1 Asymptotic analysis

- Notations
  - $O(f(x))$: upper bound.
  - $\Omega(f(x))$: lower bound.
  - $\Theta(f(x))$: both upper and lower bound.
  - Not every $g(x)$ has $\Theta(f(x))$.
    - $g(x) = (1 + \sin x)x^3 + x^2$
- Master theorem
  - $T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$
  - Case 1: $f(n) = O(n^c)$, $c < \log_b a$.
    Then $T(n) = \Theta(n^{\log_b a})$.
    - Example: $T(n) = 8T(n/2) + 10n^2$.
      $a = 8, b = 2, c = 2, \log_b a = 3 > c$.
      Thus $T(n) = \Theta(n^3)$
  - Case 2: $f(n) = \Theta(n^c \log^k n)$, $c = \log_b a$.
    Then $T(n) = \Theta(n^c log^{k+1} n)$.
    - Example: $T(n) = 2T(n/2) + 10n$.
      $a = 2, b = 2, c = 1, k = 0, \log_b a = 1 = c$.
      Thus $T(n) = \Theta(n \log n)$
  - Case 3: $f(n) = \Omega(n^c)$, $c > \log_b a$.
    Then $T(n) = \Theta(f(n))$.
    - Example: $T(n) = 2T(n/2) + n^2$.
      $a = 2, b = 2, c = 2, \log_b a = 1 < c$.
      Thus $T(n) = \Theta(n^2)$

2 Divide and conquer

- Example: peak finding
  - 1-d $[O(\lg n)]$:
    - a[m-1] > a[m]: search left.
    - a[m+1] > a[m]: search right.
  - 2-d $[O(n \lg n)]$:
    - Given mth col, find max in ith row.
    - a[i, m-1] > a[i, m]: search left.
    - a[i, m+1] > a[i, m]: search right.

3 Heap

- Priority queue
  - insert(S, x)
  - max(S)
  - extract_max(S)
  - increase_key(S, x, k)

3.1 Max heap

- Parent is greater than children (and descendants).
- build_max_heap$[O(n)]$: bottom up max_heapify.
  Denote h as height. $\frac{n}{4}$th node has height 1.

$$T(n) = \sum_{h=1}^{H} \frac{n}{2^{h+1}} h = \frac{n}{2} \sum_{h=1}^{H} \frac{h}{2^h} = O(n)$$

$\sum_{l=0}^{\infty} hx^h = \frac{x}{(1-x)^2}$ when $x < 1$. Here $x = 1/2$.

$$s = \sum_{h=0}^{\infty} hx^h$$
$$t = \int \frac{s}{x} dx = x \sum_h x^h = \frac{x}{1-x}$$
$$\frac{s}{x} = \frac{dt}{dx} = \frac{1}{(1-x)^2}$$

- max_heapify$[O(\log h)]$: sink down a node, provided this node is not max heap, but both children are max heap. This process makes the node a max heap.
- insert$[O(\log H)]$: insert to the end and bubble up.
- pop$[O(\log H)]$: replace root with last and heapify.

3.2 Median heap

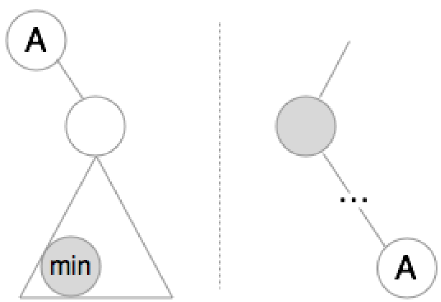
- Maintain MaxHeap on the left and MinHeap on the right.
- Insert:
  - Insert into MaxHeap, if key < root of MaxHeap.
  - Insert into MinHeap, otherwise.
  - Balance if $size_{max} - size_{min} > 1$ by pop-and-insert.

4 Tree

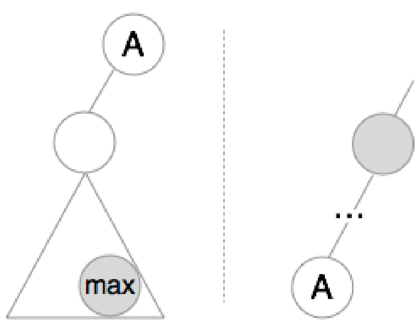
- Height of tree: length of longest path from root to leaf.
- Height of node: length of longest path from node to leaf.
- Depth of node: length from root to node.
- Augmented tree: augment nodes with some property (size, height, etc.).
- Balanced tree: $h = O(\lg n)$.
  - AVL tree
  - Red-black (2-3-4) tree
  - Skip list
  - Treap
  - 2-3 tree
  - B-tree

4.1 Binary search tree

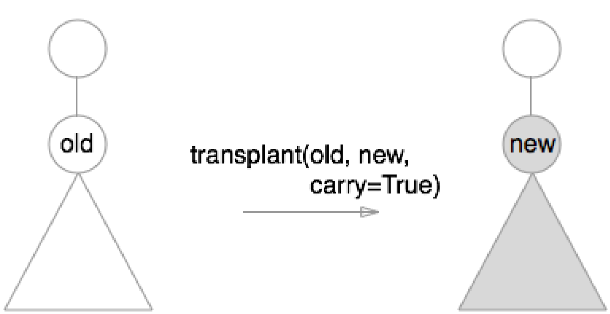
- Example: lane reservation.
- Def: left is smaller, right is larger.
- search, min, max $[O(h)]$
- succ $[O(h)]$
  - Case 1: has right child.
  - Case 2: doesn't have right child.



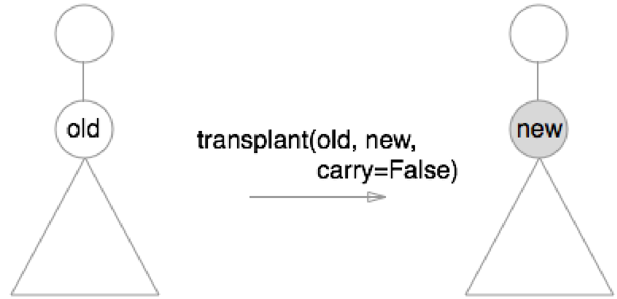
- pred $[O(h)]$



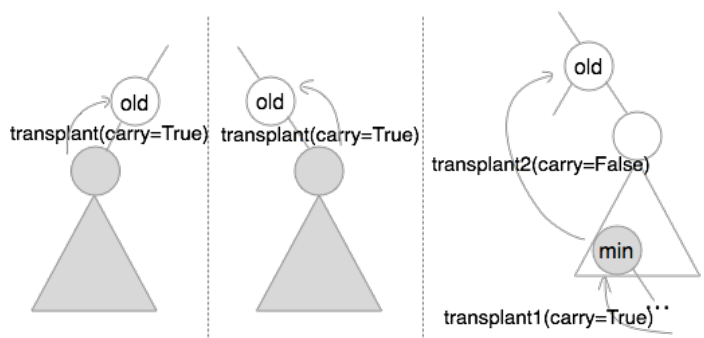
- insert
  - Probe parent.
  - Insert node as leaf.
- delete
  - Transplant
    * Carry



    * Non-carry



  - Case 1: node with single left (or nill).
  - Case 2: node with single right (or nill).
  - Case 3: node with both children.
  - Candidate strategy:
    * Case 1/2: transplant its only child.
    * Case 3, strategy 1: transplant min of right subtree. Min of right subtree falls into case 1 or 2, and needs one more transplant.
    * Case 3, strategy 2: transplant max of left subtree. Max of left subtree falls into case 1 or 2, and needs one more transplant.



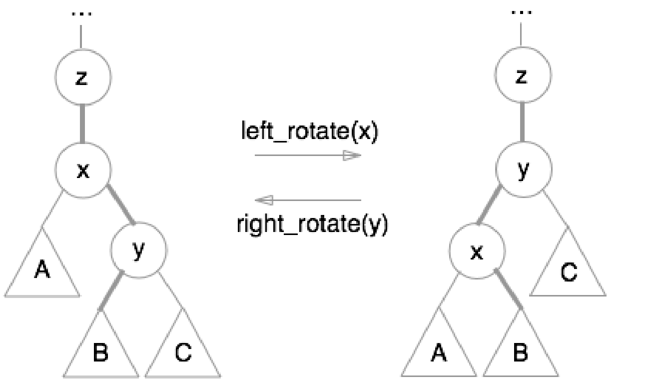
4.2 AVL tree

- Def: $|h_{left} - h_{right}| \le 1, \quad (h_{null} = -1)$.
- AVL tree is balanced tree. Proof: Denote N_h as min # of nodes to form an AVL tree, then

$$\begin{aligned} N_h &= 1 + N_{h-1} + N_{h-2} \\ &> 2N_{n-2} = \Theta(2^{h/2}) \\ h &= \Theta(\lg N_h) = O(\lg n) \end{aligned}$$

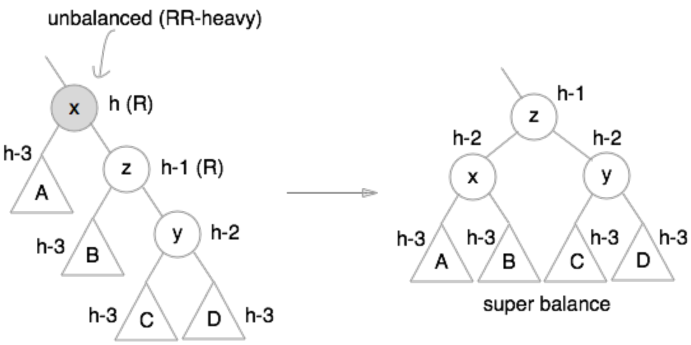
- insert
  - Intuitions:
    * left_rotate on the right-heavy node.
    * right_rotate on the left-heavy node.
    * Only ancestors of the inserted node can become unbalanced after insertion.
    * Ensure only ancestors need rotation, and keep subtrees untainted.
  - left_rotate: rotate node from parent to left child.

– right_rotate: rotate node from parent to right child.



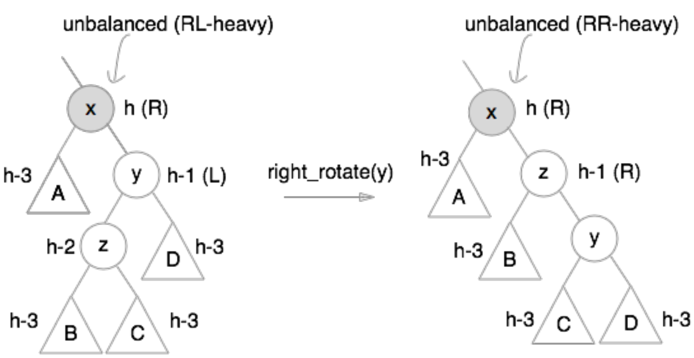
– Case 1: RR-heavy.

– Strategy 1: left rotate the second right-heavy node.



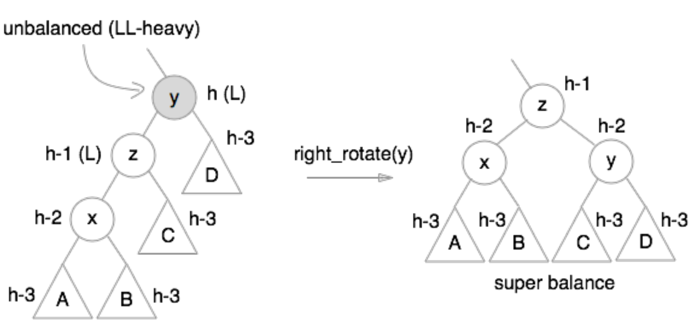
– Case 2: RL-heavy.

– Strategy 2: right rotate the left-heavy node and reduce to case 1. Brief: right_rotate then left_rotate.



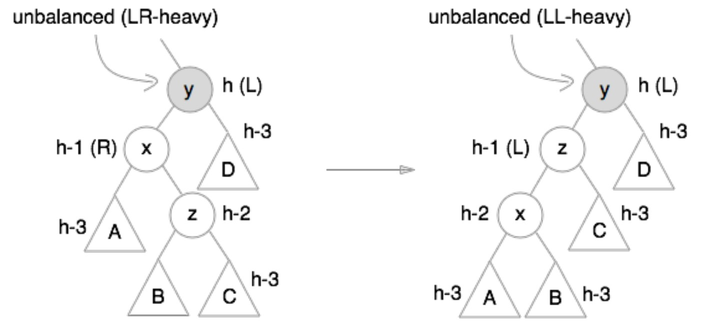
– Case 3: LL-heavy.

– Strategy 3: right rotate the second left-heavy node.

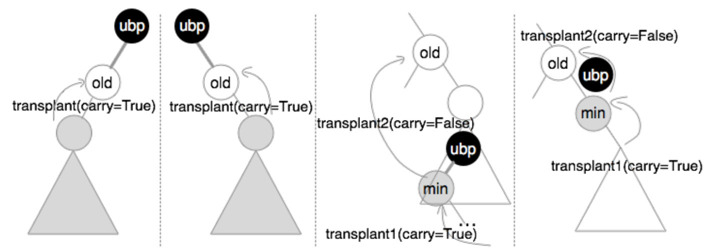


– Case 4: LR-heavy.

– Strategy 4: left rotate the right-heavy node and reduce to case 3. Brief: left_rotate then right_rotate.



- delete
  - Delete as binary search tree.
  - Unbalance point
    - ∗ Case 1: the deleted has single left (or nil).
    - ∗ Case 2: the deleted has single right (or nil).
    - ∗ Case 3: the deleted has both children, and the candidate (successor or predecessor depending on the strategy) is not son of it.
    - ∗ Case 4: the deleted has both children, and the candidate is son of it.



4.3 Red-black tree

- Properties
  - All nodes are either red or black.
  - End nodes are extended with nil leaves.
  - Root and leaves (nil's) are black.
  - Red node has black parent.
  - All simple paths from node x to a leaf have same # of black nodes.
    - ∗ black_height: # of black's to leaf, excluding start, including leaf.
- Claim: $h \leq 2\log(n+1)$. Proof:
  - Merge red's into black's (2-3-4 tree with height h').
  - # leaves = n+1
  - $2^{h'} \leq \#$ leaves $\leq 4^{h'}$
  - $h \leq 2h'$
- insert
  - Insert as red node.

- Resolve the nodes down to the root.
- TODO: cases to switch color or rotate.

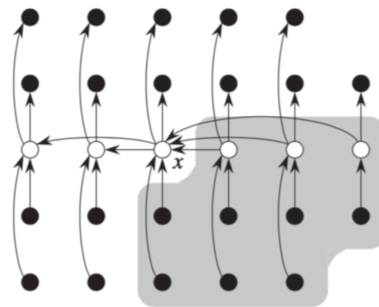
- Versus AVL tree
  - More cases to consider.
  - Asymptotically the same complexity.
  - Faster insert/delete (fewer rotations).
  - Slower lookup (greater height).

5 Sort

- Goal: output n sorted items.
- Comparison model
  - Decision tree: every comparison model can be represented as a decision tree.
  - # leaves $\geq$ # possible outcomes $= n!$
  - height $\geq \log n! = n \log n - O(n)$ (Sterling).
  - Lower bound: $\Omega(n \log n)$
- $O(n \log n)$ sorting
  - Quick sort: Pivot. Divide-and-conquer.
  - Merge sort: Divide-and-conquer.
  - Heap/BST.
- Linear-time (integer) sorting
  - Counting sort
    * L[i]: counter or list of items for integer i.
    * $O(n + k)$, where n is number of items, k is the largest integer.
  - Radix sort
    * Base b repr: $(\overline{d_{D-1}...d_1 d_0})_b$, where $D = \log_b k$.
    * Sort each digit by counting sort.
    * $O((n + b) \log_b k)$
    * $O(c\,n)$ if $k = n^c$, $b = \Theta(n)$.

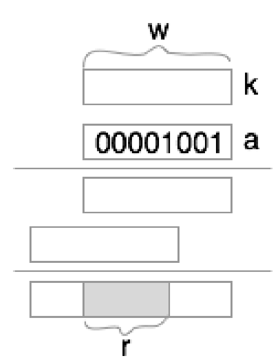
6 Order statistics

- Find kth smallest element.
- $pivot(p, q, k)$: find kth element within $[p, q)$ recursively.
  - Worst case: $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$
  - Expectation: $E[T(n)] = \Theta(n)$
    * X_i: indicates if split into $(i, n-i-1)$
    * $T_i = T(\max\{i, n-i-1\}) + \Theta(n)$
    * $T(n) = \sum X_i T_i$
    * $E[T(n)] = \sum_{i=0}^{n-1} T_i/n \leq \Theta(n) + \frac{2}{n} \sum_{k=n/2}^{n} T(k)$
    * $E[T(n)] \leq c\,n$ by mathematical induction: claim for n, prove for $n+1$.
- Median pivot
  - Recursively find median of median.
  - Choose median of median as pivot.
  - Partition like above.
  - $T(n) \leq T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$
  - $T(n) \leq \Theta(n)$ by induction.



7 Hash

- Prehash: map keys to non-negative integers.
- Hash: map the large key space to smaller space.
- Chaining: chain up the collided items.
- Simple uniform hashing
  - Assumption: a key is equally likely to be mapped into any slot. The hash value is *r.v.*. This assumption generally does NOT hold true, especially when hash function is determinant.
  - Load factor $\alpha = n/m$: expected length of chain, where n is # keys, m is # slots.
  - Complexity: $O(1 + \alpha)$.
- Hash functions:
  - Mod: $h(k) = k\%m$, where m is prime.
  - Mul: $h(k) = [(a \cdot k) \,\%\, 2^w] >> (w - r)$, where w is # bits in k (key), and r is # bits in m (# slots).



  - Universal: $h_{a,b}(k) = [(a\,k + b) \,\%\, p] \,\%\, m$, where $k \in \mathcal{K}$, $p > |\mathcal{K}|$ is picked as prime, and a, b are randomly picked within $\{0, ..., p-1\}$.
    * $Pr\{h_1(k_1) = h_2(k_2)\} = 1/m$, given k_1, k_2.

- Applications
  - Dict: random hash + dynamic table
  - String match: Rabin-Karp
    - $h(s) = (\sum ord(s_i) \cdot base^i) \,\%\, m$
    - $h(s_{(i+1):(i+l)}) = \{[h(s_{i:(i+l-1)}) - ord(s_i)] \cdot base + ord(s_{i+l})\} \,\%\, m$

7.1 Universal hashing

- Adversary model: given hash function is not random, adversary can always find a set of collided keys.
- Example: $h_{a,b}(k) = [(a\,k + b) \,\%\, p] \,\%\, m$. Each dict is instantiated with different a and b.
- Universal: $|\{h \in \mathcal{H} : h(x) = h(y), x \neq y\}| = |\mathcal{H}|/m$.
  - Collision: $Pr\{h_1(x) = h_2(y)\} = 1/m$, given x, y.
  - Theorem: given x, $E[\#collisions] < n/m = \alpha$. Proof: given h is random, denote $r.v.$ $C_{x,y} = \mathbb{1}\{h(x) = h(y)\}$, and $C_x = \sum_{y \in \mathcal{K} \setminus x} C_{x,y}$, then
    - $E[C_{x,y}] = 1/m$
    - $E[C_x] = \sum_{y \in \mathcal{K} \setminus x} E[C_{x,y}] = \frac{n-1}{m}$
- Example $h_a(k) = (\sum a_i k_i) \,\%\, m$, where $a = (\overline{a_r..a_1 a_0})_m$ and $k = (\overline{k_r..k_1 k_0})_m$. Then $|\mathcal{H}| = |\mathcal{K}| = m^{r+1}$.
  - Assume x, y differ in 0th bit, i.e. $x_0 \neq y_0$.
  - $\sum_{i=0}^{r} a_i(x_i - y_i) \equiv 0 \ (mod\ m)$
  - $a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \ (mod\ m)$
  - A theory of Finite (Galois) Field: $\forall z \not\equiv 0 \in \mathbb{Z}_m, m \in P, then\ \exists z^{-1}, s.t.\ z\,z^{-1} \equiv 1 \ (mod\ m)$.
  - $a_0 \equiv [-\sum_{i=1}^{r} a_i(x_i - y_i)](x_0 - y_0)^{-1} \ (mod\ m)$
  - # choices to collide at a_0: m^r ($a_1..a_r$ are free).

7.2 Perfect hashing

- Two-level hashing
  - Allow n_i collisions in ith slot at 1st level.
  - Nearly no collision at 2nd level if # slots $m_i = n_i^2$.
  - Worst complexity: $O(1)$.
  - Theorem: Hash n keys into $m = n^2$ slots using random h in universal hashing would lead to $E[\#collisions] < 1/2$.
    - $E[C_{x,y}] = \frac{1}{m} = \frac{1}{n^2}$
    - $E[C] = \binom{n}{2}\frac{1}{m} = \frac{n-1}{2n} < 1/2$

8 Amortization

- Dynamic table
  - Grow/shrink the table when necessary.
  - Load factor $\alpha = n/m$, n is # items, m is # slots.
  - Double the table when $\alpha > \overline{\alpha}$
  - Shrink the table when $\alpha < \underline{\alpha}$
  - Aggregate analysis on insertion
    - k insertions involve $\log k$ doublings
    - Doubling cost: $\Theta(2^1 + 2^2 + ... + 2^{\log k}) = \Theta(k)$
    - Insertion cost: $\Theta(k)$
    - Amortized cost: $\frac{1}{k}(\Theta(k) + \Theta(k)) = \Theta(1)$
- Aggregate method
- Accounting method
  - Charge ith op amortized cost $\hat{c}_i$.
  - Overcharged are stored to bank.
  - undercharged are taken from bank.
  - Design charge scheme such that $balance \geq 0$.
- Potential method
  - Design Φ_i bound with data structure D_i.
  - Requirements:
    - $\Phi_0 = 0$
    - $\Phi_i \geq 0, \forall i$
    - $\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} \geq 0$
  - Analyze $\hat{c}_i$ for different cases according to Φ_i.
  - Any Φ_i satisfies requirements will do. Better Φ_i yields tighter upper bound.
- Example: $\overline{\alpha} = 1, \underline{\alpha} = 1/4$
  - Potential

$$\Phi_i = \begin{cases} 2n_i - m_i & ,\alpha_i \geq 1/2 \\ n_i/2 - m_i & ,\alpha_i < 1/2 \end{cases}$$

9 Numbers

9.1 Catalan numbers

- Def
  - P: set of balanced parentheses.
  - $\perp \in P$ ($\perp$ means empty).
  - If $\alpha, \beta \in P$, then $(\alpha)\beta \in P$.
- $C_n = |P_n|$ with n pairs of parentheses.
  - $C_0 = C_1 = 1$.
  - $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$

9.2 High precision

- Newton's method
  - Tangent on x_i: $y = f(x_i) + f'(x_i)(x - x_i)$
- Square root
  - $x = \sqrt{a} \Rightarrow f(x) = x^2 - a = 0$
  - $x_{i+1} = x_i - f(x_i)/f'(x_i) = (x_i + a/x_i)/2$
  - Quadratic convergence: # ✓ digits $\propto$ (# iters)^2
- Multiplication (Karatsuba)

- $z = x\,y$
- $x = x_1\,r^{n/2} + x_0$
- $z = z_0 + z_2 + z_1$
  - $z_0 = x_0\,y_0$
  - $z_2 = x_2\,y_2$
  - $z_1 = (x_0 + x_1)(y_0 + y_1) - z_0 - z_2$
- $T(n) = 3T(n/2) + \Theta(n) = \Theta(n^{\log 3}) \approx \Theta(n^{1.58})$

- Advanced multiplication
  - gmpy2
  - Schonhage-Strassen: FFT, $\Theta(n\,\lg n\,\lg\lg n)$.
  - Furer: $\Theta(n\,\log n\,2^{O(\log^* n)})$
- Division
  - $x = R/b \Rightarrow f(x) = 1/x - b/R = 0$
  - $x_{i+1} = 2x_i - bx_i^2/R$
  - $R = (\overline{r_{d-1}..r_1 r_0})_2$, power of 2 is easy to divide.
  - $T(n) = O(\lg n\, n^\alpha)$, break down to mul's.
- When precision digits grow large, complexity of mul, div, sqrt becomes nearly the same as $O(n^\alpha)$.

10 Graph

- Edge
  - Tree edge: $old \rightarrow new$
  - Forward edge: given path $x_0 \rightarrow ... \rightarrow x_n$, any edge not in the path that points to descendant.
  - Backward edge: ..., to ascendant.
  - Cross edge: Between subtrees.
  - Undirected G can have tree edges and back edges.
- Search
  - Breadth first search (BFS)
  - Depth first search (DFS)
- Min span tree (DFS/BFS): search and save edges.
- Cycle detection (DFS): find back edges.
  - Back edge: Edge($v_2 \rightarrow v_1$), v_1 is ascendant.
  - v_1 is ascendant of v_2 if $(t_1^{in}, t_1^{out}) \supset (t_2^{in}, t_2^{out})$.
  - Practically we also need to check if v_1 and v_2 have the same root. It is possible that we first DFS v_1, and then v_2.
  - In short, inside or disjoint is good (acyclic).
- Topological sort (DFS):
  - Append the out nodes.
  - Reverse the order.
- Dependency management (DFS):
  - Reverse edges and run topological sort.

10.1 Dijkstra

- Single source shortest path
- Do NOT allow negative weights
- Outputs
  - $d[v]$: shortest distance from source to v.
  - $p[v]$: parent of v in the shortest path.
- Dynamic programming
  - S is safe set, within which have been reached by shortest path from source.
  - R is the remaining set.
- Optimize
  - Maintain d as priority queue, so that each time *EXTRACT-MIN* only takes $O(\log|V|)$
  - Total complexity: $O(|E| + |V|\log|V|)$
- Single source single destination
  - S_f forward safe set.
  - S_b backward safe set.
  - Stop when two sets intersect.

10.2 Bellman-Ford

- Single source shortest path that allows negative weights.
- Mark $d[v] = -\text{inf}$ to indicate negative cycle.
- Algorithm
  - Relax ALL edges for $|V| - 1$ passes.
  - Relax one more time to check negative cycle.
- Complexity $O(|V||E|)$.

11 Misc