Fundamentals of Photonics
Bahaa E. A. Saleh, Malvin Carl Teich
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CHAPTER

6

POLARIZATION AND CRYSTAL OPTICS

Augustin Jean Fresnel (1788–1827) advanced a theory of light in which waves exhibit transverse vibrations. The equations describing the partial reflection and refraction of light are named after him. Fresnel also made important contributions to the theory of light diffraction.

The polarization of light is determined by the time course of the *direction* of the electric-field vector $\mathscr{E}(\mathbf{r}, t)$. For monochromatic light, the three components of $\mathscr{E}(\mathbf{r}, t)$ vary sinusoidally with time with amplitudes and phases that are generally different, so that at each position $\mathbf{r}$ the endpoint of the vector $\mathscr{E}(\mathbf{r}, t)$ moves in a plane and traces an ellipse, as illustrated in Fig. 6.0-1(a). The plane, the orientation, and the shape of the ellipse generally vary with position.

In paraxial optics, however, light propagates along directions that lie within a narrow cone centered about the optical axis (the z axis). Waves are approximately transverse electromagnetic (TEM) and the electric-field vector therefore lies approximately in the transverse plane (the x-y plane), as illustrated in Fig. 6.0-1(b). If the medium is isotropic, the polarization ellipse is approximately the same everywhere, as illustrated in Fig. 6.0-1(b). The wave is said to be **elliptically polarized.**

The orientation and ellipticity of the ellipse determine the state of polarization of the optical wave, whereas the size of the ellipse is determined by the optical intensity. When the ellipse degenerates into a straight line or becomes a circle, the wave is said to be **linearly polarized** or **circularly polarized**, respectively.

Polarization plays an important role in the interaction of light with matter as attested to by the following examples:

- The amount of light reflected at the boundary between two materials depends on the polarization of the incident wave.
- The amount of light absorbed by certain materials is polarization dependent.
- Light scattering from matter is generally polarization sensitive.

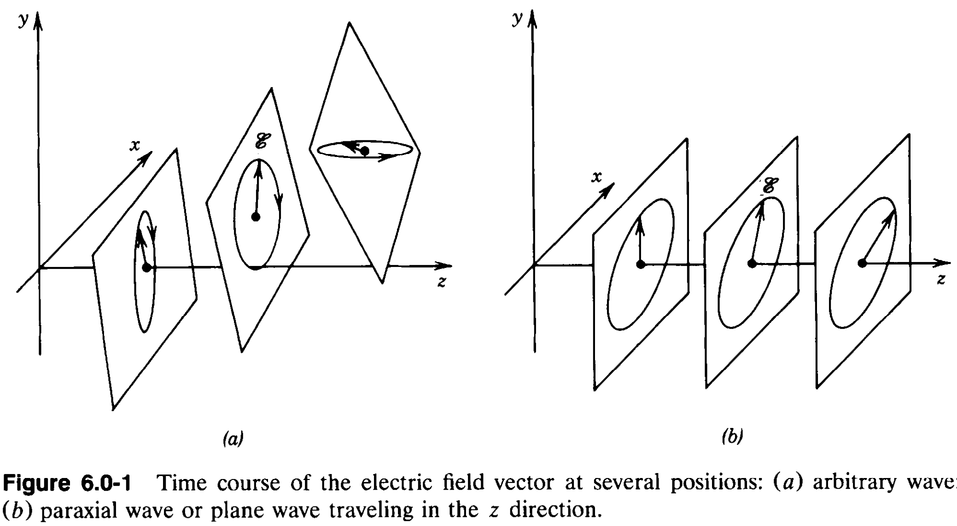


Figure 6.0-1 Time course of the electric field vector at several positions: (a) arbitrary wave; (b) paraxial wave or plane wave traveling in the z direction.

- The refractive index of anisotropic materials depends on the polarization. Waves with different polarizations therefore travel at different velocities and undergo different phase shifts, so that the polarization ellipse is modified as the wave advances (e.g., linearly polarized light can be transformed into circularly polarized light). This property is used in the design of many optical devices.
- So-called optically active materials have the natural ability to rotate the polarization plane of linearly polarized light. In the presence of a magnetic field, most materials rotate the polarization. When arranged in certain configurations, liquid crystals also act as polarization rotators.

This chapter is devoted to elementary polarization phenomena and a number of their applications. Elliptically polarized light is introduced in Sec. 6.1 using a matrix formalism that is convenient for describing polarization devices. Section 6.2 describes the effect of polarization on the reflection and refraction of light at the boundaries between dielectric media. The propagation of light through anisotropic media (crystals), optically active media, and liquid crystals are the subjects of Secs. 6.3, 6.4, and 6.5, respectively. Finally, basic polarization devices (polarizers, retarders, and rotators) are discussed in Sec. 6.6.

6.1 POLARIZATION OF LIGHT

A. Polarization

Consider a monochromatic plane wave of frequency ν traveling in the z direction with velocity c. The electric field lies in the x-y plane and is generally described by

$$\mathscr{E}(z,t) = \operatorname{Re}\left\{\mathbf{A}\exp\left[j2\pi\nu\left(t-\frac{z}{c}\right)\right]\right\}, \tag{6.1-1}$$

where the complex envelope

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}, \tag{6.1-2}$$

is a vector with complex components A_x and A_y. To describe the polarization of this wave, we trace the endpoint of the vector $\mathscr{E}(z,t)$ at each position z as a function of time.

The Polarization Ellipse

Expressing A_x and A_y in terms of their magnitudes and phases, $A_x = a_x \exp(j\varphi_x)$ and $A_y = a_y \exp(j\varphi_y)$, and substituting into (6.1-2) and (6.1-1), we obtain

$$\mathscr{E}(z,t) = \mathscr{E}_x\hat{\mathbf{x}} + \mathscr{E}_y\hat{\mathbf{y}}, \tag{6.1-3}$$

where

$$\mathscr{E}_x = a_x\cos\left[2\pi\nu\left(t-\frac{z}{c}\right)+\varphi_x\right] \tag{6.1-4a}$$

$$\mathscr{E}_y = a_y\cos\left[2\pi\nu\left(t-\frac{z}{c}\right)+\varphi_y\right] \tag{6.1-4b}$$

are the x and y components of the electric-field vector $\mathscr{E}(z,t)$. The components $\mathscr{E}_x$ and $\mathscr{E}_y$ are periodic functions of $t - z/c$ oscillating at frequency ν. Equations (6.1-4) are the parametric equations of the ellipse,

$$\frac{\mathscr{E}_x^2}{a_x^2} + \frac{\mathscr{E}_y^2}{a_y^2} - 2\cos\varphi \frac{\mathscr{E}_x \mathscr{E}_y}{a_x a_y} = \sin^2\varphi, \tag{6.1-5}$$

where $\varphi = \varphi_y - \varphi_x$ is the phase difference.

At a fixed value of z, the tip of the electric-field vector rotates periodically in the x–y plane, tracing out this ellipse. At a fixed time t, the locus of the tip of the electric-field vector follows a helical trajectory in space lying on the surface of an elliptical cylinder (see Fig. 6.1-1). The electric field rotates as the wave advances, repeating its motion periodically for each distance corresponding to a wavelength $\lambda = c/\nu$.

The state of polarization of the wave is determined by the shape of the ellipse (the direction of the major axis and the ellipticity, the ratio of the minor to the major axis of the ellipse). The shape of the ellipse therefore depends on two parameters—the ratio of the magnitudes a_y/a_x and the phase difference $\varphi = \varphi_y - \varphi_x$. The size of the ellipse, on the other hand, determines the intensity of the wave $I = (a_x^2 + a_y^2)/2\eta$, where η is the impedance of the medium.

Linearly Polarized Light

If one of the components vanishes ($a_x = 0$, for example), the light is linearly polarized in the direction of the other component (the y direction). The wave is also linearly polarized if the phase difference $\varphi = 0$ or π, since (6.1-4) gives $\mathscr{E}_y = \pm(a_y/a_x)\mathscr{E}_x$, which is the equation of a straight line of slope $\pm a_y/a_x$ (the + and − signs correspond to $\varphi = 0$ or π, respectively). In these cases the elliptical cylinder in Fig. 6.1-1(b) collapses into a plane as illustrated in Fig. 6.1-2. The wave is therefore also said to have **planar polarization.** If $a_x = a_y$, for example, the plane of polarization makes an angle 45° with the x axis. If $a_x = 0$, the plane of polarization is the y–z plane.

Circularly Polarized Light

If $\varphi = \pm\pi/2$ and $a_x = a_y = a_0$, (6.1-4) gives $\mathscr{E}_x = a_0 \cos[2\pi\nu(t - z/c) + \varphi_x]$ and $\mathscr{E}_y = \mp a_0 \sin[2\pi\nu(t - z/c) + \varphi_x]$, from which $\mathscr{E}_x^2 + \mathscr{E}_y^2 = a_0^2$, which is the equation of a circle. The elliptical cylinder in Fig. 6.1-1(b) becomes a circular cylinder and the wave is said to be circularly polarized. In the case $\varphi = +\pi/2$, the electric field at a fixed position z rotates in a clockwise direction when viewed from the direction toward which the wave is approaching. The light is then said to be **right circularly polarized.** The case $\varphi = -\pi/2$ corresponds to counterclockwise rotation and **left circularly**

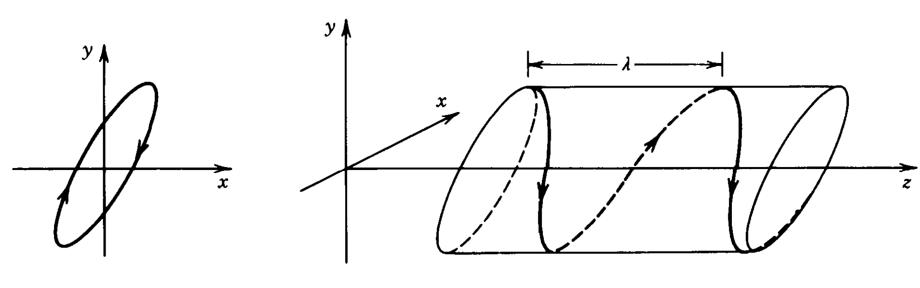


Figure 6.1-1 (a) Rotation of the endpoint of the electric-field vector in the x-y plane at a fixed position z. (b) Snapshot of the trajectory of the endpoint of the electric-field vector at a fixed time t.

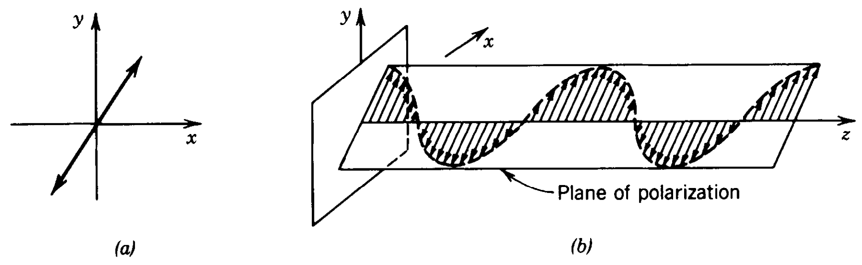


Figure 6.1-2 Linearly polarized light. (a) Time course at a fixed position z. (b) A snapshot (fixed time t).

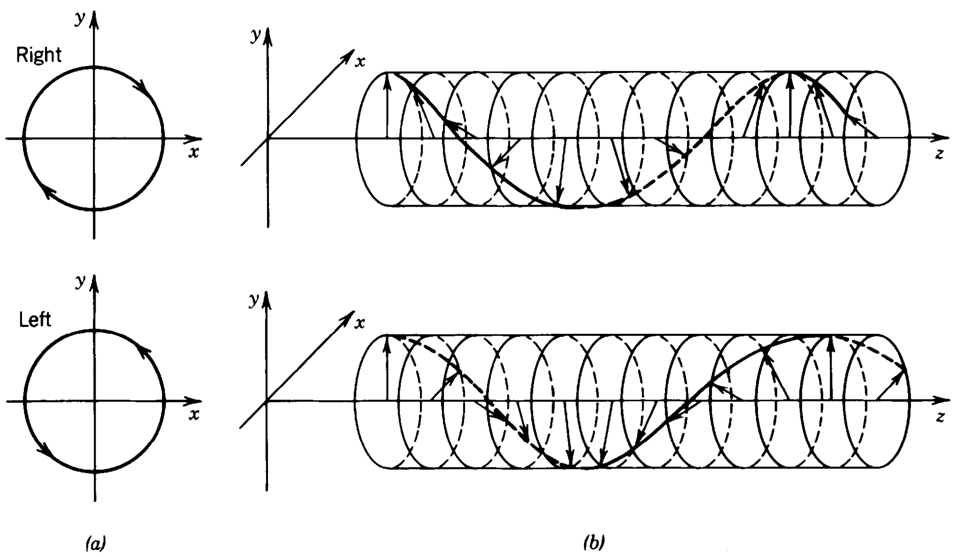


Figure 6.1-3 Trajectories of the endpoint of the electric-field vector of a circularly polarized plane wave. (a) Time course at a fixed position z. (b) A snapshot (fixed time t). The sense of rotation in (a) is opposite that in (b) because the traveling wave depends on $t - z/c$.

polarized light.† In the right circular case, a snapshot of the lines traced by the endpoints of the electric-field vectors at different positions is a right-handed helix (like a right-handed screw pointing in the direction of the wave), as illustrated in Fig. 6.1-3. For left circular polarization, a left-handed helix is followed.

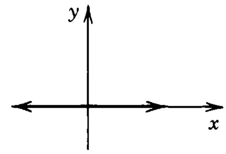
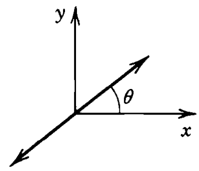
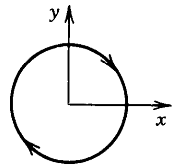
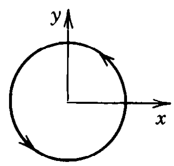
B. Matrix Representation

The Jones Vector

A monochromatic plane wave of frequency ν traveling in the z direction is completely characterized by the complex envelopes $A_x = a_x \exp(j\varphi_x)$ and $A_y = a_y \exp(j\varphi_y)$ of the x and y components of the electric field. It is convenient to write these complex

†This convention is used in most textbooks of optics. The opposite designation is used in the engineering literature: in the case of right (left) circularly polarized light, the electric-field vector at a fixed position rotates counterclockwise (clockwise) when viewed from the direction toward which the wave is approaching.

TABLE 6.1-1 Jones Vectors

Linearly polarized wave, in x direction	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
Linearly polarized wave, plane of polarization making angle θ with x axis	$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$	
Right circularly polarized	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$	
Left circularly polarized	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix}$	

quantities in the form of a column matrix

$$\mathbf{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}, \tag{6.1-6}$$

known as the **Jones vector**. Given the Jones vector, we can determine the total intensity of the wave, $I = (|A_x|^2 + |A_y|^2)/2\eta$, and use the ratio $a_y/a_x = |A_y|/|A_x|$ and the phase difference $\varphi = \varphi_y - \varphi_x = \arg\{A_y\} - \arg\{A_x\}$ to determine the orientation and shape of the polarization ellipse.

The Jones vectors for some special polarization states are provided in Table 6.1-1. The intensity in each case has been normalized so that $|A_x|^2 + |A_y|^2 = 1$ and the phase of the x component $\varphi_x = 0$.

Orthogonal Polarizations

Two polarization states represented by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ are said to be orthogonal if the inner product between $\mathbf{J}_1$ and $\mathbf{J}_2$ is zero. The inner product is defined by

$$(\mathbf{J}_1, \mathbf{J}_2) = A_{1x}A_{2x}^* + A_{1y}A_{2y}^*, \tag{6.1-7}$$

where A_{1x} and A_{1y} are the elements of $\mathbf{J}_1$ and A_{2x} and A_{2y} are the elements of $\mathbf{J}_2$. An example of orthogonal Jones vectors are the linearly polarized waves in the x and y directions. Another example is the right and left circularly polarized waves.

Expansion of Arbitrary Polarization as a Superposition of Two Orthogonal Polarizations

An arbitrary Jones vector $\mathbf{J}$ can always be analyzed as a weighted superposition of two orthogonal Jones vectors (say $\mathbf{J}_1$ and $\mathbf{J}_2$), called the expansion basis, $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are normalized such that $(\mathbf{J}_1, \mathbf{J}_1) = (\mathbf{J}_2, \mathbf{J}_2) = 1$, the expansion weights are the inner products $\alpha_1 = (\mathbf{J}, \mathbf{J}_1)$ and $\alpha_2 = (\mathbf{J}, \mathbf{J}_2)$. Using the x and y linearly polarized vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example, as an expansion basis, the expansion weights for a Jones vector of components A_x and A_y are simply $\alpha_1 = A_x$ and $\alpha_2 = A_y$. Similarly, if the right and left circularly polarized waves $(1/\sqrt{2})\begin{bmatrix} 1 \\ j \end{bmatrix}$ and $(1/\sqrt{2})\begin{bmatrix} 1 \\ -j \end{bmatrix}$ are used as an expansion basis, the expansion weights are $\alpha_1 = (1/\sqrt{2})(A_x - jA_y)$ and $\alpha_2 = (1/\sqrt{2})(A_x + jA_y)$.

EXERCISE 6.1-1

Linearly Polarized Wave as a Sum of Right and Left Circularly Polarized Waves. Show that the linearly polarized wave with plane of polarization making an angle θ with the x axis is equivalent to a superposition of right and left circularly polarized waves with weights $(1/\sqrt{2})e^{-j\theta}$ and $(1/\sqrt{2})e^{j\theta}$, respectively.

Matrix Representation of Polarization Devices

Consider the transmission of a plane wave of arbitrary polarization through an optical system that maintains the plane-wave nature of the wave, but alters its polarization, as illustrated schematically in Fig. 6.1-4. The system is assumed to be linear, so that the principle of superposition of optical fields is obeyed. Two examples of such systems are the reflection of light from a planar boundary between two media, and the transmission of light through a plate with anisotropic optical properties.

The complex envelopes of the two electric-field components of the input (incident) wave, A_{1x} and A_{1y}, and those of the output (transmitted or reflected) wave, A_{2x} and A_{2y}, are in general related by the weighted superpositions

$$\begin{aligned} A_{2x} &= T_{11}A_{1x} + T_{12}A_{1y} \\ A_{2y} &= T_{21}A_{1x} + T_{22}A_{1y}, \end{aligned} \tag{6.1-8}$$

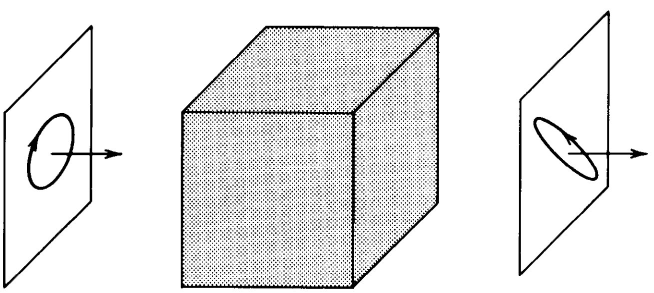


Figure 6.1-4 An optical system that alters the polarization of a plane wave.

where T_{11}, T_{12}, T_{21}, and T_{22} are constants describing the device. Equations (6.1-8) are general relations that all linear optical polarization devices must satisfy.

The linear relations in (6.1-8) may conveniently be written in matrix notation by defining a 2×2 matrix $\mathbf{T}$ with elements T_{11}, T_{12}, T_{21}, and T_{22} so that

$$\begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}. \tag{6.1-9}$$

If the input and output waves are described by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$, respectively, then (6.1-9) may be written in the compact matrix form

$$\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1. \tag{6.1-10}$$

The matrix $\mathbf{T}$, called the **Jones matrix**, describes the optical system, whereas the vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ describe the input and output waves.

The structure of the Jones matrix $\mathbf{T}$ of a given optical system determines its effect on the polarization state and intensity of the incident wave. The following is a list of the Jones matrices of some systems with simple characteristics. Physical devices that have such characteristics will be discussed subsequently in this chapter.

Linear Polarizers. The system represented by the Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \tag{6.1-11}$$

Linear Polarizer along x Direction

transforms a wave of components (A_{1x}, A_{1y}) into a wave of components $(A_{1x}, 0)$, thus polarizing the wave along the x direction, as illustrated in Fig. 6.1-5. The system is a **linear polarizer** with transmission axis pointing in the x direction.

Wave Retarders. The system represented by the matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-j\Gamma) \end{bmatrix}} \tag{6.1-12}$$

Wave-Retarder (Fast Axis along x Direction)

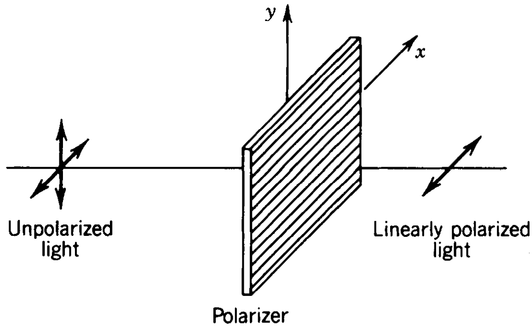


Figure 6.1-5 The linear polarizer.

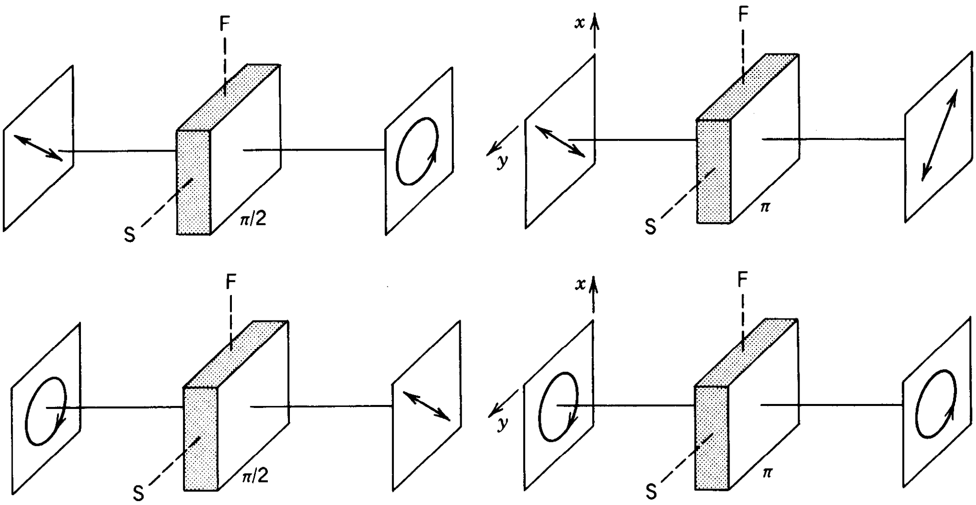


Figure 6.1-6 Operations of the quarter-wave ($\pi/2$) retarder and the half-wave (π) retarder. F and S represent the fast and slow axes of the retarder, respectively.

transforms a wave with field components (A_{1x}, A_{1y}) into another with components $(A_{1x}, e^{-j\Gamma}A_{1y})$, thus delaying the y component by a phase Γ, leaving the x component unchanged. It is therefore called a **wave retarder**. The x and y axes are called the fast and slow axes of the retarder, respectively. By simple application of matrix algebra, the following properties, illustrated in Fig. 6.1-6, may be shown:

- When $\Gamma = \pi/2$, the retarder (then called a **quarter-wave retarder**) converts linearly polarized light $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ into left circularly polarized light $\begin{bmatrix} 1 \\ -j \end{bmatrix}$, and converts right circularly polarized light $\begin{bmatrix} 1 \\ j \end{bmatrix}$ into linearly polarized light $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- When $\Gamma = \pi$, the retarder (then called a **half-wave retarder**) converts linearly polarized light $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ into linearly polarized light $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, thus rotating the plane of polarization by 90°. The half-wave retarder converts right circularly polarized light $\begin{bmatrix} 1 \\ j \end{bmatrix}$ into left circularly polarized light $\begin{bmatrix} 1 \\ -j \end{bmatrix}$.

Polarization Rotators. The Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \tag{6.1-13}$$

Polarization Rotator

represents a device that converts a linearly polarized wave $\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}$ into a linearly polarized wave $\begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix}$ where $\theta_2 = \theta_1 + \theta$. It therefore rotates the plane of polarization of a linearly polarized wave by an angle θ. The device is called a **polarization rotator**.

Cascaded Polarization Devices

The action of cascaded optical systems on polarized light may be conveniently determined by using conventional matrix multiplication formulas. A system characterized by the Jones matrix $\mathbf{T}_1$ followed by another characterized by $\mathbf{T}_2$ are equivalent to a single system characterized by the product matrix $\mathbf{T} = \mathbf{T}_2\mathbf{T}_1$. The matrix of the system through which light is transmitted first should appear to the right in the matrix product since it applies on the input Jones vector first.

EXERCISE 6.1-2

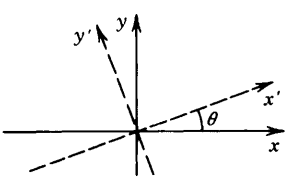
Cascaded Wave Retarders. Show that two cascaded quarter-wave retarders with parallel fast axes are equivalent to a half-wave retarder. What if the fast axes are orthogonal?

Coordinate Transformation

Elements of the Jones vectors and Jones matrices depend on the choice of the coordinate system. If these elements are known in one coordinate system, they can be determined in another coordinate system by using matrix methods. If $\mathbf{J}$ is the Jones vector in the x-y coordinate system, then in a new coordinate system x'-y', with the x' direction making an angle θ with the x direction, the Jones vector $\mathbf{J}'$ is given by

$$\mathbf{J}' = \mathbf{R}(\theta)\mathbf{J}, \tag{6.1-14}$$

where $\mathbf{R}(\theta)$ is the matrix



$$\boxed{\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.} \tag{6.1-15}$$

Coordinate Transformation Matrix

This can be shown by relating the components of the electric field in the two coordinate systems.

The Jones matrix $\mathbf{T}$, which represents an optical system, is similarly transformed into $\mathbf{T}'$, in accordance with the matrix relations

$$\mathbf{T}' = \mathbf{R}(\theta)\mathbf{T}\mathbf{R}(-\theta) \tag{6.1-16}$$

$$\mathbf{T} = \mathbf{R}(-\theta)\mathbf{T}'\mathbf{R}(\theta), \tag{6.1-17}$$

where $\mathbf{R}(-\theta)$ is given by (6.1-15) with $-\theta$ replacing θ. The matrix $\mathbf{R}(-\theta)$ is the inverse of $\mathbf{R}(\theta)$, so that $\mathbf{R}(-\theta)\mathbf{R}(\theta)$ is a unit matrix. Equation (6.1-16) can be shown by using the relation $\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1$ and the transformation $\mathbf{J}_2' = \mathbf{R}(\theta)\mathbf{J}_2 = \mathbf{R}(\theta)\mathbf{T}\mathbf{J}_1$. Since $\mathbf{J}_1 = \mathbf{R}(-\theta)\mathbf{J}_1'$, $\mathbf{J}_2' = \mathbf{R}(\theta)\mathbf{T}\mathbf{R}(-\theta)\mathbf{J}_1'$; since $\mathbf{J}_2' = \mathbf{T}'\mathbf{J}_1'$, (6.1-16) follows.

EXERCISE 6.1-3

Jones Matrix of a Polarizer. Show that the Jones matrix of a linear polarizer with a transmission axis making an angle θ with the x axis is

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}. \tag{6.1-18}$$

Linear Polarizer at Angle θ

Derive (6.1-18) using (6.1-17), (6.1-15), and (6.1-11).

Normal Modes

The normal modes of a polarization system are the states of polarization that are not changed when the wave is transmitted through the system. These states have Jones vectors satisfying

$$\mathbf{T}\mathbf{J} = \mu\mathbf{J}, \tag{6.1-19}$$

where μ is a constant. The normal modes are therefore the eigenvectors of the Jones matrix $\mathbf{T}$, and the values of μ are the corresponding eigenvalues. Since the matrix $\mathbf{T}$ is of size 2×2 there are only two independent normal modes, $\mathbf{T}\mathbf{J}_1 = \mu_1\mathbf{J}_1$ and $\mathbf{T}\mathbf{J}_2 = \mu_2\mathbf{J}_2$. If the matrix $\mathbf{T}$ is Hermitian, i.e., $T_{12} = T_{21}^*$, the normal modes are orthogonal, $(\mathbf{J}_1, \mathbf{J}_2) = 0$. The normal modes are usually used as an expansion basis, so that an arbitrary input wave $\mathbf{J}$ may be expanded as a superposition of normal modes, $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. The response of the system may be easily evaluated since $\mathbf{T}\mathbf{J} = \mathbf{T}(\alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2) = \alpha_1\mathbf{T}\mathbf{J}_1 + \alpha_2\mathbf{T}\mathbf{J}_2 = \alpha_1\mu_1\mathbf{J}_1 + \alpha_2\mu_2\mathbf{J}_2$ (see Appendix C).

EXERCISE 6.1-4

Normal Modes of Simple Polarization Systems

(a) Show that the normal modes of the linear polarizer are linearly polarized waves.
(b) Show that the normal modes of the wave retarder are linearly polarized waves.
(c) Show that the normal modes of the polarization rotator are right and left circularly polarized waves.

What are the eigenvalues of the systems above?

6.2 REFLECTION AND REFRACTION

In this section we examine the reflection and refraction of a monochromatic plane wave of arbitrary polarization incident at a planar boundary between two dielectric media. The media are assumed to be linear, homogeneous, isotropic, nondispersive, and nonmagnetic; the refractive indices are n_1 and n_2. The incident, refracted, and

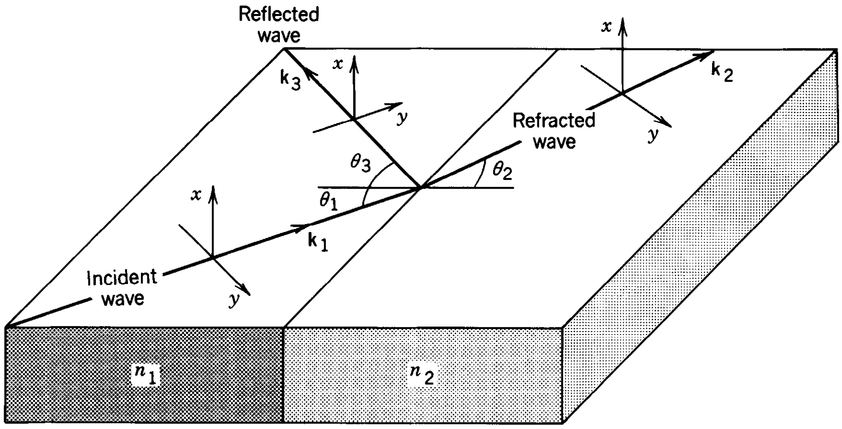


Figure 6.2-1 Reflection and refraction at the boundary between two dielectric media.

reflected waves are labeled with the subscripts 1, 2, and 3, respectively, as illustrated in Fig. 6.2-1.

As shown in Sec. 2.4A, the wavefronts of these waves are matched at the boundary if the angles of reflection and incidence are equal, $\theta_3 = \theta_1$, and the angles of refraction and incidence satisfy Snell's law,

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \tag{6.2-1}$$

To relate the amplitudes and polarizations of the three waves we associate with each wave an x-y coordinate system in a plane normal to the direction of propagation (Fig. 6.2-1). The electric-field envelopes of these waves are described by Jones vectors

$$\mathbf{J}_1 = \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}, \quad \mathbf{J}_2 = \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix}, \quad \mathbf{J}_3 = \begin{bmatrix} A_{3x} \\ A_{3y} \end{bmatrix}.$$

We proceed to determine the relations between $\mathbf{J}_2$ and $\mathbf{J}_1$ and between $\mathbf{J}_3$ and $\mathbf{J}_1$. These relations are written in the matrix form $\mathbf{J}_2 = \mathbf{t}\mathbf{J}_1$, and $\mathbf{J}_3 = \mathbf{r}\mathbf{J}_1$, where $\mathbf{t}$ and $\mathbf{r}$ are 2×2 Jones matrices describing the transmission and reflection of the wave, respectively.

Elements of the transmission and reflection matrices may be determined by using the boundary conditions required by electromagnetic theory (tangential components of $\mathbf{E}$ and $\mathbf{H}$ and normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous at the boundary). The magnetic field associated with each wave is orthogonal to the electric field and their magnitudes are related by the characteristic impedances, η_o/n_1 for the incident and reflected waves, and η_o/n_2 for the transmitted wave, where $\eta_o = (\mu_o/\epsilon_o)^{1/2}$. The result is a set of equations that are solved to obtain relations between the components of the electric fields of the three waves.

The algebraic steps involved are reduced substantially if we observe that the two normal modes for this system are linearly polarized waves with polarization along the x and y directions. This may be proved if we show that an incident, a reflected, and a refracted wave with their electric field vectors pointing in the x direction are self-consistent with the boundary conditions, and similarly for three waves linearly polarized in the y direction. This is indeed the case. The x and y polarized waves are therefore separable and independent.

The x-polarized mode is called the **transverse electric (TE)** polarization or the **orthogonal** polarization, since the electric fields are orthogonal to the plane of incidence. The y-polarized mode is called the **transverse magnetic (TM)** polarization since the magnetic field is orthogonal to the plane of incidence, or the **parallel** polarization since the electric fields are parallel to the plane of incidence. The orthogonal and parallel polarizations are also called the s and p polarizations (s for the German *senkrecht*, meaning "perpendicular").

The independence of the x and y polarizations implies that the Jones matrices **t** and **r** are diagonal,

$$\mathbf{t} = \begin{bmatrix} t_x & 0 \\ 0 & t_y \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} r_x & 0 \\ 0 & r_y \end{bmatrix},$$

so that

$$E_{2x} = t_x E_{1x}, \qquad E_{2y} = t_y E_{1y} \tag{6.2-2}$$

$$E_{3x} = r_x E_{1x}, \qquad E_{3y} = r_y E_{1y}. \tag{6.2-3}$$

The coefficients t_x and t_y are the complex amplitude transmittances for the TE and TM polarizations, respectively, and similarly for the complex amplitude reflectances r_x and r_y.

Applying the boundary conditions to the TE and TM polarizations separately gives the following expressions for the reflection and transmission coefficients, known as the **Fresnel equations**:

$$\boxed{\begin{aligned} r_x &= \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} && (6.2\text{-}4) \\ t_x &= 1 + r_x && (6.2\text{-}5) \end{aligned}}$$

Fresnel Equations (TE Polarization)

$$\boxed{\begin{aligned} r_y &= \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} && (6.2\text{-}6) \\ t_y &= \frac{n_1}{n_2}(1 + r_y). && (6.2\text{-}7) \end{aligned}}$$

Fresnel Equations (TM Polarization)

Given n_1, n_2, and θ_1, the reflection coefficients can be determined by first determining θ_2 using Snell's law, (6.2-1), from which

$$\cos\theta_2 = \left(1 - \sin^2\theta_2\right)^{1/2} = \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1\right]^{1/2}. \tag{6.2-8}$$

Since the quantities under the square roots in (6.2-8) can be negative, the reflection and transmission coefficients are in general complex. The magnitudes $|r_x|$ and $|r_y|$ and the phase shifts $\varphi_x = \arg\{r_x\}$ and $\varphi_y = \arg\{r_y\}$ are plotted as functions of the angle of incidence θ_1 in Figs. 6.2-2 to 6.2-5 for each of the two polarizations for external reflection ($n_1 < n_2$) and internal reflection ($n_1 > n_2$).

TE Polarization

The reflection coefficient r_x for the TE-polarized wave is given by (6.2-4).

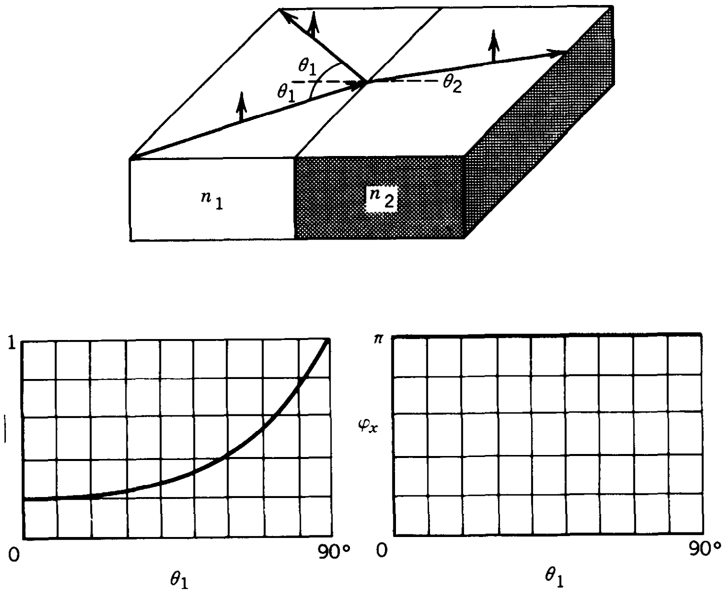


Figure 6.2-2 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for external reflection of the TE polarized wave ($n_2/n_1 = 1.5$).

- *External Reflection* ($n_1 < n_2$). The reflection coefficient r_x is always real and negative, corresponding to a phase shift $\varphi_x = \pi$. The magnitude $|r_x| = (n_2 - n_1)/(n_1 + n_2)$ at $\theta_1 = 0$ (normal incidence) and increases to unity at $\theta_1 = 90°$ (grazing incidence).
- *Internal Reflection* ($n_1 > n_2$). For small θ_1 the reflection coefficient is real and positive. Its magnitude is $(n_1 - n_2)/(n_1 + n_2)$ when $\theta_1 = 0°$, increasing gradually

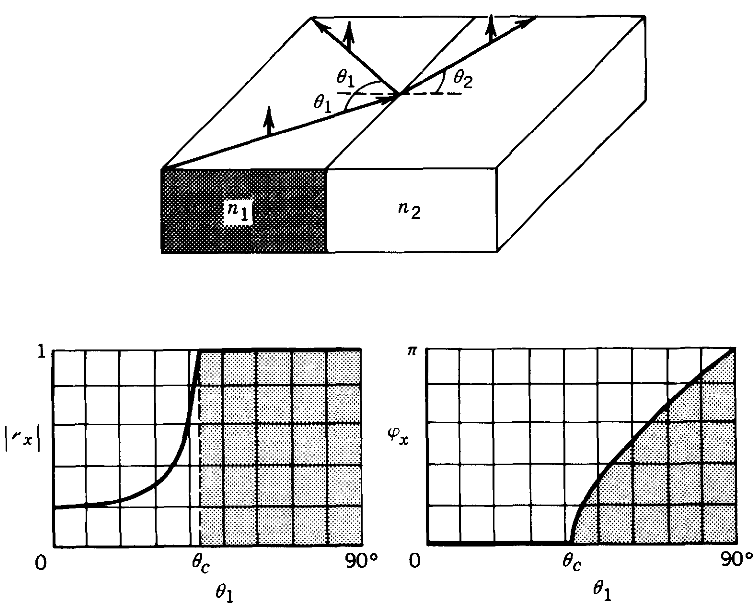


Figure 6.2-3 Magnitude and phase of the reflection coefficient for internal reflection of the TE wave ($n_1/n_2 = 1.5$).

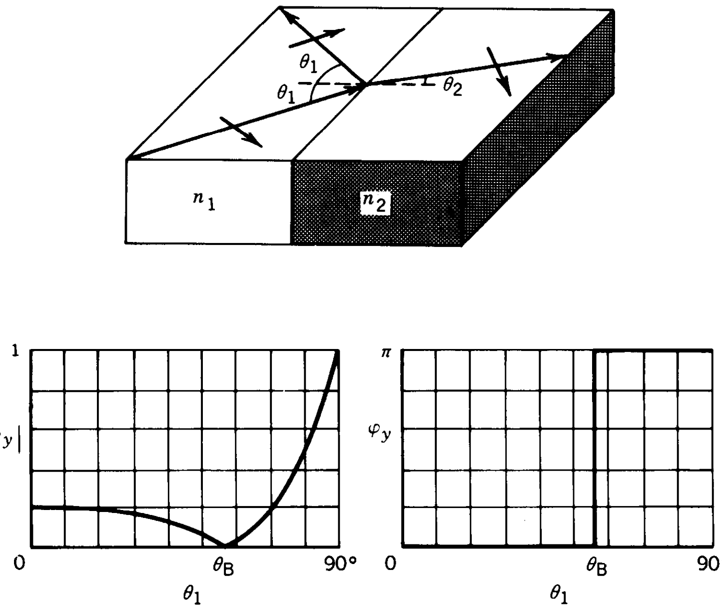


Figure 6.2-4 Magnitude and phase of the reflection coefficient for external reflection of the TM wave ($n_2/n_1 = 1.5$).

to unity when θ_1 equals the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$. For $\theta_1 > \theta_c$, the magnitude of r_x remains unity, corresponding to total internal reflection. This may be shown by using (6.2-8) to write† $\cos\theta_2 = -[1 - \sin^2\theta_1/\sin^2\theta_c]^{1/2} = -j[\sin^2\theta_1/\sin^2\theta_c - 1]^{1/2}$, and substituting into (6.2-6). Total internal reflection is accompanied by a phase shift $\varphi_x = \arg\{r_x\}$ given by

$$\boxed{\tan\frac{\varphi_x}{2} = \frac{\left(\sin^2\theta_1 - \sin^2\theta_c\right)^{1/2}}{\cos\theta_1}.} \qquad (6.2\text{-}9)$$

TE Reflection Phase Shift

The phase shift φ_x increases from 0 at $\theta_1 = \theta_c$ to π at $\theta_1 = 90°$, as illustrated in Fig. 6.2-3.

TM Polarization

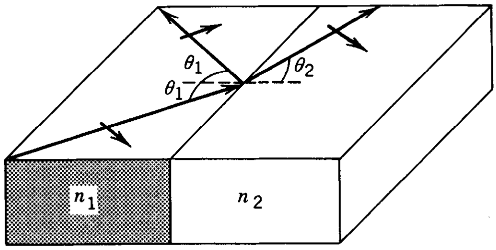
The dependence of the reflection coefficient r_y on θ_1 in (6.2-6) is similarly examined for external and internal reflections:

- *External Reflection* ($n_1 < n_2$). The reflection coefficient is real. It decreases from a positive value of $(n_2 - n_1)/(n_2 + n_1)$ at normal incidence until it vanishes at an angle $\theta_1 = \theta_B$,

$$\boxed{\theta_B = \tan^{-1}\frac{n_2}{n_1},} \qquad (6.2\text{-}10)$$

Brewster Angle

†The choice of the minus sign for the square root is consistent with the derivation that leads to the Fresnel equations.



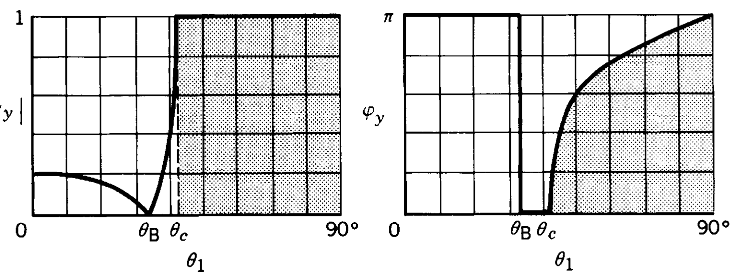


Figure 6.2-5 Magnitude and phase of the reflection coefficient for internal reflection of the TM wave ($n_1/n_2 = 1.5$).

known as the **Brewster angle**. For $\theta_1 > \theta_B$, r_y reverses sign and its magnitude increases gradually approaching unity at $\theta_1 = 90°$. The property that the TM wave is not reflected at the Brewster angle is used in making polarizers (see Sec. 6.6).

- *Internal Reflection* ($n_1 > n_2$). At $\theta_1 = 0°$, r_y is negative and has magnitude $(n_1 - n_2)/(n_1 + n_2)$. As θ_1 increases the magnitude drops until it vanishes at the Brewster angle $\theta_B = \tan^{-1}(n_2/n_1)$. As θ_1 increases beyond θ_B, r_y becomes positive and increases until it reaches unity at the critical angle θ_c. For $\theta_1 > \theta_c$ the wave undergoes total internal reflection accompanied by a phase shift $\varphi_y = \arg\{r_y\}$ given by

$$\tan\frac{\varphi_y}{2} = \frac{\left(\sin^2\theta_1 - \sin^2\theta_c\right)^{1/2}}{\cos\theta_1 \sin^2\theta_c}. \qquad (6.2\text{-}11)$$

(6.2-11)
TM Reflection
Phase Shift

EXERCISE 6.2-1

Brewster Windows. At what angle is a TM-polarized beam of light transmitted through a glass plate of refractive index $n = 1.5$ placed in air ($n = 1$) without suffering reflection losses at either surface? These plates, known as Brewster windows, are used in lasers (Fig. 6.2-6; see Sec. 14.2D).

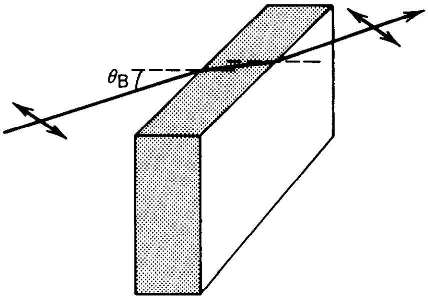


Figure 6.2-6 The Brewster window transmits TM-polarized light with no reflection loss.

Power Reflectance and Transmittance

The reflection and transmission coefficients r and t are ratios of the complex amplitudes. The power reflectance $\mathscr{R}$ and transmittance $\mathscr{T}$ are defined as the ratios of power flow (along a direction normal to the boundary) of the reflected and transmitted waves to that of the incident wave. Because the reflected and incident waves propagate in the same medium and make the same angle with the normal to the surface,

$$\mathscr{R} = |r|^2. \tag{6.2-12}$$

Conservation of power requires that

$$\mathscr{T} = 1 - \mathscr{R}. \tag{6.2-13}$$

Note, however, that $\mathscr{T} = [n_2 \cos\theta_2 / n_1 \cos\theta_1]|t|^2$ which is *not* generally equal to $|t|^2$ since the power travels at different angles. It follows that for both TE and TM polarizations, and for both external and internal reflection, the reflectance at normal incidence is

$$\boxed{\mathscr{R} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2.} \tag{6.2-14}$$

Power Reflectance at Normal Incidence

At a boundary between glass ($n = 1.5$) and air ($n = 1$), for example, $\mathscr{R} = 0.04$, so that 4% of the light is reflected at normal incidence. At the boundary between GaAs ($n = 3.6$) and air ($n = 1$), $\mathscr{R} \approx 0.32$, so that 32% of the light is reflected at normal incidence. The reflectance can be much greater or much less at oblique angles as illustrated in Fig. 6.2-7.

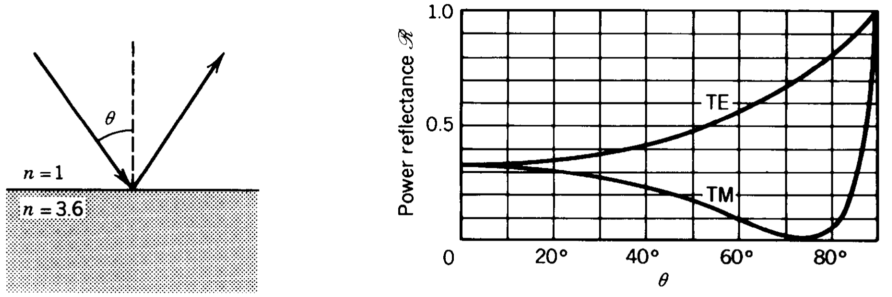


Figure 6.2-7 Power reflectance of TE and TM polarization plane waves at the boundary between air ($n = 1$) and GaAs ($n = 3.6$) as a function of the angle of incidence θ.

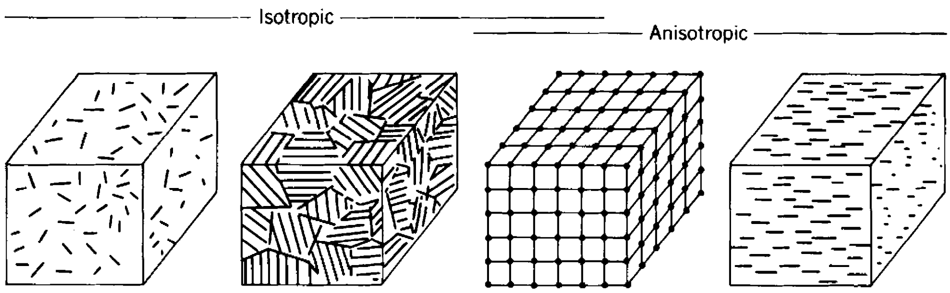


Figure 6.3-1 Positional and orientational order in different kinds of materials.

6.3 OPTICS OF ANISOTROPIC MEDIA

A dielectric medium is said to be anisotropic if its macroscopic optical properties depend on direction. The macroscopic properties of matter are of course governed by the microscopic properties: the shape and orientation of the individual molecules and the organization of their centers in space. The following is a description of the positional and orientational types of order inherent in several kinds of optical materials (see Fig. 6.3-1).

- If the molecules are located in space at totally random positions and are themselves isotropic or are oriented along totally random directions, the medium is isotropic. *Gases*, *liquids*, and *amorphous solids* are isotropic.
- If the molecules are anisotropic and their orientations are not totally random, the medium is anisotropic, even if the positions are totally random. This is the case for *liquid crystals*, which have orientational order but lack complete positional order.
- If the molecules are organized in space according to regular periodic patterns and are oriented in the same direction, as in *crystals*, the medium is in general anisotropic.
- *Polycrystalline materials* have a structure in the form of disjointed crystalline grains that are randomly oriented relative to each other. The grains are themselves generally anisotropic, but their averaged macroscopic behavior is isotropic.

A. Refractive Indices

Permittivity Tensor

In a linear anisotropic dielectric medium (a crystal, for example), each component of the electric flux density **D** is a linear combination of the three components of the electric field

$$D_i = \sum_j \epsilon_{ij} E_j, \tag{6.3-1}$$

where $i, j = 1, 2, 3$ indicate the x, y, and z components, respectively (see Sec. 5.2B). The dielectric properties of the medium are therefore characterized by a 3×3 array of nine coefficients $\{\epsilon_{ij}\}$ forming a tensor of second rank known as the **electric permittivity tensor** and denoted by the symbol $\boldsymbol{\epsilon}$. Equation (6.3-1) is usually written in the symbolic form $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$. The electric permittivity tensor is symmetrical, $\epsilon_{ij} = \epsilon_{ji}$, and is therefore characterized by only six independent numbers. For crystals of certain symmetries, some of these six coefficients vanish and some are related, so that even fewer coefficients are necessary.

Principal Axes and Principal Refractive Indices

Elements of the permittivity tensor depend on the choice of the coordinate system relative to the crystal structure. A coordinate system can always be found for which the off-diagonal elements of ϵ_{ij} vanish, so that

$$D_1 = \epsilon_1 E_1, \qquad D_2 = \epsilon_2 E_2, \qquad D_3 = \epsilon_3 E_3, \tag{6.3-2}$$

where $\epsilon_1 = \epsilon_{11}$, $\epsilon_2 = \epsilon_{22}$, and $\epsilon_3 = \epsilon_{33}$. These are the directions for which **E** and **D** are parallel. For example, if **E** points in the x direction, **D** must also point in the x direction. This coordinate system defines the **principal axes** and principal planes of the crystal. Throughout the remainder of this chapter, the coordinate system x, y, z (denoted also by the numbers $1, 2, 3$) will be assumed to lie along the crystal's principal axes. The permittivities ϵ_1, ϵ_2, and ϵ_3 correspond to refractive indices

$$n_1 = \left(\frac{\epsilon_1}{\epsilon_o}\right)^{1/2}, \qquad n_2 = \left(\frac{\epsilon_2}{\epsilon_o}\right)^{1/2}, \qquad n_3 = \left(\frac{\epsilon_3}{\epsilon_o}\right)^{1/2}, \tag{6.3-3}$$

known as the **principal refractive indices** (ϵ_o is the permittivity of free space).

Biaxial, Uniaxial, and Isotropic Crystals

In crystals with certain symmetries two of the refractive indices are equal ($n_1 = n_2$) and the crystals are called **uniaxial** crystals. The indices are usually denoted $n_1 = n_2 = n_o$ and $n_3 = n_e$. For reasons to become clear later, n_o and n_e are called the **ordinary** and **extraordinary** indices, respectively. The crystal is said to be **positive uniaxial** if $n_e > n_o$, and **negative uniaxial** if $n_e < n_o$. The z axis of a uniaxial crystal is called the **optic axis**. In other crystals (those with cubic unit cells, for example) the three indices are equal and the medium is optically isotropic. Media for which the three principal indices are different are called **biaxial**.

Impermeability Tensor

The relation between **D** and **E** can be inverted and written in the form $\mathbf{E} = \boldsymbol{\epsilon}^{-1}\mathbf{D}$, where $\boldsymbol{\epsilon}^{-1}$ is the inverse of the tensor $\boldsymbol{\epsilon}$. It is also useful to define the tensor $\boldsymbol{\eta} = \epsilon_o \boldsymbol{\epsilon}^{-1}$ called the electric **impermeability tensor** (not to be confused with the impedance of the medium), so that $\epsilon_o \mathbf{E} = \boldsymbol{\eta}\mathbf{D}$. Since $\boldsymbol{\epsilon}$ is symmetrical, $\boldsymbol{\eta}$ is also symmetrical. Both tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ share the same principal axes (directions for which **E** and **D** are parallel). In the principal coordinate system, $\boldsymbol{\eta}$ is diagonal with principal values $\epsilon_o/\epsilon_1 = 1/n_1^2$, $\epsilon_o/\epsilon_2 = 1/n_2^2$, and $\epsilon_o/\epsilon_3 = 1/n_3^2$. Either of the tensors $\boldsymbol{\epsilon}$ or $\boldsymbol{\eta}$ describes the optical properties of the crystal completely.

Geometrical Representation of Vectors and Tensors

A *vector* describes a physical variable with magnitude and direction (the electric field **E**, for example). It is represented *geometrically* by an arrow pointing in that direction with length proportional to the magnitude of the vector [Fig. 6.3-2(*a*)]. The vector is represented *numerically* by three numbers: its projections on the three axes of some coordinate system. These (components) are dependent on the choice of the coordinate system. However, the magnitude and direction of the vector in the physical space are independent of the choice of the coordinate system.

A second-rank *tensor* is a rule that relates two vectors. It is represented *numerically* in a given coordinate system by nine numbers. When the coordinate system is changed,

(a) *(b)*

Figure 6.3-2 Geometrical representation of a vector (*a*) and a symmetrical tensor (*b*).

another set of nine numbers is obtained, but the physical nature of the rule is not changed. A useful *geometrical* representation of a symmetrical second-rank tensor (the dielectric tensor $\boldsymbol{\epsilon}$, for example) is a quadratic surface (an ellipsoid) defined by [Fig. 6.3-2(*b*)]

$$\sum_{ij} \epsilon_{ij} x_i x_j = 1, \tag{6.3-4}$$

known as the **quadric representation**. This surface is invariant to the choice of the coordinate system, so that if the coordinate system is rotated, both x_i and ϵ_{ij} are altered but the ellipsoid remains intact. In the principal coordinate system ϵ_{ij} is diagonal and the ellipsoid has a particularly simple form,

$$\epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 = 1. \tag{6.3-5}$$

The ellipsoid carries all information about the tensor (six degrees of freedom). Its principal axes are those of the tensor, and its axes have half-lengths $\epsilon_1^{-1/2}$, $\epsilon_2^{-1/2}$, and $\epsilon_3^{-1/2}$.

The Index Ellipsoid

The **index ellipsoid** (also called the **optical indicatrix**) is the quadric representation of the electric impermeability tensor $\boldsymbol{\eta} = \epsilon_o \boldsymbol{\epsilon}^{-1}$,

$$\sum_{ij} \eta_{ij} x_i x_j = 1. \tag{6.3-6}$$

Using the principal axes as a coordinate system, the index ellipsoid is described by

$$\boxed{\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1,} \tag{6.3-7}$$

The Index Ellipsoid

where $1/n_1^2$, $1/n_2^2$, and $1/n_3^2$ are the principal values of $\boldsymbol{\eta}$.

The optical properties of the crystal (the directions of the principal axes and the values of the principal refractive indices) are therefore described completely by the index ellipsoid (Fig. 6.3-3). The index ellipsoid of a uniaxial crystal is an ellipsoid of revolution and that of an optically isotropic medium is a sphere.

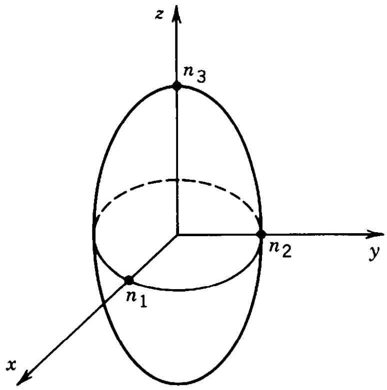


Figure 6.3-3 The index ellipsoid. The coordinates (x, y, z) are the principal axes and (n_1, n_2, n_3) are the principal refractive indices of the crystal.

B. Propagation Along a Principal Axis

The rules that govern the propagation of light in crystals under general conditions are rather complicated. However, they become relatively simple if the light is a plane wave traveling along one of the principal axes of the crystal. We begin with this case.

Normal Modes

Let x–y–z be a coordinate system in the directions of the principal axes of a crystal. A plane wave traveling in the z direction and linearly polarized in the x direction travels with phase velocity c_o/n_1 (wave number $k = n_1 k_o$) without changing its polarization. The reason is that the electric field then has only one component E_1 in the x direction, so that **D** is also in the x direction, $D_1 = \epsilon_1 E_1$, and the wave equation derived from Maxwell's equations will have a velocity $(\mu_o \epsilon_1)^{-1/2} = c_o/n_1$. A wave with linear polarization along the y direction similarly travels with phase velocity c_o/n_2 and "experiences" a refractive index n_2. Thus the normal modes for propagation in the z direction are the linearly polarized waves in the x and y directions. Other cases in which the wave propagates along one of the principal axes and is linearly polarized along another are treated similarly, as illustrated in Fig. 6.3-4.

Polarization Along an Arbitrary Direction

What if the wave travels along one principal axis (the z axis, for example) and is linearly polarized along an arbitrary direction in the x-y plane? This case can be

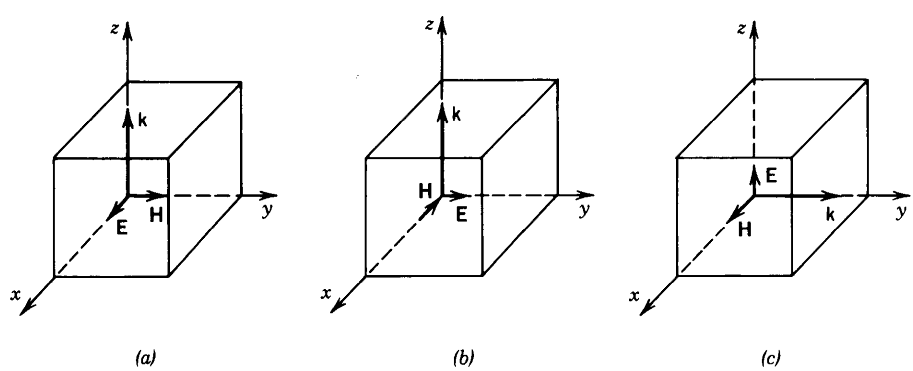


Figure 6.3-4 A wave traveling along a principal axis and polarized along another principal axis has a phase velocity c_o/n_1, c_o/n_2, or c_o/n_3, if the electric field vector points in the x, y, or z directions, respectively. (a) $k = n_1 k_o$; (b) $k = n_2 k_o$; (c) $k = n_3 k_o$.

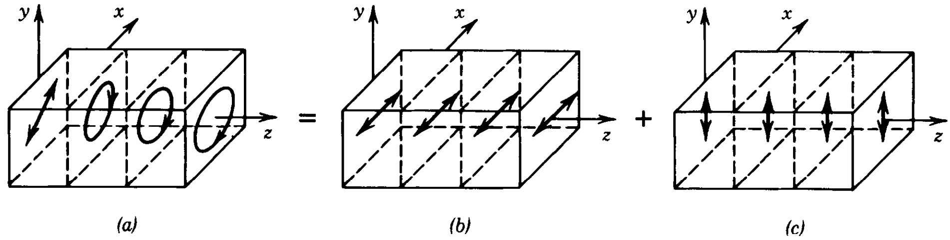


Figure 6.3-5 A linearly polarized wave at 45° in the $z = 0$ plane is analyzed as a superposition of two linearly polarized components in the x and y directions (normal modes), which travel at velocities c_o/n_1 and c_o/n_2. As a result of phase retardation, the wave is converted into an elliptically polarized wave.

addressed by analyzing the wave as a sum of the normal modes, the linearly polarized waves in the x and y directions. Since these two components travel with different velocities, c_o/n_1 and c_o/n_2, they undergo different phase shifts, $\varphi_x = n_1 k_o d$ and $\varphi_y = n_2 k_o d$, after propagating a distance d. Their phase retardation is therefore $\varphi = \varphi_y - \varphi_x = (n_2 - n_1)k_o d$. When the two components are combined, they form an elliptically polarized wave, as explained in Sec. 6.1 and illustrated in Fig. 6.3-5. The crystal can therefore be used as a **wave retarder**—a device in which two orthogonal polarizations travel at different phase velocities, so that one is retarded with respect to the other.

C. Propagation in an Arbitrary Direction

We now consider the general case of a plane wave traveling in an anisotropic crystal in an arbitrary direction defined by the unit vector $\hat{\mathbf{u}}$. The analysis is lengthy but the final results are simple. We will show that the two normal modes are linearly polarized waves. The refractive indices n_a and n_b and the directions of polarization of these modes may be determined by use of the following procedure based on the index ellipsoid. An analysis leading to a proof of this procedure will be subsequently provided.

Index-Ellipsoid Construction for Determining the Normal Modes

The following is a geometrical construction for determining the polarizations and refractive indices n_a and n_b of the normal modes of a wave traveling in the direction of the unit vector $\hat{\mathbf{u}}$ in an anisotropic material with the index ellipsoid $x_1^2/n_1^2 + x_2^2/n_2^2 + x_3^2/n_3^2 = 1$, illustrated in Fig. 6.3-6.

- Draw a plane passing through the origin of the index ellipsoid, normal to $\hat{\mathbf{u}}$. The intersection of the plane with the ellipsoid is an ellipse, called the index ellipse.
- The half-lengths of the major and minor axes of the index ellipse are the refractive indices n_a and n_b of the two normal modes.
- The directions of the major and minor axes of the index ellipse are the directions of the vectors $\mathbf{D}_a$ and $\mathbf{D}_b$ for the normal modes. These directions are orthogonal.
- The vectors $\mathbf{E}_a$ and $\mathbf{E}_b$ may be determined from $\mathbf{D}_a$ and $\mathbf{D}_b$ by use of (6.3-2).

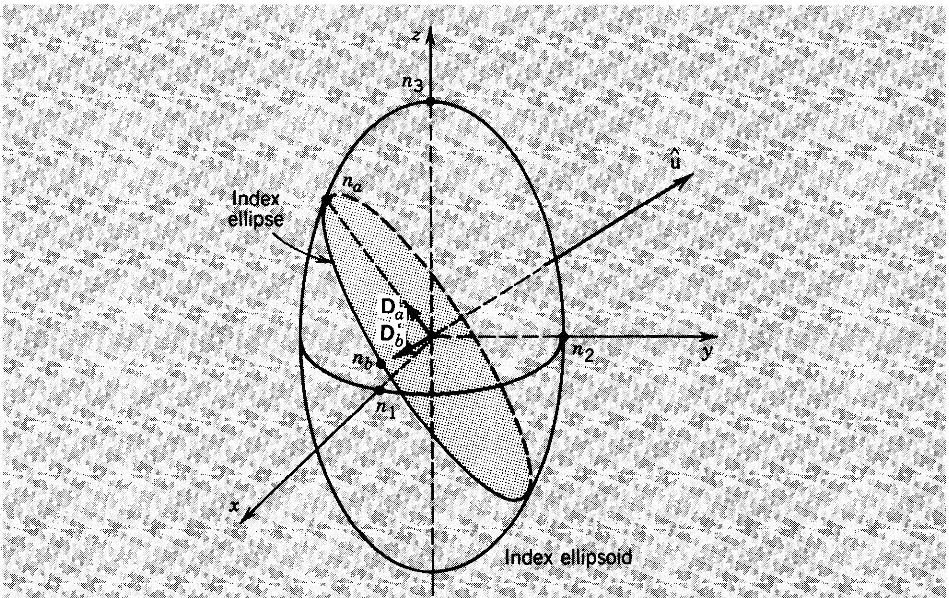


Figure 6.3-6 Normal modes determined from the index ellipsoid.

The Dispersion Relation

To determine the normal modes for a plane wave traveling in the direction $\hat{\mathbf{u}}$, we use Maxwell's equations (5.3-2) to (5.3-5) and the medium equation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$. Since all fields are assumed to vary with the position $\mathbf{r}$ as $\exp(-j\mathbf{k}\cdot\mathbf{r})$, where $\mathbf{k} = k\hat{\mathbf{u}}$, Maxwell's equations (5.3-2) and (5.3-3) reduce to

$$\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D} \tag{6.3-8}$$

$$\mathbf{k} \times \mathbf{E} = \omega \mu_o \mathbf{H}. \tag{6.3-9}$$

It follows from (6.3-8) that $\mathbf{D}$ is normal to both $\mathbf{k}$ and $\mathbf{H}$. Equation (6.3-9) similarly indicates that $\mathbf{H}$ is normal to both $\mathbf{k}$ and $\mathbf{E}$. These geometrical conditions are illustrated in Fig. 6.3-7, which also shows the Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ (direction of power

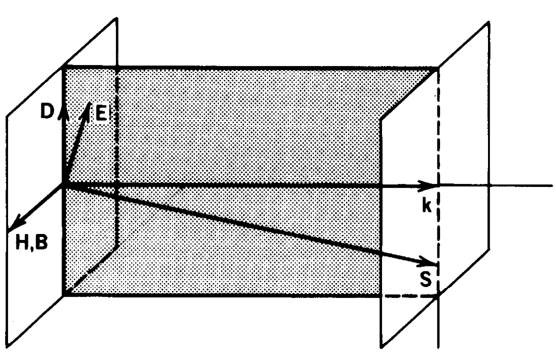


Figure 6.3-7 The vectors **D**, **E**, **k**, and **S** all lie in one plane to which **H** and **B** are normal. $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{E} \perp \mathbf{S}$.

flow), which is orthogonal to both **E** and **H**. Thus **D**, **E**, **k**, and **S** lie in one plane to which **H** and **B** are normal. In this plane $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{S} \perp \mathbf{E}$; but **D** is not necessarily parallel to **E**, and **S** is not necessarily parallel to **k**.

Substituting (6.3-8) into (6.3-9) and using $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$, we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu_o \boldsymbol{\epsilon} \mathbf{E} = \mathbf{0}. \tag{6.3-10}$$

This vector equation, which **E** must satisfy, translates to three linear homogeneous equations for the components E_1, E_2, and E_3 along the principal axes, written in the matrix form

$$\begin{bmatrix} n_1^2 k_o^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_o^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_o^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.3-11}$$

where (k_1, k_2, k_3) are the components of **k**, $k_o = \omega/c_o$, and (n_1, n_2, n_3) are the principal refractive indices given by (6.3-3). The condition that these equations have a nontrivial solution is obtained by setting the determinant of the matrix to zero. The result is an equation relating ω to k_1, k_2, and k_3 of the form $\omega = \omega(k_1, k_2, k_3)$, where $\omega(k_1, k_2, k_3)$ is a nonlinear function. This relation, known as the **dispersion relation**, is the equation of a surface in the k_1, k_2, k_3 space, known as the **normal surface** or the **k surface**. The intersection of the direction $\hat{\mathbf{u}}$ with the **k** surface determines the vector **k** whose magnitude $k = n\omega/c_o$ provides the refractive index n. There are two intersections corresponding to the two normal modes of each direction.

The **k** surface is a centrosymmetric surface made of two sheets, each corresponding to a solution (a normal mode). It can be shown that the **k** surface intersects each of the principal planes in an ellipse and a circle, as illustrated in Fig. 6.3-8. For biaxial crystals ($n_1 < n_2 < n_3$), the two sheets meet at four points defining two optic axes. In the uniaxial case ($n_1 = n_2 = n_o$, $n_3 = n_e$), the two sheets become a sphere and an ellipsoid of revolution meeting at only two points defining a single optic axis, the z axis. In the isotropic case ($n_1 = n_2 = n_3 = n$), the two sheets degenerate into one sphere.

The intersection of the direction $\hat{\mathbf{u}} = (u_1, u_2, u_3)$ with the **k** surface corresponds to a wavenumber k satisfying

$$\sum_{j=1,2,3} \frac{u_j^2 k^2}{k^2 - n_j^2 k_o^2} = 1. \tag{6.3-12}$$

This is a fourth-order equation in k (or second order in k^2). It has four solutions $\pm k_a$ and $\pm k_b$, of which only the two positive values are meaningful, since the negative values represent a reversed direction of propagation. The problem is therefore solved: the wave numbers of the normal modes are k_a and k_b and the refractive indices are $n_a = k_a/k_o$ and $n_b = k_b/k_o$.

To determine the directions of polarization of the two normal modes, we determine the components $(k_1, k_2, k_3) = (ku_1, ku_2, ku_3)$ and the elements of the matrix in (6.3-11) for each of the two wavenumbers $k = k_a$ and k_b. We then solve two of the three equations in (6.3-11) to determine the ratios E_1/E_3 and E_2/E_3, from which we determine the direction of the corresponding electric field **E**.

*Proof of the Index-Ellipsoid Construction for Determining the Normal Modes

Since we already know that **D** lies in a plane normal to $\hat{\mathbf{u}}$, it is convenient to aim at finding **D** of the normal modes by rewriting (6.3-10) in terms of **D**. Using $\mathbf{E} = \boldsymbol{\epsilon}^{-1}\mathbf{D}$,

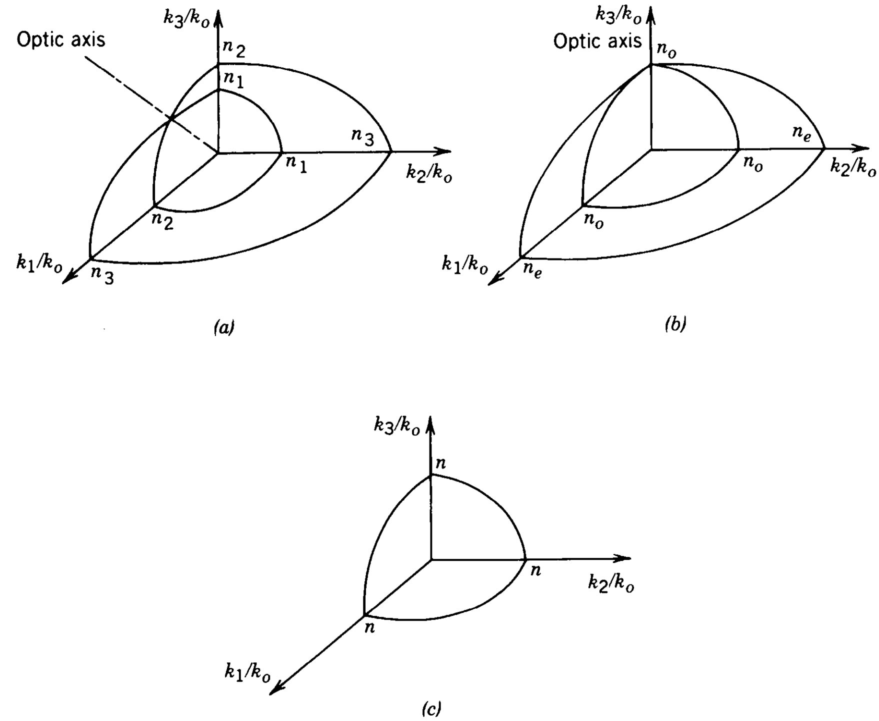


Figure 6.3-8 One octant of the **k** surface for (*a*) a biaxial crystal ($n_1 < n_2 < n_3$); (*b*) a uniaxial crystal ($n_1 = n_2 = n_o$, $n_3 = n_e$); and (*c*) an isotropic crystal ($n_1 = n_2 = n_3 = n$).

$\boldsymbol{\eta} = \epsilon_o \boldsymbol{\epsilon}^{-1}$, $\mathbf{k} = k\hat{\mathbf{u}}$, $n = k/k_o$, and $k_o^2 = \omega^2 \mu_o \epsilon_o$, (6.3-10) gives

$$-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta}\mathbf{D}) = \frac{1}{n^2}\mathbf{D}. \tag{6.3-13}$$

For each of the indices n_a and n_b of the normal modes, we determine the corresponding vector **D** by solving (6.3-13).

The operation $-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta}\mathbf{D})$ may be interpreted as a projection of the vector $\boldsymbol{\eta}\mathbf{D}$ onto a plane normal to $\hat{\mathbf{u}}$. We may therefore write (6.3-13) in the form

$$\mathbf{P}_u \boldsymbol{\eta}\mathbf{D} = \frac{1}{n^2}\mathbf{D}, \tag{6.3-14}$$

where $\mathbf{P}_u$ is an operator representing the projection operation. Equation (6.3-14) is an eigenvalue equation for the operator $\mathbf{P}_u\boldsymbol{\eta}$, with $1/n^2$ the eigenvalue and **D** the eigenvector. There are two eigenvalues, $1/n_a^2$ and $1/n_b^2$, and two corresponding eigenvectors, $\mathbf{D}_a$ and $\mathbf{D}_b$, representing the two normal modes.

The eigenvalue problem (6.3-14) has a simple geometrical interpretation. The tensor $\boldsymbol{\eta}$ is represented geometrically by its quadric representation—the index ellipsoid. The operator $\mathbf{P}_u\boldsymbol{\eta}$ represents projection onto a plane normal to $\hat{\mathbf{u}}$. Solving the eigenvalue problem in (6.3-14) is equivalent to finding the principal axes of the ellipse formed by the intersection of the plane normal to $\hat{\mathbf{u}}$ with the index ellipsoid. This proves the validity of the geometrical construction described earlier for using the index ellipsoid to determine the normal modes.

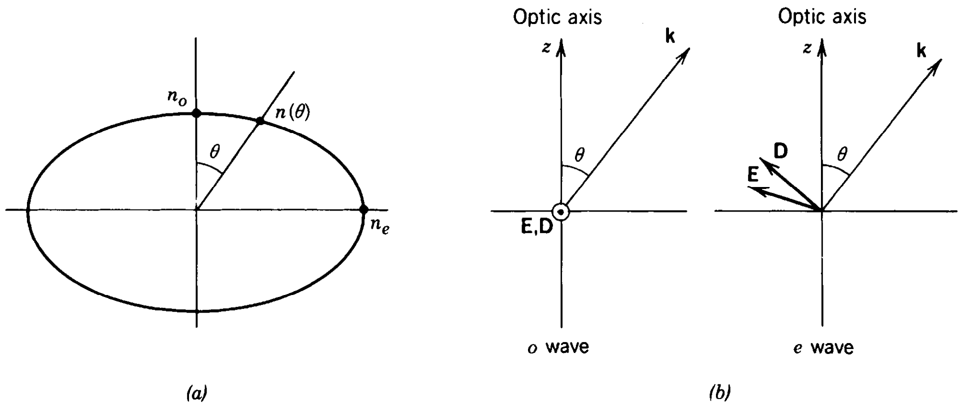


Figure 6.3-9 (*a*) Variation of the refractive index $n(\theta)$ of the extraordinary wave with θ (the angle between the direction of propagation and the optic axis). (*b*) The **E** and **D** vectors for the ordinary wave (*o* wave) and the extraordinary wave (*e* wave). The circle with a dot at the center signifies that the direction of the vector is out of the plane of the paper, toward the reader.

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$) the index ellipsoid is an ellipsoid of revolution. For a wave traveling at an angle θ with the optic axis the index ellipse has half-lengths n_o and $n(\theta)$, where

$$\boxed{\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2},} \tag{6.3-15}$$

Refractive Index of the Extraordinary Wave

so that the normal modes have refractive indices $n_a = n_o$ and $n_b = n(\theta)$. The first mode, called the **ordinary wave**, has a refractive index n_o regardless of θ. The second mode, called the **extraordinary wave**, has a refractive index $n(\theta)$ varying from n_o when $\theta = 0°$, to n_e when $\theta = 90°$, in accordance with the ellipse shown in Fig. 6.3-9(*a*). The vector **D** of the ordinary wave is normal to the plane defined by the optic axis (z axis) and the direction of wave propagation **k**, and the vectors **D** and **E** are parallel. The extraordinary wave, on the other hand, has a vector **D** in the k–z plane, which is normal to **k**, and **E** is not parallel to **D**. These vectors are illustrated in Fig. 6.3-9(*b*).

D. Rays, Wavefronts, and Energy Transport

The nature of waves in anisotropic media is best explained by examining the **k** surface $\omega = \omega(k_1, k_2, k_3)$ obtained by equating the determinant of the matrix in (6.3-11) to zero as illustrated in Fig. 6.3-8. The **k** surface describes the variation of the phase velocity $c = \omega/k$ with the direction $\hat{\mathbf{u}}$. The distance from the origin to the **k** surface in the direction of $\hat{\mathbf{u}}$ is therefore inversely proportional to the phase velocity.

The group velocity may also be determined from the **k** surface. In analogy with the group velocity $v = d\omega/dk$, which describes the velocity with which light pulses (wavepackets) travel (see Sec. 5.6), the group velocity for *rays* (localized beams, or spatial wavepackets) is the vector $\mathbf{v} = \nabla_k \omega(\mathbf{k})$, the gradient of ω with respect to **k**. Since the **k** surface is the surface $\omega(k_1, k_2, k_3) =$ constant, **v** must be normal to the **k** surface. Thus rays travel along directions normal to the **k** surface.

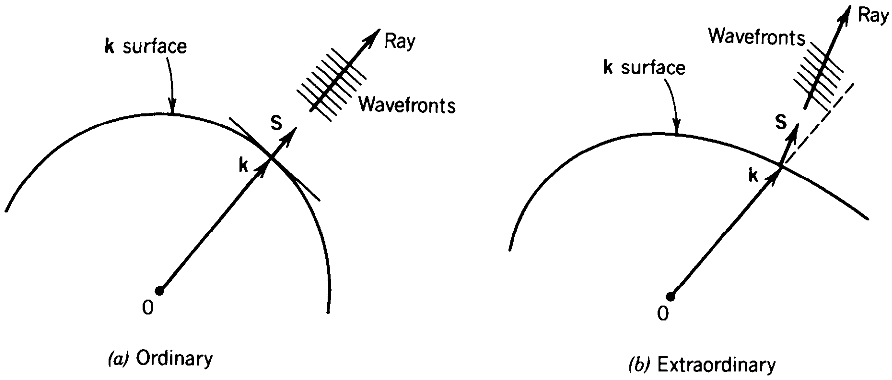


Figure 6.3-10 Rays and wavefronts for (*a*) spherical **k** surface, and (*b*) nonspherical **k** surface.

The Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ is also normal to the **k** surface. This can be shown by assuming a fixed ω and two vectors **k** and $\mathbf{k} + \Delta\mathbf{k}$ lying on the **k** surface. By taking the differential of (6.3-9) and (6.3-8) and using certain vector identities, it can be shown that $\Delta\mathbf{k} \cdot \mathbf{S} = 0$, so that **S** is normal to the **k** surface. Consequently, **S** is also parallel to the group velocity vector **v**. The wavefronts are perpendicular to the wavevector **k** (since the phase of the wave is $\mathbf{k} \cdot \mathbf{r}$). The wavefront normals are therefore parallel to the wavevector **k**.

If the **k** surface is a sphere, as in isotropic media, for example, the vectors **v**, **S**, and **k** are all parallel, indicating that rays are parallel to the wavefront normal **k** and energy flows in the same direction, as illustrated in Fig. 6.3-10(*a*). On the other hand, if the **k** surface is not normal to the wavevector **k**, as illustrated in Fig. 6.3-10(*b*), the rays and the direction of energy transport are not orthogonal to the wavefronts. Rays then have the "extraordinary" property of traveling at an oblique angle with their wavefronts [Fig. 6.3-10(*b*)].

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$), the equation of the **k** surface $\omega = \omega(k_1, k_2, k_3)$ simplifies to

$$\left(k^2 - n_o^2 k_o^2\right)\left(\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} - k_o^2\right) = 0, \tag{6.3-16}$$

which has two solutions: a sphere,

$$k = n_o k_o, \tag{6.3-17}$$

and an ellipsoid of revolution,

$$\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} = k_o^2. \tag{6.3-18}$$

Because of symmetry about the z axis (optic axis), there is no loss of generality in assuming that the vector **k** lies in the y–z plane. Its direction is then characterized by the angle θ with the optic axis. It is therefore convenient to draw the k-surfaces only in the y–z plane—a circle and an ellipse, as shown in Fig. 6.3-11.

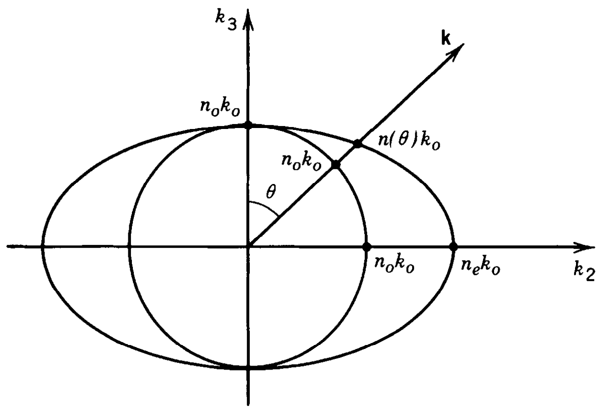


Figure 6.3-11 Intersection of the **k** surface with the y–z plane for a uniaxial crystal.

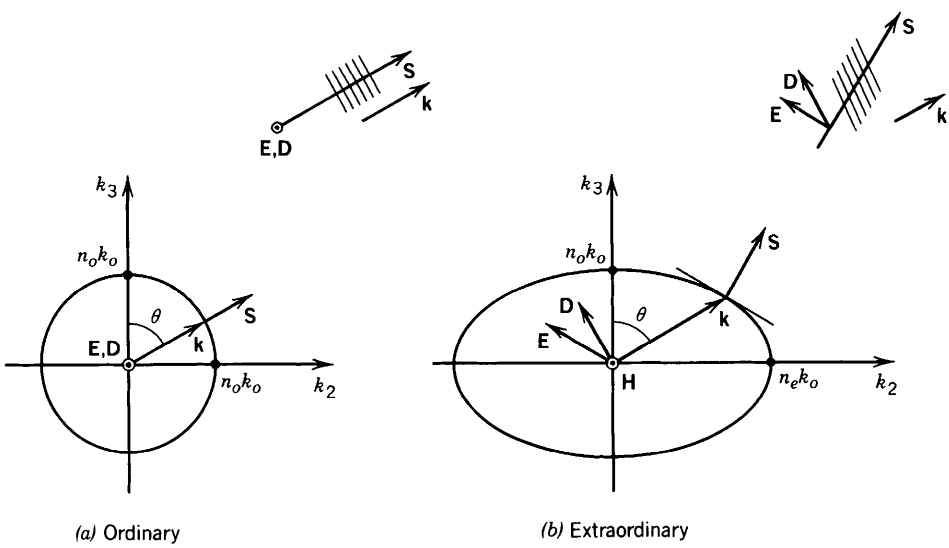


Figure 6.3-12 The normal modes for a plane wave traveling in a direction **k** at an angle θ with the optic axis z of a uniaxial crystal are: (a) An ordinary wave of refractive index n_o polarized in a direction normal to the k–z plane. (b) An extraordinary wave of refractive index $n(\theta)$ [given by (6.3-15)] polarized in the k–z plane along a direction tangential to the ellipse (the **k** surface) at the point of its intersection with **k**. This wave is "extraordinary" in the following ways: **D** is not parallel to **E** but both lie in the k–z plane; **S** is not parallel to **k** so that power does not flow along the direction of **k**; rays are not normal to wavefronts and the wave travels "sideways."

Given the direction $\hat{\mathbf{u}}$ of the vector **k**, the wavenumber k is determined by finding the intersection with the **k** surfaces. The two solutions define the two normal modes, the ordinary and extraordinary waves. The ordinary wave has a wavenumber $k = n_o k_o$ regardless of direction, whereas the extraordinary wave has a wavenumber $n(\theta) k_o$, where $n(\theta)$ is given by (6.3-15), confirming earlier results obtained from the index-ellipsoid geometrical construction. The directions of rays, wavefronts, energy flow, and field vectors **E** and **D** for the ordinary and extraordinary waves in a uniaxial crystal are illustrated in Fig. 6.3-12.

E. Double Refraction

Refraction of Plane Waves

We now examine the refraction of a plane wave at the boundary between an isotropic medium (say air, $n = 1$) and an anisotropic medium (a crystal). The key principle is that the wavefronts of the incident wave and the refracted wave must be matched at the boundary. Because the anisotropic medium supports two modes of distinctly different phase velocities, one expects that for each incident wave there are two refracted waves with two different directions and different polarizations. The effect is called **double refraction** or **birefringence**.

The phase-matching condition requires that

$$k_o \sin\theta_1 = k \sin\theta, \tag{6.3-19}$$

where θ_1 and θ are the angles of incidence and refraction. In an anisotropic medium, however, the wave number $k = n(\theta)k_o$ is itself a function of θ, so that

$$\sin\theta_1 = n(\theta)\sin\theta, \tag{6.3-20}$$

a modified Snell's law. To solve (6.3-19), we draw the intersection of the **k** surface with the plane of incidence and search for an angle θ for which (6.3-19) is satisfied. Two solutions, corresponding to the two normal modes, are expected. The polarization state of the incident light governs the distribution of energy among the two refracted waves.

Take, for example, a uniaxial crystal and a plane of incidence parallel to the optic axis. The **k** surfaces intersect the plane of incidence in a circle and an ellipse (Fig. 6.3-13). The two refracted waves that satisfy the phase-matching condition are:

- An ordinary wave of orthogonal polarization (TE) at an angle $\theta = \theta_o$ for which

$$\sin\theta_1 = n_o \sin\theta_o;$$

- An extraordinary wave of parallel polarization (TM) at an angle $\theta = \theta_e$, for which

$$\sin\theta_1 = n(\theta_e)\sin\theta_e,$$

where $n(\theta)$ is given by (6.3-15).

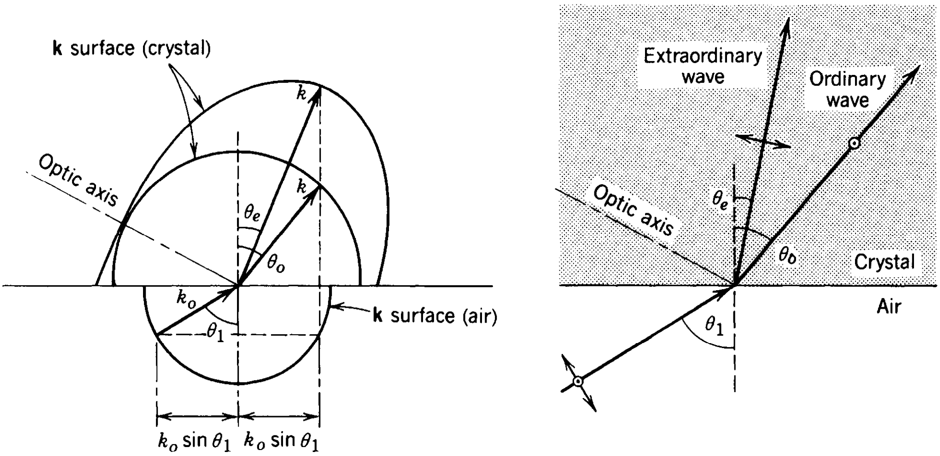


Figure 6.3-13 Determination of the angles of refraction by matching projections of the **k** vectors in air and in a uniaxial crystal.

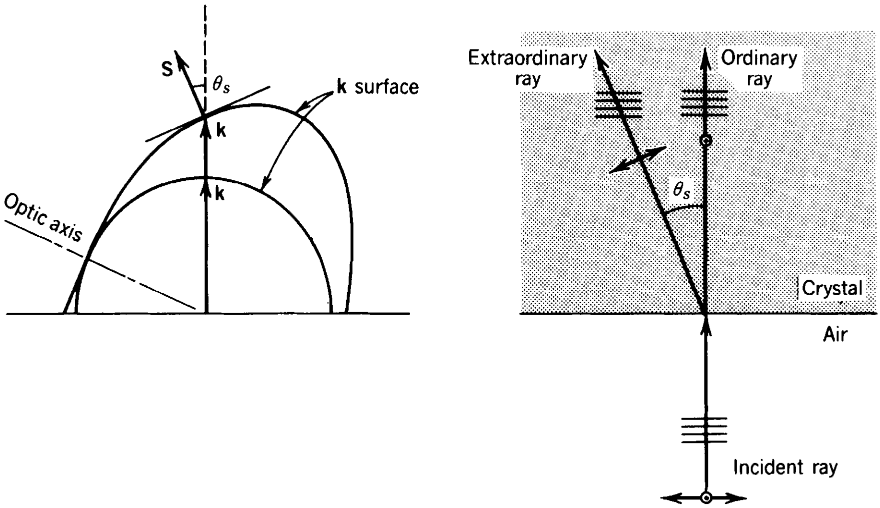


Figure 6.3-14 Double refraction at normal incidence.

If the incident wave carries the two polarizations, the two refracted waves will emerge.

Refraction of Rays

The previous analysis dealt with the refraction of plane waves. The refraction of rays is different since rays in an anisotropic medium do not necessarily travel in a direction normal to the wavefronts. In air, before entering the crystal, the wavefronts are normal to the rays. The refracted wave must have a wavevector satisfying the phase-matching condition, so that Snell's law (6.3-20) applies, with the angle of refraction θ determining the direction of **k**. Since the direction of **k** is not the direction of the ray, Snell's law is not applicable to rays.

An example that dramatizes the deviation from Snell's law is that of normal incidence at a uniaxial crystal whose optic axis is neither parallel nor perpendicular to the crystal boundary. The incident wave has a **k** vector normal to the boundary. To ensure phase matching, the refracted waves must also have wavevectors in the same direction. Intersections with the **k** surface yield two points corresponding to two waves. The ordinary ray is parallel to **k**. But the extraordinary ray points in the direction of the normal to the **k** surface, at an angle θ_s with the normal to the crystal boundary, as illustrated in Fig. 6.3-14. Thus normal incidence creates oblique refraction. Note, however, that the principle of phase matching is still maintained; wavefronts of both

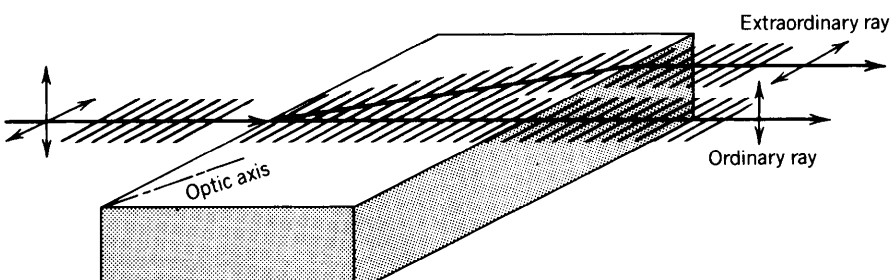


Figure 6.3-15 Double refraction through an anisotropic plate. The plate serves as a polarizing beamsplitter.

refracted rays are parallel to the crystal boundary and to the wavefront of the incident ray.

When light rays are transmitted through a plate of anisotropic material as described above, the two rays refracted at the first surface refract at the second surface, creating two laterally separated rays with orthogonal polarizations, as illustrated in Fig. 6.3-15.

6.4 OPTICAL ACTIVITY AND FARADAY EFFECT

A. Optical Activity

Certain materials act naturally as polarization rotators, a property known as optical activity. Their normal modes are circularly polarized, instead of linearly polarized waves; the waves with right- and left-circular polarizations travel at different phase velocities. Optical activity is found in materials in which the molecules have an inherently helical character. Examples are quartz, selenium, tellurium, and tellurium oxide (TeO_2). Many organic materials exhibit optical activity. The rotatory power and the sense of rotation are also sensitive to the chemical structure and concentration of solutions (this effect has been used, for example, to measure sugar content in solutions).

It will be shown subsequently that an optically active medium with right- and left-circular-polarization phase velocities c_o/n_+ and c_o/n_- acts as a polarization rotator with an angle of rotation $\pi(n_- - n_+)d/\lambda_o$ proportional to the distance d. The rotatory power (angle per unit length) of the optically active medium is therefore

$$\boxed{\rho = \frac{\pi(n_- - n_+)}{\lambda_o}.} \tag{6.4-1}$$

Rotatory Power

The direction of rotation of the polarization plane is in the same sense as that of the circularly polarized component of the greater phase velocity (smaller refractive index). If $n_+ < n_-$, ρ is positive and the rotation is in the same direction as the electric field vector of the right circularly polarized wave [clockwise when viewed from the direction toward which the wave is approaching, as illustrated in Fig. 6.4-1(a)].

The optically active medium is a spatially dispersive medium since the relation between $\mathbf{D}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ is not local. $\mathbf{D}(\mathbf{r})$ at position $\mathbf{r}$ is determined not only by $\mathbf{E}(\mathbf{r})$, but also by $\mathbf{E}(\mathbf{r}')$ at points $\mathbf{r}'$ in the immediate vicinity of $\mathbf{r}$ [since it is dependent on the derivatives in $\nabla \times \mathbf{E}(\mathbf{r})$]. Spatial dispersiveness is analogous to temporal dispersiveness, which is caused by the noninstantaneous response of the medium (see Sec. 5.2B).

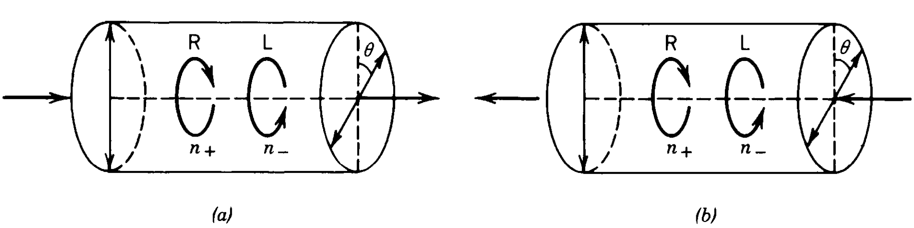


Figure 6.4-1 (a) Rotation of the plane of polarization in an optically active medium is a result of the difference in the velocities of the two circular polarizations. In this illustration, the right circularly polarized wave (R) is faster than the left circularly polarized wave (L), i.e., $n_+ < n_-$, so that ρ is positive. (b) If the wave in (a) is reflected after traversing the medium, the plane of polarization rotates in the opposite direction and the wave retraces itself.

Equation (6.4-1) may be obtained by decomposing the linearly polarized wave into a sum of right and left circularly polarized waves of equal amplitudes (see Exercise 6.1-1),

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{2}e^{-j\theta}\begin{bmatrix} 1 \\ j \end{bmatrix} + \frac{1}{2}e^{j\theta}\begin{bmatrix} 1 \\ -j \end{bmatrix},$$

where θ is the initial angle of the polarization plane. After a distance d of propagation in the medium, phase shifts $\varphi_+ = 2\pi n_+ d/\lambda_o$ and $\varphi_- = 2\pi n_- d/\lambda_o$, respectively, are encountered by the right and left circularly polarized waves, so that the new Jones vector is

$$\frac{1}{2}e^{-j\theta}e^{-j\varphi_+}\begin{bmatrix} 1 \\ j \end{bmatrix} + \frac{1}{2}e^{j\theta}e^{-j\varphi_-}\begin{bmatrix} 1 \\ -j \end{bmatrix} = e^{-j\varphi_o}\begin{bmatrix} \cos\left(\theta - \dfrac{\varphi}{2}\right) \\ \sin\left(\theta - \dfrac{\varphi}{2}\right) \end{bmatrix},$$

where $\varphi_o = \frac{1}{2}(\varphi_+ + \varphi_-)$ and $\varphi = \varphi_- - \varphi_+ = 2\pi(n_- - n_+)d/\lambda_o$. This Jones vector represents a linearly polarized wave with the plane of polarization rotated by an angle $\varphi/2 = \pi(n_- - n_+)d/\lambda_o$, as indicated above.

Medium Equations

We now show that a dielectric medium characterized by the medium equation

$$\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E} + \epsilon_o \xi\, j\omega\mathbf{B} = \boldsymbol{\epsilon}\mathbf{E} - \epsilon_o \xi \nabla \times \mathbf{E}, \qquad (6.4\text{-}2)$$

where ξ is a constant, is optically active. This medium relation arises in molecular structures with a helical character. In these structures, a time-varying magnetic flux density $\mathbf{B}$ induces a circulating current that sets up an electric dipole moment (and hence polarization) proportional to $j\omega\mathbf{B} = -\nabla \times \mathbf{E}$, which is responsible for the last term in (6.4-2).

The optically active medium is a spatially dispersive medium since the relation between $\mathbf{D}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ is not local. $\mathbf{D}(\mathbf{r})$ at position $\mathbf{r}$ is determined not only by $\mathbf{E}(\mathbf{r})$, but also by $\mathbf{E}(\mathbf{r}')$ at points $\mathbf{r}'$ in the immediate vicinity of $\mathbf{r}$ [since it is dependent on the derivatives in $\nabla \times \mathbf{E}(\mathbf{r})$]. Spatial dispersiveness is analogous to temporal dispersiveness, which is caused by the noninstantaneous response of the medium (see Sec. 5.2B).

We proceed to show that the two normal modes of a medium satisfying (6.4-2) are circularly polarized waves and we determine the velocities c_o/n_+ and c_o/n_- in terms of the constant ξ.

Normal Modes of the Optically Active Medium

Consider the propagation of a plane wave $\mathbf{E}(\mathbf{r}) = \mathbf{E}\exp(-j\mathbf{k}\cdot\mathbf{r})$ in a medium satisfying (6.4-2). Setting $\mathbf{D}(\mathbf{r}) = \mathbf{D}\exp(-j\mathbf{k}\cdot\mathbf{r})$, (6.4-2) yields

$$\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E} + j\epsilon_o \mathbf{G} \times \mathbf{E}, \qquad (6.4\text{-}3)$$

where

$$\mathbf{G} = \xi\mathbf{k} \qquad (6.4\text{-}4)$$

is known as the **gyration vector**. Clearly, the vector $\mathbf{D}$ is not parallel to $\mathbf{E}$ since the vector $\mathbf{G} \times \mathbf{E}$ in (6.4-3) is perpendicular to $\mathbf{E}$. The relation between $\mathbf{D}$ and $\mathbf{E}$ is therefore dependent on the wavevector $\mathbf{k}$, which is not surprising since the medium is spatially dispersive. (This is analogous to the dependence of the dielectric properties of a temporally dispersive medium on ω.)

For simplicity, we assume that ϵ has uniaxial symmetry (with indices n_o and n_e), use the principal axes of the tensor ϵ as a coordinate system, and consider only waves propagating along the optic axis. The first term in (6.4-3) then corresponds to propagation of an ordinary wave of refractive index n_o.

To prove that the normal modes are circularly polarized, consider the two circularly polarized waves of electric-field vectors $\mathbf{E} = (E_0, \pm jE_0, 0)$ and wavevector $\mathbf{k} = (0, 0, k)$. The $+$ and $-$ signs correspond to right and left circularly polarized cases, respectively. Substituting in (6.4-3), we obtain $\mathbf{D} = (D_0, \pm jD_0, 0)$, where $D_0 = \epsilon_o(n_o^2 \pm G)E_0$. It follows that $\mathbf{D} = \epsilon_o n_{\pm}^2 \mathbf{E}$, where

$$\boxed{n_{\pm} = \left(n_o^2 \pm G\right)^{1/2},} \tag{6.4-5}$$

so that for either of the two circularly polarized waves the vector $\mathbf{D}$ is parallel to the vector $\mathbf{E}$. Equation (6.3-10) is satisfied if the wavenumber $k = n_{\pm}k_0$. Thus the right and left circularly polarized waves propagate, without change of their state of polarization, with refractive indices n_+ and n_-, respectively. They *are* the normal modes for this medium.

EXERCISE 6.4-1

Rotatory Power of an Optically Active Medium. Show that if $G \ll n_o$, the rotatory power of an optically active medium (rotation of the polarization plane per unit length) is approximately given by

$$\rho \approx -\frac{\pi G}{\lambda_o n_o}. \tag{6.4-6}$$

The rotatory power is strongly dependent on the wavelength. Since G is proportional to k, as indicated by (6.4-4), it is inversely proportional to the wavelength λ_o. Thus the rotatory power in (6.4-6) is inversely proportional to λ_o^2. In addition, the refractive index n_o is itself wavelength dependent. The rotatory power ρ of quartz is ≈ 31 deg/mm at $\lambda_o = 500$ nm and ≈ 22 deg/mm at 600 nm; for silver thiogallate ($AgGaS_2$) ρ is ≈ 700 deg/mm at 490 nm and ≈ 500 deg/mm at 500 nm.

B. Faraday Effect

Certain materials act as polarization rotators when placed in a static magnetic field, a property known as the Faraday effect. The angle of rotation is proportional to the distance, and the rotatory power ρ (angle per unit length) is proportional to the component B of the magnetic flux density in the direction of wave propagation,

$$\rho = VB, \tag{6.4-7}$$

where V is known as the **Verdet constant.**

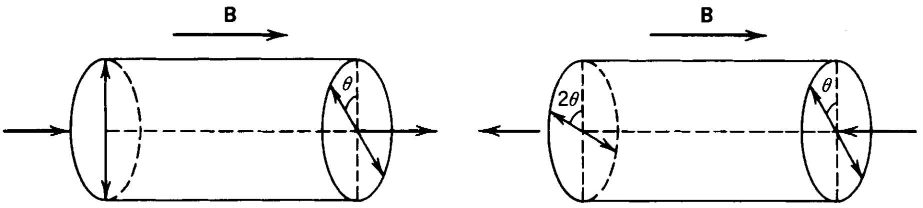


Figure 6.4-2 Polarization rotation in a medium exhibiting the Faraday effect. The sense of rotation is invariant to the direction of travel of the wave.

The sense of rotation is governed by the direction of the magnetic field: for $V > 0$, the rotation is in the direction of a right-handed screw pointing in the direction of the magnetic field. In contradistinction to optical activity, the sense of rotation does not reverse with the reversal of the direction of propagation of the wave (Fig. 6.4-2). When a wave travels through a Faraday rotator, reflects back onto itself, and travels once more through the rotator in the opposite direction, it undergoes twice the rotation.

The medium equation for materials exhibiting the Faraday effect is

$$\mathbf{D} = \epsilon \mathbf{E} + j\epsilon_o \gamma \mathbf{B} \times \mathbf{E}, \tag{6.4-8}$$

where $\mathbf{B}$ is the magnetic flux density and γ is a constant of the medium that is called the **magnetogyration coefficient**. This relation originates from the interaction of the static magnetic field $\mathbf{B}$ with the motion of electrons in the molecules under the influence of the optical electric field $\mathbf{E}$.

To establish an analogy between the Faraday effect and optical activity (6.4-8) is written as

$$\mathbf{D} = \epsilon \mathbf{E} + j\epsilon_o \mathbf{G} \times \mathbf{E}, \tag{6.4-9}$$

where

$$\mathbf{G} = \gamma \mathbf{B}. \tag{6.4-10}$$

Equation (6.4-9) is identical to (6.4-3) with the vector $\mathbf{G} = \gamma\mathbf{B}$ in Faraday rotators playing the role of the gyration vector $\mathbf{G} = \xi\mathbf{k}$ in optically active media. Note that in the Faraday effect $\mathbf{G}$ is independent of $\mathbf{k}$, so that reversal of the direction of propagation does not reverse the sense of rotation of the polarization plane. This property can be used to make optical isolators, as explained in Sec. 6.6.

With this analogy, and using (6.4-6), we conclude that the rotatory power of the Faraday medium is $\rho \approx -\pi G/\lambda_o n_o = -\pi\gamma B/\lambda_o n_o$, from which the Verdet constant (the rotatory power per unit magnetic flux density) is

$$\boxed{V \approx -\frac{\pi\gamma}{\lambda_o n_o}.} \tag{6.4-11}$$

Clearly, the Verdet constant is a function of the wavelength λ_o.

Materials that exhibit the Faraday effect include glasses, yttrium–iron–garnet (YIG), terbium–gallium–garnet (TGG), and terbium–aluminum–garnet (TbAlG). The Verdet constant V of TbAlG is $V = -1.16$ min/cm-Oe at $\lambda_o = 500$ nm.

6.5 OPTICS OF LIQUID CRYSTALS

Liquid Crystals

The liquid-crystal state is a state of matter in which the elongated (typically cigar-shaped) molecules have orientational order (like crystals) but lack positional order (like liquids). There are three types (phases) of liquid crystals, as illustrated in Fig. 6.5-1:

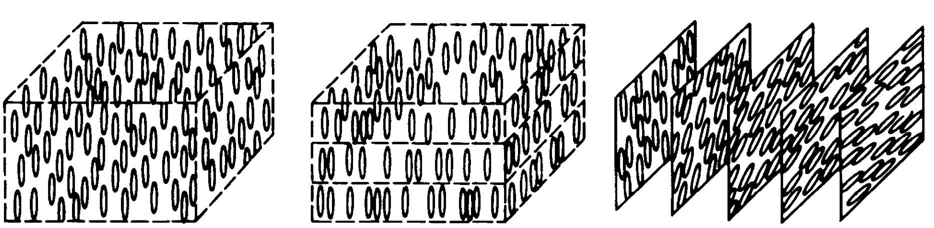
- In **nematic** liquid crystals the molecules tend to be parallel but their positions are random.
- In **smectic** liquid crystals the molecules are parallel, but their centers are stacked in parallel layers within which they have random positions, so that they have positional order in only one dimension.
- The **cholesteric** phase is a distorted form of the nematic phase in which the orientation undergoes helical rotation about an axis.

Liquid crystallinity is a *fluid* state of matter. The molecules change orientation when subjected to a force. For example, when a thin layer of liquid crystal is placed between two parallel glass plates the molecular orientation is changed if the plates are rubbed; the molecules orient themselves along the direction of rubbing.

Twisted nematic liquid crystals are nematic liquid crystals on which a twist, similar to the twist that exists naturally in the cholesteric phase, is imposed by external forces (for example, by placing a thin layer of the liquid crystal material between two glass plates polished in perpendicular directions as shown in Fig. 6.5-2). Because twisted nematic liquid crystals have enjoyed the greatest number of applications in photonics (in liquid-crystal displays, for example), this section is devoted to their optical properties. The electro-optic properties of twisted nematic liquid crystals, and their use as optical modulators and switches, are described in Chap. 18.

Optical Properties of Twisted Nematic Liquid Crystals

The twisted nematic liquid crystal is an optically *inhomogeneous anisotropic medium* that acts locally as a uniaxial crystal, with the optic axis parallel to the molecular



(a) (b) (c)

Figure 6.5-1 Molecular organizations of different types of liquid crystals: (a) nematic; (b) smectic; (c) cholesteric.

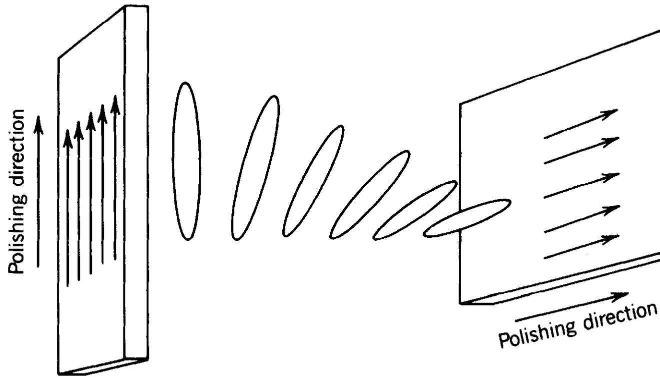


Figure 6.5-2 Molecular orientations of the twisted nematic liquid crystal.

direction. The optical properties are conveniently studied by dividing the material into thin layers perpendicular to the axis of twist, each of which acts as a uniaxial crystal, with the optic axis rotating gradually in a helical fashion (Fig. 6.5-3). The cumulative effects of these layers on the transmitted wave is determined. We proceed to show that under certain conditions the twisted nematic liquid crystal acts as a polarization rotator, with the polarization plane rotating in alignment with the molecular twist.

Consider the propagation of light along the axis of twist (the z axis) of a twisted nematic liquid crystal and assume that the twist angle varies linearly with z,

$$\theta = \alpha z, \tag{6.5-1}$$

where α is the twist coefficient (degrees per unit length). The optic axis is therefore parallel to the x-y plane and makes an angle θ with the x direction. The ordinary and extraordinary indices are n_o and n_e (typically, $n_e > n_o$), and the phase retardation coefficient (retardation per unit length) is

$$\beta = (n_e - n_o)k_o. \tag{6.5-2}$$

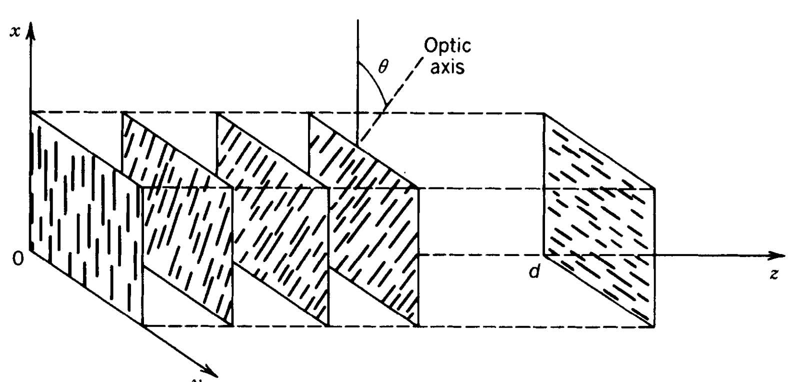


Figure 6.5-3 Propagation of light in a twisted nematic liquid crystal. In this diagram the angle of twist is 90°.

The liquid crystal cell is described completely by the twist coefficient α and the retardation coefficient β.

In practice, β is much greater than α, so that many cycles of phase retardation are introduced before the optic axis rotates appreciably. We show below that if the incident wave at $z = 0$ is linearly polarized in the x direction, then when $\beta \gg \alpha$, the wave maintains its linearly polarized state, but the plane of polarization rotates in alignment with the molecular twist, so that the angle of rotation is $\theta = \alpha z$ and the total rotation in a crystal of length d is the angle of twist αd. The liquid crystal cell then serves as a polarization rotator with rotatory power α. The polarization rotation property of the twisted nematic liquid crystal is useful for making display devices, as explained in Sec. 18.3.

Proof. We proceed to show that the twisted nematic liquid crystal acts as a polarization rotator if $\beta \gg \alpha$. We divide the width d of the cell into N incremental layers of equal widths $\Delta z = d/N$. The mth layer located at distance $z = z_m = m\,\Delta z$, $m = 1, 2, \ldots, N$, is a wave retarder whose slow axis (the optic axis) makes an angle $\theta_m = m\Delta\theta$ with the x axis, where $\Delta\theta = \alpha\Delta z$. It therefore has a Jones matrix

$$\mathbf{T}_m = \mathbf{R}(-\theta_m)\mathbf{T}_r\mathbf{R}(\theta_m), \tag{6.5-3}$$

where

$$\mathbf{T}_r = \begin{bmatrix} \exp(-jn_e k_o\,\Delta z) & 0 \\ 0 & \exp(-jn_o k_o\,\Delta z) \end{bmatrix} \tag{6.5-4}$$

is the Jones matrix of a retarder with axis in the x direction and $\mathbf{R}(\theta)$ is the coordinate rotation matrix in (6.1-15) [see (6.1-17)].

It is convenient to rewrite $\mathbf{T}_r$ in terms of the phase retardation coefficient $\beta = (n_e - n_o)k_o$,

$$\mathbf{T}_r = \exp(-j\varphi\,\Delta z)\begin{bmatrix} \exp\left(-j\beta\dfrac{\Delta z}{2}\right) & 0 \\ 0 & \exp\left(j\beta\dfrac{\Delta z}{2}\right) \end{bmatrix}, \tag{6.5-5}$$

where $\varphi = (n_o + n_e)k_o/2$. Since multiplying the Jones vector by a constant phase factor does not affect the state of polarization, we shall simply ignore the prefactor $\exp(-j\varphi\,\Delta z)$ in (6.5-5).

The overall Jones matrix of the device is the product

$$\mathbf{T} = \prod_{m=1}^{N} \mathbf{T}_m = \prod_{m=1}^{N} \mathbf{R}(-\theta_m)\mathbf{T}_r\mathbf{R}(\theta_m). \tag{6.5-6}$$

Using (6.5-3) and noting that $\mathbf{R}(\theta_m)\mathbf{R}(-\theta_{m-1}) = \mathbf{R}(\theta_m - \theta_{m-1}) = \mathbf{R}(\Delta\theta)$, we obtain

$$\mathbf{T} = \mathbf{R}(-\theta_N)\left[\mathbf{T}_r\mathbf{R}(\Delta\theta)\right]^{N-1}\mathbf{T}_r\mathbf{R}(\theta_1). \tag{6.5-7}$$

Substituting from (6.5-5) and (6.1-15)

$$\mathbf{T}_r\mathbf{R}(\Delta\theta) = \begin{bmatrix} \exp\left(-j\beta\frac{\Delta z}{2}\right) & 0 \\ 0 & \exp\left(j\beta\frac{\Delta z}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\alpha\Delta z & \sin\alpha\Delta z \\ -\sin\alpha\Delta z & \cos\alpha\Delta z \end{bmatrix}. \quad (6.5\text{-}8)$$

Using (6.5-7) and (6.5-8), the Jones matrix $\mathbf{T}$ of the device can, in principle, be determined in terms of the parameters α, β, and $d = N\Delta z$.

When $\alpha \ll \beta$, we can assume that the incremental rotation matrix $\mathbf{R}(\Delta\theta)$ is approximately an identity matrix and obtain

$$\mathbf{T} \approx \mathbf{R}(-\theta_N)[\mathbf{T}_r]^N\mathbf{R}(\theta_1) = \mathbf{R}(-\alpha N\Delta z)\begin{bmatrix} \exp\left(-j\beta\frac{\Delta z}{2}\right) & 0 \\ 0 & \exp\left(j\beta\frac{\Delta z}{2}\right) \end{bmatrix}^N$$

$$= \mathbf{R}(-\alpha N\Delta z)\begin{bmatrix} \exp\left(-j\beta N\frac{\Delta z}{2}\right) & 0 \\ 0 & \exp\left(j\beta N\frac{\Delta z}{2}\right) \end{bmatrix}.$$

In the limit as $N \to \infty$, $\Delta z \to 0$, and $N\Delta z \to d$,

$$\mathbf{T} = \mathbf{R}(-\alpha d)\begin{bmatrix} \exp\left(-j\beta\frac{d}{2}\right) & 0 \\ 0 & \exp\left(j\beta\frac{d}{2}\right) \end{bmatrix}. \quad (6.5\text{-}9)$$

This Jones matrix represents a wave retarder of retardation βd with the slow axis along the x direction, followed by a polarization rotator with rotation angle αd. If the original wave is linearly polarized along the x direction the wave retarder provides only a phase shift; the device then simply rotates the polarization by an angle αd equal to the twist angle.

6.6 POLARIZATION DEVICES

This section is a brief description of a number of devices that are used to modify the state of polarization of light. The basic principles of most of these devices have been discussed earlier in this chapter.

A. Polarizers

A polarizer is a device that transmits the component of the electric field in the direction of its transmission axis and blocks the orthogonal component. This preferential treatment of the two components of the electric field is achieved by selective absorption, selective reflection from an isotropic medium, or selective reflection/refraction at the boundary of an anisotropic medium.

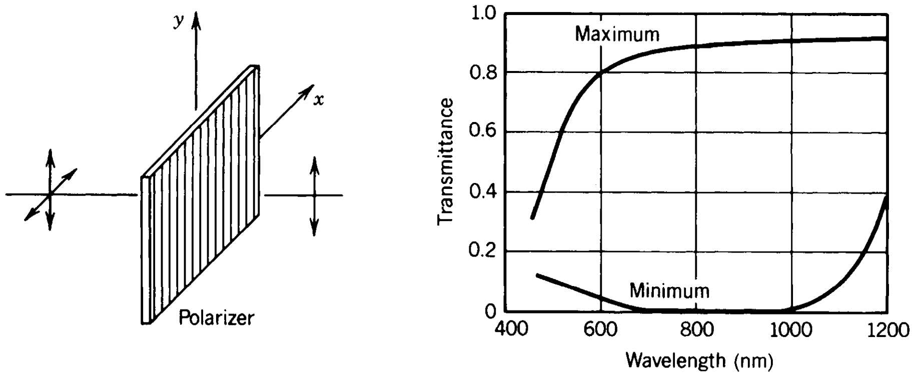


Figure 6.6-1 Power transmittances of a typical dichroic polarizer with the polarization plane of the light aligned for maximum and minimum transmittance.

Polarization by Selective Absorption (Dichroism)

The absorption of light by certain anisotropic materials, called **dichroic materials**, depends on the direction of the electric field (Fig. 6.6-1). These materials have anisotropic molecular structures whose response is sensitive to the direction of the applied field. The most common dichroic material is the Polaroid H-sheet (basically a sheet of polyvinyl alcohol heated and stretched in a certain direction then impregnated with iodine atoms).

Polarization by Selective Reflection

The reflection of light from the boundary between two dielectric isotropic materials is polarization dependent (see Sec. 6.2). At the Brewster angle of incidence, light of TM polarization is not reflected (i.e., is totally refracted). At this angle, only the TE component of the incident light is reflected, so that the reflector serves as a polarizer (Fig. 6.6-2).

Polarization by Selective Refraction in Anisotropic Media (Polarizing Beamsplitters)

When light refracts at the surface of an anisotropic crystal the two polarizations refract at different angles and are spatially separated (see Sec. 6.3E and Fig. 6.3-15). This is an excellent way of obtaining polarized light from unpolarized light. The device usually

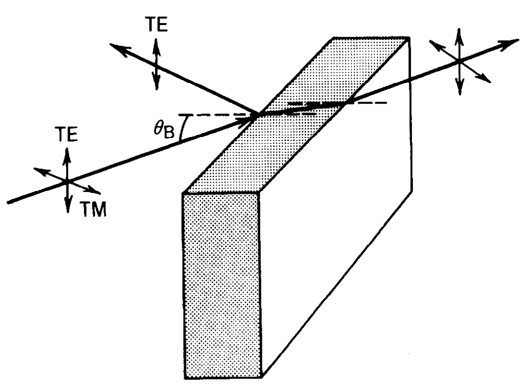


Figure 6.6-2 Brewster-angle polarizer.

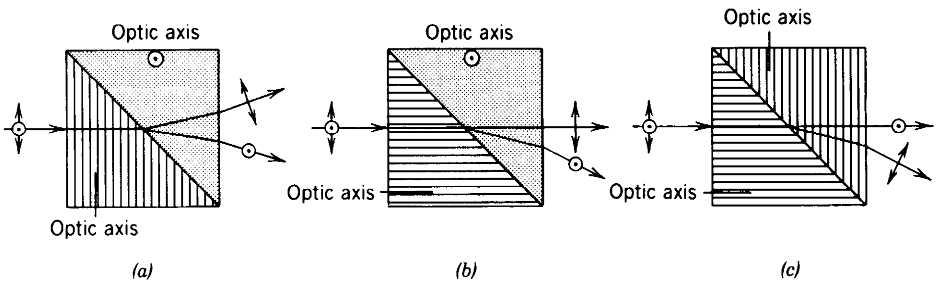


Figure 6.6-3 Polarizing prisms: (*a*) Wollaston prism; (*b*) Rochon prism; (*c*) Sénarmont prism. The directions and polarizations of the exiting waves differ in the three cases. In this illustration, the crystals are negative uniaxial (e.g., calcite).

takes the form of two cemented prisms made of anisotropic (uniaxial) crystals in different orientations, as illustrated by the examples in Fig. 6.6-3. These prisms serve as **polarizing beamsplitters.**

B. Wave Retarders

The wave retarder is characterized by its retardation Γ and its fast and slow axes (see Sec. 6.1B). The normal modes are linearly polarized waves polarized in the directions of the axes, and the velocities are different. Upon transmission through the retarder, a relative phase shift Γ between these modes ensues.

Wave retarders are often made of anisotropic materials. As explained in Sec. 6.3B, when light travels along a principal axis of a crystal (say the z axis), the normal modes are linearly polarized waves pointing along the two other principal axes (x and y axes). The two modes travel with the principal refractive indices n_1 and n_2. If $n_1 < n_2$, the x axis is the fast axis. If the plate has a thickness d, the phase retardation is $\Gamma = (n_2 - n_1)k_o d = 2\pi(n_2 - n_1)d/\lambda_o$. The retardation is directly proportional to the thickness d and inversely proportional to the wavelength λ_o (note, however, that $n_2 - n_1$ itself is wavelength dependent).

The refractive indices of mica, for example, are 1.599 and 1.594 at $\lambda_o = 633$ nm, so that $\Gamma/d \approx 15.8\pi$ rad/mm. A 63.3-μm thin sheet is a half-wave retarder ($\Gamma \approx \pi$).

Control of Light Intensity by Use of a Wave Retarder and Two Polarizers

The power (or intensity) transmittance of a system constructed from a wave retarder of retardation Γ placed between two crossed polarizers, at 45° with respect to the retarder's axes, as shown in Fig. 6.6-4, is

$$\mathscr{T} = \sin^2\frac{\Gamma}{2}. \tag{6.6-1}$$

This may be obtained by use of Jones matrices or by examining the polarization ellipse of the retarded light as a function of Γ and determining the component in the direction of the output polarizer, as illustrated in Fig. 6.6-4. If $\Gamma = 0$, no light is transmitted since the polarizers are orthogonal. If $\Gamma = \pi$, all the light is transmitted since the retarder rotates the polarization 90°, making it match the transmission axis of the second polarizer.

The intensity of the transmitted light can be controlled by altering the retardation Γ (for example, by changing the indices n_1 and n_2). This is the basic principle underlying the electro-optic modulators discussed in Chap. 18.

Furthermore, since Γ depends on d, slight variations in the thickness of a sample can be monitored by examining the pattern of the transmitted light. Also since Γ is

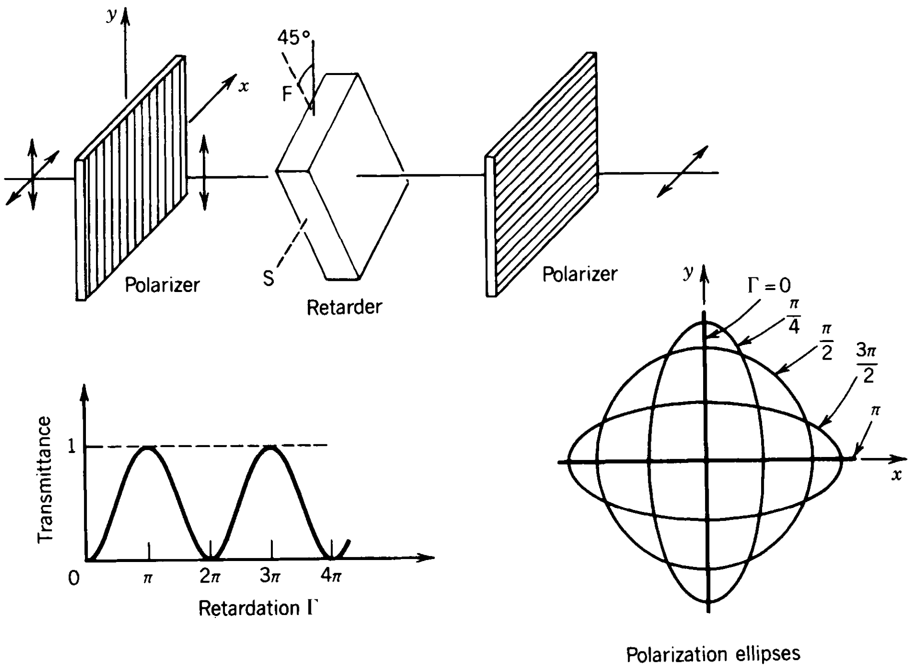


Figure 6.6-4 Controlling light intensity by use of a wave retarder with variable retardation Γ between two crossed polarizers.

wavelength dependent, the transmittance of the system is frequency sensitive. The system therefore serves as a filter, but the selectivity is not very sharp. Other configurations using wave retarders and polarizers can be used to construct narrowband transmission filters.

C. Polarization Rotators

A polarization rotator rotates the plane of polarization of linearly polarized light by a fixed angle, maintaining its linearly polarized nature. Optically active media and materials exhibiting the Faraday effect act as polarization rotators, as shown in Sec. 6.4. The twisted nematic liquid crystal also acts as a polarization rotator under certain conditions, as shown in Sec. 6.5.

If a polarization rotator is placed between two polarizers, the amount of transmitted light depends on the rotation angle. The intensity of light can be controlled (modulated) if the angle of rotation is controlled by some external means (e.g., by varying the magnetic flux density applied to a Faraday rotator, or by changing the molecular orientation of a liquid crystal by means of an applied electric field). Electro-optic modulation of light and liquid-crystal display devices are discussed in Chap. 18.

Optical Isolators

An optical isolator is a device that transmits light in only one direction, thus acting as a "one-way valve." Optical isolators are useful in preventing reflected light from returning back to the source. This type of feedback can have deleterious effects on the operation of certain light sources (semiconductor lasers, for example).

A system made of a polarizing beamsplitter followed by a quarter-wave retarder acts as an isolator. Light traveling in the forward direction is polarized by the cube, then circularly polarized by the retarder. Upon reflection from a mirror beyond the retarder,

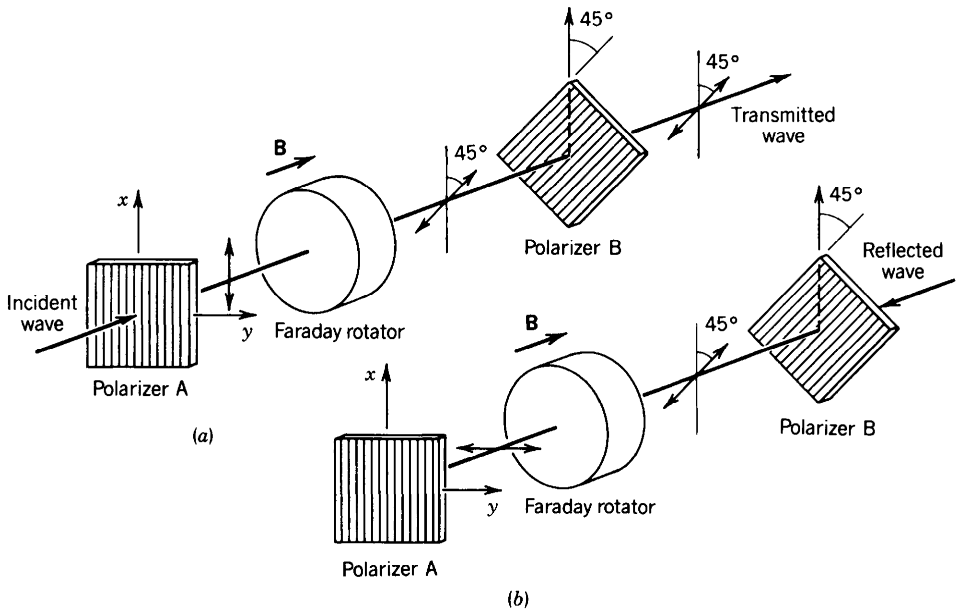


Figure 6.6-5 An optical isolator using a Faraday rotator transmits light in one direction, as in (*a*), and blocks light in the opposite direction, as in (*b*).

the sense of rotation is reversed (left to right, or vice versa), so that upon transmission back through the retarder it becomes polarized in the orthogonal direction and is therefore blocked by the polarizing cube (see Problem 6.1-6). Although this type of isolator can offer attenuation of the backward wave up to 30 dB (0.1%), it operates only over a narrow wavelength range.

A Faraday rotator placed between two polarizers making a 45° angle with each other can also be used as an optical isolator. The magnetic flux density applied to the rotator is adjusted so that the polarization is rotated by 45° in the direction of a right-handed screw pointing in the z direction [Fig. 6.6-5(*a*)]. Light traveling from left to right crosses polarizer A, rotates 45°, and is transmitted through polarizer B. However, light traveling in the opposite direction [Fig. 6.6-5(*b*)], although it crosses polarizer B, rotates an additional 45° and is blocked by polarizer A. A Faraday rotator cannot be replaced by an optically active or liquid-crystal polarization rotator since, in those devices, the sense of rotation is such that the polarization of the reflected wave retraces that of the incident wave and is therefore transmitted back through the polarizers to the source. Faraday-rotator isolators made of yttrium–iron–garnet (YIG) or terbium–gallium–garnet (TGG), for example, can offer an attenuation of the backward wave up to 90 dB, over a relatively wide wavelength range.

READING LIST

General

See also the list of general books on optics in Chapter 1.

D. S. Kliger, J. W. Lewis, and C. E. Randall, *Polarized Light in Optics and Spectroscopy*, Academic Press, Boston, 1990.

J. F. Nye, *Physical Properties of Crystals: Their Representation by Tensors and Matrices*, Oxford University Press, New York, 1967, 2nd ed. 1984.

R. M. A. Azzam and N. M. Bashara, *Ellipsometry and Polarized Light*, North-Holland, Amsterdam, 1977.

W. Swindell, *Polarized Light*, Dowden, Hutchinson & Ross, Stroudsburg, PA, 1975.

B. A. Robson, *The Theory of Polarization Phenomena*, Clarendon Press, Oxford, 1974.

D. Clarke and J. F. Grainger, *Polarized Light and Optical Measurement*, Pergamon Press, Oxford, 1971.

P. Gay, *An Introduction to Crystal Optics*, Longmans, London, 1967.

L. Velluz, M. LeGrand, and M. Grosjean, *Optical Circular Dichroism*, Academic Press, New York, 1965.

E. A. Wood, *Crystals and Light: An Introduction to Optical Crystallography*, Van Nostrand, Princeton, NJ, 1964.

W. A. Shurcliff and S. S. Ballard, *Polarized Light*, Van Nostrand, Princeton, NJ, 1964.

W. A. Shurcliff, *Polarized Light: Production and Use*, Harvard University Press, Cambridge, MA, 1962.

Books on Liquid Crystals

J. L. Ericksen and D. Kinderlehrer, eds., *Theory and Applications of Liquid Crystals*, Springer-Verlag, New York, 1987.

L. M. Blinov, *Electro-Optical and Magneto-Optical Properties of Liquid Crystals*, Wiley, New York, 1983.

W. H. de Jeu, *Physical Properties of Liquid Crystalline Materials*, Gordon and Breach, New York, 1980.

P.-G. de Gennes, *The Physics of Liquid Crystals*, Clarendon Press, Oxford, 1974, 1979.

S. Chandrasekhar, *Liquid Crystals*, Cambridge University Press, New York, 1977.

G. Meier, E. Sackmann, and J. G. Grabmaier, *Applications of Liquid Crystals*, Springer-Verlag, Berlin, 1975.

Articles

J. M. Bennett and H. E. Bennett, Polarization, in *Handbook of Optics*, W. G. Driscoll, ed., McGraw-Hill, New York, 1978.

V. M. Agranovich and V. L. Ginzburg, Crystal Optics with Spatial Dispersion, in *Progress in Optics*, vol. 9, E. Wolf, ed., North-Holland, Amsterdam, 1971.

PROBLEMS

6.1-1 **Orthogonal Polarizations.** Show that if two elliptically polarized states are orthogonal, the major axes of their ellipses are perpendicular and the senses of rotation are opposite.

6.1-2 **Rotating a Polarization Rotator.** Show that the Jones matrix of a polarization rotator is invariant to rotation of the coordinate system.

6.1-3 **The Half-Wave Retarder.** Linearly polarized light is transmitted through a half-wave retarder. If the polarization plane makes an angle θ with the fast axis of the retarder, show that the transmitted light is linearly polarized at an angle $-\theta$, i.e., rotates by an angle 2θ. Why is the half-wave retarder not equivalent to a polarization rotator?

6.1-4 **Wave Retarders in Tandem.** Write down the Jones matrices for:
(a) A $\pi/2$ wave retarder with the fast axis at 0°.
(b) A π wave retarder with the fast axis at 45°.
(c) A $\pi/2$ wave retarder with the fast axis at 90°.
If these three retarders are placed in tandem, show that the resulting device introduces a 90° rotation with a phase shift $\pi/2$.

6.1-5 **Reflection of Circularly Polarized Light.** Show that circularly polarized light changes handedness (right becomes left, and vice versa) upon reflection from a mirror.

6.1-6 **Optical Isolators.** An optical isolator transmits light traveling in one direction and blocks it in the opposite direction. Show that isolation of the light reflected by a planar mirror may be achieved by using a combination of a linear polarizer and a quarter-wave retarder with axes at 45° with respect to the transmission axis of the polarizer.

6.2-1 **Reflectance of Glass.** A plane wave is incident from air ($n = 1$) onto a glass plate ($n = 1.5$) at an angle of incidence 45°. Determine the intensity reflectances of the TE and TM waves. What is the average reflectance for unpolarized light (light carrying TE and TM waves of equal intensities)?

6.2-2 **Refraction at the Brewster Angle.** Show that at the Brewster angle of incidence the directions of the reflected and refracted waves are orthogonal. The electric field of the refracted TM wave is then parallel to the direction of the reflected wave.

6.2-3 **Retardation Associated with Total Internal Reflection.** Determine the phase retardation between the TE and TM waves introduced by total internal reflection at the boundary between glass ($n = 1.5$) and air ($n = 1$) at an angle of incidence $\theta = 1.2\theta_c$, where θ_c is the critical angle.

6.2-4 **Goos–Hänchen Shift.** Two TE plane waves undergo total internal reflection at angles θ and $\theta + d\theta$, where $d\theta$ is an incremental angle. If the phase retardation introduced between the reflected waves is written in the form $d\varphi = \xi\, d\theta$, determine an expression for the coefficient ξ. Sketch the interference patterns of the two incident waves and the two reflected waves and verify that they are shifted by a lateral distance proportional to ξ. When the incident wave is a beam (composed of many plane-wave components), the reflected beam is displaced laterally by a distance proportional to ξ. This effect is known as the Goos–Hänchen effect.

6.2-5 **Reflection from an Absorptive Medium.** Use Maxwell's equations and appropriate boundary conditions to show that the complex amplitude reflectance at the boundary between free space and a medium with refractive index n and absorption coefficient α at normal incidence is $r = [(n - j\alpha c/2\omega) - 1]/[(n - j\alpha c/2\omega) + 1]$.

6.3-1 **Maximum Retardation in Quartz.** Quartz is a positive uniaxial crystal with $n_e = 1.553$ and $n_o = 1.544$. (a) Determine the retardation per mm at $\lambda_o = 633$ nm when the crystal is oriented such that retardation is maximized. (b) At what thickness(es) does the crystal act as a quarter-wave retarder?

6.3-2 **Maximum Extraordinary Effect.** Determine the direction of propagation for which the angle between the wavevector $\mathbf{k}$ and the Poynting vector $\mathbf{S}$ (also the direction of ray propagation) in quartz ($n_e = 1.553$ and $n_o = 1.544$) is maximum.

6.3-3 **Double Refraction.** A plane wave is incident from free space onto a quartz crystal ($n_e = 1.553$ and $n_o = 1.544$) at an angle of incidence 30°. The optic axis is in the plane of incidence and is perpendicular to the direction of the incident wave. Determine the directions of the wavevectors and the rays of the two refracted waves.

6.3-4 **Lateral Shift in Double Refraction.** What is the optimum geometry for maximizing the lateral shift between the refracted ordinary and extraordinary beams in a positive uniaxial crystal? Indicate all pertinent angles and directions.

6.3-5 **Transmission Through a $LiNbO_3$ Plate.** Examine the transmission of an unpolarized He–Ne laser beam ($\lambda_o = 633$ nm) through a $LiNbO_3$ ($n_e = 2.29$, $n_o = 2.20$)

plate of thickness 1 cm, cut such that its optic axis makes an angle 45° with the normal to the plate. Determine the lateral shift and the retardation between the ordinary and extraordinary beams.

*6.3-6 **Conical Refraction.** When the wavevector **k** points along an optic axis of a biaxial crystal an unusual situation occurs. The two sheets of the **k** surface meet and the surface can be approximated by a conical surface. A ray is incident normal to the surface of a biaxial crystal with one of its optic axes also normal to the surface. Show that multiple refraction occurs with the refracted rays forming a cone. This effect is known as conical refraction. What happens when the conical rays refract from the parallel surface of the crystal into air?

6.6-1 **Circular Dichroism.** Determine the Jones matrix for a device that converts light with any state of polarization into right circularly polarized light. Certain materials have different absorption coefficients for right and left circularly polarized light, a property known as **circular dichroism**.

6.6-2 **Polarization Rotation by a Sequence of Linear Polarizers.** A wave that is linearly polarized in the x direction is transmitted through a sequence of N linear polarizers whose transmission axes are inclined by angles $m\theta$ ($m = 1, 2, \ldots, N$; $\theta = \pi/2N$) with respect to the x axis. Show that the transmitted light is linearly polarized in the y direction but its amplitude is reduced by the factor $\cos^N\theta$. What happens in the limit $N \to \infty$? *Hint*: Use Jones matrices and note that

$$\mathbf{R}[(m+1)\theta]\mathbf{R}(-m\theta) = \mathbf{R}(\theta),$$

where $\mathbf{R}(\theta)$ is the coordinate transformation matrix.