Knuth-Bendix Completion

Equality Saturation SS25

Michael Schifferer

Example Proof

Axioms:
A1: $0+X = X$
A2: $X = X+0$
A3: $-X+X = 0$
A4: $(X+Y)+Z = X+(Y+Z)$

Goal: $--a = a$

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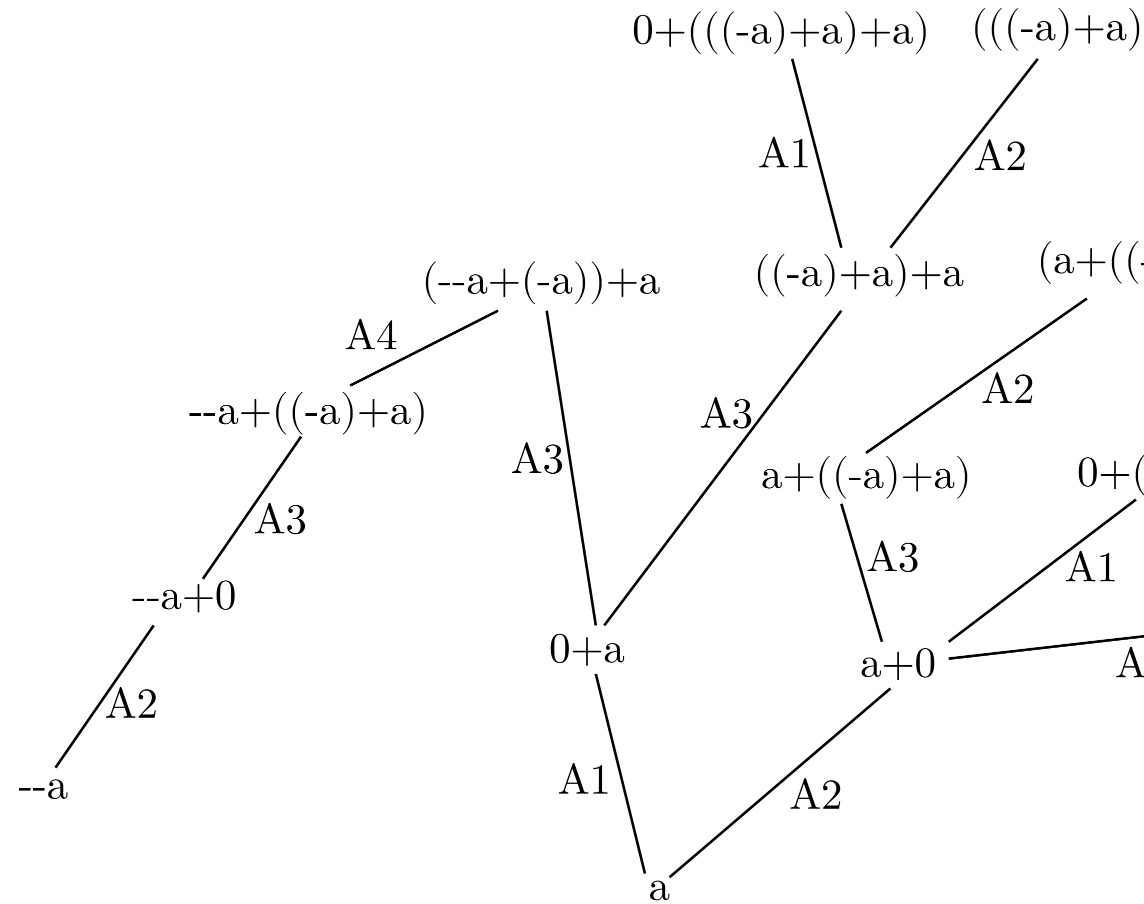

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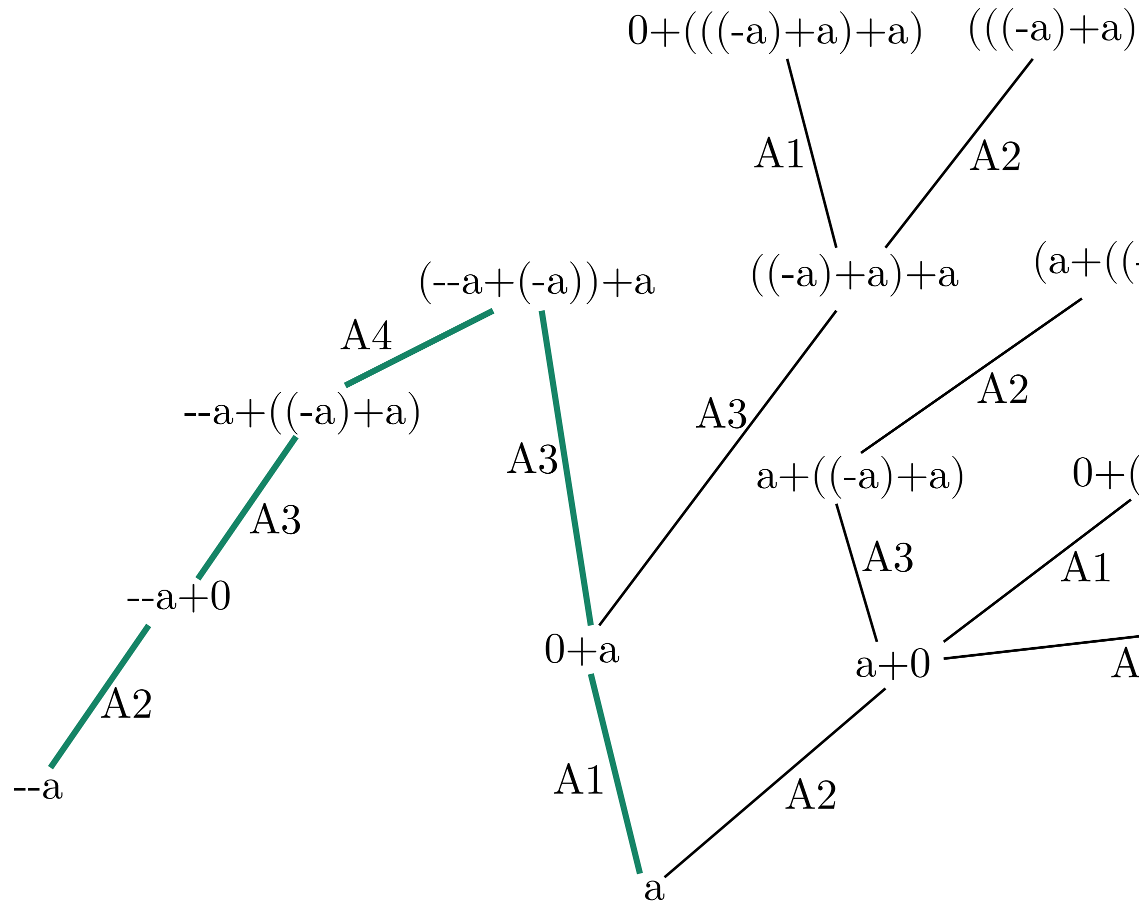

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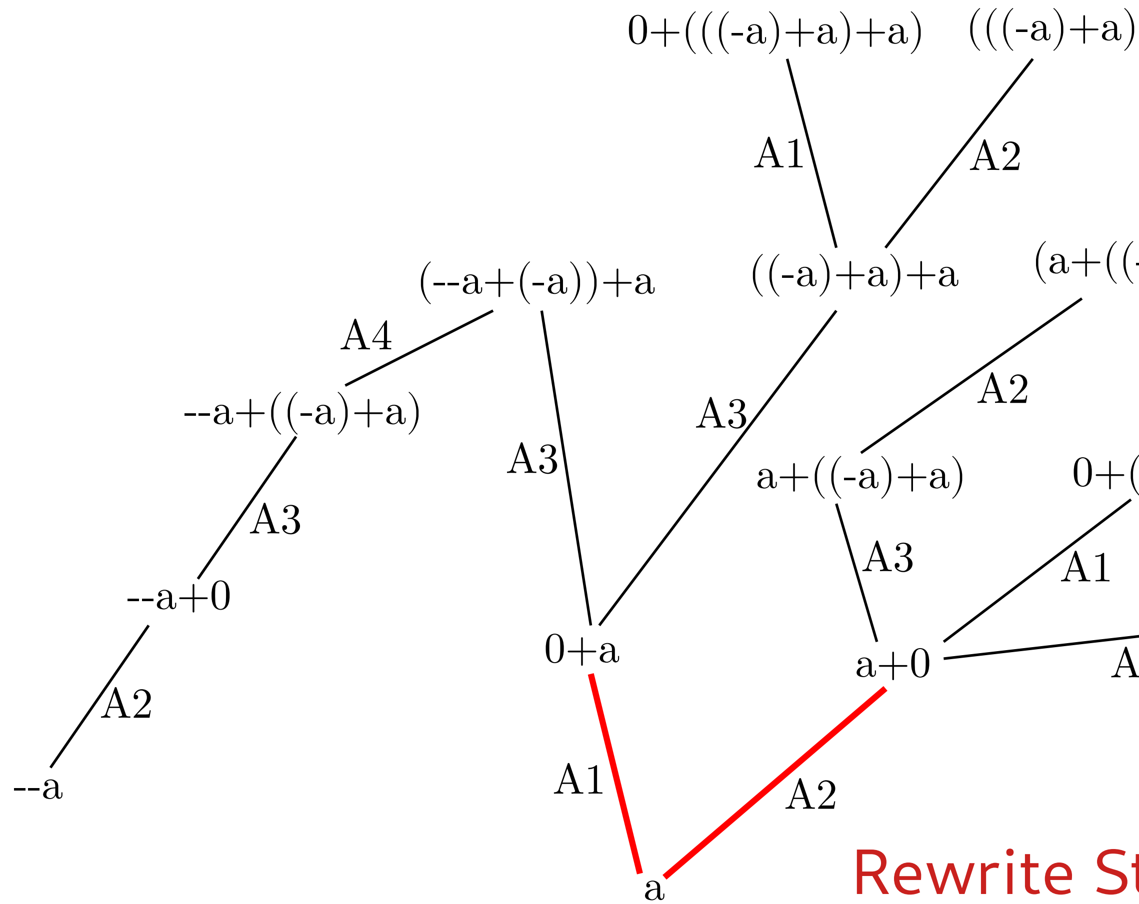

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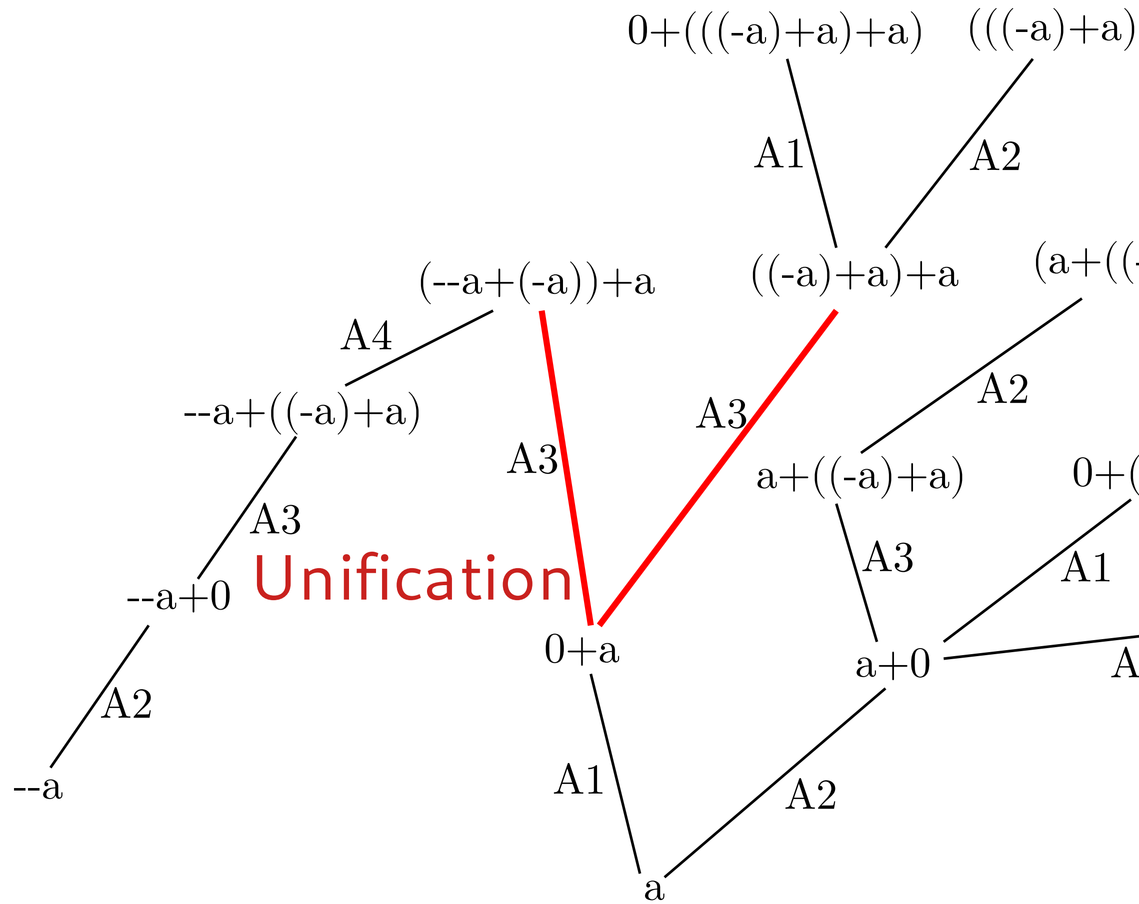

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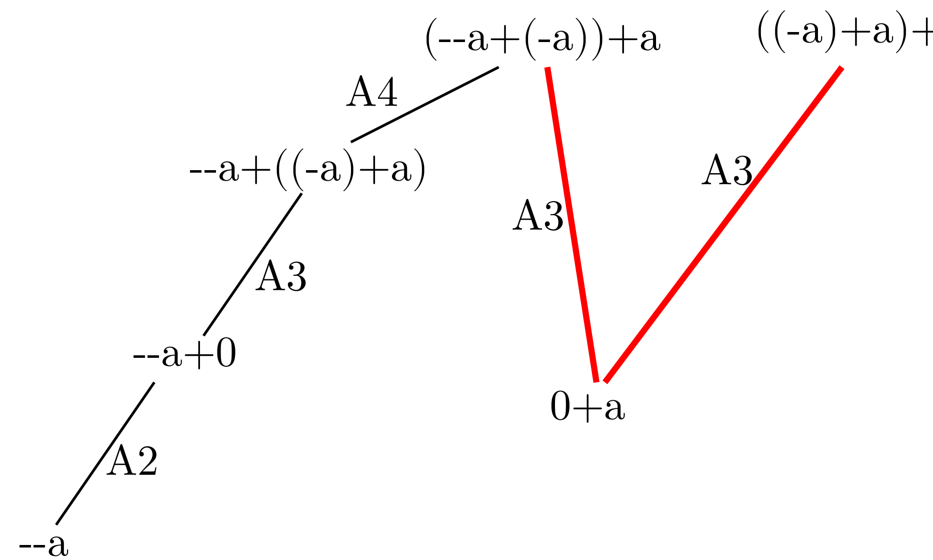
Unification:

1. Matching yields $0+a = (-X+X)+a$



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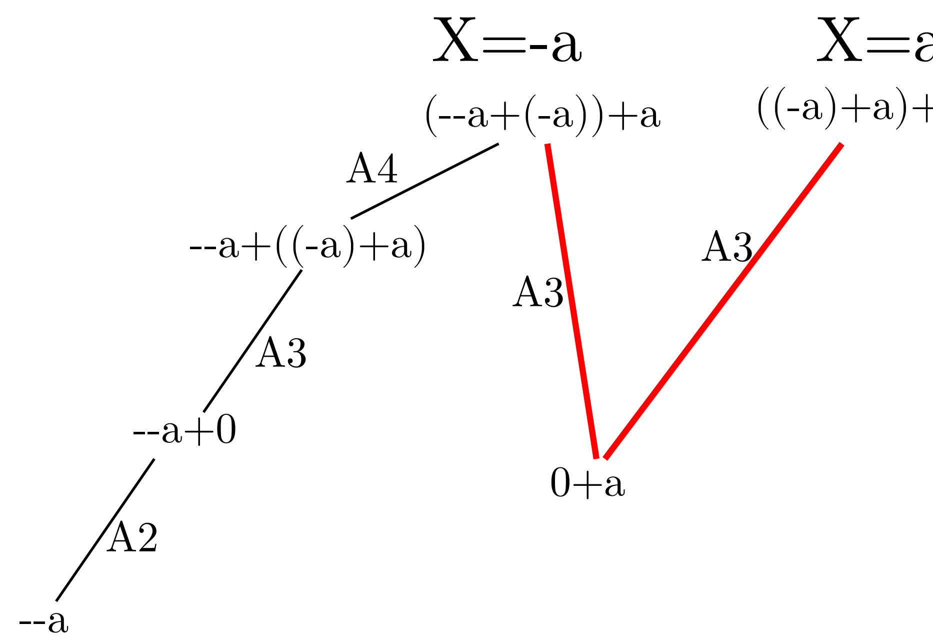
Unification:

1. Matching yields $0+a = (-X+X)+a$
2. Instantiation required



Solution: Confluent Rewrite System

Rewrite System

Axioms(undirected):
A1: 0+X = X
A2: X = X+0
A3: -X+X = 0
A4: (X+Y)+Z = X+(Y+Z)

Ordering

Rewrite Rules(d
R1: 0+X $\rightarrow$ X
R2: X+0 $\rightarrow$ X
R3: -X+X $\rightarrow$ 0
R4: (X+Y)+Z -

Knuth-Bendix Ordering

Given terms s, t, admissible weight function w, and ordering $>$ on functi
symbols, we have $\mathrm{s} >_{\mathrm{kbo}} \mathrm{t}$ iff

Read: Number of occurrences of x in s

$\#(\mathrm{x}, \mathrm{s}) \geq \#(\mathrm{x}, \mathrm{t})$ for all variables x

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KBO

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\begin{array}{l}
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Fixes Unification problem!

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1) $w(s) > w(t)$ or
2) $w(s) = w(t)$ and
   a) $t = x$, $s = f^n(x)$ e.g. -$0 >_{kbo} 0$, --$a >_{kbo} a$

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a) $t = x$, $s = f^n(x)$ or

b) $s = f(s_1,\ldots,s_n)$, $t = g(t_1,\ldots,t_m)$ and $f > g$ e.g. $2*2 >_{kbo} 2+2$

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   b) $s = f(s_1,\ldots,s_n)$, $t = g(t_1,\ldots,t_m)$ and $f > g$ or

   c) $s = f(s_1,\ldots,s_n)$, $t = f(t1,\ldots,t_n)$ and $(s_1,\ldots,s_n)\ (>_{kbo})_{lex}\ (t_1,\ldots,t_n)$

      e.g. $(X+Y)+Z >_{kbo} X+(Y+Z)$

Knuth-Bendix Ordering

Properties:

- Well-foundedness: No infinite descending chains
- Monotonicity: $\mathrm{s} > \mathrm{t} \Rightarrow \mathrm{f(s)} > \mathrm{f(t)}$ for all f
- Stability under substitution: $\mathrm{s} > \mathrm{t} \Rightarrow \mathrm{s}\sigma > \mathrm{t}\sigma$ for all σ
- Subterm property: $\mathrm{s} > \mathrm{s'}$ for all proper subterms s

Knuth-Bendix Ordering

Properties:

- Well-foundedness: No infinite descending chains
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Properties guarantee **Termination** of the rewrite system

Example Proof

Rewrite Rules:

R1: $0+X \to X$

R2: $X+0 \to X$

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R4: $(X+Y)+Z \to X+(Y+Z)$

Goal: $--a = a$

Proof:

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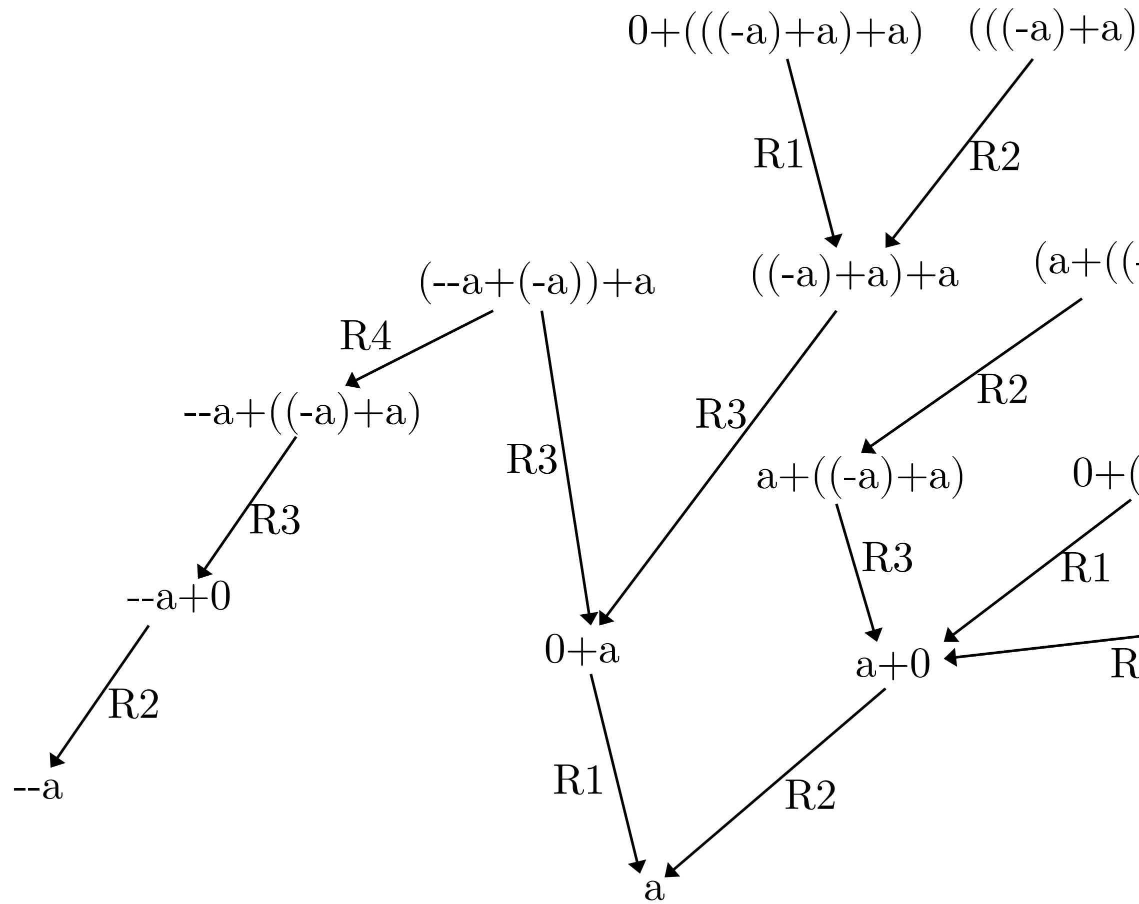
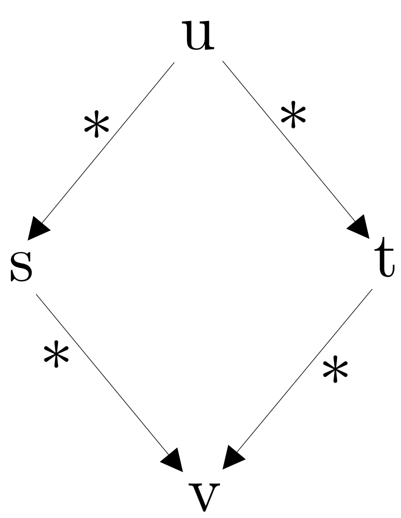


Solution: Confluent Rewrite System

Confluence

Formally: $\forall u,s,t.\ u \xrightarrow{*} s \wedge u \xrightarrow{*} t \Rightarrow \exists v.\ s \xrightarrow{*} v \wedge t \xrightarrow{*} v$



Fixes Strategy problem!

Confluence

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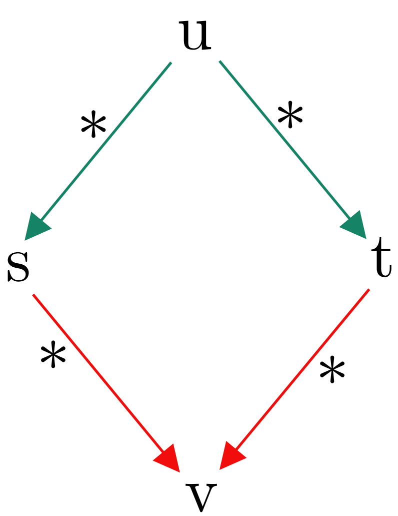

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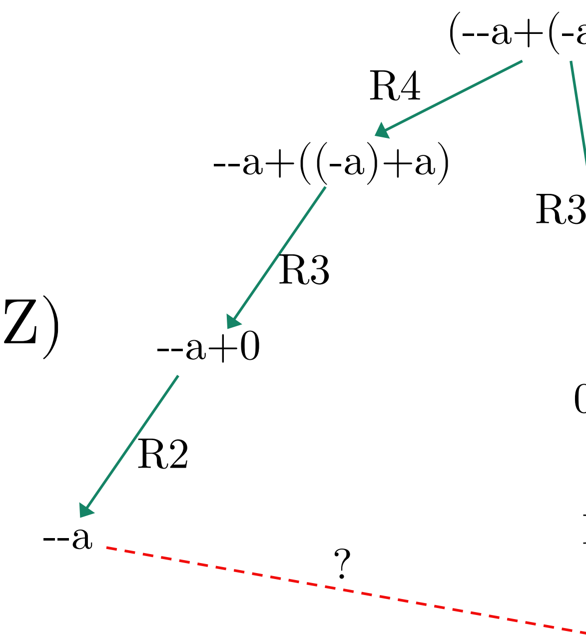

Knuth-Bendix Completion

(The data consists of: the axiom set, a set of equations initially containing the given axioms; and the rule set, an initially em
of rewrite rules.)

```
while the axiom set is not empty do
begin
   Select and remove an axiom from the axiom set;

   Normalize the selected axiom;

   if the normalized axiom is not of the form x = x then
   begin
      Order the axiom using the reduction ordering
      and introduce it as a new rule in the rule set;

      Superpose the new rule on all existing rules (including itself)
      and introduce each critical pair into the axiom set;
   end
end.
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Superposition, Normalization, Ordering

Superposing $\mathrm{R4}$ *on* $\mathrm{R3}$ yields:

$$\begin{tikzcd} & \text{(-X+X)+Z} \arrow[dl, "\mathrm{R4}"'] \arrow[dr, "\mathrm{R3}"] & \\ \text{-X+(X+Z)} & & \text{0+Z} \end{tikzcd}$$

Rewrite Rules:
R1: 0+X $\rightarrow$ X
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Superposition, Normalization, Ordering

Normalizing using existing rules yields:

$$\begin{tikzcd}
& \text{(-X+X)+Z} \arrow[dl, "R4"'] \arrow[dr, "R3"] & \\
\text{-X+(X+Z)} & & \text{0+Z} \arrow[d, "R1"] \\
& & \text{Z}
\end{tikzcd}$$

-X+(X+Z) $=$ Z as a **critical pair**

Rewrite Rules:
R1: 0+X $\rightarrow$ X
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Superposition, Normalization, Ordering

Normalizing using existing rules yields:

$$\begin{tikzcd}
& (-X+X)+Z \arrow[dl, "R4"'] \arrow[dr, "R3"] & \\
-X+(X+Z) \arrow[drr, dashed, "R5"'] & & 0+Z \arrow[d, "R1"] \\
& & Z
\end{tikzcd}$$

$$-X+(X+Z) = Z \xRightarrow{\mathit{KBO}} \text{R5: -X}$$

Rewrite Rules:
R1: $0+X \to X$
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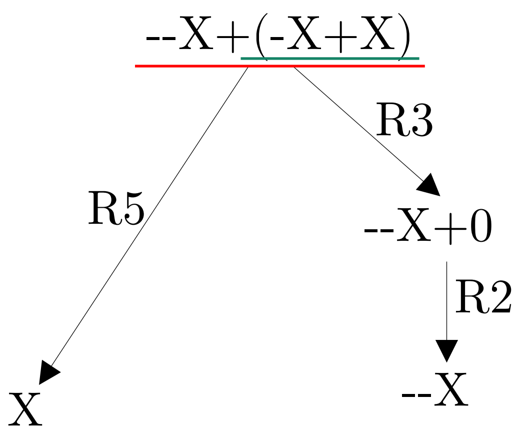

$\mathrm{X} = \text{--X}$ as a **critical pair**

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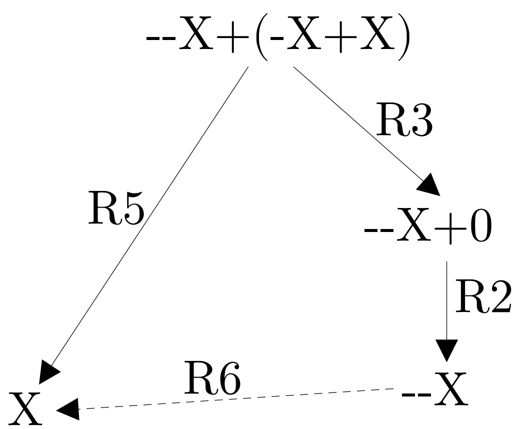
Superposition, Normalization, Ordering

Normalization and ordering yields:



$$X = \text{--}X \xrightarrow{\textit{KBO}} \text{R6: --}X \rightarrow$$

Rewrite Rules:
R1: 0+X → X
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R5: -X+(X+Z) → Z

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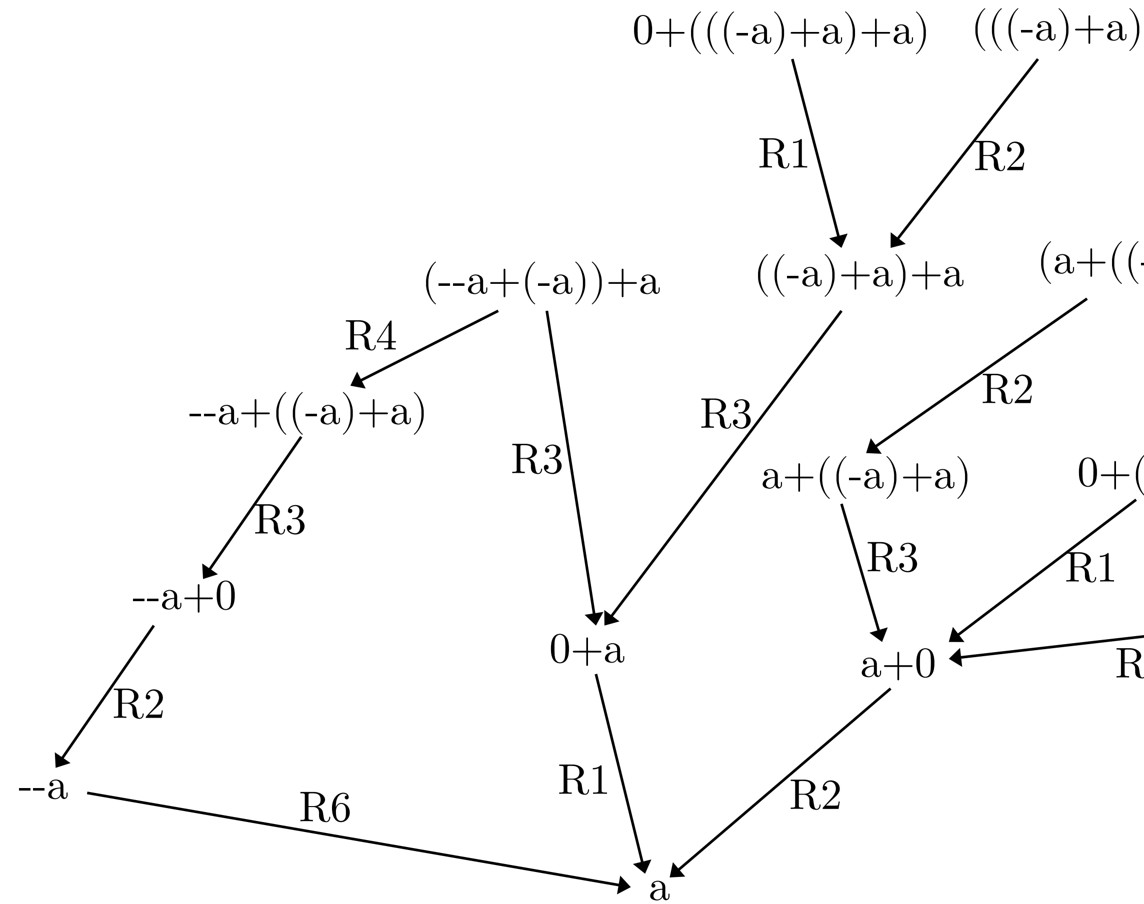
Proof:
$--a = a$ R6



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Proof:
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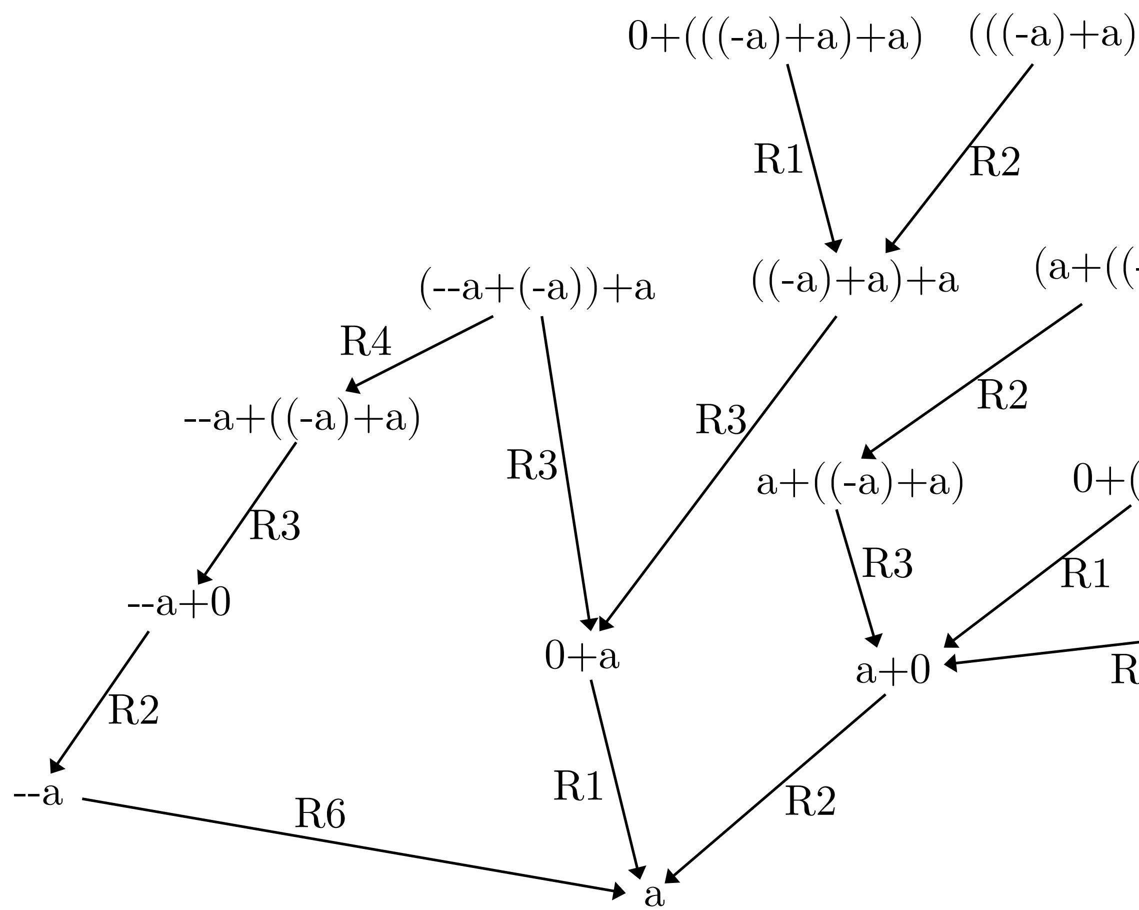
Still not confluent!



Knuth-Bendix Completion

The algorithm may

- Terminate, when confluence is reached
  - i.e. all critical pairs are trivial after normalization
- Fail, when encountering an unorientable axiom
  - e.g. $\mathrm{x} + \mathrm{y} = \mathrm{y} + \mathrm{x}$
- Diverge, if no finite confluent TRS exists

Limitations

- Failure
  - Unfailing Knuth-Bendix Completion
- Divergence
  - Becomes a semi-decision procedure
- Finding $>$ and w for KBO
- Computationally expensive as Number of critical pairs explodes
- Partial functions e.g. division by 0
  - Special encoding can help

Questions?

Admissible Weight Function

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature,
let $>$ be a strict partial ordering ("precedence") on Ω,
let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a weight function,
such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$;
$w(c) \geq w_0$ for all constants $c/0 \in \Omega$.

If $w(f) = 0$ for some $f/1 \in \Omega$, then $f \geq g$ for all $g \in \Omega$.

w can be extended to terms as follows:

$$w(t) = \sum_{x \in \mathrm{Var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t).$$

https://rg1-teaching.mpi-inf.mpg.de/eqlogic-ss03/v8.pdf

Term ordering. We alw
with all functions havi
ordered so that more
occuring functions are

https://smallbone.se/papers/twee.pdf