MODULE-1

FUNDAMENTALS OF LOGICS

Basic connectives and truth tables, Logical equivalence – The laws of Logic, Logical implication – Rules of Inference. Fundamentals of Logic contd.: The Use of Quantifiers-Quantifiers, Definitions, and the Proofs of Theorems.

Basic Connectives and Truth Tables

Definitions:

- A *proposition* (or *statements*) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both. we use the lowercase letters of the alphabet (such as p, q, and r) to represent these *propositions*.
- The truth value of a proposition is *true*, denoted by T, if it is a true proposition, and the truth value of a proposition is *false*, denoted by F, if it is a false proposition.
- Let p be a proposition. The *negation* of p, denoted by "$\neg p$" (also denoted by $\bar{p}$), is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.
  Example: Let p: Michael's PC runs Linux, then $\neg p$ will be "Michael's PC does not run Linux."

TABLE 1

The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

TABLE 2

The Truth Table for the Conjunction of two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3

The Truth Table for the Disjunction of two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- New propositions, called *compound propositions*, are formed from existing propositions using logical operators. These logical operators are also called *connectives*.

  - Let p and q be propositions. The *conjunction* of p and q, denoted by "$p \wedge q$", is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.
    Example: Let p: Today is Friday, q: It is raining today, then, $p \wedge q$: Today is Friday and it is raining today.
  - Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.
    Example: Let p: Today is Friday, q: It is raining today, then, $p \vee q$: Today is Friday or it is raining today.

- Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$ sometimes by $p \veebar q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.
- Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or antecedent or premise) and q is called the *conclusion* (or consequence).
  Example: Let p: Maria learns discrete mathematics, q: Maria will find a good job, then, $p \rightarrow q$: If Maria learns discrete mathematics, then she will find a good job.

TABLE 4

The Truth Table for the Exclusive or of two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5

The Truth Table for the Conditional Statement

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Consider the conditional statement $p \rightarrow q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \rightarrow p$ is called the *converse* of $p \rightarrow q$. The *contrapositive* of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called the *inverse* of $p \rightarrow q$.
  Example: Let $p \rightarrow q$: If it is raining, then the home team wins. Then,
  $q \rightarrow p$: If the home team wins, then it is raining
  $\neg q \rightarrow \neg p$: If the home team does not win, then it is not raining
  $\neg p \rightarrow \neg q$: If it is not raining, then the home team does not win.
- Let p and q be propositions. The *biconditional* statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.
  Example: Let p: You can take the flight, q: You buy a ticket, then, $p \leftrightarrow q$: You can take the flight if and only if you buy a ticket.

TABLE 6

The Truth Table for the Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- A bit can be used to represent a truth value, because there are only two truth values, namely, 1 and 0. We will use a "1" bit to represent true and a "0" bit to represent false. That is, 1 represents T (true), 0 represents F (false).

Table

p	$\neg p$
0	1
1	0

Table

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Problems:

1. Let p, q and r denote the following statements about a particular triangle ABC.
   p: Triangle ABC is Isosceles.
   q: Triangle ABC is Equilateral.
   r: Triangle ABC is Equiangular.

Translate each of the following into English statements.

a) $q \rightarrow p$
b) $\neg p \rightarrow \neg q$
c) $q \leftrightarrow r$
d) $p \wedge \neg q$
e) $r \rightarrow p$

Sol:

a) If triangle ABC is equilateral, then it is isosceles.
b) If triangle ABC is not isosceles, then it is not equilateral.
c) Triangle ABC is equilateral, if and only if it is equiangular.
d) Triangle ABC is equilateral, but it is not equilateral.
e) If triangle ABC is equiangular, then it is isosceles.

2. Let *s*, *t*, and *u* denote the following primitive statements:
   s: Phyllis goes out for a walk.
   t: The moon is out.
   u: It is snowing.

(i) Translate each of the symbolic form into an English sentence.

a) $(t \wedge \neg u) \rightarrow s$
b) $t \rightarrow (\neg u \rightarrow s)$
c) $\neg(s \leftrightarrow (u \vee t))$

Sol: a) If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) If the moon is out, then if it is not snowing Phyllis goes out for a walk. [So $\neg u \rightarrow s$ is understood to mean $(\neg u) \rightarrow s$ as opposed to $\neg(u \rightarrow s)$.]

c) It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

(ii) Write the following in symbolic form.

a) Phyllis will go out walking if and only if the moon is out.
b) If it is snowing and the moon is not out, then Phyllis will not go out for a walk.
c) It is snowing but Phyllis will still go out for a walk.

Sol: a) Here the words "if and only if" indicate that we are dealing with a biconditional. In symbolic form this becomes $s \leftrightarrow t$.

b) This compound statement is an implication where the hypothesis is also a compound statement. One may express this statement in symbolic form as $(u \wedge \neg t) \rightarrow \neg s$.

c) Now we come across a new connective — namely, but. In our study of logic we shall follow the convention that the connectives *but* and *and* convey the same meaning. Consequently, this sentence may be represented as $u \wedge s$.

3. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore", respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$ b) $p \wedge q$ c) $\neg p \vee q$ d) $p \rightarrow \neg q$ e) $\neg q \rightarrow p$ f) $\neg p \rightarrow \neg q$
g) $p \leftrightarrow \neg q$ h) $\neg p \wedge (p \vee \neg q)$.

Sol: a) $\neg q$: Sharks have not been spotted near the shore.

b) $p \wedge q$: Swimming at the New Jersey shore is allowed, and sharks have been spotted near the shore.

c) $\neg p \vee q$: Swimming at the New Jersey shore is not allowed, or sharks have been spotted near the shore.

d) $p \rightarrow \neg q$: If swimming at the New Jersey shore is allowed, then sharks have not been spotted near the shore.

e) $\neg q \rightarrow p$: If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed.

f) $\neg p \rightarrow \neg q$: If swimming at the New Jersey shore is not allowed, then sharks have not been spotted near the shore.

g) $p \leftrightarrow \neg q$: Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore.

h) $\neg p \wedge (p \vee \neg q)$: Swimming at the New Jersey shore is not allowed, and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore.

4. Let p, q, and r be the propositions

p: Grizzly bears have been seen in the area.
q: Hiking is safe on the trail.
r: Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives (including negations).

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.
b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.
d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

e) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

Sol: a) $r \wedge \neg p$

b) $\neg p \wedge q \wedge r$

c) $r \longrightarrow (q \leftrightarrow \neg p)$

d) $\neg q \wedge \neg p \wedge r$

e) $(p \wedge r) \longrightarrow \neg q.$

5. Construct a truth table for each of these compound propositions.

a) $\neg(p \vee \neg q) \rightarrow \neg p$

b) $p \rightarrow (q \rightarrow r)$

c) $(p \vee \neg q) \rightarrow q$

d) $(p \wedge (p \rightarrow q)) \rightarrow q$

Sol:

a)

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee \neg q}$	$\boldsymbol{\neg(p \vee \neg q) \rightarrow \neg p}$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

b)

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
F	F	F	T	T
F	F	T	T	T
F	T	F	F	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

c)

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\neg q}$	$\boldsymbol{p \vee \neg q}$	$\boldsymbol{(p \vee \neg q) \rightarrow q}$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

d)

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

6. Write the truth tables for the compound statement.

(i) $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$.

(ii) $q \wedge (\neg r \to p)$.

Sol:

(i)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	F	F	T	F
F	T	T	T	T	T	T
T	F	F	F	T	T	F
T	F	T	F	T	T	T
T	T	F	F	T	T	F
T	T	T	T	T	T	T

(ii)

p	q	r	$\neg r$	$\neg r \to p$	$q \wedge (\neg r \to p)$
F	F	F	T	F	F
F	F	T	F	T	F
F	T	F	T	F	F
F	T	T	F	T	T
T	F	F	T	T	F
T	F	T	F	T	F
T	T	F	T	T	T
T	T	T	F	T	T

Home work

1. Let p and q be the propositions, p: It is below freezing, q: It is snowing. Write the logical connectives.

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

2. Construct a truth table for each of these compound propositions.

a) $q \leftrightarrow (\neg p \vee \neg q)$
b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$
c) $[(p \wedge q) \vee (\neg r)] \leftrightarrow p$

Logical equivalence

❖ A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology* $(T_0 or\ T)$. A compound proposition that is always false is called a *contradiction* $(F_0 or\ F)$. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

❖ The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology, i.e p and q are called *logically equivalent*, when p is true (respectively, false) if and only if the q is true (respectively, false). The notation $p \equiv q$ or $p \Leftrightarrow q$ denotes that p and q are logically equivalent.

❖ De Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$

❖ *Dual:* Let *s* be a statement. If *s* contains no logical connectives other than ∧ and ∨, then the *dual* of *s*, denoted s^d, is the statement obtained from *s* by replacing each occurrence of ∧ and ∨ by ∨ and ∧, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

Example: Given the primitive statements p, q, r and the compound statement

$$s: (p \wedge \neg q) \vee (r \wedge T_0),$$

then, dual of *s* is $s^d: (p \vee \neg q) \wedge (r \vee F_0)$.

❖ *The Principle of Duality.* Let *s* and *t* be statements that contain no logical connectives other than ∧ and ∨. If $s \equiv t$, then $s^d \equiv t^d$.

TABLE Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TABLE Logical Equivalences.

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

1. Show that the implication $\neg(p \to q) \to \neg q$ is a tautology.

Sol: The truth table for this implication

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

We see that the truth values of the implication is all T's, hence it is a tautology.

2. Show that

(i) $p \to (p \vee q)$ is a tautology

(ii) $p \wedge (\neg p \wedge q)$ is a contradiction.

Sol:

p	q	$p \vee q$	$p \to (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
F	F	F	T	T	F	F
F	T	T	T	T	T	F
T	F	T	T	F	F	F
T	T	T	T	F	F	F

3. Prove that for any propositions p, q, r, the compound proposition is a tautology $[(p \to q) \land (q \to r)] \to (p \to r)$.

Sol:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \land (q \to r)$	$p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

4. Show that $\neg (p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.

Sol:

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

5. Show that $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ are logically equivalent.

Sol:

p	q	r	$q \land r$	$p \lor (q \land r)$	$p \lor q$	$p \lor r$	$(p \lor q) \land (p \lor r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

6. Prove that for any propositions p, q, r, $[(p \lor q) \to r] \Leftrightarrow [(p \to r) \land (q \to r)]$.

Sol:

p	q	r	$p \lor q$	$(p \lor q) \to r$	$p \to r$	$q \to r$	$(p \to r) \land (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

7. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence.

Sol: $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ *by the logical equivalence involving conditional statements*

$\equiv \neg(\neg p) \wedge \neg q$ *by the second De-Morgan law*

$\equiv p \wedge \neg q$ *by the double negation law.*

8. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence.

Sol: $\neg\big(p \vee (\neg p \wedge q)\big) \equiv \neg p \wedge \neg(\neg p \wedge q)$ *by the second De-Morgan law*

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ *by the first De-Morgan law*

$\equiv \neg p \wedge (p \vee \neg q)$ *by the double negation law*

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ *by the second distribution law*

$\equiv F \vee (\neg p \wedge \neg q)$ *because* $\neg p \wedge p \equiv F$

$\equiv (\neg p \wedge \neg q) \vee F$ *by the commutative law for disjunction*

$\equiv \neg p \wedge \neg q$ *by the identity law for F.*

9. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology by developing a series of logical equivalence.

Sol: $(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ *by the logical equivalence involving conditional statements*

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ *by the first De-Morgan law*

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ *by the associative and commutative law for disjunction*

$\equiv T \vee T$ *by negation law*

$\equiv T$ *by domination law.*

10. Negate and simplify the compound statement $(p \vee q) \rightarrow r$ by developing a series of logical equivalence.

Sol: $(p \vee q) \rightarrow r \equiv \neg(p \vee q) \vee r$ *by the logical equivalence involving conditional statements*

$\equiv \neg[\neg(p \vee q) \vee r]$ *by taking the negation*

$\equiv \neg\neg(p \vee q) \wedge \neg r$ *by the first De-Morgan law*

$\equiv (p \vee q) \wedge \neg r$ *by the double negation law.*

Homework

Let p, q, r denote the primitive statements.

1. Use truth tables to verify the following logical equivalence.

(i) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$.

(ii) $[p \rightarrow (q \vee r)] \equiv [\neg r \rightarrow (p \rightarrow q)]$.

(iii) $(p \vee q) \wedge \neg(\neg p \wedge q) \equiv p$.

(iv) $\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \equiv q \wedge r$.

Logical implication: Rules of Inference

- If p, q are arbitrary statements such that $p \to q$ is a tautology, then we say that p *logically implies* q and we write $p \Rightarrow q$ to denote this situation.
- $p \nRightarrow q$ indicate that $p \to q$ is *not* a tautology, so the given implication (namely, $p \to q$) is *not* a logical implication.
- If $p \equiv q$, then the statement $p \leftrightarrow q$ is a tautology, so the statements p, q have same truth values. Under these conditions the statements $p \to q$, $q \to p$ are tautologies and we have $p \Rightarrow q$ and $q \Rightarrow p$. Conversely, if $p \Rightarrow q$ and $q \Rightarrow p$ then, $p \equiv q$.

Rules of inference:

- An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*.
- An argument is *valid* if the truth of all its premises implies that the conclusion is true.
- An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables.
- An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.
- From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $(p_1 \land p_2 \land \cdots \land p_n) \to q$ is a tautology.

Rule of Inference	Related Logical Implication	Name of Rule
1) p $p \to q$ $\therefore q$	$[p \land (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$	Law of the Syllogism
3) $p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \land \neg q] \to \neg p$	Modus Tollens
4) p q $\therefore p \land q$		Rule of Conjunction
5) $p \lor q$ $\neg p$ $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Rule of Disjunctive Syllogism
6) $\neg p \to F_0$ $\therefore p$	$(\neg p \to F_0) \to p$	Rule of Contradiction
7) $p \land q$ $\therefore p$	$(p \land q) \to p$	Rule of Conjunctive Simplification
8) p $\therefore p \lor q$	$p \to p \lor q$	Rule of Disjunctive Amplification
9) $p \land q$ $p \to (q \to r)$ $\therefore r$	$[(p \land q) \land [p \to (q \to r)]] \to r$	Rule of Conditional Proof
10) $p \to r$ $q \to r$ $\therefore (p \lor q) \to r$	$[(p \to r) \land (q \to r)] \to [(p \lor q) \to r]$	Rule for Proof by Cases
11) $p \to q$ $r \to s$ $p \lor r$ $\therefore q \lor s$	$[(p \to q) \land (r \to s) \land (p \lor r)] \to (q \lor s)$	Rule of the Constructive Dilemma
12) $p \to q$ $r \to s$ $\neg q \lor \neg s$ $\therefore \neg p \lor \neg r$	$[(p \to q) \land (r \to s) \land (\neg q \lor \neg s)] \to (\neg p \lor \neg r)$	Rule of the Destructive Dilemma

1. Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

Sol: Let the propositions be,
p -“It is sunny this afternoon,” q - “It is colder than yesterday,” r - “We will go swimming,” s - “We will take a canoe trip,” and t - “We will be home by sunset.”
Then the premises become $\neg p \land q$, $r \to p$, $\neg r \to s$, and $s \to t$.
The conclusion is simply “t”.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7) .

2. Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Sol: Let the proposition be, p -"You send me an e-mail message," q - "I will finish writing the program," r - "I will go to sleep early," and s - "I will wake up feeling refreshed."
Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$.
The desired conclusion is $\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Law of syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Law of syllogism using (4) and (5)

3. Establish the validity of the argument

$$\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ \underline{\neg s} \\ \therefore\ t. \end{array}$$

Sol:

Step	Reason
1. $p \wedge q$	Premise
2. p	Rule of Conjunctive Simplification using (1)
3. $p \rightarrow (r \wedge q)$	Premise
4. $r \wedge q$	Rule of Detachment using (2), (3)
5. r	Rule of Conjunctive Simplification using (4)
6. $r \rightarrow (s \vee t)$	Premise
7. $s \vee t$	Rule of Detachment using (5), (6)
8. $\neg s$	Premise
9. $\therefore t$	Rule of Disjunctive Syllogism using (7), (8).

4. Establish the validity of the argument

$$\begin{array}{l} p \to (q \to r) \\ \neg q \to \neg p \\ \underline{p \qquad\qquad} \\ \therefore\ r. \end{array}$$

Sol: <u>Step</u> — <u>Reason</u>

1. p — Premise
2. $\neg q \to \neg p$ — Premise
3. $p \to q$ — $p \to q \equiv \neg q \to \neg p$ using (2)
4. q — Rule of Detachment using (1), (3)
5. $p \wedge q$ — Rule of Conjunction using (1), (4)
6. $p \to (q \to r)$ — Premise
7. $(p \wedge q) \to r$ — $p \to (q \to r) \equiv (p \wedge q) \to r$ using (6)
8. $\therefore r$ — Rule of Detachment using (5), (7)

5. Prove that the following are valid arguments

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \to r \\ \underline{\neg r \qquad\qquad} \\ \therefore\ p \end{array}$$

Sol: <u>Step</u> — <u>Reason</u>

1. $q \to r$ — Premise
2. $\neg r$ — Premise
3. $\neg q$ — Modus tollens using (1) and (2)
4. $\neg p \leftrightarrow q$ — Premise
5. $(\neg p \to q) \wedge (q \to \neg p)$ — by the logical equivalence involving biconditional statements using (4)
6. $\neg p \to q$ — Rule of Conjunctive Simplification using (5)
7. $\neg\neg p$ — Modus tollens using (3) and (6)
8. $\therefore\ p$ — Rule of double negation law for (7)

6. Provide the reasons for the steps verifying the following argument.

$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ \underline{p \wedge t \qquad\qquad} \\ \therefore\ u. \end{array}$$

<u>Step</u> — **Sol:** <u>Reason</u>

1. $p \to q$ — Premise
2. $q \to (r \wedge s)$ — Premise
3. $p \to (r \wedge s)$ — Law of syllogism using (1) and (2)
4. $p \wedge t$ — Premise
5. p — Conjunctive simplification using (4)
6. $r \wedge s$ — Rule of detachment using (5) and (3)
7. r — Rule of conjunctive simplification using (6)

8. $\neg r \vee (\neg t \vee u)$	Premise
9. $\neg(r \wedge t) \vee u$	Associative law of $\vee$ and De Morgans Law using (8)
10. t	Rule of conjunctive simplification using (4)
11. $r \wedge t$	Rule of conjunction using (7) and (10)
12. $\therefore u$	Law of double negation and Rule of Disjunctive syllogism using (9) and (11).

Home work

1. Show that the premises "If Rochelle gets the supervisor's position and works hard then she gets a raise", "If she gets a raise, then she'll buy a new car", "She has not purchased a new car", lead to the conclusion "Rochelle did not get the supervisor's position or she did not work hard".
2. Establish the validity of the argument

(i)
$$\begin{array}{l} p \to r \\ \neg p \to q \\ \underline{q \to s} \\ \therefore \neg r \to s. \end{array}$$

(ii)
$$\begin{array}{l} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \to \neg t} \\ \therefore p \end{array}$$

(iii)
$$\begin{array}{l} (\neg p \vee \neg q) \to (r \wedge s) \\ r \to t \\ \underline{\neg t} \\ \therefore p \end{array}$$

The Use of Quantifiers

❖ A declarative sentence is an *open statement* if
(i) it contains one or more variables, and
(ii) it is not a statement, but
(iii) it becomes a statement when the variables in it are replaced by certain allowable choice.

Example: "The number $x + 2$ is an even integer" is an open statement that contains the single variable x.

❖ "Certain allowable choice" constitutes the *universe* or *the universe of discourse* for the open statement.

❖ The open statement is denoted by $p(x)$ $[q(x)\ or\ r(x), etc]$ and $q(x, y)$ to represent open statement that contains two variables.
Example: $p(x)$: The number $x + 2$ is an even integer.
$\neg p(x)$: The number $x + 2$ is not an even integer.
$q(x, y)$: $x - y$ is an even integer.

❖ The variable x in each open statements $p(x)$ is called the free variable.

❖ *Quantification* expresses the extent to which an open statement is true over a range of elements. In English, the words *for all*, *some*, *many*, *none*, and *few* are used in quantifications.

❖ Two types of quantifiers: *existential quantifiers* and *universal quantifiers*.

❖ The *existential quantifiers*: "For some x", "For at least one x" or "There exists an x such that." We use the notation $\exists x$.
Ex: "For some x, $p(x)$" in symbolic form "$\exists x\ p(x)$."

❖ The *universal quantifiers*: "For all x", "For any x", "For each x", "For every x". We use the notation $\forall x$. "For all x,y", "For any x,y", "For each x, y", "For every x,y" We use the notation $\forall x, y$.

❖Let $p(x)$, $q(x)$ be open statements defined for a given universe.

➢ The open statements $p(x)$ and $q(x)$ are called (logically) *equivalent*, denoted as $\forall x\ [p(x) \equiv q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe (i.e. $p(a) \equiv q(a)$ for each a in the universe).

➢ If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe, we say $p(x)$ *logically implies* $q(x)$ and denoted as $\forall x\ [p(x) \rightarrow q(x)]$.

❖For open statements $p(x), q(x)$ defined for a prescribed universe and the universally quantified statement $\forall x\ [p(x) \rightarrow q(x)]$, we define:

1) The *contrapositive* as $\forall x\ [\neg q(x) \rightarrow \neg p(x)]$
2) The *converse* as $\forall x\ [q(x) \rightarrow p(x)]$
3) The *inverse* as $\forall x\ [\neg p(x) \rightarrow \neg q(x)]$.

❖ *Theorems* are the statements of mathematical interest, statements that are known to be true.

❖A theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion.

❖Sometimes the term theorem is used only to describe major results that have many and varied consequences. Certain of these consequences that follow rather immediately from a theorem are termed *corollaries*.

❖Less important theorems sometimes are called *propositions*.

❖We demonstrate that a theorem is true with a proof. A *proof* is a valid argument that establishes the truth of a theorem.

❖The statements used in a proof can include *axioms* (or *postulates*), which are statements we assume to be true, the premises, if any, of the theorem, and previously proven theorems.

❖A less important theorem that is helpful in the proof of other results is called a *lemma* (plural lemmas or lemmata).

❖A *conjecture* is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Table

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Table Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x:

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$$
$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x) \vee \exists x\, q(x)]$$
$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\, p(x) \wedge \forall x\, q(x)]$$
$$[\forall x\, p(x) \vee \forall x\, q(x)] \Rightarrow \forall x\,[p(x) \vee q(x)]$$

Table Rules for Negating Statements with One Quantifier

$$\neg[\forall x\, p(x)] \Leftrightarrow \exists x\, \neg p(x)$$
$$\neg[\exists x\, p(x)] \Leftrightarrow \forall x\, \neg p(x)$$
$$\neg[\forall x\, \neg p(x)] \Leftrightarrow \exists x\, \neg\neg p(x) \Leftrightarrow \exists x\, p(x)$$
$$\neg[\exists x\, \neg p(x)] \Leftrightarrow \forall x\, \neg\neg p(x) \Leftrightarrow \forall x\, p(x)$$

❖ **The Rule of universal specification:**

If $p(x)$ is an open statement for a given universe, and if $\forall x\, p(x)$ is true, then $p(a)$ is true for each a in the universe.

❖ **The Rule of universal generalization:**

If an open statement $p(x)$ is proved to be true when x is replaced by any *arbitrarily chosen* element c from our universe, then the universally quantified statements $\forall x\, p(x)$ is true.

Note: The rule extends beyond a single variable.

For example, we have an open statement $q(x, y)$ that is proved to be true when x and y are replaced by any *arbitrarily chosen* element from the same universe, then the universally quantified statements $\forall x, y\ \ q(x, y)$ is true.

Definition I: The integer n is *even* if there exists an integer a such that $n = 2a$, and n is *odd* if there exists an integer a such that $n = 2a + 1$.

Problems

1. Consider the open statements. $p(t)$: t has two sides of equal length. $q(t)$: t is an isosceles triangle. $r(t)$: t has two angles of equal measure. Then the arguments:
In the triangle XYZ there is no pair of angles of equal measure.
If a triangle has two sides of equal length, then it is isosceles.
If a triangle is isosceles, then it has two angles of equal measure.
Therefore triangle XYZ has no two sides of equal length.
Write the arguments symbolically and validate the arguments.

Sol: Symbolically-

$$\begin{array}{l} \neg r(c) \\ \forall t\,[p(t) \to q(t)] \\ \forall t\,[q(t) \to r(t)] \\ \hline \therefore \neg p(c) \end{array}$$

Sol: Step	Reason
1) $\forall t\,[p(t) \to q(t)]$	Premise
2) $p(c) \to q(c)$	Rule of universal specification using (1)
3) $\forall t\,[q(t) \to r(t)]$	Premise
4) $q(c) \to r(c)$	Rule of universal specification using (3)
5) $p(c) \to r(c)$	Law of the syllogism using (2) and (4)
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	Modus Tollens using (5) and (6)

2. Let $j(x)$, $s(x)$, and $p(x)$ be open statements that are defined for a given universe. Establish the validity of the argument.

$$\begin{array}{l} \forall x\,[(j(x) \vee s(x)) \to \neg p(x)] \\ p(m) \\ \hline \therefore \neg s(m) \end{array}$$

Sol: Step	Reason
1) $\forall x\,[(j(x) \vee s(x)) \to \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $(j(m) \vee s(m)) \to \neg p(m)$	Rule of universal specification using (1)
4) $p(m) \to \neg(j(m) \vee s(m))$	Law of double negation,$(q \to t) \equiv (\neg t \to \neg q)$ using (3)
5) $p(m) \to (\neg j(m) \wedge \neg s(m))$	DeMorgan's law using (4)
6) $\neg j(m) \wedge \neg s(m)$	Modus Ponens using (2) and (5)
7) $\neg s(m)$	Rule of Conjunctive simplification using (6)

3. Let $p(x)$, $q(x)$, and $r(x)$ be open statements that are defined for a given universe. Establish the validity of the argument.

$$\begin{array}{l} \forall x\,[p(x) \to q(x)] \\ \forall x\,[q(x) \to r(x)] \\ \hline \therefore \forall x\,[p(x) \to r(x)] \end{array}$$

Sol: Step	Reason
1) $\forall x\,[p(x) \to q(x)]$	Premise
2) $p(c) \to q(c)$	Rule of universal specification using (1)
3) $\forall x\,[q(x) \to r(x)]$	Premise
4) $q(c) \to r(c)$	Rule of universal specification using (3)
5) $p(c) \to r(c)$	Law of the syllogism using (2) and (4)
6) $\therefore \forall x\,[p(x) \to r(x)]$	Rule of universal generalization using (5)

4. Prove that the following argument is valid.

$$\begin{array}{l} \forall x, [p(x) \to \{q(x) \wedge r(x)\}] \\ \forall x, [p(x) \wedge s(x)] \\ \hline \therefore \forall x, [r(x) \wedge s(x)] \end{array}$$

Sol: Step — Reason

1) $\forall x, [p(x) \to \{q(x) \land r(x)\}]$ — Premise
2) $p(c) \to \{q(c) \land r(c)\}$ — Rule of universal specification using (1)
3) $\forall x, [p(x) \land s(x)]$ — Premise
4) $p(c) \land s(c)$ — Rule of universal specification using (3)
5) $p(c)$ — Rule of Conjunctive simplification using (4)
6) $q(c) \land r(c)$ — Modus Ponens using (2) and (5)
7) $r(c)$ — Law of commutative and Rule of Conjunctive simplification using (6)
8) $s(c)$ — Law of commutative and Rule of Conjunctive simplification using (4)
9) $r(c) \land s(c)$ — Rule of conjunction using (7) and (8)
10) $\therefore \forall x, [r(x) \land s(x)]$ — Rule of universal generalization using (9)

Theorem: For all integers k and l**,** if k, l are both odd, then $k+l$ is even.
Proof: Since k and l are odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a, b, by Definition I. Then, $k + l = (2a + 1) + (2b + 1) = 2(a + b + 1)$.
Since a, b are integers, $a + b + 1 = c$ is an integer, $k + l = 2c$. By Definition I, $k + l$ is even.

Theorem: For all integers k and l**,** if k, l are both odd, then their product kl is also odd.
Proof: Since k and l are odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a, b, by Definition I.
Then, $kl = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$.
Since a, b are integers, $2ab + a + b = c$ is an integer, $kl = 2c + 1$. By Definition I, kl is odd.

Theorem: Give (i) a direct proof (ii) an indirect proof (iii) a contradiction proof, of the following statement. "If m is an even integer, then $m + 7$ is odd."
Proof: (i)Since m is even, we have $m = 2a$ for some integer a.
Then, $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a + 3) + 1$. Since $a + 3$ is an integer, we know that $m + 7$ is odd (by Definition I).
(ii)Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer b and $m = b - 7 = 2b - 8 + 1 = 2(b - 4) + 1$, where $b - 4$ is an integer. Hence m is odd. (because $\forall m[p(m) \to q(m)] \equiv \forall m[\neg q(m) \to \neg p(m)]$).
(iii)Now assume that m is even and that $m + 7$ is also even. Then $m + 7$ even implies that $m + 7 = 2c$ for some integer c. And $m = 2c - 7 = 2c - 8 + 1 = 2(c - 4) + 1$ with $c - 4$ an integer, so m is odd, which is a contradiction. Since the assumption is false, its negation is true. Hence $m + 7$ is odd.

Theorem: Give (i) an indirect proof (ii) a contradiction proof, of the following statement. "For every integer n, if n^2 is odd, then n is odd."
Proof: Let n be any integer.
Then the given statement reads $p \to q$, where $p: n^2$ is odd and $q: n$ is odd.
(i) We prove by contraposition i..e $\neg q \to \neg p$ is true.

Assume that $\neg q$ is true, that is assume that n is not an odd integer. Then, $n = 2k$, where k is an integer. Consequently, $n^2 = (2k)^2 = 2(2k^2)$, so that n^2 is not odd. That is, p is false or equivalently, $\neg p$ is true. This proves the contrapositive statement $\neg q \to \neg p$.

(ii) Assume that $p \to q$ is false, that is, assume that p is true and q is false. Now, q is false means: n is even, so that $n = 2k$ for some integer k. This yields $n^2 = (2k)^2 = 4k^2$ from which it is evident that n^2 is even, that is, p is false. This contradicts the assumption that p is true. This proves $p \to q$ is true (for any integer n).

Theorem: For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$.

Proof: Consider the negation of the conclusion that is, $0 < x \le 5$ and $0 < y \le 5$. Under these circumstances we find that $0 = 0.0 < x\,.\,y \le 5\,.\,5 = 25$**,** so the product xy does not exceed 25**.** Hence the proof. (Proof by contrapositive: $\forall x\ [\neg q(x) \to \neg p(x)]$).

Homework

1. For the universe of all people, consider the open statements $m(x)\!: x$ is a mathematics professor; $c(x)\!: x$ has studied calculus. Consider the argument.

All mathematics professors have studied calculus.
Leona is a mathematics professor.
Therefore Leona has studied calculus.

Write the symbolic form of argument and validate.

2.Find whether the following argument is valid:

No engineering student of first or second semester studies logic.
Anil is an engineering student who studied Logic.

∴ Anil is not in second semester.

3. Give a direct proof for each of the following.

a) for all integers k and l, if k, l are both even, then $k + l$ is even.

b) for all integers k and l, if k, l are both even, then kl is even.

4. Give (i) a direct proof (ii) an indirect proof (iii) a contradiction proof, of the following statement. “If n is an odd integer, then $n + 11$ is an even integer.”