a,b}(k) = [(a\,k + b) \,\%\, p] \,\%\, m$. Each dict is instantiated with different a and b.
- Universal: $|\{h \in \mathcal{H} : h(x) = h(y), x \neq y\}| = |\mathcal{H}|/m$.
  - Collision: $Pr\{h_1(x) = h_2(y)\} = 1/m$, given x, y.
  - Theorem: given x, $E[\#collisions] < n/m = \alpha$.
    Proof: given h is random, denote *r.v.* $C_{x,y} = \mathbb{1}\{h(x) = h(y)\}$, and $C_x = \sum_{y \in \mathcal{K} \setminus x} C_{x,y}$, then
    - $E[C_{x,y}] = 1/m$
    - $E[C_x] = \sum_{y \in \mathcal{K} \setminus x} E[C_{x,y}] = \frac{n-1}{m}$
- Example $h_a(k) = (\sum a_i k_i) \,\%\, m$, where $a = (\overline{a_r..a_1 a_0})_m$ and $k = (\overline{k_r..k_1 k_0})_m$. Then $|\mathcal{H}| = |\mathcal{K}| = m^{r+1}$.
  - Assume x, y differ in 0th bit, i.e. $x_0 \neq y_0$.
  - $\sum_{i=0}^{r} a_i(x_i - y_i) \equiv 0 \ (mod\ m)$
  - $a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \ (mod\ m)$
  - A theory of Finite (Galois) Field: $\forall z \not\equiv 0 \in \mathbb{Z}_m, m \in P, then\ \exists z^{-1}, s.t.\ z\,z^{-1} \equiv 1 \ (mod\ m)$.
  - $a_0 \equiv [-\sum_{i=1}^{r} a_i(x_i - y_i)](x_0 - y_0)^{-1} \ (mod\ m)$
  - # choices to collide at a_0: m^r ($a_1..a_r$ are free).

7.2 Perfect hashing

- Two-level hashing
  - Allow n_i collisions in ith slot at 1st level.
  - Nearly no collision at 2nd level if # slots $m_i = n_i^2$.
  - Worst complexity: $O(1)$.
  - Theorem: Hash n keys into $m = n^2$ slots using random h in universal hashing would lead to $E[\#collisions] < 1/2$.
    - $E[C_{x,y}] = \frac{1}{m} = \frac{1}{n^2}$
    - $E[C] = \binom{n}{2}\frac{1}{m} = \frac{n-1}{2n} < 1/2$

8 Amortization

- Dynamic table
  - Grow/shrink the table when necessary.
  - Load factor $\alpha = n/m$, n is # items, m is # slots.
  - Double the table when $\alpha > \overline{\alpha}$
  - Shrink the table when $\alpha < \underline{\alpha}$
  - Aggregate analysis on insertion
    - k insertions involve $\log k$ doublings
    - Doubling cost: $\Theta(2^1 + 2^2 + ... + 2^{\log k}) = \Theta(k)$
    - Insertion cost: $\Theta(k)$
    - Amortized cost: $\frac{1}{k}(\Theta(k) + \Theta(k)) = \Theta(1)$
- Aggregate method
- Accounting method
  - Charge ith op amortized cost $\hat{c}_i$.
  - Overcharged are stored to bank.
  - undercharged are taken from bank.
  - Design charge scheme such that $balance \geq 0$.
- Potential method
  - Design Φ_i bound with data structure D_i.
  - Requirements:
    - $\Phi_0 = 0$
    - $\Phi_i \geq 0, \forall i$
    - $\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} \geq 0$
  - Analyze $\hat{c}_i$ for different cases according to Φ_i.
  - Any Φ_i satisfies requirements will do. Better Φ_i yields tighter upper bound.
- Example: $\overline{\alpha} = 1, \underline{\alpha} = 1/4$
  - Potential
$$\Phi_i = \begin{cases} 2n_i - m_i & , \alpha_i \geq 1/2 \\ n_i/2 - m_i & , \alpha_i < 1/2 \end{cases}$$

9 Numbers

9.1 Catalan numbers

- Def
  - P: set of balanced parentheses.
  - $\perp \in P$ ($\perp$ means empty).
  - If $\alpha, \beta \in P$, then $(\alpha)\beta \in P$.
- $C_n = |P_n|$ with n pairs of parentheses.
  - $C_0 = C_1 = 1$.
  - $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$

9.2 High precision

- Newton's method
  - Tangent on x_i: $y = f(x_i) + f'(x_i)(x - x_i)$
- Square root
  - $x = \sqrt{a} \Rightarrow f(x) = x^2 - a = 0$
  - $x_{i+1} = x_i - f(x_i)/f'(x_i) = (x_i + a/x_i)/2$
  - Quadratic convergence: # ✓ digits $\propto$ (# iters)^2
- Multiplication (Karatsuba)

  - $z = x\,y$
  - $x = x_1\,r^{n/2} + x_0$
  - $z = z_0 + z_2 + z_1$
    * $z_0 = x_0\,y_0$
    * $z_2 = x_2\,y_2$
    * $z_1 = (x_0 + x_1)(y_0 + y_1) - z_0 - z_2$
  - $T(n) = 3T(n/2) + \Theta(n) = \Theta(n^{\log 3}) \approx \Theta(n^{1.58})$
- Advanced multiplication
  - gmpy2
  - Schonhage-Strassen: FFT, $\Theta(n \lg n \lg\lg n)$.
  - Furer: $\Theta(n \log n\, 2^{O(\log^* n)})$
- Division
  - $x = R/b \Rightarrow f(x) = 1/x - b/R = 0$
  - $x_{i+1} = 2x_i - bx_i^2/R$
  - $R = (\overline{r_{d-1}..r_1 r_0})_2$, power of 2 is easy to divide.
  - $T(n) = O(\lg n\, n^\alpha)$, break down to mul's.
- When precision digits grow large, complexity of mul, div, sqrt becomes nearly the same as $O(n^\alpha)$.

10 Graph

- Edge
  - Tree edge: $old \to new$
  - Forward edge: given path $x_0 \to ... \to x_n$, any edge not in the path that points to descendant.
  - Backward edge: ..., to ascendant.
  - Cross edge: Between subtrees.
  - Undirected G can have tree edges and back edges.
- Search
  - Breadth first search (BFS)
  - Depth first search (DFS)
- Min span tree (DFS/BFS): search and save edges.
- Cycle detection (DFS): find back edges.
  - Back edge: Edge($v_2 \to v_1$), v_1 is ascendant.
  - v_1 is ascendant of v_2 if $(t_1^{in}, t_1^{out}) \supset (t_2^{in}, t_2^{out})$.
  - Practically we also need to check if v_1 and v_2 have the same root. It is possible that we first DFS v_1, and then v_2.
  - In short, inside or disjoint is good (acyclic).
- Topological sort (DFS):
  - Append the out nodes.
  - Reverse the order.
- Dependency management (DFS):
  - Reverse edges and run topological sort.

10.1 Dijkstra

- Single source shortest path
- Do NOT allow negative weights
- Outputs
  - $d[v]$: shortest distance from source to v.
  - $p[v]$: parent of v in the shortest path.
- Dynamic programming
  - S is the safe set, within which are the nodes that have been reached with shortest path from the source.
  - R is the remaining set.
- Optimizaiton
  - Maintain d as priority queue, so that each time $EXTRACT\text{-}MIN$ only takes $O(\log |V|)$
    * This requires an update of the heap element, which could be achieved by a $sift\_down()$ and a *bubble_up*() of the updated element.
    * Alternatively, we can simply push the new element, and check if element is outdated when we pop an element from the queue. The element is fresh if it does not exist in the safe set yet.
  - Total complexity: $O(|E| + |V| \log |V|)$
- Single source single destination
  - S_f forward safe set.
  - S_b backward safe set.
  - Stop when two sets intersect.

10.2 Bellman-Ford

- Single source shortest path that allows negative weights.
- Mark $d[v] = -\text{inf}$ to indicate negative cycle.
- Algorithm
  - Relax ALL edges for $|V| - 1$ passes.
  - Relax one more time to check negative cycle.
- Complexity $O(|V||E|)$.

11 Misc