

11

PHOTON OPTICS

11.1 THE PHOTON

- A. Photon Energy
- B. Photon Position
- C. Photon Momentum
- D. Photon Polarization
- E. Photon Interference
- F. Photon Time

11.2 PHOTON STREAMS

- A. Mean Photon Flux
- B. Randomness of Photon Flux
- C. Photon-Number Statistics
- D. Random Partitioning of Photon Streams

*11.3 QUANTUM STATES OF LIGHT

- A. Coherent-State Light
- B. Squeezed-State Light



Max Planck (1858–1947) suggested that the emission and absorption of light by matter occur in quanta of energy.



Albert Einstein (1879–1955) advanced the hypothesis that light itself consists of quanta of energy.

Electromagnetic optics (Chap. 5) provides the most complete treatment of light within the confines of **classical optics**. It encompasses wave optics, which in turn encompasses ray optics (Fig. 11.0-1). Although classical electromagnetic theory is capable of providing explanations for a great many effects in optics, as attested to by the earlier chapters in this book, it nevertheless fails to account for certain optical phenomena. This failure, which became evident about the turn of this century, ultimately led to the formulation of a quantum electromagnetic theory known as **quantum electrodynamics**. For optical phenomena, this theory is also referred to as **quantum optics**. Quantum electrodynamics (QED) is more general than classical electrodynamics and it is today accepted as a theory that is useful for explaining virtually all known optical phenomena.

In the framework of QED, the electric and magnetic fields \mathbf{E} and \mathbf{H} are mathematically treated as operators in a vector space. They are assumed to satisfy certain operator equations and commutation relations that govern their time dynamics and their interdependence. The equations of QED are required to accurately describe the interactions of electromagnetic fields with matter in the same way that Maxwell's equations are used in classical electrodynamics. The use of QED can lead to results that are characteristically quantum in nature and cannot be explained classically.

The formal treatment of QED is beyond the scope of this book. Nevertheless, it is possible to derive many of the quantum-mechanical properties of light and its interaction with matter by supplementing electromagnetic optics with a few simple relationships drawn from QED that represent the corpuscularity, localization, and fluctuations of electromagnetic fields and energy. This set of rules, which we call **photon optics**, permits us to deal with optical phenomena that are beyond the reach of classical theory, while retaining classical optics as a limiting case. However, photon optics is not intended to be a theory that is capable of providing an explanation for all optical effects.

In Sec. 11.1 we introduce the concept of the photon and its properties in the form of a number of rules that govern the behavior of photon energy, momentum, polarization, position, time, and interference. These rules take the form of deceptively simple relationships with far-reaching consequences. This is followed, in Sec. 11.2, by a

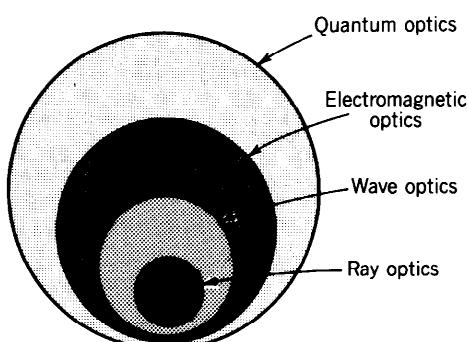


Figure 11.0-1 The theory of quantum optics provides an explanation for virtually all optical phenomena. It is more general than electromagnetic optics, which was shown earlier to encompass wave optics and ray optics.

discussion of the properties of photon streams. The number of photons emitted by a light source in a given time is almost always random, with statistical properties that depend on the nature of the source. The photon-number statistics for several important optical sources, including the laser and thermal radiators, are discussed. The effects of simple optical components (such as a beamsplitter and a filter) on the randomness of a photon stream are also examined. In Sec. 11.3 we use quantum optics to discuss the random fluctuations of the magnitude and phase of the electromagnetic field and to provide a brief introduction to coherent and squeezed states of light. The interaction of photons with atoms is discussed in Chap. 12.

11.1 THE PHOTON

Light consists of particles called **photons**. A photon has zero rest mass and carries electromagnetic energy and momentum. It also carries an intrinsic angular momentum (or spin) that governs its polarization properties. The photon travels at the speed of light in vacuum (c_o); its speed is retarded in matter. Photons also have a wavelike character that determines their localization properties in space and the rules by which they interfere and diffract.

The notion of the photon initially grew out of an attempt by Planck to resolve a long-standing riddle concerning the spectrum of blackbody radiation. He finally achieved this goal by quantizing the allowed energy values of each of the electromagnetic modes in a cavity from which radiation was emanating (this subject is discussed in Chap. 12). The concept of the photon and the rules of photon optics are introduced in this section by considering light inside an optical resonator (a cavity). This is a convenient choice because it restricts the space under consideration to a simple geometry. The presence of the resonator turns out not to be an important restriction in the argument; the results can be shown to be independent of its presence.

Electromagnetic-Optics Theory of Light in a Resonator

In accordance with electromagnetic optics, light inside a lossless resonator of volume V is completely characterized by an electromagnetic field that takes the form of a sum of discrete orthogonal modes of different frequencies, different spatial distributions, and different polarizations. The electric field vector is $\mathcal{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}(\mathbf{r}, t)\}$, where

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{q}} A_{\mathbf{q}} U_{\mathbf{q}}(\mathbf{r}) \exp(j2\pi\nu_{\mathbf{q}}t) \hat{\mathbf{e}}_{\mathbf{q}}. \quad (11.1-1)$$

The q th mode has complex amplitude $A_{\mathbf{q}}$, frequency $\nu_{\mathbf{q}}$, polarization along the direction of the unit vector $\hat{\mathbf{e}}_{\mathbf{q}}$, and a spatial distribution characterized by the complex function $U_{\mathbf{q}}(\mathbf{r})$, which is normalized such that $\int_V |U_{\mathbf{q}}(\mathbf{r})|^2 d\mathbf{r} = 1$. The choice of the expansion functions $U_{\mathbf{q}}(\mathbf{r})$ and $\hat{\mathbf{e}}_{\mathbf{q}}$ is not unique.

In a cubic resonator of dimension d , one convenient choice of the spatial expansion functions is the set of standing waves

$$U_{\mathbf{q}}(\mathbf{r}) = \left(\frac{2}{d}\right)^{3/2} \sin\frac{q_x\pi x}{d} \sin\frac{q_y\pi y}{d} \sin\frac{q_z\pi z}{d}, \quad (11.1-2)$$

where q_x , q_y , and q_z are integers denoted collectively by the index $\mathbf{q} = (q_x, q_y, q_z)$ [see Sec. 9.1 and Fig. 11.1-1(a)]. The energy contained in the mode is

$$E_{\mathbf{q}} = \frac{1}{2}\epsilon \int_V \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}^*(\mathbf{r}, t) d\mathbf{r} = \frac{1}{2}\epsilon |A_{\mathbf{q}}|^2.$$

In classical electromagnetic theory, the energy E_q can assume an arbitrary nonnegative value, no matter how small. The total energy is the sum of the energies in all the modes.

Photon-Optics Theory of Light in a Resonator

The electromagnetic-optics theory described above is maintained in photon optics, but a restriction is placed on the energy that is allowed to be carried by each mode. Rather than assuming a continuous range, the energy of a mode is restricted to a discrete set of values equally separated by a fixed energy. The energy of a mode is said to be quantized, with only integral units of this fixed energy allowed. Each unit of energy is carried by a photon.

Light in a resonator is comprised of a set of modes, each containing an integral number of identical photons. Characteristics of the mode, such as its frequency, spatial distribution, direction of propagation, and polarization, are assigned to the photon.

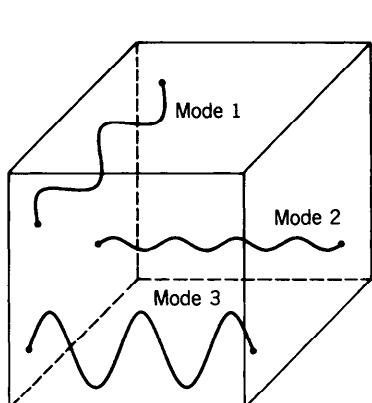
A. Photon Energy

Photon optics provides that the energy of an electromagnetic mode is quantized to discrete levels separated by the energy of a photon (Fig. 11.1-1). The energy of a photon in a mode of frequency ν is

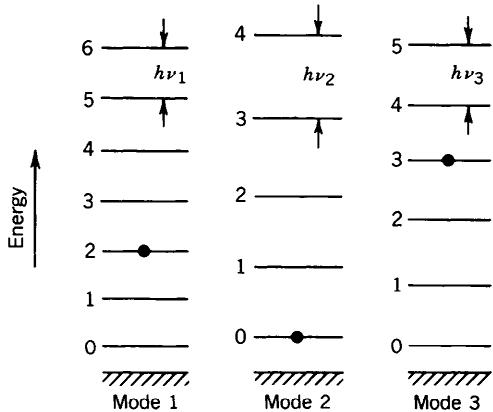
$$E = h\nu = \hbar\omega, \quad (11.1-3)$$

Photon Energy

where $h = 6.63 \times 10^{-34}$ J-s is **Planck's constant** and $\hbar = h/2\pi$. Energy may be added to, or taken from, this mode only in units of $h\nu$.



(a)



(b)

Figure 11.1-1 (a) Three modes of different frequencies and directions in a cubic resonator. (b) Allowed energies of three modes of frequencies ν_1 , ν_2 , and ν_3 . The solid circles indicate the number of photons in each mode; modes 1, 2, and 3 contain 2, 0, and 3 photons, respectively.

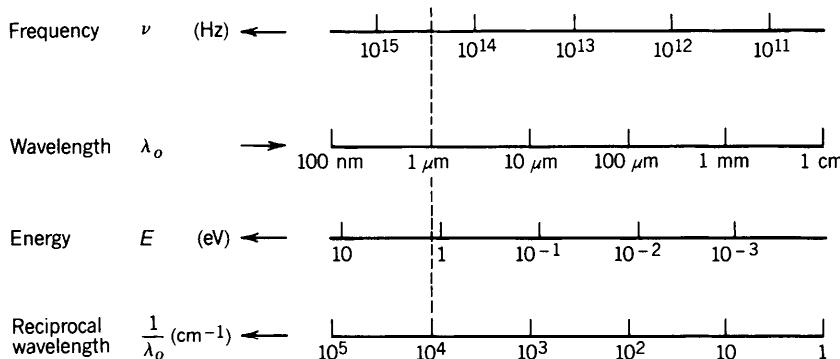


Figure 11.1-2 Relationships between photon frequency ν (Hz), wavelength λ_0 , energy E (eV), and reciprocal wavelength $1/\lambda_0$ (cm^{-1}). A photon of wavelength 1 cm has reciprocal wavelength 1 cm^{-1} . A photon of frequency $\nu = 3 \times 10^{14}$ Hz has wavelength $\lambda_0 = 1 \mu\text{m}$, energy 1.24 eV, and reciprocal wavelength $10,000 \text{ cm}^{-1}$.

A mode containing zero photons nevertheless carries an energy $E_0 = \frac{1}{2}h\nu$, which is called the zero-point energy. When it carries n photons, therefore, the mode has total energy

$$E_n = (n + \frac{1}{2})h\nu, \quad n = 0, 1, 2, \dots \quad (11.1-4)$$

In most experiments the zero-point energy is not directly observable because only energy differences [such as $E_{n2} - E_{n1}$ in (11.1-4)] are measured. The presence of the zero-point energy can, however, be manifested in subtle ways when matter is exposed to static fields. It plays a crucial role in the process of spontaneous emission from an atom, as discussed in Chap. 12.

The order of magnitude of photon energy is easily estimated. An infrared photon of wavelength $\lambda_0 = 1 \mu\text{m}$ has frequency 3×10^{14} Hz since $\lambda_0\nu = c_0$ in vacuum. Its energy is thus $h\nu = 1.99 \times 10^{-19} \text{ J} = 1.24 \text{ eV}$ (electron volts), which is the same as the kinetic energy of an electron that has been accelerated through a potential difference of 1.24 V. The conversion formula between wavelength (μm) and photon energy (eV) is therefore simply $\lambda_0(\mu\text{m}) = 1.24/E(\text{eV})$.

As another example, a microwave photon with a wavelength of 1 cm has an energy that is 10^4 times smaller, $h\nu = 1.24 \times 10^{-4} \text{ eV}$. The reciprocal wavelength is often also used as a unit of energy. It is specified in cm^{-1} , also called wavenumbers (1 cm^{-1} corresponds to $1.24 \times 10^{-4} \text{ eV}$ and 1 eV corresponds to 8068.1 cm^{-1}). The relationship between photon frequency, wavelength, energy, and reciprocal wavelength is illustrated in Fig. 11.1-2.

Because photons of higher frequency carry larger energy, the particle nature of light becomes increasingly important as the frequency of the radiation increases. Furthermore, wavelike effects such as diffraction and interference become more difficult to discern as the wavelength becomes shorter. X-rays and gamma-rays almost always behave like collections of particles, in contrast to radio waves, which almost always behave like waves. The frequency of light in the optical region is such that both particle-like and wavelike behavior occur, thus spurring the need for photon optics.

B. Photon Position

Associated with each photon is a wave described by the complex wavefunction $AU(\mathbf{r}) \exp(j2\pi\nu t)\hat{\mathbf{e}}$ of the mode. However, when a photon impinges on a detector of small area dA located normal to the direction of propagation at the position \mathbf{r} , its

indivisibility causes it to be either wholly detected or not detected at all. The location at which the photon is registered is not precisely determined. It is governed by the optical intensity $I(\mathbf{r}) \propto |U(\mathbf{r})|^2$, in accordance with the following probabilistic law:

The probability $p(\mathbf{r}) dA$ of observing a photon at a point \mathbf{r} within an incremental area dA , at any time, is proportional to the local optical intensity $I(\mathbf{r}) \propto |U(\mathbf{r})|^2$, i.e.,

$$p(\mathbf{r}) dA \propto I(\mathbf{r}) dA. \quad (11.1-5)$$

Photon Position

The photon is more likely to be found at those locations where the intensity is high. A photon in a mode described by a standing wave with the intensity distribution $I(x, y, z) \propto \sin^2(\pi z/d)$, where $0 \leq z \leq d$, for example, is most likely to be detected at $z = d/2$, but will never be detected at $z = 0$ or $z = d$. In contrast to waves, which are extended in space, and particles, which are localized, optical photons behave as extended *and* localized entities. This behavior is called **wave-particle duality**. The localized nature of photons becomes evident when they are detected.

EXERCISE 11.1-1

Photons in a Gaussian Beam

- (a) Consider a single photon described by a Gaussian beam (i.e., a $\text{TEM}_{0,0}$ mode of a spherical-mirror resonator; see Secs. 3.1B, 5.4A, and 9.2B). What is the probability of detecting the photon at a point within a circle whose radius is the waist radius of the beam W_0 ? Recall that at the waist ($z = 0$), $I(\rho, z = 0) \propto \exp(-2\rho^2/W_0^2)$, where ρ is the radial coordinate.
- (b) If the beam carries a large number N of independent photons, estimate the average number of photons that lie within this circle.

Transmission of a Single Photon Through a Beamsplitter

An ideal beamsplitter is an optical device that losslessly splits a beam of light into two beams emerging at right angles. It is characterized by a transmittance \mathcal{T} and a reflectance $\mathcal{R} = 1 - \mathcal{T}$. The intensity of the transmitted wave I_t and the intensity of the reflected wave I_r can be calculated from the intensity of the incident wave I using the electromagnetic relations $I_r = (1 - \mathcal{T})I$ and $I_t = \mathcal{T}I$.

Because a photon is indivisible, it must choose between the two possible directions permitted by the beamsplitter. A single photon incident on it follows one of the two possible paths in accordance with the probabilistic photon-position rule (11.1-5). The probability that the photon is transmitted is proportional to I_t and is therefore equal to the transmittance \mathcal{T} . The probability that it is reflected is $1 - \mathcal{T}$. From a probability point of view, the problem is identical to that of flipping a coin. Figure 11.1-3 illustrates the process.

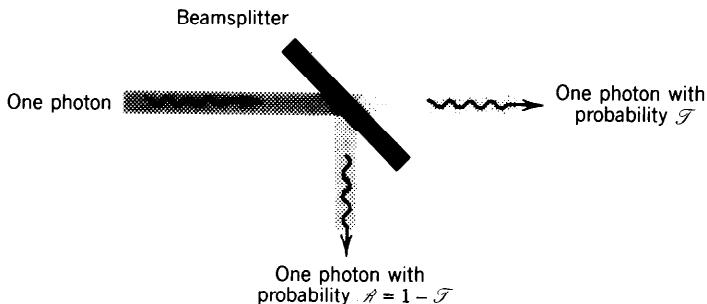


Figure 11.1-3 Probabilistic reflection or transmission of a photon at a beamsplitter.

C. Photon Momentum

The momentum of a photon is related to the wavevector of its associated wavefunction by the following rule:

A photon in a mode described by the plane wave

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{A} \exp(-j\mathbf{k} \cdot \mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}$$

has a momentum vector

$$\mathbf{p} = \hbar\mathbf{k}. \quad (11.1-6)$$

The photon travels in the direction of the wavevector and the magnitude of the momentum is $p = \hbar k = \hbar 2\pi/\lambda$, i.e.,

$$p = \frac{\hbar}{\lambda}. \quad (11.1-7)$$

Electromagnetic optics leads to the same energy-momentum relationship $\mathbf{p} = (E/c)\hat{\mathbf{k}}$ for a plane wave, where p is the momentum content per unit volume of the wave, E is the energy content per unit volume, and $\hat{\mathbf{k}}$ is a unit vector in the direction of \mathbf{k} . Of course, the concept of the photon does not exist in electromagnetic optics, so that the expressions in (11.1-6) and (11.1-7) containing \hbar are unique to photon optics.

*Momentum of a Localized Wave

A wave more general than a plane wave, with a complex wavefunction of the form $AU(\mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}$, can be expanded as a sum of plane waves of different wavevectors by using the techniques of Fourier optics (see Chap. 4). The component with wavevector \mathbf{k} may be written in the form $A(\mathbf{k}) \exp(-j\mathbf{k} \cdot \mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}$, where $A(\mathbf{k})$ is its amplitude.

The momentum of a photon described by an arbitrary complex wavefunction $AU(\mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}$ is uncertain. It has the value

$$\mathbf{p} = \hbar\mathbf{k},$$

with probability proportional to $|A(\mathbf{k})|^2$, where $A(\mathbf{k})$ is the amplitude of the plane-wave Fourier component of $U(\mathbf{r})$ with wavevector \mathbf{k} .

If $f(x, y) = U(x, y, 0)$ is the complex amplitude at the $z = 0$ plane, the plane-wave Fourier component of wavevector $\mathbf{k} = (k_x, k_y, k_z)$ has an amplitude $A(\mathbf{k}) = F(k_x/2\pi, k_y/2\pi)$, where $F(\nu_x, \nu_y)$ is the two-dimensional Fourier transform of $f(x, y)$ (see Chap. 4). Because the functions $f(x, y)$ and $F(\nu_x, \nu_y)$ are a Fourier transform pair, their widths are inversely related and satisfy the duration–bandwidth relation (see Appendix A, (A.2-6)). The uncertainty relation between the position of the photon and the direction of its momentum is established because the position of the photon at the $z = 0$ plane is probabilistically determined by $|U(\mathbf{r})|^2 = |f(x, y)|^2$, and the direction of its momentum is probabilistically determined by $|A(\mathbf{k})|^2 = |F(k_x/2\pi, k_y/2\pi)|^2$. Thus if, at the plane $z = 0$, σ_x is the position uncertainty in the x direction, and $\sigma_\theta = \sin^{-1}(\sigma_{k_x}/k) \approx (\lambda/2\pi)\sigma_{k_x}$ is the angular uncertainty about the z axis (assumed $\ll 1$), then the uncertainty relation $\sigma_x\sigma_{k_x} \geq \frac{1}{2}$ is equivalent to $\sigma_x\sigma_\theta \geq \lambda/4\pi$.

A plane-wave photon has a known momentum (fixed direction and magnitude), so that $\sigma_\theta = 0$, but its position is totally uncertain ($\sigma_x = \infty$); it is equally likely to be detected anywhere in the $z = 0$ plane. When a plane-wave photon passes through an aperture, its position is localized, at the expense of a spread in the direction of its momentum. The position–momentum uncertainty therefore parallels the theory of diffraction described in Chap. 4. At the other extreme from the plane wave is the spherical-wave photon. It is well localized in position (at the center of the wave), but its momentum has a direction that is totally uncertain.

Radiation Pressure

Because momentum is conserved, its association with a photon means that the emitting atom experiences a recoil of magnitude $h\nu/c$. Furthermore, the momentum associated with a photon can be transferred to objects of finite mass, giving rise to a force and causing mechanical motion. As an example, light beams can be used to deflect atomic beams traveling perpendicular to the photons. The term **radiation pressure** is often used to describe this phenomenon (pressure is force/area).

EXERCISE 11.1-2

Photon-Momentum Recoil. Calculate the recoil velocity imparted to a ^{198}Hg atom that has emitted a photon of energy 4.88 eV. Compare this with the root-mean-square thermal velocity v of the atom at $T = 300$ K (obtained by setting the average kinetic energy equal to the average thermal energy, $\frac{1}{2}mv^2 = \frac{3}{2}k_B T$).

D. Photon Polarization

As indicated earlier, light is characterized as a sum of modes of different frequencies, directions, and polarizations.

The polarization of a photon is that of its mode.

The choice of a particular set of modes is not unique, however. This important concept is best explained by examining the polarization properties of light from the perspective of photon optics.

Linearly Polarized Photons

Consider light described by a superposition of two plane-wave modes propagating in the z direction, one linearly polarized in the x direction and the other linearly

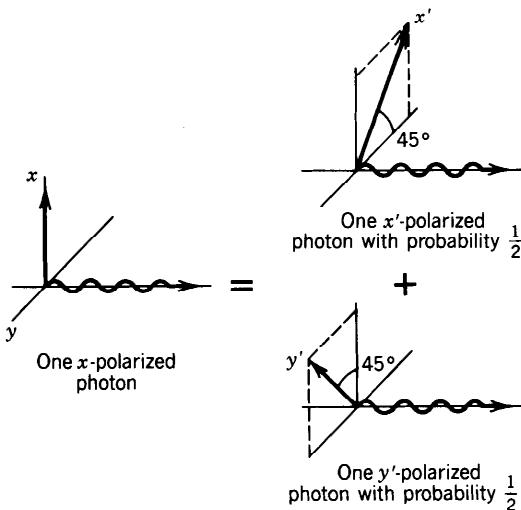


Figure 11.1-4 Probabilistic outcomes for a linearly polarized photon.

polarized in the \$y\$ direction:

$$\mathbf{E}(\mathbf{r}, t) = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}) \exp(-jkz) \exp(j2\pi\nu t).$$

However, the very same electromagnetic field may also be represented in a different coordinate system \$(x', y')\$ (e.g., one that makes a \$45^\circ\$ angle with the initial coordinate system). Thus we can equally well view the field in terms of two modes carrying photons polarized along the \$x'\$ and \$y'\$ directions, i.e.,

$$\mathbf{E}(\mathbf{r}, t) = (A_{x'} \hat{\mathbf{x}}' + A_{y'} \hat{\mathbf{y}}') \exp(-jkz) \exp(j2\pi\nu t),$$

where

$$A_{x'} = \frac{1}{\sqrt{2}}(A_x - A_y), \quad A_{y'} = \frac{1}{\sqrt{2}}(A_x + A_y).$$

If we know that the \$x\$-polarized mode is occupied by a photon, and the \$y\$-polarized mode is empty, what can be said about the possibility of finding a photon polarized along the \$x'\$ direction? This question is addressed in photon optics by invoking the usual probabilistic approach. The probabilities of finding a photon with \$x\$, \$y\$, \$x'\$, or \$y'\$ polarization are proportional to the intensities \$|A_x|^2\$, \$|A_y|^2\$, \$|A_{x'}|^2\$, and \$|A_{y'}|^2\$, respectively. In our example \$|A_x|^2 = 1\$, \$|A_y|^2 = 0\$, so that \$|A_{x'}|^2 = |A_{y'}|^2 = \frac{1}{2}\$. Therefore, given that there is one photon polarized along the \$x\$ direction and no photon polarized along the \$y\$ direction, the probabilities of finding a photon polarized along the \$x'\$ or \$y'\$ directions are both \$\frac{1}{2}\$. This is illustrated schematically in Fig. 11.1-4.

EXAMPLE 11.1-1. Transmission of a Linearly Polarized Photon Through a Polarizer.

Consider a plane wave, linearly polarized at an angle \$\theta\$ with respect to the \$x\$ axis, directed onto a polarizer which has its transmission axis along the \$x\$ direction (see Fig. 11.1-5). The polarizer transmits light that is linearly polarized in the \$x\$ direction but blocks light that is linearly polarized in the \$y\$ direction. It is known from classical polarization optics that the intensity of the transmitted light \$I_t = I_i \cos^2 \theta\$, where \$I_i\$ is the intensity of the incident light (see Sec. 6.1B). What happens if only a single photon impinges on the polarizer? If the photon is polarized along the \$x\$ axis, it always passes through. If it is

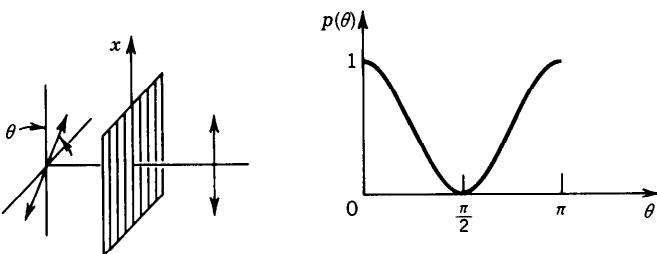


Figure 11.1-5 Probability of observing a linearly polarized photon after transmission through a polarizer at an angle θ .

polarized along the y axis, it is always blocked. The probability for the passage of the photon is determined by the classical intensity I_t . Thus the probability of passage of a photon polarized at an angle θ with the polarizer is $p(\theta) = \cos^2 \theta$. The probability that the photon is blocked is therefore $1 - p(\theta) = \sin^2 \theta$.

Circularly Polarized Photons

A modal expansion in terms of two circularly polarized plane-wave modes, one right-handed and one left-handed, can also be used, i.e.,

$$\mathbf{E}(\mathbf{r}, t) = [A_R \hat{\mathbf{e}}_R + A_L \hat{\mathbf{e}}_L] \exp(-jkz) \exp(j2\pi\nu t),$$

where $\hat{\mathbf{e}}_R = (1/\sqrt{2})(\hat{\mathbf{x}} + j\hat{\mathbf{y}})$ and $\hat{\mathbf{e}}_L = (1/\sqrt{2})(\hat{\mathbf{x}} - j\hat{\mathbf{y}})$ (see Sec. 6.1B). These modes carry right-handed and left-handed circularly polarized photons, respectively. Again, the probabilities of finding a photon with these polarizations are proportional to the intensities $|A_R|^2$ and $|A_L|^2$. As illustrated in Fig. 11.1-6, a linearly polarized photon is equivalent to the superposition of a right-handed and a left-handed circularly polarized photon, each with probability $\frac{1}{2}$. Conversely, when a circularly polarized photon is passed through a linear polarizer, the probability of detecting it is $\frac{1}{2}$.

Photon Spin

Photons possess intrinsic angular momentum (spin). The magnitude of the **photon spin** is quantized to the two values

$$S = \pm \hbar.$$

(11.1-8)
Photon Spin

Right-handed (left-handed) circularly polarized photons have their spin vector parallel (antiparallel) to their momentum vector. Linearly polarized photons have an equal

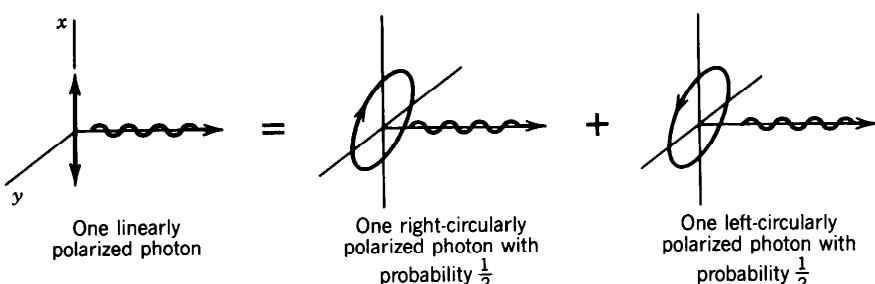


Figure 11.1-6 A linearly polarized photon is equivalent to the superposition of a right- and left-circularly polarized photon, each with probability $\frac{1}{2}$.

probability of exhibiting parallel and antiparallel spin. In the same way that photons can transfer linear momentum to an object, circularly polarized photons can exert a torque on an object. For example, a circularly polarized photon will exert a torque on a half-wave plate of quartz.

E. Photon Interference

Young's two-pinhole interference experiment is generally invoked to demonstrate the wave nature of light (see Exercise 2.5-2 on page 67). However, Young's experiment can be carried out even when there is only a single photon in the apparatus at a given time. The outcome of this experiment can be understood in the context of photon optics by using the photon-position rule. The intensity at the observation plane is calculated using electromagnetic (wave) optics and the result is converted to a probability density function that specifies the random position of the detected photon. The interference arises from phase differences in the two paths.

Consider a plane wave illuminating a screen with two pinholes, as shown in Fig. 11.1-7. This generates two spherical waves that interfere at the observation plane. In the Fresnel approximation these produce a sinusoidal intensity given by (see Exercise 2.5-2)

$$I(x) = 2I_0 \left(1 + \cos \frac{2\pi\theta x}{\lambda} \right), \quad (11.1-9)$$

where I_0 is the intensity of each of the waves at the observation plane, λ is the wavelength, and θ is the angle subtended by the two pinholes at the observation plane (Fig. 11.1-7). The line that joins the holes defines the x axis. The result in (11.1-9) describes the intensity pattern that is experimentally observed when the incident light is strong.

Now if only a single photon is present in the apparatus, the probability of detecting it at position x is proportional to $I(x)$, in accordance with (11.1-5). It is most likely to be detected at those values of x for which $I(x)$ is maximum. It will never be detected at values for which $I(x) = 0$. If a histogram of the locations of the detected photon is constructed by repeating the experiment many times, as Taylor did in 1909, the classical interference pattern obtained by carrying out the experiment once with a strong beam of light emerges. The interference pattern represents the probability distribution of the position at which the photon is observed.

The occurrence of interference results from the extended nature of the photon, which permits it to pass through *both* holes of the apparatus. This gives it knowledge of the entire geometry of the experiment when it reaches the observation plane, where it

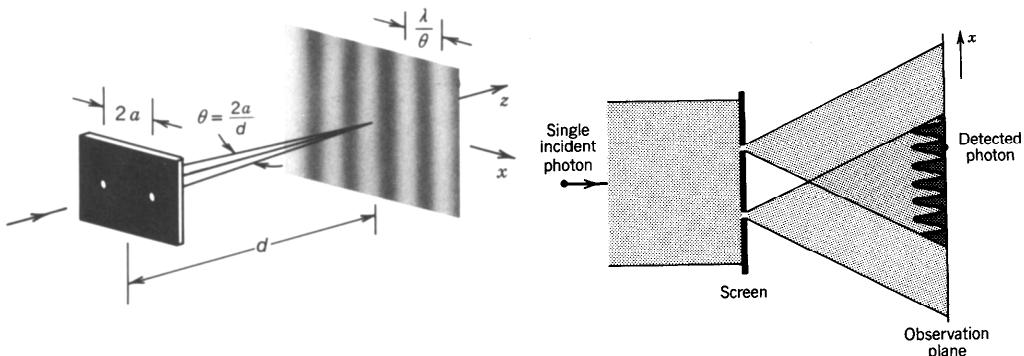


Figure 11.1-7 Young's two-pinhole experiment with a single photon. The interference pattern $I(x)$ is proportional to the probability density of detecting the photon at position x .

is detected as a single entity. If one of the holes were to be covered, the interference pattern would disappear because the photon was forced to pass through the other hole, depriving it of knowledge of the whole apparatus.

EXERCISE 11.1-3

Photon in a Mach-Zehnder Interferometer. Consider a plane wave of light of wavelength λ that is split into two parts at a beamsplitter (see Sec. 11.1B) and recombined in a Mach-Zehnder interferometer, as shown in Fig. 11.1-8 [see also Fig. 2.5-3(a)].

If the wave contains only a single photon, plot the probability of finding the photon at the detector as a function of d/λ (for $0 \leq d/\lambda \leq 1$), where d is the difference between the two optical paths of the light. Assume that the mirrors and beamsplitters are perfectly flat and lossless, and that the beamsplitters have a 50% reflectance. Where might the photon be located when the probability of finding it at the detector is not unity?

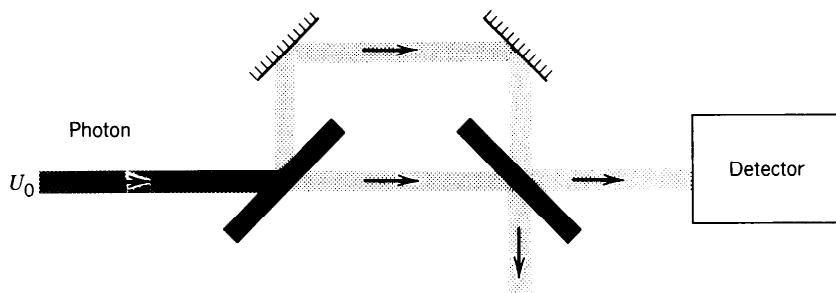


Figure 11.1-8 Mach-Zehnder interferometer.

F. Photon Time

The modal expansion provided in (11.1-1) represents monochromatic (single-frequency) modes which are “eternal” harmonic functions of time. A photon in a monochromatic mode is equally likely to be detected at any time. However, as indicated previously, a modal expansion of the radiation inside (or outside) a resonator is not unique. A more general expansion may be made in terms of polychromatic modes (time-localized wavepackets, for example). The probability of detecting the photon described by the complex wavefunction $U(\mathbf{r}, t)$ (see Sec. 2.6A) at any position, in the incremental time interval between t and $t + dt$, is proportional to $I(\mathbf{r}, t) dt \propto |U(\mathbf{r}, t)|^2 dt$.

The photon-position rule presented in (11.1-5) may therefore be generalized to include photon time localization:

The probability of observing a photon at a point \mathbf{r} within the incremental area $d\mathbf{A}$, and during the incremental time interval dt following time t , is proportional to the intensity of the mode at \mathbf{r} and t , i.e.,

$$p(\mathbf{r}, t) d\mathbf{A} dt \propto I(\mathbf{r}, t) d\mathbf{A} dt \propto |U(\mathbf{r}, t)|^2 d\mathbf{A} dt. \quad (11.1-10)$$

Photon Position
and Time

Time–Energy Uncertainty

The time during which a photon in a monochromatic mode of frequency ν may be detected is totally uncertain, whereas the value of its frequency ν (and its energy $h\nu$) is absolutely certain. On the other hand, a photon in a wavepacket mode with an intensity function $I(t)$ of duration σ_t must be localized within this time. Bounding the photon time in this way engenders an uncertainty in the photon's frequency (and energy) as a result of the properties of the Fourier transform. The result is a “polychromatic” photon. The frequency uncertainty is readily determined by Fourier expanding $U(t)$ in terms of its harmonic components,

$$U(t) = \int_{-\infty}^{\infty} V(\nu) \exp(j2\pi\nu t) d\nu \quad (11.1-11)$$

where $V(\nu)$ is the Fourier transform of $U(t)$ (see Sec. A.1, Appendix A). The \mathbf{r} dependence has been suppressed for simplicity. The width σ_ν of $|V(\nu)|^2$ represents the spectral width. If σ_t is the rms width of the function $|U(t)|^2$ (i.e., the power-rms width), then σ_t and σ_ν must satisfy the duration–bandwidth reciprocity relation $\sigma_\nu \sigma_t \geq 1/4\pi$, or $\sigma_\omega \sigma_t \geq \frac{1}{2}$ (see Sec. A.2, Appendix A for the definitions of σ_t and σ_ν that lead to this uncertainty relation).

The energy of the photon $h\nu$ then cannot be specified to an accuracy better than $\sigma_E = h\sigma_\nu$. It follows that the energy uncertainty of a photon, and the time during which it may be detected, must satisfy

$$\sigma_E \sigma_t \geq \frac{\hbar}{2},$$

(11.1-12)

Time–Energy Uncertainty

known as the **time–energy uncertainty relation**. This relation is analogous to that between position and wavenumber (momentum), which sets a limit on the precision with which the position and momentum of a photon can be simultaneously specified. The average energy \bar{E} of this polychromatic photon is $\bar{E} = h\bar{\nu} = \hbar\bar{\omega}$.

To summarize: A monochromatic photon ($\sigma_\nu \rightarrow 0$) has an eternal duration within which it can be observed ($\sigma_t \rightarrow \infty$). In contrast, a photon associated with an optical wavepacket is localized in time and is therefore polychromatic with a corresponding energy uncertainty. Thus a wavepacket photon can be viewed as a confined traveling packet of energy.

EXERCISE 11.1-4

Single Photon in a Gaussian Wavepacket. Consider a plane-wave wavepacket (see Sec. 2.6A) containing a single photon traveling in the z direction, with complex wavefunction

$$U(\mathbf{r}, t) = \alpha \left(t - \frac{z}{c} \right)$$

where

$$\alpha(t) = \exp\left(-\frac{t^2}{4\tau^2}\right) \exp(j2\pi\nu_0 t).$$

- (a) Show that the uncertainties in its time and z position are $\sigma_t = \tau$ and $\sigma_z = c\sigma_t$, respectively.

- (b) Show that the uncertainties in its energy and momentum satisfy the minimum uncertainty relations

$$\sigma_E \sigma_t = \frac{\hbar}{2} \quad (11.1-13)$$

$$\sigma_z \sigma_p = \frac{\hbar}{2}. \quad (11.1-14)$$

Equation (11.1-14) is the minimum-uncertainty limit of the **Heisenberg position-momentum uncertainty relation** [see (A.2-7) in Appendix A].

Summary

Electromagnetic radiation may be described as a sum of modes, e.g., monochromatic uniform plane waves of the form

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{q}} A_{\mathbf{q}} \exp(-j\mathbf{k}_{\mathbf{q}} \cdot \mathbf{r}) \exp(j2\pi\nu_{\mathbf{q}} t) \hat{\mathbf{e}}_{\mathbf{q}}.$$

Each plane wave has two orthogonal polarization states (e.g., vertical/horizontal-linearly polarized, right/left-circularly polarized, etc.) represented by the vectors $\hat{\mathbf{e}}_{\mathbf{q}}$. When the energy of a mode is measured, the result is an integer (in general, random) number of energy quanta (photons). Each of the photons associated with the mode \mathbf{q} has the following properties:

- Energy $E = \hbar\nu_{\mathbf{q}}$.
- Momentum $\mathbf{p} = \hbar\mathbf{k}$.
- Spin $S = \pm\hbar$, if it is circularly polarized.
- The photon is equally likely to be found anywhere in space, and at any time, since the wavefunction of the mode is a monochromatic plane wave.

The choice of modes is not unique. A modal expansion in terms of nonmonochromatic (quasimonochromatic), nonplanar waves,

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{q}} A_{\mathbf{q}} U_{\mathbf{q}}(\mathbf{r}, t) \hat{\mathbf{e}}_{\mathbf{q}},$$

is also possible. The photons associated with the mode \mathbf{q} then have the following properties:

- Photon position and time are governed by the complex wavefunction $U_{\mathbf{q}}(\mathbf{r}, t)$. The probability of detecting a photon in the incremental time between t and $t + dt$, in an incremental area dA at position \mathbf{r} , is proportional to $|U_{\mathbf{q}}(\mathbf{r}, t)|^2 dA dt$.
- If $U_{\mathbf{q}}(\mathbf{r}, t)$ has a finite time duration σ_t , i.e., if the photon is localized in time, then the photon energy $\hbar\nu_{\mathbf{q}}$ has an uncertainty $\hbar\sigma_t \geq \hbar/4\pi\sigma_t$.
- If $U_{\mathbf{q}}(\mathbf{r}, t)$ has a finite spatial extent in the transverse ($z = 0$) plane, i.e., if the photon is localized in the x direction, for example, then the direction of photon momentum is uncertain. The spread in photon momentum can be determined by analyzing $U_{\mathbf{q}}(\mathbf{r}, t)$ as a sum of plane waves, the wave with wavevector \mathbf{k} corresponding to photon momentum $\hbar\mathbf{k}$. Localization of the photon in the transverse plane results in a spread of the uncertainty of the photon-momentum direction.

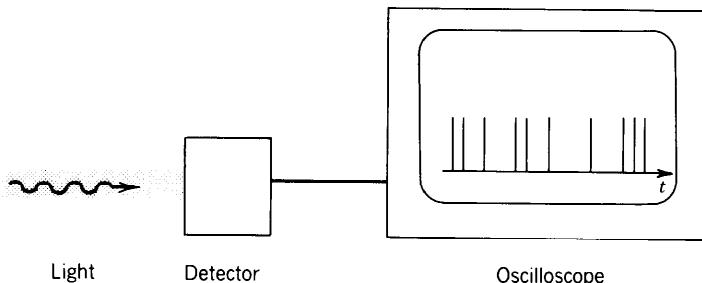


Figure 11.2-1 Photon registrations at random localized instants of time.

11.2 PHOTON STREAMS

In Sec. 11.1 we concentrated on the properties and behavior of single photons. We now consider the properties of collections of photons. As a result of the processes by which photons are created (e.g., emissions from atoms; see Chap. 12), the number of photons occupying a mode is generally random. The probability distribution obeyed by the photon number is governed by the quantum state of the mode, which is determined by the nature of the light source (see Sec. 11.3). Real photon streams often contain numerous propagating modes, each carrying a random number of photons.

If an experiment is carried out in which a weak stream of photons falls on a light-sensitive surface, the photons are registered (detected) at random localized instants of time and at random points in space, in accordance with (11.1-10). This space-time process can be discerned by viewing an object with the naked eye in a dimly lit room.

The time course of such photon registrations can be highlighted by looking at the temporal and spatial behavior separately. Consider the use of a detector that integrates light over a finite area, as illustrated in Fig. 11.2-1. The probability of detecting a photon in the incremental time interval between t and $t + dt$ is proportional to the optical power $P(t)$ at the time t . The photons will be registered at random times.

On the other hand, the spatial pattern of photon registrations is readily manifested by using a detector that integrates over a fixed exposure time T (e.g., photographic film). In accordance with (11.1-10), the probability of observing a photon in an incremental area dA surrounding the point \mathbf{r} is proportional to the integrated local intensity $\int_0^T I(\mathbf{r}, t) dt$. This is illustrated by the “grainy” photographic image of Max Planck provided in Fig. 11.2-2. This image was obtained by rephotographing, under

Figure 11.2-2 Random photon registrations with a spatial density that follows the local optical intensity. This image of Max Planck taken with a weak stream of photons should be compared with the photograph on page 384 taken with intense light.

very low light conditions, the picture of Max Planck shown on page 384. Each of the white dots represents a random photon registration; the density of registrations follows the local intensity.

A. Mean Photon Flux

We begin by introducing a number of definitions that relate the mean photon flux to classical electromagnetic intensity, power, and energy. These definitions are related to the probability law (11.1-10) governing the position and time at which a single photon is observed. We then discuss the randomness of the photon flux and the photon-number statistics for different sources of light. Finally, we consider the random partitioning of a photon stream.

Mean Photon-Flux Density

Monochromatic light of frequency ν and classical intensity $I(\mathbf{r})$ (watts/cm²) carries a mean **photon-flux density**

$$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\nu}, \quad (11.2-1)$$

Mean Photon-Flux
Density

where $h\nu$ is the energy of each photon. This equation converts a classical measure (with units of energy/s·cm²) into a quantum measure (with units of photons/s·cm²). For quasimonochromatic light of central frequency $\bar{\nu}$, all photons have approximately the same energy $h\bar{\nu}$, so that the mean photon-flux density is approximately

$$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\bar{\nu}}. \quad (11.2-2)$$

Typical values of $\phi(\mathbf{r})$ for some common sources of light are provided in Table 11.2-1. It is clear from these numbers that trillions of photons rain down on each square centimeter of us each second.

TABLE 11.2-1 Mean Photon-Flux Density for Several Light Sources

Source	Mean Photon-Flux Density (photons/s·cm ²)
Starlight	10^6
Moonlight	10^8
Twilight	10^{10}
Indoor light	10^{12}
Sunlight	10^{14}
Laser light (10-mW He-Ne laser beam at $\lambda_o = 633$ nm focused to a $20\text{-}\mu\text{m}$ -diameter spot)	10^{22}

Mean Photon Flux

The mean photon flux Φ (with units of photons/s) is obtained by integrating the mean photon-flux density over a specified area,

$$\Phi = \int_A \phi(\mathbf{r}) dA = \frac{P}{h\bar{\nu}}, \quad (11.2-3)$$

Mean Photon Flux

where again $h\bar{\nu}$ is the average energy of a photon, and

$$P = \int_A I(\mathbf{r}) dA \quad (11.2-4)$$

is the optical power (watts). As an example, 1 nW of optical power, at a wavelength $\lambda_o = 0.2 \mu\text{m}$, delivers to an object an average photon flux $\Phi \approx 10^9$ photons per second. Roughly speaking, one photon will therefore strike the object every nanosecond, i.e.,

$$1 \text{ nW at } \lambda_o = 0.2 \mu\text{m} \rightarrow 1 \text{ photon/ns}. \quad (11.2-5)$$

A $\lambda_o = 1 \mu\text{m}$ photon carries one-fifth of the energy, so that 1 nW corresponds to an average of 5 photons/ns.

Mean Number of Photons

The mean number of photons \bar{n} detected in the area A and the time interval T is obtained by multiplying the photon flux Φ by the time duration,

$$\bar{n} = \Phi T = \frac{E}{h\bar{\nu}}, \quad (11.2-6)$$

Mean Photon Number

where $E = PT$ is the optical energy (joules).

To summarize, the relations between the classical and quantum measures are:

Classical		Quantum	
Optical intensity	$I(\mathbf{r})$	Photon-flux density	$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\bar{\nu}}$
Optical power	P	Photon flux	$\Phi = \frac{P}{h\bar{\nu}}$
Optical energy	E	Photon number	$\bar{n} = \frac{E}{h\bar{\nu}}$

Spectral Densities of Photon Flux

For polychromatic light of broad bandwidth, it is useful to define spectral densities of the classical intensity, power, and energy, and their quantum counterparts: spectral

photon-flux density, spectral photon flux, and spectral photon number:

Classical	Quantum
I_ν (W/cm ² -Hz)	$\phi_\nu = \frac{I_\nu}{h\nu}$ (photons/s-cm ² -Hz)
P_ν (W/Hz)	$\Phi_\nu = \frac{P_\nu}{h\nu}$ (photons/s-Hz)
E_ν (J/Hz)	$\bar{n}_\nu = \frac{E_\nu}{h\nu}$ (photons/Hz)

For example, $P_\nu d\nu$ represents the optical power in the frequency range ν to $\nu + d\nu$, and $\Phi_\nu d\nu$ represents the flux of photons whose frequency lies between ν and $\nu + d\nu$.

Time-Varying Light

If the light intensity is time varying, the photon-flux density is a function of time,

$$\phi(\mathbf{r}, t) = \frac{I(\mathbf{r}, t)}{h\nu}. \quad (11.2-7)$$

The optical power and the photon flux are also, then, functions of time:

$$\Phi(t) = \int_A \phi(\mathbf{r}, t) dA = \frac{P(t)}{h\nu}, \quad (11.2-8)$$

Mean Photon Flux

where

$$P(t) = \int_A I(\mathbf{r}, t) dA. \quad (11.2-9)$$

The mean number of photons registered in a time interval between $t = 0$ and $t = T$ also varies with time. It is obtained by integrating the photon flux,

$$\bar{n} = \int_0^T \Phi(t) dt = \frac{E}{h\nu}, \quad (11.2-10)$$

Mean Photon Number

where

$$E = \int_0^T P(t) dt = \int_0^T \int_A I(\mathbf{r}, t) dA dt. \quad (11.2-11)$$

is the optical energy (the intensity integrated over time and area).

B. Randomness of Photon Flux

Even if the classical intensity $I(\mathbf{r}, t)$ is constant, the time of arrival and position of registration of a single photon are governed by probabilistic laws, as we have seen in Sec. 11.1 (see Fig. 11.2-1). If a source provides exactly one photon, the probability density of detecting that photon at the space-time point (\mathbf{r}, t) is proportional to $I(\mathbf{r}, t)$, in accordance with (11.1-10). We shall see in this section that the classical electromag-

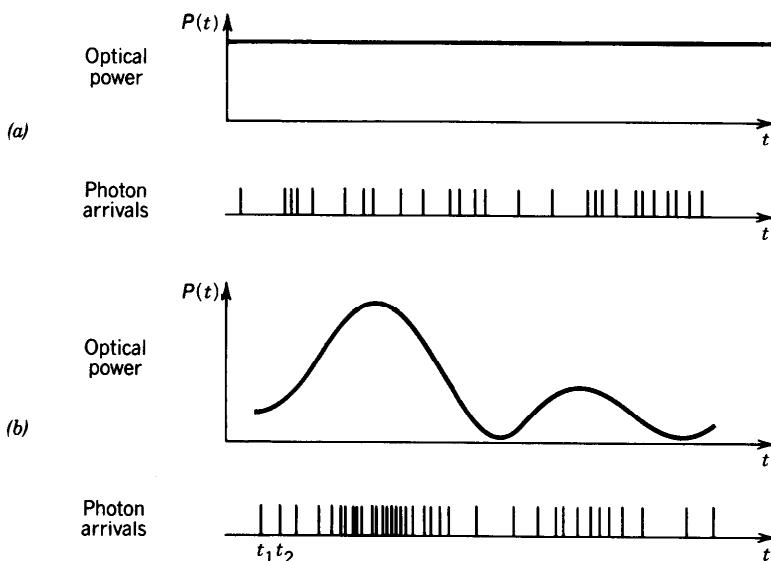


Figure 11.2-3 (a) Constant optical power and the corresponding random photon arrival times. (b) Time-varying optical power and the corresponding random photon arrival times.

nagnetic intensity $I(\mathbf{r}, t)$ governs the behavior of photon streams as well as single photons. The interpretation ascribed to $I(\mathbf{r}, t)$ differs, however. For photon streams, the classical intensity $I(\mathbf{r}, t)$ determines the mean photon-flux density $\phi(\mathbf{r}, t)$. The properties of the light source determine the fluctuations in $\phi(\mathbf{r}, t)$.

If the optical power $P(t)$ varies with time, the density of random times at which the associated photons are detected generally follows the function $P(t)$, as schematically illustrated in Fig. 11.2-3. The mean flux $\Phi(t)$ is $P(t)/h\nu$, but the actual times at which the photons are detected are random. Where the power is large, there are, on the average, more photons; where the power is small, the photons are sparse. Even when P is constant, the times at which the photons are detected is random, with behavior determined by the source (Figs. 11.2-3(a) and 11.2-4). For example, at $\lambda_o = 1.24 \mu\text{m}$, 1 nW carries an *average* of 6.25 photons/ns, or 0.00625 photons every picosecond. Of course, only integral numbers of photons may be detected. An average of 0.00625 photons/ps means that if 10^5 time intervals (each of duration $T = 1 \text{ ps}$) were examined, most of the time intervals would be empty (no photons), about 625 intervals would contain one photon, and very few intervals would contain two or more photons.

The image of Max Planck in Fig. 11.2-2 shows the same behavior in the spatial domain. The locations of the detected photons generally follow the classical intensity distribution, with a high density of photons where the intensity is large and low photon density where the intensity is small. But there is considerable graininess (noise) in the image. Fluctuations in the photon-flux density are most discernible when its mean value is small, as in the case of Fig. 11.2-2. When the mean photon-flux density becomes large everywhere in the image, the graininess disappears and the classical intensity distribution is recovered, as seen in the picture of Max Planck on page 384.

The study of the randomness of photon numbers is important for applications such as noise in weak images and optical information transmission. In a fiber-optic communication system, for example, information is carried on a photon stream (see Sec. 22.3). Only the mean number of photons emitted by the source is controlled at the transmitter. The actual number of emitted photons is unpredictable, the nature of the source

determining the form of its randomness. The unpredictability of the photon number results in errors in the transmission of information.

C. Photon-Number Statistics

The statistical distribution of the number of photons depends on the nature of the light source and must generally be treated by use of the quantum theory of light, as described briefly in Sec. 11.3. However, under certain conditions, the arrival of photons may be regarded as the independent occurrences of a sequence of random events at a rate equal to the photon flux, which is proportional to the optical power. The optical power may be deterministic (as in coherent light) or random (as in partially coherent light). For partially coherent light, the power fluctuations are correlated, so that the arrival of photons is no longer a sequence of independent events; the photon statistics are then significantly different.

Coherent Light

Consider light of constant optical power P . The corresponding mean photon flux $\Phi = P/h\nu$ (photons/s) is also constant, but the actual times of registration of the photons are random as shown in Fig. 11.2-4. Given a time interval of duration T , let the number of detected photons be n . We already know that the mean value of n is $\bar{n} = \Phi T = PT/h\nu$. We wish to obtain an expression for the probability distribution $p(n)$, i.e., the probability $p(0)$ of detecting no photons, the probability $p(1)$ of detecting one photon, and so on.

An expression for the probability distribution $p(n)$ can be derived under the assumption that the registrations of photons are statistically independent. The result is the Poisson distribution

$$p(n) = \frac{\bar{n}^n \exp(-\bar{n})}{n!}, \quad n = 0, 1, 2, \dots \quad (11.2-12)$$

Poisson Distribution

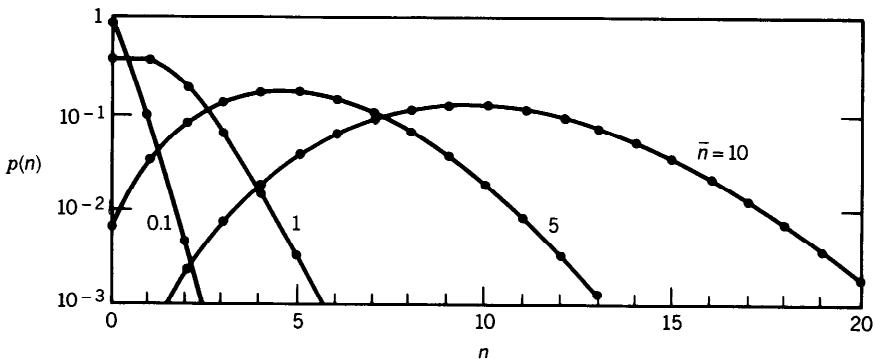
This distribution, known as the **Poisson distribution**, is displayed on a semilogarithmic plot in Fig. 11.2-5 for several values of the mean \bar{n} . The curves become progressively broader as \bar{n} increases.

Derivation of the Poisson Distribution

Divide the time interval T into a large number N of subintervals of sufficiently small width T/N each, such that each interval carries one photon with probability $p = \bar{n}/N$ and no photons with probability $1 - p$. The probability of finding n independent



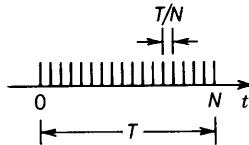
Figure 11.2-4 Random arrival of photons in a light beam of power P within intervals of duration T . Although the optical power is constant, the number n of photons arriving within each interval is random.

**Figure 11.2-5** Poisson distribution $p(n)$ of the photon number n .

photons in the N intervals, like the flips of a biased coin, then follows the binomial distribution

$$p(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n},$$

$$= \frac{N!}{n!(N-n)!} \left(\frac{\bar{n}}{N}\right)^n \left(1 - \frac{\bar{n}}{N}\right)^{N-n}$$



In the limit as $N \rightarrow \infty$, $N!/(N-n)! N^n \rightarrow 1$, and $[1 - (\bar{n}/N)]^{N-n} \rightarrow \exp(-\bar{n})$, so that (11.2-12) is obtained.

Mean and Variance

Two important parameters characterize any random number n —its mean value,

$$\bar{n} = \sum_{n=0}^{\infty} np(n), \quad (11.2-13)$$

and its variance

$$\sigma_n^2 = \sum_{n=0}^{\infty} (n - \bar{n})^2 p(n), \quad (11.2-14)$$

which is the average of the squared deviation from the mean. The standard deviation σ_n (the square root of the variance) is a measure of the width of the distribution. The quantities $p(n)$, \bar{n} , and σ_n are collectively called the photon-number statistics. Although the function $p(n)$ contains more information than just its mean and variance, these are useful measures.

It is not difficult to show [by use of (11.2-12) in (11.2-13) and (11.2-14)] that the mean of the Poisson distribution is indeed \bar{n} and its variance is equal to its mean,

$\sigma_n^2 = \bar{n}. \quad (11.2-15)$

Variance of the
Poisson Distribution

For example, when $\bar{n} = 100$, $\sigma_n = 10$; i.e., the generation of 100 photons is accompanied by an inaccuracy of about ± 10 photons.

The Poisson photon-number distribution applies for many light sources, including an ideal laser emitting a beam of monochromatic coherent light in a single mode (see Chap. 14). This distribution corresponds to a quantum state of light known as the coherent state (see Sec. 11.3A).

Signal-to-Noise Ratio

The randomness of the number of photons constitutes a fundamental source of noise that we have to contend with when using light to transmit a signal. Representing the mean of the signal as \bar{n} and its noise by the root mean square value σ_n , a useful measure of the performance of light as an information-carrying medium is the signal-to-noise ratio (SNR). The SNR of the random number n is defined as

$$\text{SNR} \equiv \frac{(\text{mean})^2}{\text{variance}} = \frac{\bar{n}^2}{\sigma_n^2}. \quad (11.2-16)$$

For the Poisson distribution

$$\boxed{\text{SNR} = \bar{n}}, \quad (11.2-17)$$

Poisson Photon-Number
Signal-to-Noise Ratio

i.e., the signal-to-noise ratio increases without limit as the mean photon number increases.

Although the SNR is a useful measure of the randomness of a signal, in some applications it is necessary to know the probability distribution itself. For example, if one communicates by sending a mean number of photons $\bar{n} = 20$, according to (11.2-12) the probability that no photons are received is $p(0) \approx 2 \times 10^{-9}$. This represents a probability of error in the transmission of information. This topic is addressed in Chap. 22.

Thermal Light

When the photon arrival times are correlated, the photon number statistics obey distributions other than the Poisson. This is the case for thermal light. Consider an optical resonator whose walls are maintained at temperature T kelvins (K), so that photons are emitted into the modes of the resonator. In accordance with the laws of statistical mechanics, under conditions of thermal equilibrium the probability distribution for the electromagnetic energy E_n in one of its modes satisfies the **Boltzmann probability distribution**

$$\boxed{P(E_n) \propto \exp\left(-\frac{E_n}{k_B T}\right)}. \quad (11.2-18)$$

Boltzmann Distribution

Here k_B is **Boltzmann's constant** ($k_B = 1.38 \times 10^{-23}$ J/K). The energy associated with each mode is random. Higher energies are relatively less probable than lower energies, in accordance with a simple exponential law governed by the quantity $k_B T$. The smaller the value of $k_B T$, the less likely are higher energies. At room temperature ($T = 300$ K), $k_B T = 0.026$ eV, which is equivalent to 208 cm^{-1} . The Boltzmann

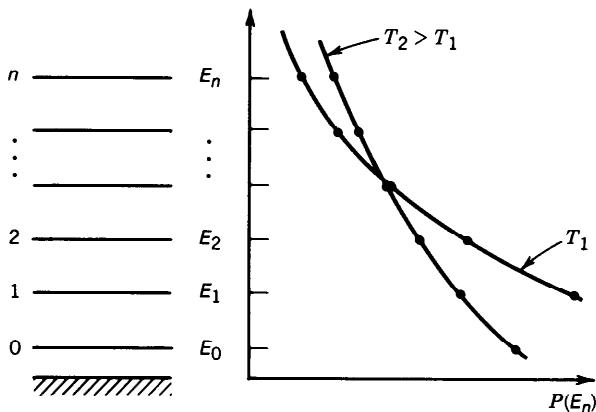


Figure 11.2-6 Boltzmann probability distribution $P(E_n)$ versus energy E_n .

distribution for this single mode is sketched in Fig. 11.2-6 with temperature as a parameter.

It follows from (11.2-18) and the photon-energy quantization relation given by $E_n = (n + \frac{1}{2})\hbar\nu$ that the probability of finding n photons in a single mode of a resonator in thermal equilibrium is given by

$$\begin{aligned} p(n) &\propto \exp\left(-\frac{n\hbar\nu}{k_B T}\right) \\ &= \left[\exp\left(-\frac{\hbar\nu}{k_B T}\right)\right]^n, \quad n = 0, 1, 2, \dots \end{aligned} \quad (11.2-19)$$

Using the condition that the probability distribution must have a sum equal to unity, i.e., $\sum_{n=0}^{\infty} p(n) = 1$, the normalization constant is determined to be $[1 - \exp(-\hbar\nu/k_B T)]$. The zero-point energy $E_0 = \frac{1}{2}\hbar\nu$ disappears into the normalization and does not affect the results, in accordance with the discussion in Sec. 11.1A.

The result is most simply written in terms of its mean \bar{n} as

$$p(n) = \frac{1}{\bar{n} + 1} \left(\frac{\bar{n}}{\bar{n} + 1}\right)^n,$$

$(11.2-20)$
 Bose-Einstein
 Distribution

where

$$\bar{n} = \frac{1}{\exp(\hbar\nu/k_B T) - 1}, \quad (11.2-21)$$

as determined from (11.2-13). In the parlance of probability theory, this distribution is called the **geometric distribution** since $p(n)$ is a geometrically decreasing function of n . In physics it is referred to as the **Bose-Einstein probability distribution**.

The Bose-Einstein distribution is displayed on a semilogarithmic plot in Fig. 11.2-7, for several values of \bar{n} (or equivalently, for several values of the temperature T). Its exponential character is evident in the straight-line behavior in the plot. Comparing

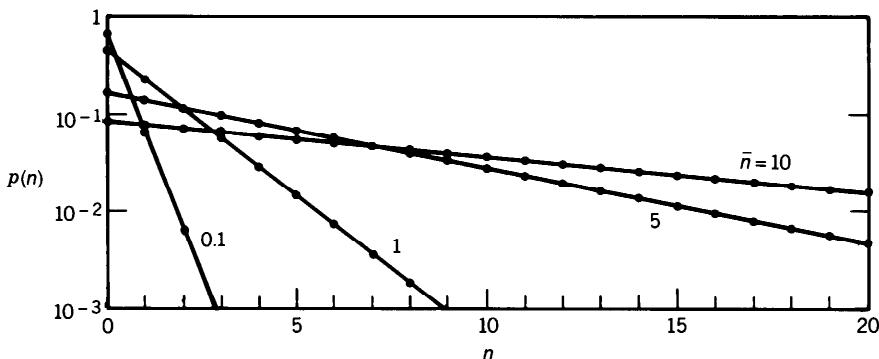


Figure 11.2-7 Bose-Einstein distribution $p(n)$ of the photon number n .

Figs. 11.2-7 with 11.2-5 demonstrates that the photon-number distribution for thermal light is far broader than that for coherent light.

Using (11.2-14), the photon-number variance turns out to be

$$\sigma_n^2 = \bar{n} + \bar{n}^2. \quad (11.2-22)$$

Bose-Einstein Variance

Comparing this expression to the variance for the Poisson distribution, which is simply \bar{n} , we see that thermal light has a larger variance corresponding to more uncertainty and a greater range of fluctuations of the photon number. The signal-to-noise ratio of the Bose-Einstein distribution is

$$\text{SNR} = \frac{\bar{n}}{\bar{n} + 1};$$

it is always smaller than unity no matter how large the optical power. The amplitude and phase of thermal light behave like random quantities, as described in Chapter 10. This randomness results in a broadening of the photon-number distribution. Indeed, this form of light is too noisy to be used in high-data-rate information transmission.

EXERCISE 11.2-1

Average Energy in a Resonator Mode. Show that the average energy of a resonator mode of frequency ν , under conditions of thermal equilibrium at temperature T , is given by

$$\bar{E} = k_B T \frac{h\nu/k_B T}{\exp(h\nu/k_B T) - 1}. \quad (11.2-23)$$

Sketch the dependence of \bar{E} on ν for several values of $k_B T/h$. Use a Taylor series expansion of the denominator to obtain a simplified approximate expression for \bar{E} in the limit $h\nu/k_B T \ll 1$. Explain the result on a physical basis.

***Other Sources of Light**

As mentioned earlier, for a certain class of light sources the photon arrivals can be regarded as a sequence of independent events, arriving at a rate proportional to the optical power. For coherent light, the power is deterministic, and the photon number obeys the Poisson distribution $p(n) = \mathcal{W}^n e^{-\mathcal{W}} / n!$, where

$$\mathcal{W} = \frac{1}{h\nu} \int_0^T P(t) dt = \frac{1}{h\nu} \int_0^T \int_A I(\mathbf{r}, t) dA dt. \quad (11.2-24)$$

The integrated optical power normalized to units of photon number, \mathcal{W} , is a constant representing the mean photon number \bar{n} .

When the intensity $I(\mathbf{r}, t)$ itself fluctuates randomly in time and/or space, the optical power $P(t)$ also undergoes random fluctuations [see Fig. 11.2-3(b)], and its integral \mathcal{W} is therefore also random. As a result, not only is the photon number random but so is its mean \mathcal{W} . Because of this added source of randomness, the photon-number statistics for partially coherent light will differ from the Poisson distribution. If the fluctuations in the mean photon number \mathcal{W} are described by a probability density function $p(\mathcal{W})$, the unconditional probability distribution for partially coherent light may be obtained by averaging the conditional Poisson distribution $p(n|\mathcal{W}) = \mathcal{W}^n e^{-\mathcal{W}} / n!$ over all permitted values of \mathcal{W} , each weighted by its probability density $p(\mathcal{W})$. The resultant photon-number distribution is then

$$p(n) = \int_0^\infty \frac{\mathcal{W}^n e^{-\mathcal{W}}}{n!} p(\mathcal{W}) d\mathcal{W}, \quad (11.2-25)$$

Mandel's Formula

which is known as Mandel's formula. Equation (11.2-25) is also referred to as the doubly stochastic Poisson counting distribution because of the two sources of randomness that contribute to it: the photons themselves (which behave in Poisson fashion) and the intensity fluctuations arising from the noncoherent nature of the light (which must be specified).

Note that this theory of photon statistics is applicable only to a certain class of light (called classical light); a more general theory based on a quantum description of the state of light is described briefly in Sec. 11.3.

The photon-number mean and variance for partially coherent light, which can be derived by using (11.2-13) and (11.2-14) in conjunction with (11.2-25), are

$$\bar{n} = \bar{\mathcal{W}} \quad (11.2-26)$$

and

$$\sigma_n^2 = \bar{n} + \sigma_{\mathcal{W}}^2, \quad (11.2-27)$$

respectively. Here $\sigma_{\mathcal{W}}^2$ signifies the variance of \mathcal{W} . Note that the variance of the photon number is the sum of two contributions—the first term is the basic contribution of the Poisson distribution, and the second is an additional contribution due to the classical fluctuations of the optical power.

In one important example of statistical fluctuations, the normalized integrated optical power \mathcal{W} obeys the exponential probability density function

$$p(\mathcal{W}) = \begin{cases} \frac{1}{\mathcal{W}} \exp\left(-\frac{\mathcal{W}}{\mathcal{W}}\right), & \mathcal{W} \geq 0 \\ 0, & \mathcal{W} < 0. \end{cases} \quad (11.2-28)$$

This distribution is applicable to quasi-monochromatic spatially coherent light, when the real and imaginary components of the complex amplitude are independent and have normal (Gaussian) probability distributions. The spectral width must be sufficiently small so that the coherence time τ_c is much greater than the counting time T , and the coherence area A_c must be much larger than the area of the detector A . The photon-number distribution $p(n)$ corresponding to (11.2-28) can be obtained by substitution into (11.2-25) and evaluation of the integral. The result turns out to be the Bose-Einstein distribution given in (11.2-20). The Gaussian-distributed optical field therefore has photon statistics identical to those of single-mode thermal light. When the area A and the time T are not small, the statistics are modified; they describe multimode thermal light (see Probs. 11.2-5 to 11.2-7).

D. Random Partitioning of Photon Streams

A photon stream is said to be partitioned when it is subjected to the removal of some of its photons. The photons removed may be either diverted or destroyed. The process is called random partitioning when they are diverted and random deletion when they are destroyed. There are numerous ways in which this can occur. Perhaps the simplest example of random partitioning is provided by an ideal lossless beamsplitter. Photons are randomly selected to join either of the two emerging streams (see Fig. 11.2-8). An example of random deletion is provided by the action of an optical absorption filter on a light beam. Photons are randomly selected either to pass through the filter or to be destroyed (and converted into heat).

We restrict our treatment to situations in which the possibility of each photon being removed behaves in accordance with an independent random (Bernoulli) trial. In terms of the beamsplitter, this is satisfied if a photon stream impinges on only one of the input ports (Fig. 11.2-8). This eliminates the possibility of interference, which, in general, invalidates the independent-trial assumption. Although the results derived below are couched in terms of random partitioning, they apply equally well to random deletion.

Consider a lossless beamsplitter with transmittance \mathcal{T} and reflectance $\mathcal{R} = 1 - \mathcal{T}$. In electromagnetic optics, the intensity of the transmitted wave I_t is related to the intensity of the incident wave I by $I_t = \mathcal{T}I$. The result of a single photon impinging on a beamsplitter was examined in Sec. 11.1B; it was shown that the probability of transmission is equal to the transmittance \mathcal{T} . We now proceed to calculate the

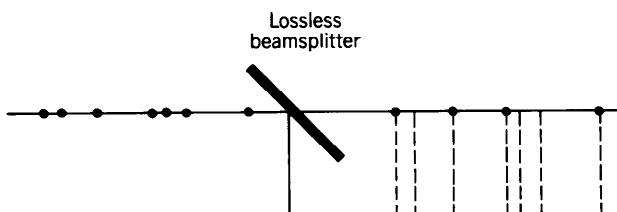


Figure 11.2-8 Random partitioning of photons by a beamsplitter.

outcome when a photon stream of mean flux Φ is incident, so that a mean number of photons $\bar{n} = \Phi T$ strikes the beamsplitter in the time interval T .

In accordance with (11.2-6), the mean number of photons in a beam is proportional to the optical energy. The mean number of transmitted and reflected photons in this time must therefore be $\mathcal{T}\bar{n}$ and $(1 - \mathcal{T})\bar{n}$, respectively. We now consider a more general question: what happens to the photon-number statistics $p(n)$ of the photon stream on partitioning by a beamsplitter?

A single photon falling on the beamsplitter is transmitted with probability \mathcal{T} and reflected with probability $1 - \mathcal{T}$ (see Fig. 11.1-3). If the incident beam contains precisely n photons, the probability $p(m)$ that m photons are transmitted is the same as that of flipping a coin n times, where the probability of achieving a head (being transmitted) is \mathcal{T} . From elementary probability theory we know that the outcome is the binomial distribution

$$p(m) = \binom{n}{m} \mathcal{T}^m (1 - \mathcal{T})^{n-m}, \quad m = 0, 1, \dots, n, \quad (11.2-29)$$

where $\binom{n}{m} = n! / m!(n - m)!$. The mean number of transmitted photons is easily shown to be

$$\bar{m} = \mathcal{T}n. \quad (11.2-30)$$

The variance for the binomial distribution is given by

$$\sigma_m^2 = \mathcal{T}(1 - \mathcal{T})n = (1 - \mathcal{T})\bar{m}. \quad (11.2-31)$$

Because of the symmetry of the problem, the results for the reflected beam are obtained immediately. As the average number of transmitted photons \bar{m} increases, the signal-to-noise ratio, represented by $\bar{m}^2/\sigma_m^2 = \bar{m}/(1 - \mathcal{T})$ increases. Therefore, for large intensities, the photons will be partitioned between the two streams in good accord with \mathcal{T} and $(1 - \mathcal{T})$, indicating that the laws of classical optics are recovered.

The expressions provided above are useful because they permit us to calculate the effect of a beamsplitter on photons obeying various photon-number statistics. The solution is obtained by recognizing that in these cases the number of photons n at the input to the beamsplitter is random rather than fixed. Let the probability that there are exactly n photons present be $p_0(n)$. If we treat the photons as independent events, the photon-number probability distribution in the transmitted stream will be a weighted sum of binomial distributions, with n taking on the random value n . The weighting is in accordance with the probability that n photons were present. The probability of finding m photons transmitted through the beamsplitter, when the input photon-number distribution is $p_0(n)$, is therefore given by $p(m) = \sum_n p(m|n)p_0(n)$, where $p(m|n) = \binom{n}{m} \mathcal{T}^m (1 - \mathcal{T})^{n-m}$ is the binomial distribution. Explicitly, then,

$$p(m) = \sum_{n=m}^{\infty} \binom{n}{m} \mathcal{T}^m (1 - \mathcal{T})^{n-m} p_0(n), \quad m = 0, 1, 2, \dots. \quad (11.2-32)$$

Photon-Number
Statistics under
Random Partitioning

When $p_0(n)$ is the Poisson distribution (coherent light) or the Bose-Einstein distribution (thermal light), the results turn out to be quite simple: $p(m)$ has exactly the same form of photon-number distribution as $p_0(n)$. These distributions retain their form under random partitioning. Thus single-mode laser light transmitted through a beamsplitter remains Poisson and thermal light remains Bose-Einstein, but of course

with a reduced photon-number mean. Light with a deterministic number of photons (see Sec. 11.3B), on the other hand, does not retain its form under random partitioning, and this unfortunate property accounts for its lack of robustness.

The signal-to-noise ratio of m is easily calculated for photon streams that have undergone partitioning or deletion. For coherent light and single-mode thermal light, the results are

$$\text{SNR} = \begin{cases} \mathcal{T}\bar{n} & \text{coherent light} \\ \frac{\mathcal{T}\bar{n}}{\mathcal{T}\bar{n} + 1} & \text{thermal light.} \end{cases} \quad (11.2-33)$$

Since $\mathcal{T} \leq 1$ it is clear that random partitioning decreases the signal-to-noise ratio. Another way of stating this is that random partitioning introduces noise. The effect is most severe for deterministic photon-number light.

The same results are also applicable to the *detection* of photons. If every photon has an independent chance of being detected, then out of n incident photons, m photons would be detected where $p(m)$ is related to $p_0(n)$ by (11.2-32). This result will be useful in the theory of photon detection (Chap. 17).

*11.3 QUANTUM STATES OF LIGHT

The position, momentum, and number of photons in an electromagnetic mode are generally random quantities. In this section it will be shown that the electric field itself is also generally random. Consider a plane-wave monochromatic electromagnetic mode in a volume V , described by the electric field $\text{Re}\{\mathbf{E}(\mathbf{r}, t)\}$, where

$$\mathbf{E}(\mathbf{r}, t) = A \exp(-j\mathbf{k} \cdot \mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}.$$

According to classical electromagnetic optics, the energy of the mode is fixed at $\frac{1}{2}\epsilon|A|^2V$. We define a complex variable a , such that $\frac{1}{2}\epsilon|A|^2V = h\nu|a|^2$, which allows $|a|^2$ to be interpreted as the energy of the mode in units of photon number. The electric field may then be written as

$$\mathbf{E}(\mathbf{r}, t) = \left(\frac{2h\nu}{\epsilon V} \right)^{1/2} a \exp(-j\mathbf{k} \cdot \mathbf{r}) \exp(j2\pi\nu t) \hat{\mathbf{e}}, \quad (11.3-1)$$

where the complex variable a determines the complex amplitude of the field.

In classical electromagnetic optics, $a \exp(j2\pi\nu t)$ is a rotating phasor whose projection on the real axis determines the sinusoidal electric field (see Fig. 11.3-1). The real and imaginary parts $x = \text{Re}\{a\}$ and $p = \text{Im}\{a\}$ are called the quadrature components of the phasor a because they are a quarter cycle (90°) out of phase with each other. They determine the amplitude and phase of the sine wave that represents the temporal variation of the electric field. The rotating phasor $a \exp(j2\pi\nu t)$ also describes the motion of a harmonic oscillator; the real component x is proportional to position and the imaginary component p to momentum. From a mathematical point of view, a classical monochromatic mode of the electromagnetic field and a classical harmonic oscillator behave identically.

Similarly, a quantum monochromatic electromagnetic mode and a one-dimensional quantum-mechanical harmonic oscillator have identical behavior. We therefore review the quantum theory of a simple harmonic oscillator before proceeding.

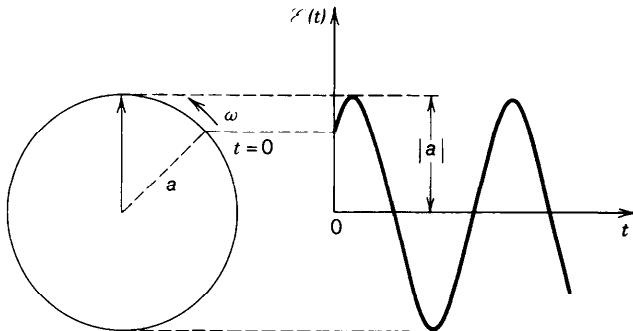


Figure 11.3-1 The real and imaginary parts of the variable $a \exp(j2\pi\nu t)$, which governs the complex amplitude of a classical electromagnetic field of frequency ν . The time dynamics are identical to those of a harmonic oscillator of angular frequency $\omega = 2\pi\nu$.

Quantum Theory of the Harmonic Oscillator

A particle of mass m , position x , momentum p , and potential energy $V(x) = \frac{1}{2}\kappa x^2$, where κ is the elastic constant, is a harmonic oscillator of total energy $\frac{1}{2}p^2/m + \frac{1}{2}\kappa x^2$ and oscillation frequency $\omega = (\kappa/m)^{1/2}$. In accordance with quantum mechanics its behavior may be described by a complex wavefunction $\psi(x)$ satisfying the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x), \quad (11.3-2)$$

where E is the particle energy. For the harmonic oscillator the solutions of the Schrödinger equation give rise to discrete values of energy given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\nu, \quad n = 0, 1, 2, \dots; \quad (11.3-3)$$

adjacent energy levels are separated by a quantum of energy $\hbar\nu = \hbar\omega$. The corresponding eigenfunctions $\psi_n(x)$ are normalized Hermite–Gaussian functions,

$$\psi_n(x) = (2^n n!)^{-1/2} \left(\frac{2m\omega}{\hbar}\right)^{1/4} H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] \exp\left(-\frac{m\omega x^2}{2\hbar}\right),$$

where $H_n(x)$ is the Hermite polynomial of order n [see (3.3-5) to (3.3-7) and (3.3-10)].

An arbitrary wavefunction $\psi(x)$ may be expanded in terms of the orthonormal eigenfunctions $\{\psi_n(x)\}$ as the superposition $\psi(x) = \sum_n c_n \psi_n(x)$. Given the wavefunction $\psi(x)$, which determines the state of the system, the behavior of the particle may be determined as follows:

- The probability $p(n)$ that the harmonic oscillator carries n quanta of energy is given by the coefficient $|c_n|^2$.
- The probability density of finding the particle at the position x is given by $|\psi(x)|^2$.
- The probability density that the momentum of the particle is p is given by $|\phi(p)|^2$, where $\phi(p)$ is proportional to the inverse Fourier transform of $\psi(x)$ evaluated at the frequency p/\hbar ,

$$\phi(p) = \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} \psi(x) \exp\left(j2\pi \frac{p}{\hbar} x\right) dx. \quad (11.3-4)$$

The Fourier transform relation between the variables x and p/\hbar implies a Heisenberg position-momentum uncertainty relation

$$\frac{\sigma_x \sigma_p}{\hbar} \geq \frac{1}{4\pi} \quad \text{or} \quad \sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Analogy Between an Optical Mode and a Harmonic Oscillator

The energy of an electromagnetic mode is $\hbar\nu|\mathbf{a}|^2 = \hbar\nu(x^2 + p^2)$. The analogy with a harmonic oscillator of energy $\frac{1}{2}(p^2/m + \kappa x^2)$ is established by effecting the substitutions

$$x = (2\hbar\nu)^{-1/2} \omega x \quad \text{and} \quad p = (2\hbar\nu)^{-1/2} p.$$

The mode energy then becomes $\frac{1}{2}(p^2 + \omega^2 x^2)$, which is the same as the energy of a harmonic oscillator of mass $m = 1$ (for which $\omega = \sqrt{\kappa}$). Because the analogy is complete, we conclude that the energy of a quantum electromagnetic mode, like that of a quantum-mechanical harmonic oscillator, is quantized to the values $(n + \frac{1}{2})\hbar\nu$, as suggested earlier. With the use of proper scaling factors, the behavior of the position x and momentum p of the harmonic oscillator also describe the quadrature components of the electromagnetic field x and p .

Summary

An electromagnetic mode of frequency ν is described by a complex wavefunction $\psi(x)$ that governs the uncertainties of the quadrature components x and p and the statistics of the number of photons in the mode.

- The probability $p(n)$ that the mode contains n photons is given by $|c_n|^2$, where the c_n are coefficients of the expansion of $\psi(x)$ in terms of the eigenfunctions $\psi_n(x)$,

$$\psi(x) = \sum_n c_n \psi_n(x).$$

- The probability densities of the quadrature components x and p are given by the functions $|\psi(x)|^2$ and $|\phi(p)|^2$, where $\psi(\cdot)$ and $\phi(\cdot)$ are related by

$$\phi(p) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi(x) \exp(j2px) dx. \quad (11.3-5)$$

- If $\psi(x)$ is known, then $\phi(p)$ may be calculated and the probability densities of x and p determined. The complex wavefunction $\psi(x)$ therefore determines the uncertainties of the quadrature components of the complex amplitude.

As shown in Sec. A.2 of Appendix A, the Fourier transform relation between $\psi(x)$ and $\phi(p)$ indicates that there is an uncertainty relation between the power-rms widths of the quadrature components given by

$$\sigma_x \sigma_p \geq \frac{1}{4}.$$

(11.3-6)

Quadrature Uncertainty

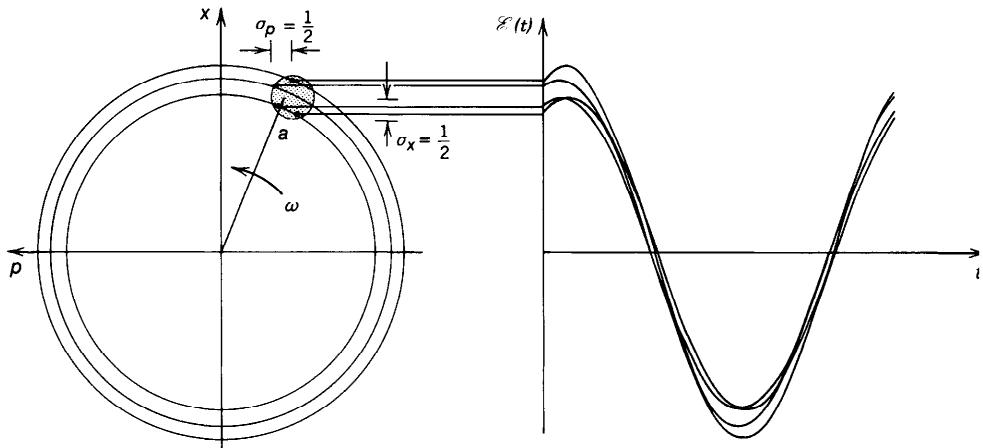


Figure 11.3-2 Uncertainties for the coherent state. Representative values of $\mathcal{E}(t)$ & $a \exp(j2\pi\nu t)$ are drawn by choosing arbitrary points within the uncertainty circle. The coefficient of proportionality is chosen to be unity.

The real and imaginary components of the electric field cannot both be determined simultaneously with arbitrary precision.

A. Coherent-State Light

The uncertainty product $\sigma_x \sigma_p$ attains its minimum value of $\frac{1}{4}$ when the function $\psi(x)$ is Gaussian (see Sec. A.2 of Appendix A). In that case

$$\psi(x) \propto \exp[-(x - \alpha_x)^2], \quad (11.3-7)$$

whereupon its Fourier transform is also Gaussian, so that

$$\phi(p) \propto \exp[-(p - \alpha_p)^2]. \quad (11.3-8)$$

Here, α_x and α_p are arbitrary values that represent the means of x and p . The quadrature uncertainties, determined from $|\psi(x)|^2$ and $|\phi(p)|^2$, are then given by

$$\sigma_x = \sigma_p = \frac{1}{2}. \quad (11.3-9)$$

Under these conditions the electromagnetic field is said to be in a coherent state. The one-standard-deviation range of uncertainty in the quadrature components x and p , as well as in the complex amplitude a and in the electric field $\mathcal{E}(t)$, are illustrated in Fig. 11.3-2 for coherent-state light. The squared-magnitude $|c_n|^2$ of the coefficient of the expansion of $\psi(x)$ in the Hermite-Gaussian basis equals $\bar{n}^n \exp(-\bar{n})/n!$, where $\bar{n} = \alpha_x^2 + \alpha_p^2$; thus the photon-counting probability $p(n)$ is Poisson. Unlike its status in electromagnetic optics, in the context of photon optics coherent light is *not* deterministic.

The uncertainty of the coherent state is most pronounced when α_x and α_p are small. The time behavior of the electric field is illustrated in Fig. 11.3-3 in the limit when $\alpha_x = \alpha_p = 0$. This corresponds to the case when the mode contains zero photons and has only the residual zero-point energy $\frac{1}{2}\hbar\nu$; this is called the vacuum state.

B. Squeezed-State Light

Quadrature-Squeezed Light

Although the uncertainty product $\sigma_x \sigma_p$ cannot be reduced below its minimum value of $\frac{1}{4}$, the uncertainty of one of the quadrature components may be reduced (squeezed) below $\frac{1}{2}$, of course at the expense of an increased uncertainty in the other component.

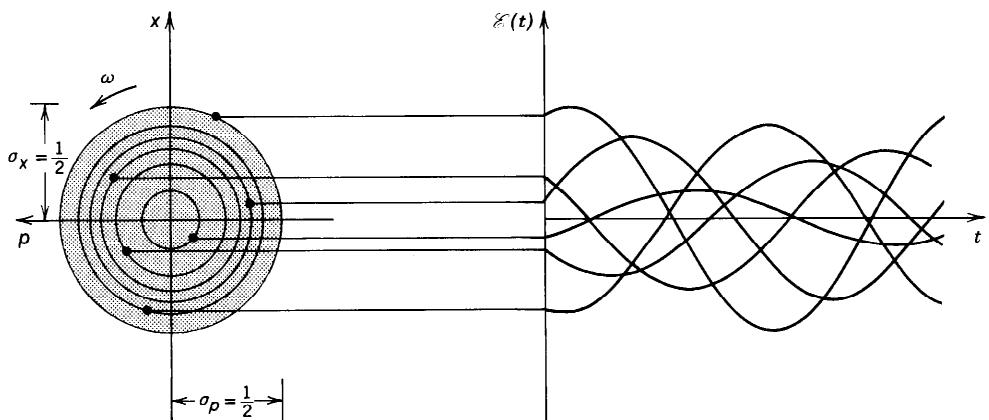


Figure 11.3-3 Representative uncertainties for the vacuum state.

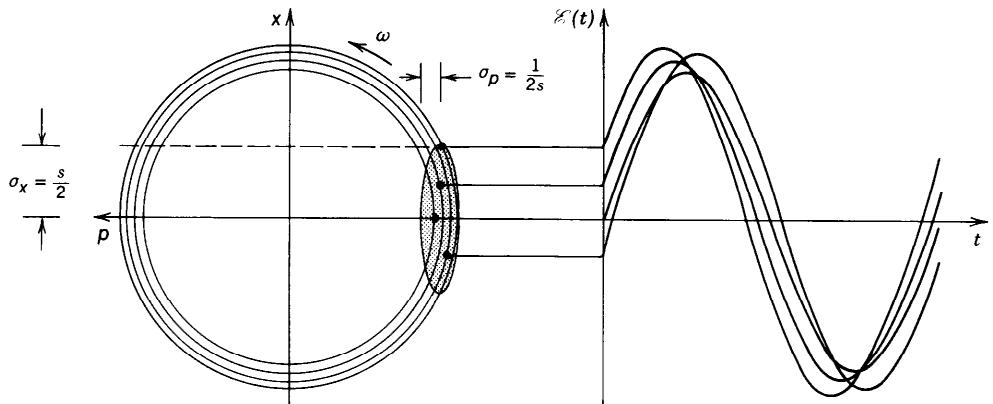


Figure 11.3-4 Representative uncertainties for a quadrature-squeezed state.

The light is then said to be **quadrature squeezed**. For example, a state for which $\psi(x)$ is a Gaussian function with a (stretched) width $\sigma_x = s/2$ ($s > 1$) corresponds to a Gaussian $\phi(p)$ with a (squeezed) width $\sigma_p = 1/2s$. The product $\sigma_x\sigma_p$ maintains its minimum value of $\frac{1}{4}$, but the uncertainty circle of the phasor a is squeezed into an elliptical form, as shown in Fig. 11.3-4. The asymmetry in the uncertainties of the two quadrature components is manifested in the time course of the electric field by periodic occurrences of increased uncertainty followed, each quarter cycle later, by occurrences of decreased uncertainty. If the field were to be measured only at those times when its uncertainty is minimal, its noise would be reduced below that of the coherent state. The selection of those times may be achieved by heterodyning the squeezed field with a coherent optical field of appropriate phase (see Sec. 22.5). Because of its reduced noisiness, squeezed light is useful in precision measurements and in information transmission.

Photon-Number-Squeezed Light

Quadrature-squeezed light exhibits an uncertainty in one of its quadrature components that is reduced relative to that of the coherent state. Another form of nonclassical light is **photon-number-squeezed** or **sub-Poisson** light. It has a photon-number variance that is squeezed below the coherent-state (Poisson) value, i.e., $\sigma_n^2 < \bar{n}$. Photon-number fluctuations obeying this relation are nonclassical since (11.2-27) cannot be satisfied.

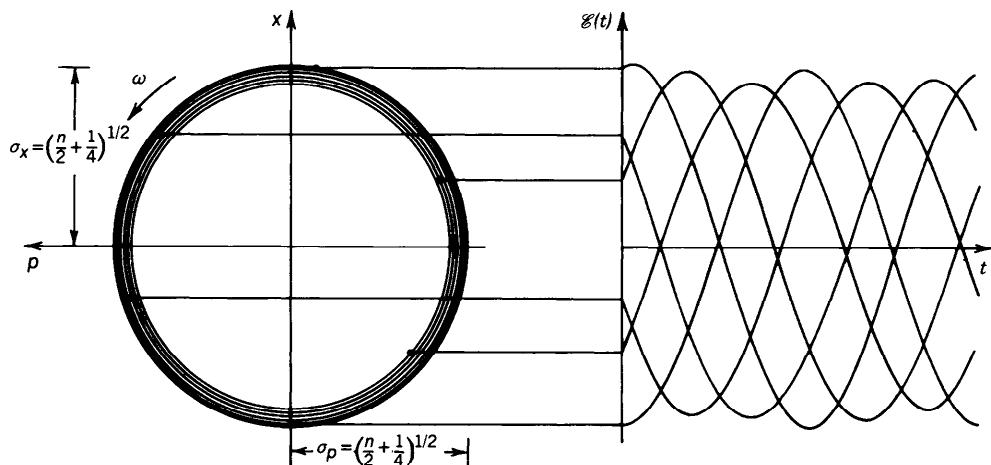


Figure 11.3-5 Representative uncertainties for the number state. This state is photon-number squeezed but not quadrature squeezed.

Photon-number squeezed light, like quadrature-squeezed light, has applications in precision measurements and information transmission. It can be generated by specially designed semiconductor injection lasers.

As an example of photon-number-squeezed light, consider an electromagnetic mode described by the harmonic oscillator eigenstate $\psi(x) = \psi_{n_0}(x)$. This is called a number state because $p(n) = |c_n|^2 = 1$ for $n = n_0$, while all other coefficients (c_n for $n \neq n_0$) vanish, so that the number of photons carried by the mode is precisely n_0 . Number-state light therefore has a deterministic photon number. The mean photon number is obviously $\bar{n} = n_0$ and the variance is zero (since there are no photon-number fluctuations). The case $n_0 = 1$ corresponds to the presence of precisely one photon.

The uncertainties of number-state light are shown in Fig. 11.3-5. Although the quadrature components, as well as the phasor magnitude and phase, are all uncertain, the photon number is absolutely certain. The question arises as to whether it is possible to carry out experiments requiring a fixed number of photons by using coherent-state light in a selective manner. Could this, for example, be achieved by monitoring the photons from a coherent source in successive time intervals, and then using the photons only in those time intervals where the desired photon number is observed? The problem with this approach is that it is difficult to observe the photons without annihilating them. One way of circumventing the problem is to generate photons in correlated pairs by means of a process such as parametric downconversion (see Secs. 19.2C and 19.4C). With two “copies” of a photon stream, one can be observed and used to indicate or control the photon number in the other.

READING LIST

Books on Quantum Mechanics

- L. E. Ballentine, *Quantum Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1990.
- W. Greiner, *Quantum Mechanics*, Springer-Verlag, New York, 1989.
- A. Yariv, *Introduction to the Theory and Applications of Quantum Mechanics*, Wiley, New York, 1982.
- L. I. Schiff, *Quantum Mechanics*, McGraw-Hill, New York, 3rd ed. 1968.

- R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, vol. 3, *Quantum Mechanics*, Addison-Wesley, Reading, MA, 1965.
- R. P. Feynman, *Quantum Electrodynamics*, W. A. Benjamin, New York, 1962.
- A. Messiah, *Quantum Mechanics*, vols. 1 and 2, North-Holland/Wiley, Amsterdam/New York, 1961/1962.
- E. Merzbacher, *Quantum Mechanics*, Wiley, New York, 1961.
- R. M. Eisberg, *Fundamentals of Modern Physics*, Wiley, New York, 1961.
- P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford University Press, New York, 4th ed. 1958.
- L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Addison-Wesley, Reading, MA, 1958.

Books on Quantum Optics

- J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena*, Reidel, Dordrecht, The Netherlands, 2nd ed. 1991.
- P. Meystre and M. Sargent III, *Elements of Quantum Optics*, Springer-Verlag, New York, 1990.
- W. H. Louisell, *Quantum Statistical Properties of Radiation*, Wiley, New York, 1973, 1990.
- C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms*, Wiley, New York, 1989.
- E. R. Pike and H. Walther, eds., *Photons and Quantum Fluctuations*, Adam Hilger, Bristol, England, 1988.
- F. Haake, L. M. Narducci, and D. F. Walls, eds., *Coherence, Cooperation, and Fluctuations*, Cambridge University Press, New York, 1986.
- E. R. Pike and S. Sarkar, eds., *Frontiers in Quantum Optics*, Adam Hilger, Bristol, England, 1986.
- R. P. Feynman, *QED*, Princeton University Press, Princeton, NJ, 1985.
- J. Peřina, *Coherence of Light*, Reidel, Dordrecht, The Netherlands, 2nd ed. 1985.
- R. Loudon, *The Quantum Theory of Light*, Oxford University Press, New York, 2nd ed. 1983.
- R. L. Knight and L. Allen, *Concepts of Quantum Optics*, Pergamon Press, Oxford, 1983.
- E. Goldin, *Waves and Photons: An Introduction to Quantum Optics*, Wiley, New York, 1982.
- H. Haken, *Light: Waves, Photons, Atoms*, vol. 1, North-Holland, Amsterdam, 1981.
- D. Marcuse, *Principles of Quantum Electronics*, Academic Press, New York, 1980.
- B. Saleh, *Photoelectron Statistics*, Springer-Verlag, New York, 1978.
- H. M. Nussenzveig, *Introduction to Quantum Optics*, Gordon and Breach, New York, 1973.
- D. Marcuse, *Engineering Quantum Electrodynamics*, Harcourt, Brace & World, New York, 1970.
- J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*, W. A. Benjamin, New York, 1968.
- C. DeWitt, A. Blandin, and C. Cohen-Tannoudji, eds., *Quantum Optics and Electronics*, Gordon and Breach, New York, 1965.
- W. H. Louisell, *Radiation and Noise in Quantum Electronics*, McGraw-Hill, New York, 1964.
- W. Heitler, *The Quantum Theory of Radiation*, Clarendon Press, Oxford, England, 3rd ed. 1954.

Special Journal Issues

- Special issue on the statistical efficiency of natural and artificial vision, Part II, *Journal of the Optical Society of America A*, vol. 5, no. 4, 1988.
- Special issue on the statistical efficiency of natural and artificial vision, *Journal of the Optical Society of America A*, vol. 4, no. 12, 1987.
- Special issue on squeezed states of the electromagnetic field, *Journal of the Optical Society of America B*, vol. 4, no. 10, 1987.
- Special issue on squeezed light, *Journal of Modern Optics*, vol. 34, no. 6/7, 1987.
- Special issue on quantum-limited imaging and image processing, *Journal of the Optical Society of America A*, vol. 3, no. 12, 1986.

Special issue on the mechanical effects of light, *Journal of the Optical Society of America B*, vol. 2, no. 11, 1985.

Articles

- M. C. Teich and B. E. A. Saleh, Squeezed and Antibunched Light, *Physics Today*, vol. 43, no. 6, pp. 26–34, 1990.
- M. C. Teich and B. E. A. Saleh, Squeezed States of Light, *Quantum Optics*, vol. 1, pp. 153–191, 1989.
- M. C. Teich and B. E. A. Saleh, Photon Bunching and Antibunching, in *Progress in Optics*, vol. 26, E. Wolf, ed., North-Holland, Amsterdam, 1988, pp. 1–104.
- R. E. Slusher and B. Yurke, Squeezed Light, *Scientific American*, vol. 258, no. 5, pp. 50–56, 1988.
- M. D. Levenson and R. M. Shelby, Deamplification of Quantum Noise and Quantum Nondemolition Detection in Optical Fibers, *Optics News*, vol. 14, no. 1, pp. 7–12, 1988.
- R. W. Henry and S. C. Glotzer, A Squeezed-State Primer, *American Journal of Physics*, vol. 56, pp. 318–328, 1988.
- G. Leuchs, Squeezing the Quantum Fluctuations of Light, *Contemporary Physics*, vol. 29, pp. 299–314, 1988.
- R. Loudon and P. L. Knight, Squeezed Light, *Journal of Modern Optics*, vol. 34, pp. 709–759, 1987.
- M.-A. Bouchiat and L. Pottier, Optical Experiments and Weak Interactions, *Science*, vol. 234, pp. 1203–1210, 1986.
- D. F. Walls, Squeezed States of Light, *Nature*, vol. 306, pp. 141–146, 1983.
- H. Paul, Photon Antibunching, *Reviews of Modern Physics*, vol. 54, pp. 1061–1102, 1982.
- E. Wolf, Einstein's Researches on the Nature of Light, *Optics News*, vol. 5, no. 1, pp. 24–39, 1979.
- S. Weinberg, Light as a Fundamental Particle, *Physics Today*, vol. 28, no. 6, pp. 32–37, 1975.
- M. O. Scully and M. Sargent III, The Concept of the Photon, *Physics Today*, vol. 25, no. 3, pp. 38–47, 1972.
- H. Risken, Statistical Properties of Laser Light, in *Progress in Optics*, vol. 8, E. Wolf, ed., North-Holland, Amsterdam, 1970.
- L. Mandel and E. Wolf, eds., *Selected Papers on Coherence and Fluctuations of Light*, vols. 1 and 2, Dover, New York, 1970.
- L. Mandel and E. Wolf, Coherence Properties of Optical Fields, *Reviews of Modern Physics*, vol. 37, pp. 231–287, 1965.
- L. Mandel, Fluctuations of Light Beams, in *Progress in Optics*, vol. 2, E. Wolf, ed., North-Holland, Amsterdam, 1963.

PROBLEMS

- 11.1-1 **Photon Energy.** (a) What voltage should be applied to accelerate an electron from zero velocity in order that it acquire the same energy as a photon of wavelength $\lambda_o = 0.87 \mu\text{m}$?
 (b) A photon of wavelength $1.06 \mu\text{m}$ is combined with a photon of wavelength $10.6 \mu\text{m}$ to create a photon whose energy is the sum of the energies of the two photons. What is the wavelength of the resultant photon? Photon interactions of this type are discussed in Chap. 19.
- 11.1-2 **Position of a Single Photon at a Screen.** Consider a monochromatic light beam of wavelength λ_o falling on an infinite screen in the plane $z = 0$, with an intensity $I(\rho) = I_0 \exp(-\rho/\rho_0)$, where $\rho = (x^2 + y^2)^{1/2}$. Assume that the intensity of the source is reduced to a level at which only a single photon strikes the screen.
 (a) Find the probability that the photon strikes the screen within a radius ρ_0 of the origin.

- (b) If the beam contains exactly 10^6 photons, on the average how many photons strike within a circle of radius ρ_0 ?
- 11.1-3 **Momentum of a Free Photon.** Compare the total momentum of the photons in a 10-J laser pulse with that of a 1-g mass moving at a velocity of 1 cm/s and with an electron moving at a velocity $c_o/10$.
- *11.1-4 **Momentum of a Photon in a Gaussian Beam.** (a) What is the probability that the momentum vector of a photon associated with a Gaussian beam of waist radius W_0 lies within the beam divergence angle θ_0 ? Refer to Sec. 3.1 for definitions. (b) Does the relation $p = E/c_o$ hold in this case?
- 11.1-5 **Levitation by Light Pressure.** Consider an isolated hydrogen atom of mass 1.66×10^{-27} kg.
 (a) Find the gravitational force on this hydrogen atom near the surface of the earth (assume that at sea level the gravitational acceleration constant $g = 9.8$ m/s²).
 (b) Let an upwardly directed laser beam emitting 1-eV photons be focused in such a way that the full momentum of each of its photons is transferred to the atom. Find the average upward force on the atom provided by one photon striking it each second.
 (c) Find the number of photons that must strike the atom per second and the corresponding optical power for it not to fall under the effect of gravity, given idealized conditions in vacuum.
 (d) How many photons per second would be required to keep the atom from falling if it were perfectly reflecting?
- *11.1-6 **Single Photon in a Fabry–Perot Resonator.** Consider a Fabry–Perot resonator of length $d = 1$ cm containing nonabsorbing material of refractive index $n = 1.5$ and perfectly reflecting mirrors. Assume that there is exactly one photon in the mode described by the standing wave $\sin(10^5\pi x/d)$.
 (a) Determine the photon wavelength and energy (in eV).
 (b) Estimate the uncertainty in the photon's position and momentum (magnitude and direction). Compare with the value obtained from the relation $\sigma_p \sigma_x \approx \hbar/2$.
- 11.1-7 **Single-Photon Beating (Time Interference).** Consider a detector illuminated by a polychromatic plane wave consisting of two plane-parallel superposed monochromatic waves represented by
- $$U_1(t) = \sqrt{I_1} \exp(j2\pi\nu_1 t) \quad \text{and} \quad U_2(t) = \sqrt{I_2} \exp(j2\pi\nu_2 t),$$
- with frequencies ν_1 and ν_2 and intensities I_1 and I_2 , respectively. According to wave optics (see Sec. 2.6B), the intensity of this wave is given by $I(t) = I_1 + I_2 + 2(I_1 I_2)^{1/2} \cos[2\pi(\nu_2 - \nu_1)t]$. Assume that the two constituent plane waves have equal intensities ($I_1 = I_2$). Assume also that the wave is sufficiently weak that only a single polychromatic photon reaches the detector during the time interval $T = 1/|\nu_2 - \nu_1|$.
 (a) Plot the probability density $p(t)$ for the detection time of the photon for $0 \leq t \leq 1/|\nu_2 - \nu_1|$. At what time instant during T is the probability zero that the photon will be detected?
 (b) An attempt to discover from which of the two constituent waves the photon came would entail an energy measurement to a precision better than

$$\sigma_E < \hbar|\nu_2 - \nu_1|.$$

Use the time–energy uncertainty relation to show that the time required for such

a measurement would be of the order of the beat-frequency period so that the very process of measurement would wash out the interference.

- 11.1-8 **Photon Momentum Exchange at a Beamsplitter.** Consider a single photon, in a mode described by a plane wave, impinging on a lossless beamsplitter. What is the momentum vector of the photon before it impinges on the mirror? What are the possible values of the photon's momentum vector, and the probabilities of observing these values, after the beamsplitter?
- 11.2-1 **Photon Flux.** Show that the power of a monochromatic optical beam that carries an average of one photon per optical cycle is inversely proportional to the squared wavelength.
- 11.2-2 **The Poisson Distribution.** Verify that the Poisson probability distribution given by (11.2-12) is normalized to unity and has mean \bar{n} and variance $\sigma_n^2 = \bar{n}$.
- 11.2-3 **Photon Statistics of a Coherent Gaussian Beam.** Assume that a 100-pW He-Ne single-mode laser emits light at 633 nm in a $\text{TEM}_{0,0}$ Gaussian beam (see Chap. 3).
- What is the mean number of photons crossing a circle of radius equal to the waist radius of the beam W_0 in a time $T = 100$ ns?
 - What is the root-mean-square value of the number of photon counts in (a)?
 - What is the probability that no photons are counted in (a)?
- 11.2-4 **The Bose-Einstein Distribution.** (a) Verify that the Bose-Einstein probability distribution given by (11.2-20) is normalized and has a mean \bar{n} and variance $\sigma_n^2 = \bar{n} + \bar{n}^2$.
 (b) If a beam of photons obeying Bose-Einstein statistics contains an average of $\Phi = 1$ photon per nanosecond, what is the probability that zero photons will be detected in a 20-ns time interval?
- *11.2-5 **The Negative-Binomial Distribution.** It is well known in the literature of probability theory that the sum of M identically distributed random variables, each with a geometric (Bose-Einstein) distribution, obeys the negative binomial distribution

$$p(n) = \binom{n+M-1}{n} \frac{(\bar{n}/M)^n}{(1+\bar{n}/M)^{n+M}}.$$

Verify that the negative-binomial distribution reduces to the Bose-Einstein distribution for $M = 1$ and to the Poisson distribution as $M \rightarrow \infty$.

- *11.2-6 **Photon Statistics for Multimode Thermal Light in a Cavity.** Consider M modes of thermal radiation sufficiently close to each other in frequency that each can be considered to be occupied in accordance with a Bose-Einstein distribution of the same mean photon number $1/[\exp(h\nu/k_B T) - 1]$. Show that the variance of the *total* number of photons n is related to its mean by

$$\sigma_n^2 = \bar{n} + \frac{\bar{n}^2}{M},$$

indicating that multimode thermal light has less variance than does single-mode thermal light. The presence of the multiple modes provides averaging, thereby reducing the noisiness of the light.

- *11.2-7 **Photon Statistics for a Beam of Multimode Thermal Light.** A multimode thermal light source that carries M identical modes, each with exponentially distributed (random) integrated rate, has a probability density $p(W)$ describable by the gamma distribution

$$p(W) = \frac{1}{(M-1)!} \left(\frac{M}{\langle W \rangle} \right)^M W^{M-1} \exp\left(-\frac{MW}{\langle W \rangle}\right), \quad W \geq 0.$$

Use Mandel's formula (11.2-25) to show that the resulting photon-number distribution assumes the form of the negative-binomial distribution defined in Problem 11.2-5.

- *11.2-8 **Mean and Variance of the Doubly Stochastic Poisson Distribution.** Prove (11.2-26) and (11.2-27).
- 11.2-9 **Random Partitioning of Coherent Light.** (a) Use (11.2-32) to show that the photon-number distribution of randomly partitioned coherent light retains its Poisson form.
 (b) Show explicitly that the mean photon number for light reflected from a lossless beamsplitter is $(1 - T)\bar{n}$.
 (c) Prove (11.2-33) for coherent light.
- 11.2-10 **Random Partitioning of Single-Mode Thermal Light.** (a) Use (11.2-32) to show that the photon-number distribution of randomly partitioned single-mode thermal light retains its Bose-Einstein form.
 (b) Show explicitly that the mean photon number for light reflected from a lossless beamsplitter is $(1 - T)\bar{n}$.
 (c) Prove (11.2-34) for single-mode thermal light.
- *11.2-11 **Exponential Decay of Mean Photon Number in an Absorber.** (a) Consider an absorptive material of thickness d and absorption coefficient α (cm^{-1}). If the average number of photons that enters the material is \bar{n}_0 , write a differential equation to find the average number of photons $\bar{n}(x)$ at position x , where x is the depth into the filter ($0 \leq x \leq d$).
 (b) Solve the differential equation. State the reason that your result is the exponential intensity decay law obtained from electromagnetic optics (Sec. 5.5A).
 (c) Write an expression for the photon-number distribution at an arbitrary position x in the absorber, $p(n)$, when coherent light is incident on it.
 (d) What is the probability of survival of a single photon incident on the absorber?
- *11.3-1 **Statistics of the Binomial Photon-Number Distribution.** The binomial probability distribution may be written $p(n) = [M!/(M-n)!n!]p^n(1-p)^{M-n}$. It describes certain photon-number-squeezed sources of light.
 (a) Indicate a possible mechanism for converting number-state light into light described by binomial photon statistics.
 (b) Prove that the binomial probability distribution is normalized to unity.
 (c) Find the count mean \bar{n} and the count variance σ_n^2 of the binomial probability distribution in terms of its two parameters, p and M .
 (d) Find an expression for the SNR in terms of \bar{n} and p . Evaluate it for the limiting cases $p \rightarrow 0$ and $p \rightarrow 1$. To what kinds of light do these two limits correspond?
- *11.3-2 **Noisiness of a Hypothetical Photon Source.** Consider a hypothetical light source that produces a photon stream with a photon-number distribution that is

discrete-uniform, given by

$$p(n) = \begin{cases} \frac{1}{2\bar{n} + 1}, & 0 \leq n \leq 2\bar{n} \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Verify that the distribution is normalized to unity and has mean \bar{n} . Calculate the photon-number variance σ_n^2 and the signal-to-noise ratio (SNR) and compare them to those of the Bose-Einstein and Poisson distributions of the same mean.

(b) In terms of SNR, would this source be quieter or noisier than an ideal single-mode laser when $\bar{n} < 2$? When $\bar{n} = 2$? When $\bar{n} > 2$?

(c) By what factor is the SNR for this light larger than that for single-mode thermal light?

[Useful formulas:

$$1 + 2 + 3 + \cdots + j = \frac{j(j + 1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + j^2 = \frac{j(j + 1)(2j + 1)}{6}.$$